On Isolated Singularities of Fractional Semi-Linear Elliptic Equations*

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Abstract

In this paper, we study the local behavior of nonnegative solutions of fractional semi-linear equations $(-\Delta)^\sigma u = u^p$ with an isolated singularity, where $\sigma \in (0,1)$ and $\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. We first use blow up method and a Liouville type theorem to derive an upper bound. Then we establish a monotonicity formula and a sufficient condition for removable singularity to give a classification of the isolated singularities. When $\sigma = 1$, this classification result has been proved by Gidas and Spruck (Comm. Pure Appl. Math. 34: 525-598, 1981).

Mathematics Subject Classification (2010): 35B09; 35B40; 35J70; 35R11

1 Introduction and Main results

The purpose of this paper is to study the local behavior of nonnegative solutions of

$$(-\Delta)^\sigma u = u^p \qquad \text{in } B_1\backslash\{0\} \tag{1.1}$$

with an isolated singularity at the origin, where the punctured unit ball $B_1\backslash\{0\} \subset \mathbb{R}^n$ with $n \geq 2$, $\sigma \in (0,1)$ and $(-\Delta)^\sigma$ is the fractional Laplacian.

When $\sigma = 1$, the isolated singularity of nonnegative solutions for (1.1) has been very well understand, see Lions [26] for $1 < p < \frac{n}{n-2}$, Aviles [4] for $p = \frac{n}{n-2}$, Gidas-Spruck [19] for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, Caffarelli-Gidas-Spruck [7] for $\frac{n}{n-2} \leq p \leq \frac{n+2}{n-2}$, Korevaar-Mazzeo-Pacard-Schoen [25] for $p = \frac{n+2}{n-2}$, and Bidaut-Véron and Véron [5] for $p > \frac{n+2}{n-2}$.

The semi-linear equation (1.1) involving the fractional Laplacian has attracted a great deal of interest since they are of central importance in many fields, such as

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see [1–3, 8, 10, 12–15, 17, 20–24] and references therein. Recently, the existence of singular solutions of equation (1.1) with prescribed isolated singularities for the critical exponent $p = \frac{n+2\sigma}{n-2\sigma}$ were studied in [1, 2, 14, 15] and for the subcritical regime $\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$ were studied in [1, 3]. Solutions of (1.1) with an isolated singularity are the simplest cases of those singular solutions. In a recent paper [8], Caffarelli, Jin, Sire and Xiong study the local behavior of nonnegative solution of (1.1) with $p = \frac{n+2\sigma}{n-2\sigma}$. More precisely, let u be a nonnegative solution of (1.1) with $p = \frac{n+2\sigma}{n-2\sigma}$ and suppose that the origin is not a removable singularity. Then, near the origin

$$c_1|x|^{-\frac{n-2\sigma}{2}} \le u(x) \le c_2|x|^{-\frac{n-2\sigma}{2}}, \tag{1.2}$$

where c_1, c_2 are positive constants.

In this paper, we are interested in the local behavior of nonnegative solutions of (1.1) with $\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. For the classical case $\sigma = 1$, this has been proved in the pioneering paper [19] by Gidas and Spruck.

We study the equation (1.1) via the well known extension theorem for the fractional Laplacian $(-\Delta)^\sigma$ established by Caffarelli-Silvestre [9]. We use capital letters, such as $X = (x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, to denote points in $\mathbb{R}^{n+1}_+$. We also denote $\mathcal{B}_R$ as the ball in $\mathbb{R}^{n+1}$ with radius R and center at the origin, $\mathcal{B}_R^+$ as the upper half-ball $\mathcal{B}_R \cap \mathbb{R}^{n+1}_+$, and $\partial^0\mathcal{B}_R^+$ as the flat part of $\partial\mathcal{B}_R^+$ which is the ball B_R in $\mathbb{R}^n$. Then the problem (1.1) is equivalent to the following extension problem

$$\begin{cases} -\text{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial U}{\partial \nu^\sigma}(x,0) = U^p(x,0) & \text{on } \partial^0\mathcal{B}_1^+ \backslash \{0\}, \end{cases} \tag{1.3}$$

where $\frac{\partial U}{\partial \nu^\sigma}(x,0) := -\lim_{t\to 0^+} t^{1-2\sigma}\partial_t U(x,t)$. By [9], we only need to analyze the behavior of the traces

$$u(x) := U(x,0)$$

of the nonnegative solutions $U(x,t)$ of (1.3) near the origin, from which we can get the behavior of solutions of (1.1) near the origin.

We say that U is a nonnegative solution of (1.3) if U is in the weighted Sobolev space $W^{1,2}(t^{1-2\sigma}, \mathcal{B}_1^+ \backslash \overline{\mathcal{B}}_\epsilon^+)$ for every $\epsilon > 0$, $U \ge 0$, and it satisies (1.3) in the sense of distribution away from 0, i.e., for every nonnegative $\Phi \in C_c^\infty\left((\mathcal{B}_1^+ \cup \partial^0\mathcal{B}_1^+)\backslash\{0\}\right)$,

$$\int_{\mathcal{B}_1^+} t^{1-2\sigma}\nabla U \nabla\Phi = \int_{\partial^0\mathcal{B}_1^+} U^p\Phi. \tag{1.4}$$

See [23] for more details on this definition. Then it follows from the regularity result in [23] that U is locally Hölder continuous in $\overline{\mathcal{B}}_1^+ \backslash \{0\}$. We say that the origin 0 is a removable singularity of solution U of (1.3) if $U(x,0)$ can be extended as a continuous function near the origin, otherwise we say that the origin 0 is a non-removable singularity. Our main result is the following

Theorem 1.1. *Let U be a nonnegative solution of* (1.3)*. Assume*

$$\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}.$$

Then either the singularity near 0 is removable, or there exist two positive constants c_1 *and* c_2 *such that*

$$c_1|x|^{-\frac{2\sigma}{p-1}} \le u(x) \le c_2|x|^{-\frac{2\sigma}{p-1}}. \tag{1.5}$$

Remark 1.1. *We point out that, if* (1.5) *holds, then the Harnack inequality* (3.2) *implies that*

$$C_1|X|^{-\frac{2\sigma}{p-1}} \le U(X) \le C_2|X|^{-\frac{2\sigma}{p-1}}$$

holds as well, for some positive constants C_1 *and* C_2*.*

For the classical case $\sigma = 1$, Theorem 1.1 were proved in [19] by Gidas and Spruck. We may also see [7] for this classical case. The similar upper bound in (1.5) obtained in [19] for the classical case is very complicated and technical, here we use the blow up method and a Liouville type theorem to prove the upper bound in (1.5). To obtain the lower bound, there are some extra difficulties, one of which is that the Pohozaev identity is not available. More precisely, for the critical case $p = \frac{n+2\sigma}{n-2\sigma}$, the Pohozaev integral $P(U, R)$ is independent of R by the Pohozaev identity (see [8] for more detalis on $P(U, R)$). In [8], the authors make use of this property of the Pohozaev integral to prove the lower bound, however, this does not hold in the subcritical case. We will establish a useful monotonicity formula to overcome this difficulty. The others would be those extra techniques to get the estimates of U from those of its trace u.

The paper is organized as follow. In Section 2, we recall three propositions: a Liouville theorem, a Harnack inequality and a Sobolev inequality. Section 3 is devoted to the proof of Theorem 1.1. We first derive an upper bound and a special form of Harnack inequality. Then we establish a monotonicity formula and a sufficient condition of removable singularity to prove Theorem 1.1.

2 Preliminaries

In this section, we introduce some notations and some propositions which will be used in our arguments. We denote $\mathcal{B}_R$ as the ball in $\mathbb{R}^{n+1}$ with radius R and center 0, and B_R as the ball in $\mathbb{R}^n$ with radius R and center 0. We also denote $\mathcal{B}_R^+$ as the upper half-ball $\mathcal{B}_R \cap \mathbb{R}^{n+1}_+$, $\partial^+\mathcal{B}_R^+ = \partial\mathcal{B}_R^+ \cap \mathbb{R}^{n+1}_+$ as the positive part of $\partial\mathcal{B}_R^+$, and $\partial^0\mathcal{B}_R^+ = \partial\mathcal{B}_R^+ \backslash \partial^+\mathcal{B}_R^+$ as the flat part of $\partial\mathcal{B}_R^+$ which is the ball B_R in $\mathbb{R}^n$.

We say $U \in W^{1,2}_{loc}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+})$ if $U \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+)$ for all $R > 0$, and $U \in W^{1,2}_{loc}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+} \backslash \{0\})$ if $U \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+ \backslash \overline{\mathcal{B}}_\epsilon^+)$ for all $R > \epsilon > 0$.

We next recall three propositions, which will be used frequently in our paper. For convenience, we state them here. Their proofs can be found in [23]. The first one is a Liouville type theorem.

Proposition 2.1. *Let* $U \in W^{1,2}_{loc}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+})$ *be a nonnegative weak solution of*

$$\begin{cases} -div(t^{1-2\sigma}\nabla U) = 0 & in\ \mathbb{R}^{n+1}_+, \\ \frac{\partial U}{\partial \nu^\sigma}(x, 0) = U^p(x, 0) & on\ \mathbb{R}^n, \end{cases} \tag{2.1}$$

with

$$1 \le p < \frac{n+2\sigma}{n-2\sigma}.$$

Then

$$U(x,t) \equiv 0.$$

The second one is a Harnack inequality, see also [6].

Proposition 2.2. *Let $U \in W^{1,2}_{loc}(t^{1-2\sigma}, \mathcal{B}_1^+)$ be a nonnegative weak solution of*

$$\begin{cases} -div(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathcal{B}_1^+, \\ \frac{\partial U}{\partial \nu^\sigma}(x,0) = a(x)U(x,0) & \text{on } \partial^0\mathcal{B}_1^+, \end{cases} \tag{2.2}$$

If $a \in L^q(B_1)$ for some $q > \frac{n}{2\sigma}$, then we have

$$\sup_{\mathcal{B}_{1/2}^+} U \le C \inf_{\mathcal{B}_{1/2}^+} U, \tag{2.3}$$

where C depends only on n, σ and $\|a\|_{L^q(B_1)}$.

The last one is a Sobolev type inequality.

Proposition 2.3. *Let $D = \Omega \times (0, R) \subset \mathbb{R}^n \times \mathbb{R}_+$ with $R > 0$ and $\partial\Omega$ Lipschitz. Then there exists $C_{n,\sigma} > 0$ depending only on n and σ such that*

$$\|U(\cdot, 0)\|_{L^{2n/(n-2\sigma)}(\Omega)} \le C_{n,\sigma} \|\nabla U\|_{L^2(t^{1-2\sigma}, D)}$$

for all $U \in C_c^\infty(D \cup \partial^0 D)$.

3 Classification of Isolated Singularities

In this section, we investigate the local behavior of nonnegative solutions of (1.3) and classify their isolated singularities. We first prove an upper bound and a special form of Harnack inequality for nonnegative solutions with a possible isolated singularity. We remark that this result will be of basic importance in classifying the isolated singularities.

Proposition 3.1. *Let U be a nonnegative solution of (1.3), with $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Then,*

(1) there exists a positive constant c independent of U such that

$$u(x) \le c|x|^{-\frac{2\sigma}{p-1}} \qquad \text{in } B_{1/2}; \tag{3.1}$$

(2) (Harnack inequality) for all $0 < r < 1/8$, we have

$$\sup_{\mathcal{B}_{2r}^+ \setminus \overline{\mathcal{B}}_{r/2}^+} U \le C \inf_{\mathcal{B}_{2r}^+ \setminus \overline{\mathcal{B}}_{r/2}^+} U, \tag{3.2}$$

where C is a positive constant independent of r and U. In particular, for all $0 < r < 1/8$, we have

$$\sup_{\partial^+\mathcal{B}_r^+} U \le C \inf_{\partial^+\mathcal{B}_r^+} U, \tag{3.3}$$

where C is a positive constant independent of r and U.

Proof. Suppose by contradiction that there exists a sequence of points $\{x_k\} \subset B_{1/2}$ and a sequence of solutions $\{U_k\}$ of (1.3), such that

$$|x_k|^{\frac{2\sigma}{p-1}} u_k(x_k) \to +\infty \qquad \text{as } k \to \infty. \tag{3.4}$$

As in [8], we define

$$v_k(x) := \left(\frac{|x_k|}{2} - |x - x_k|\right)^{\frac{2\sigma}{p-1}} u_k(x), \qquad |x - x_k| \le \frac{|x_k|}{2}.$$

Take $\bar{x}_k$ satisfy $|\bar{x}_k - x_k| < \frac{|x_k|}{2}$ and

$$v_k(\bar{x}_k) = \max_{|x - x_k| \le \frac{|x_k|}{2}} v_k(x).$$

Let

$$2\mu_k := \frac{|x_k|}{2} - |\bar{x}_k - x_k|.$$

Then

$$0 < 2\mu_k \le \frac{|x_k|}{2} \quad \text{and} \quad \frac{|x_k|}{2} - |x - x_k| \ge \mu_k \quad \forall \ |x - \bar{x}_k| \le \mu_k.$$

By the definition of v_k, we have

$$(2\mu_k)^{\frac{2\sigma}{p-1}} u_k(\bar{x}_k) = v_k(\bar{x}_k) \ge v_k(x) \ge (\mu_k)^{\frac{2\sigma}{p-1}} u_k(x) \quad \forall \ |x - \bar{x}_k| \le \mu_k.$$

Hence, we obtain

$$2^{\frac{2\sigma}{p-1}} u_k(\bar{x}_k) \ge u_k(x) \quad \forall \ |x - \bar{x}_k| \le \mu_k. \tag{3.5}$$

Moreover, by (3.4), we also have

$$(2\mu_k)^{\frac{2\sigma}{p-1}} u_k(\bar{x}_k) = v_k(\bar{x}_k) \ge v_k(x_k) = \left(\frac{|x_k|}{2}\right)^{\frac{2\sigma}{p-1}} u_k(x_k) \to +\infty \quad \text{as } k \to \infty. \tag{3.6}$$

Now, we define

$$W_k(y,t) := \frac{1}{u_k(\bar{x}_k)} U_k\left(\bar{x}_k + \frac{y}{u_k(\bar{x}_k)^{\frac{p-1}{2\sigma}}}, \frac{t}{u_k(\bar{x}_k)^{\frac{p-1}{2\sigma}}}\right), \qquad (y,t) \in \Omega_k,$$

where

$$\Omega_k := \left\{(y,t) \in \mathbb{R}^{n+1}_+ \middle| \left(\bar{x}_k + \frac{y}{u_k(\bar{x}_k)^{\frac{p-1}{2\sigma}}}, \frac{t}{u_k(\bar{x}_k)^{\frac{p-1}{2\sigma}}}\right) \in \mathcal{B}_1^+ \backslash \{0\}\right\}.$$

Let $w_k(y) := W_k(y,0)$. Then W_k satisfies $w_k(0) = 1$ and

$$\begin{cases} -\mathrm{div}(t^{1-2\sigma} \nabla W_k) = 0 & \text{in } \Omega_k, \\ \frac{\partial W_k}{\partial \nu^\sigma}(x,0) = w_k^p & \text{on } \partial^0 \Omega_k. \end{cases} \tag{3.7}$$

Furthermore, by (3.5) and (3.6), we have

$$w_k(y) \le 2^{\frac{2\sigma}{p-1}} \qquad \text{in } B_{R_k}$$

with

$$R_k := \mu_k u(\bar{x}_k)^{\frac{p-1}{2\sigma}} \to +\infty \quad \text{as } k \to \infty.$$

By Proposition 2.2, for any $T > 0$, we have

$$0 \le W_k \le C(T) \quad \text{in } B_{R_k/2} \times [0, T),$$

where the constant $C(T)$ depends only on n, σ and T. By Corollary 2.10 and Theorem 2.15 in [23] there exists $\alpha > 0$ such that for every $R > 1$,

$$\|W_k\|_{W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+)} + \|W_k\|_{C^\alpha(\overline{\mathcal{B}}_R^+)} + \|w_k\|_{C^{2,\alpha}(\overline{B}_R)} \le C(R),$$

where $C(R)$ is independent of k. Therefore, there is a subsequence of $k \to \infty$, still denoted by itself, and a nonnegative function $W \in W^{1,2}_{loc}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+}) \cap C^\alpha_{loc}(\overline{\mathbb{R}^{n+1}_+})$ such that, as $k \to \infty$,

$$\begin{cases} W_k \to W & \text{weakly in} \quad W^{1,2}_{loc}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+}), \\ W_k \to W & \text{in} \quad C^{\alpha/2}_{loc}(\overline{\mathbb{R}^{n+1}_+}), \\ w_k \to w & \text{in} \quad C^2_{loc}(\mathbb{R}^n), \end{cases}$$

where $w(y) = W(y, 0)$. Moreover, W satisfies $w(0) = 1$ and

$$\begin{cases} -\mathrm{div}(t^{1-2\sigma} \nabla W) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \frac{\partial W}{\partial \nu^\sigma}(x, 0) = w^p & \text{on } \mathbb{R}^n. \end{cases} \tag{3.8}$$

This contradicts Proposition 2.1 and proves part (1) of the proposition.

Now we prove the Harnack inequality, which is actually a consequence of the upper bound (3.1). Let

$$V_r(X) = U(rX)$$

for each $r \in (0, \frac{1}{8}]$ and for $\frac{1}{4} \le |X| \le 4$. Obviously, V_r satisfies

$$\begin{cases} -\mathrm{div}(t^{1-2\sigma} \nabla V_r) = 0 & \text{in } \mathcal{B}_4 \backslash \overline{\mathcal{B}}_{1/4}, \\ \frac{\partial V_r}{\partial \nu^\sigma}(x, 0) = a_r(x) v_r(x) & \text{on } B_4 \backslash \overline{B}_{1/4}, \end{cases} \tag{3.9}$$

where $v_r(x) = V_r(x, 0)$ and $a_r(x) = r^{2\sigma} (u(rx))^{p-1}$. It follows (3.1) that

$$|a_r(x)| \le C \qquad \text{for all } 1/4 \le |x| \le 4,$$

where C is a positive constant independent of r and U. By the Harnack inequality in Proposition 2.2 and the standard Harnack inequality for uniformly elliptic equations, we have

$$\sup_{\frac{1}{2} \le |X| \le 2} V_r(X) \le C \inf_{\frac{1}{2} \le |X| \le 2} V_r(X),$$

where C is another positive constant independent of r and U. Hence, we get (3.2). □

In order to prove the lower bound in (1.5), we need to establish a monotonicity formula for the nonnegative solutions U of (1.3). More precisely, take a nonnegative solutions U of (1.3), let $0 < r < 1$ and define

$$\begin{aligned} E(r;U) :=& r^{2\frac{(p+1)\sigma}{p-1}-n}\left[r\int_{\partial^+\mathcal{B}_r^+} t^{1-2\sigma}|\frac{\partial U}{\partial \nu}|^2 + \frac{2\sigma}{p-1}\int_{\partial^+\mathcal{B}_r^+} t^{1-2\sigma}\frac{\partial U}{\partial \nu}U\right] \\ &+ \frac{1}{2}\frac{2\sigma}{p-1}\left(\frac{4\sigma}{p-1}-(n-2\sigma)\right) r^{2\frac{(p+1)\sigma}{p-1}-n-1}\int_{\partial^+\mathcal{B}_r^+} t^{1-2\sigma}U^2 \\ &- r^{2\frac{(p+1)\sigma}{p-1}-n+1}\left[\frac{1}{2}\int_{\partial^+\mathcal{B}_r^+} t^{1-2\sigma}|\nabla U|^2 - \frac{1}{p+1}\int_{\partial B_r} u^{p+1}\right]. \end{aligned}$$

Then, we have the following monotonicity formula.

Proposition 3.2. *Let U be a nonnegative solution of* (1.3) *with $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Then, $E(r;U)$ is non-decreasing in $r \in (0,1)$.*

Proof. Take standard polar coordinates in $\mathbb{R}^{n+1}_+$: $X = (x,t) = r\theta$, where $r = |X|$ and $\theta = \frac{X}{|X|}$. Let $\theta_1 = \frac{t}{|X|}$ denote the component of θ in the t direction and

$$\mathbb{S}^n_+ = \{X \in \mathbb{R}^{n+1}_+ : r = 1, \theta_1 > 0\}$$

denote the upper unit half-sphere.

We use the classical change of variable in Fowler [18],

$$V(s,\theta) = r^{\frac{2\sigma}{p-1}}U(r,\theta), \quad s = \ln r.$$

Direct calculations show that V satisfies

$$\begin{cases} V_{ss} - J_1 V_s - J_2 V + \theta_1^{2\sigma-1}\mathrm{div}_\theta(\theta_1^{1-2\sigma}\nabla_\theta V) = 0 & \text{in } (-\infty,0)\times\mathbb{S}^n_+, \\ -\lim_{\theta_1\to 0^+}\theta_1^{1-2\sigma}\partial_{\theta_1}V = V^p & \text{on } (-\infty,0)\times\partial\mathbb{S}^n_+, \end{cases} \tag{3.10}$$

where

$$J_1 = \frac{4\sigma}{p-1} - (N-2\sigma), \quad J_2 = \frac{2\sigma}{p-1}\left(n - 2\sigma - \frac{2\sigma}{p-1}\right).$$

Multiplying (3.10) by V_s and integrating, we have

$$\begin{aligned} &\int_{\mathbb{S}^n_+}\theta_1^{1-2\sigma}V_{ss}V_s - J_2\int_{\mathbb{S}^n_+}\theta_1^{1-2\sigma}VV_s - \int_{\mathbb{S}^n_+}\theta_1^{1-2\sigma}\nabla_\theta V\cdot\nabla_\theta V_s + \int_{\partial\mathbb{S}^n_+}V^pV_s \\ &= J_1\int_{\mathbb{S}^n_+}\theta_1^{1-2\sigma}(V_s)^2. \end{aligned} \tag{3.11}$$

For any $s \in (-\infty, 0)$, we define

$$\begin{aligned} \widetilde{E}(s) :=& \frac{1}{2}\int_{\mathbb{S}^n_+}\theta_1^{1-2\sigma}(V_s)^2 - \frac{J_2}{2}\int_{\mathbb{S}^n_+}\theta_1^{1-2\sigma}V^2 - \frac{1}{2}\int_{\mathbb{S}^n_+}\theta_1^{1-2\sigma}|\nabla_\theta V|^2 \\ &+ \frac{1}{p+1}\int_{\partial\mathbb{S}^n_+}V^{p+1}. \end{aligned}$$

Then, by (3.11), we get

$$\frac{d}{ds}\widetilde{E}(s) = J_1 \int_{\mathbb{S}^n_+} \theta_1^{1-2\sigma}(V_s)^2 \geq 0. \tag{3.12}$$

Here we have used the fact $J_1 > 0$ because $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Hence, $\widetilde{E}(s)$ is non-decreasing in $s \in (-\infty, 0)$.

Now, rescaling back, we have

$$\begin{aligned}
&\int_{\mathbb{S}^n_+} \theta_1^{1-2\sigma}(V_s)^2 \\
&= \int_{\mathbb{S}^n_+} \theta_1^{1-2\sigma}\left(\frac{2\sigma}{p-1} r^{\frac{2\sigma}{p-1}-1} U + r^{\frac{2\sigma}{p-1}} U_r\right)^2 r^2 \\
&= r^{2\frac{(p+1)\sigma}{p-1}-n} \int_{\partial^+\mathcal{B}_r^+} t^{1-2\sigma}\left(\frac{4\sigma^2}{(p-1)^2} r^{-1} U^2 + \frac{4\sigma}{p-1} U \frac{\partial U}{\partial \nu} + r|\frac{\partial U}{\partial \nu}|^2\right),
\end{aligned}$$

$$\int_{\mathbb{S}^n_+} \theta_1^{1-2\sigma}|\nabla_\theta V|^2 = r^{2\frac{(p+1)\sigma}{p-1}-n+1} \int_{\partial^+\mathcal{B}_r^+} t^{1-2\sigma}\left(|\nabla U|^2 - |\frac{\partial U}{\partial \nu}|^2\right),$$

$$\int_{\mathbb{S}^n_+} \theta_1^{1-2\sigma} V^2 = r^{2\frac{(p+1)\sigma}{p-1}-n-1} \int_{\partial^+\mathcal{B}_r^+} t^{1-2\sigma} U^2,$$

$$\int_{\partial\mathbb{S}^n_+} V^{p+1} = r^{2\frac{(p+1)\sigma}{p-1}-n+1} \int_{\partial B_r} u^{p+1}.$$

Substituting these into (3.12) and noting that $s = \ln r$ is non-decreasing in r, we easily obtain that $E(r; U)$ is also non-decreasing in $r \in (0, 1)$. □

By the monotonicity of $E(r; U)$ we prove the following proposition, which will play an essential role in deriving the lower bound in (1.5).

Proposition 3.3. *Let U be a nonnegative solution of* (1.3) *with $\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. If*

$$\liminf_{|x|\to 0} |x|^{\frac{2\sigma}{p-1}} u(x) = 0,$$

then

$$\lim_{|x|\to 0} |x|^{\frac{2\sigma}{p-1}} u(x) = 0.$$

Proof. Suppose by contradiction that

$$\liminf_{|x|\to 0} |x|^{\frac{2\sigma}{p-1}} u(x) = 0 \quad \text{and} \quad \limsup_{|x|\to 0} |x|^{\frac{2\sigma}{p-1}} u(x) = C > 0.$$

Therefore, there exist two sequences of points $\{x_i\}$ and $\{y_i\}$ satisfying

$$x_i \to 0, \quad y_i \to 0 \quad \text{as } i \to \infty,$$

such that

$$|x_i|^{\frac{2\sigma}{p-1}}u(x_i)\to 0 \quad\text{and}\quad |y_i|^{\frac{2\sigma}{p-1}}u(y_i)\to C>0 \quad\text{as } i\to\infty.$$

Let $g(r)=r^{\frac{2\sigma}{p-1}}\bar u(r)$, where $\bar u(r)=\frac{1}{|\partial B_r|}\int_{\partial B_r}u$ denotes the spherical average of u over ∂B_r. Then, by the Harnack inequality (3.3), we have

$$\liminf_{r\to 0} g(r)=0 \quad\text{and}\quad \limsup_{r\to 0} g(r)=C>0.$$

Hence, there exists a sequence of local minimum points r_i of $g(r)$ with

$$\lim_{i\to\infty} r_i=0 \quad\text{and}\quad \lim_{i\to\infty} g(r_i)=0.$$

Let

$$V_i(X)=\frac{U(r_iX)}{U(r_ie_1)},$$

where $e_1=(1,0,\cdots,0)$. It follows from the Harnack inequality (3.2) that V_i is locally uniformly bounded away from the origin and satisfies

$$\begin{cases}-\mathrm{div}(t^{1-2\sigma}\nabla V_i)=0 & \text{in } \mathbb{R}^{n+1}_+,\\ \frac{\partial V_i}{\partial\nu^\sigma}(x,0)=\left(r_i^{\frac{2\sigma}{p-1}}U(r_ie_1)\right)^{p-1}V_i^p(x,0) & \text{on } \mathbb{R}^n\backslash\{0\}.\end{cases}\tag{3.13}$$

Note that by the Harnack inequality (3.3), $r_i^{\frac{2\sigma}{p-1}}U(r_ie_1)\to 0$ as $i\to\infty$. Then by Corollary 2.10 and Theorem 2.15 in [23] that there exists $\alpha>0$ such that for every $R>1>r>0$

$$\|V_i\|_{W^{1,2}(t^{1-2\sigma},\mathcal{B}_R^+\backslash\overline{\mathcal{B}}_r^+)}+\|V_i\|_{C^\alpha(\mathcal{B}_R^+\backslash\overline{\mathcal{B}}_r^+)}+\|v_i\|_{C^{2,\alpha}(B_R\backslash B_r)}\le C(R,r),$$

where $v_i(x)=V_i(x,0)$ and $C(R,r)$ is independent of i. Then after passing to a subsequence, $\{V_i\}$ converges to a nonnegative function $V\in W^{1,2}_{loc}(t^{1-2\sigma},\overline{\mathbb{R}^{n+1}_+}\backslash\{0\})\cap C^\alpha_{loc}(\overline{\mathbb{R}^{n+1}_+}\backslash\{0\})$ satisfying

$$\begin{cases}-\mathrm{div}(t^{1-2\sigma}\nabla V)=0 & \text{in } \mathbb{R}^{n+1}_+,\\ \frac{\partial V}{\partial\nu^\sigma}(x,0)=0 & \text{on } \mathbb{R}^n\backslash\{0\}.\end{cases}\tag{3.14}$$

By a Bôcher type theorem in [23], we have

$$V(X)=\frac{a}{|X|^{n-2\sigma}}+b,$$

where a,b are nonnegative constants. Recall that r_i are local minimum of $g(r)$ for every i and note that

$$r^{\frac{2\sigma}{p-1}}\bar v_i(r)=r^{\frac{2\sigma}{p-1}}\frac{1}{|\partial B_r|}\int_{\partial B_r}v_i=\frac{1}{U(r_ie_1)}r^{\frac{2\sigma}{p-1}}\bar u(r_ir)=\frac{1}{U(r_ie_1)r_i^{\frac{2\sigma}{p-1}}}g(r_ir).$$

Hence, for every i, we have

$$\frac{d}{dr}\left[r^{\frac{2\sigma}{p-1}}\bar{v}_i(r)\right]\Bigg|_{r=1} = \frac{r_i}{U(r_ie_1)r_i^{\frac{2\sigma}{p-1}}}g'(r_i) = 0. \tag{3.15}$$

Let $v(x) = V(x,0)$. Then we know that $v_i(x) \to v(x)$ in $C^2_{loc}(\mathbb{R}^n\backslash\{0\})$. By (3.15), we obtain

$$\frac{d}{dr}\left[r^{\frac{2\sigma}{p-1}}\bar{v}(r)\right]\Bigg|_{r=1} = 0,$$

which implies that

$$a\left(\frac{2\sigma}{p-1} - (n-2\sigma)\right) + \frac{2\sigma b}{p-1} = 0. \tag{3.16}$$

On the other hand, by $V(e_1) = 1$, we have

$$a + b = 1. \tag{3.17}$$

Combine (3.16) with (3.17), we get

$$a = \frac{2\sigma}{(p-1)(n-2\sigma)} \quad \text{and} \quad b = 1 - \frac{2\sigma}{(p-1)(n-2\sigma)}.$$

Since $\frac{n}{n-2\sigma} < p$, we have $0 < a, b < 1$. Now we compute $E(r;U)$.

It follows from Proposition 2.19 in [23] that $|\nabla_x V_i|$ and $|t^{1-2\sigma}\partial_t V_i|$ are locally uniformly bounded in $C^\beta_{loc}(\overline{\mathbb{R}^{n+1}_+}\backslash\{0\})$ for some $\beta > 0$. Hence, there exists a constant $C > 0$ such that

$$|\nabla_x U(X)| \le Cr_i^{-1}U(r_ie_1) = o(1)r_i^{-\frac{2\sigma}{p-1}-1} \qquad \text{for all } |X| = r_i$$

and

$$|t^{1-2\sigma}\partial_t U(X)| \le Cr_i^{-2\sigma}U(r_ie_1) = o(1)r_i^{-\frac{2\sigma}{p-1}-2\sigma} \qquad \text{for all } |X| = r_i.$$

Thus, by a direct computation, we can get

$$\lim_{i\to\infty} E(r_i;U) = 0.$$

By the monotonicity of $E(r;U)$, we obtain

$$E(r;U) \ge 0 \quad \text{for all } r \in (0,1).$$

On the other hand, by the scaling invariance of $E(r;U)$, for every i, we have

$$0 \le E(r_i;U) = E\left(1; r_i^{\frac{2\sigma}{p-1}}U(r_iX)\right) = E\left(1; r_i^{\frac{2\sigma}{p-1}}U(r_ie_1)V_i\right).$$

Hence, for every i, we have

$$
\begin{aligned}
0 \leq & \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} |\frac{\partial V_i}{\partial \nu}|^2 + \frac{2\sigma}{p-1} \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} \frac{\partial V_i}{\partial \nu} V_i \\
& + \frac{1}{2} \frac{2\sigma}{p-1} \left(\frac{4\sigma}{p-1} - (n-2\sigma) \right) \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} V_i^2 \\
& - \frac{1}{2} \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} |\nabla V_i|^2 + \frac{1}{p+1} \int_{\partial B_1} \left(r_i^{\frac{2\sigma}{p-1}} U(r_i e_1) \right)^{p-1} V_i^{p+1}.
\end{aligned}
$$

Letting $i \to \infty$, we obtain

$$
\begin{aligned}
0 \leq & \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} |\frac{\partial V}{\partial \nu}|^2 + \frac{2\sigma}{p-1} \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} \frac{\partial V}{\partial \nu} V \\
& + \frac{1}{2} \frac{2\sigma}{p-1} \left(\frac{4\sigma}{p-1} - (n-2\sigma) \right) \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} V^2 - \frac{1}{2} \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} |\nabla V|^2 \\
= & \, a^2 (n-2\sigma)^2 \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} - a(n-2\sigma) \frac{2\sigma}{p-1} \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} \\
& + \frac{1}{2} \frac{2\sigma}{p-1} \left(\frac{4\sigma}{p-1} - (n-2\sigma) \right) \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} - \frac{1}{2} a^2 (n-2\sigma)^2 \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} \\
= & \, \frac{\sigma}{p-1} \left(\frac{2\sigma}{p-1} - (n-2\sigma) \right) \int_{\partial^+\mathcal{B}_1^+} t^{1-2\sigma} < 0.
\end{aligned}
$$

Note that in the last inequality we have used the fact $\frac{2\sigma}{p-1} - (n-2\sigma) < 0$ because $\frac{n}{n-2\sigma} < p$. Obviously, we get a contradiction. □

To characterize the "order" of an isolated singularity we establish the following sufficient condition for removability of isolated singularities. For its proof, we adapt the arguments from [19], but there are extra difficulties. Such as, we need extra efforts to derive the estimates of U from its trace u.

Proposition 3.4. *Let U be a nonnegative solution of* (1.3) *with $\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. If*

$$
\int_{\epsilon \leq |x| \leq 1} u^{\frac{(p-1)n}{2\sigma}} \leq c < +\infty \tag{3.18}
$$

with c independent of ϵ, then the singularity at the origin is removable, i.e., u(x) can be extended to a continuous solution in the entire ball B_1.

Proof. Let

$$
p_0 = \frac{n-2\sigma}{2\sigma} \left(p - \frac{n}{n-2\sigma} \right) \quad \text{and} \quad q_0 = \frac{1}{2}(p_0 + 1) = \frac{1}{2} \frac{n-2\sigma}{2\sigma} (p-1).
$$

Following Serrin [28], we define, for $q \geq q_0$, $l > 0$

$$
F(u) = \begin{cases} u^q & \text{for } 0 < u \leq l, \\ \frac{1}{q_0} \left[q l^{q-q_0} u^{q_0} + (q_0 - q) l^q \right] & \text{for } l \leq u, \end{cases}
$$

and

$$G(u) = F(u)F'(u) - q.$$

Clearly, F is a C^1 function of u and G is a piecewise smooth function of u with a corner at $u = l$. Moreover, since $p_0 > 0$ (implied by $\frac{n}{n-2\sigma} < p$), F, G satisfy

$$F \le \frac{q}{q_0} l^{q-q_0} u^{q_0}, \qquad uF' \le qF, \tag{3.19}$$

$$|G| \le FF', \tag{3.20}$$

$$G' \ge \begin{cases} \frac{1}{q} p F'^2 & \text{for } 0 < u < l, \\ \frac{1}{q_0} p_0 F'^2 & \text{for } l \le u. \end{cases} \tag{3.21}$$

For any $0 < R < 1$, let η and $\bar{\eta}$ be nonnegative C^∞ function with $0 \le \eta, \bar{\eta} \le 1$ in $\mathcal{B}_R = \{X \in \mathbb{R}^{n+1} : |X| < R\}$, η having compact support in $\mathcal{B}_R$, and $\bar{\eta}$ vanishing in some neighborhood of the origin. Take $(\eta\bar{\eta})^2 G(U)$ as a test function into (1.4), we have

$$\begin{aligned} &\int_{\mathcal{B}_R^+} t^{1-2\sigma} (\eta\bar{\eta})^2 G'(U) |\nabla U|^2 + 2 \int_{\mathcal{B}_R^+} t^{1-2\sigma} \eta\bar{\eta} G(U) \nabla U \cdot \nabla(\eta\bar{\eta}) \\ &= \int_{B_R} (\eta\bar{\eta})^2 G(u) u^p. \end{aligned} \tag{3.22}$$

Using (3.19) – (3.21) and simplifying we obtain from (3.22)

$$\int_{\mathcal{B}_R^+} t^{1-2\sigma} (\eta\bar{\eta})^2 |\nabla(F(U))|^2 \le C(q) \left\{ \int_{\mathcal{B}_R^+} t^{1-2\sigma} |\nabla(\eta\bar{\eta})|^2 F^2 + \int_{B_R} (\eta\bar{\eta})^2 u^{p-1} F^2 \right\}. \tag{3.23}$$

By Hölder and Proposition 2.3, we have

$$\begin{aligned} \int_{B_R} (\eta\bar{\eta})^2 u^{p-1} F^2 &\le \left(\int_{B_R} (u^{p-1})^{\frac{n}{2\sigma}} \right)^{\frac{2\sigma}{n}} \left(\int_{B_R} (\eta\bar{\eta} F)^{\frac{2n}{n-2\sigma}} \right)^{\frac{n-2\sigma}{n}} \\ &\le C_{n,\sigma}^2 \left(\int_{B_R} (u^{p-1})^{\frac{n}{2\sigma}} \right)^{\frac{2\sigma}{n}} \left(\int_{\mathcal{B}_R^+} t^{1-2\sigma} |\nabla(\eta\bar{\eta} F)|^2 \right) \\ &\le C_{n,\sigma}^2 \|u^{p-1}\|_{L^{\frac{n}{2\sigma}}(B_R)} \left\{ \int_{\mathcal{B}_R^+} t^{1-2\sigma} (\eta\bar{\eta})^2 |\nabla F|^2 \right. \\ &\qquad \left. + \int_{\mathcal{B}_R^+} t^{1-2\sigma} |\nabla(\eta\bar{\eta})|^2 F^2 \right\}. \end{aligned}$$

By the assumption (3.18), we can choose R small enough (depending on q, n and σ) such that

$$\|u^{p-1}\|_{L^{\frac{n}{2\sigma}}(B_R)} \le \frac{1}{2} \frac{1}{C(q) C_{n,\sigma}^2},$$

Hence, from (3.23), we obtain

$$\int_{\mathcal{B}_R^+} t^{1-2\sigma} (\eta\bar{\eta})^2 |\nabla(F(U))|^2 \le C(q) \int_{\mathcal{B}_R^+} t^{1-2\sigma} |\nabla(\eta\bar{\eta})|^2 F^2 \tag{3.24}$$

with a new constant $C(q)$. Therefore, we have from (3.24)

$$\int_{\mathcal{B}_R^+} t^{1-2\sigma}(\eta\bar{\eta})^2|\nabla(F(U))|^2 \le C(q)\Big\{\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\eta|^2F^2 + \int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\bar{\eta}|^2F^2\Big\} \tag{3.25}$$

and

$$\Big(\int_{B_R} (\eta\bar{\eta}F)^{\frac{2n}{n-2\sigma}}\Big)^{\frac{n-2\sigma}{n}} \le C(q)\Big\{\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\eta|^2F^2 + \int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\bar{\eta}|^2F^2\Big\}. \tag{3.26}$$

For any $\epsilon > 0$ small enough, we choose $\bar{\eta}_\epsilon$ satisfy

$$\bar{\eta}_\epsilon(X) = \begin{cases} 0 & \text{for } |X| \le \epsilon, \\ 1 & \text{for } 2\epsilon \le |X| < R, \end{cases} \tag{3.27}$$

and $|\nabla\bar{\eta}_\epsilon(X)| \le \frac{c}{\epsilon}$ for all $X \in \mathcal{B}_R$. By Hölder inequality

$$\begin{aligned} \int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\bar{\eta}_\epsilon|^2F^2 &\le \Big(\int_{\mathcal{B}_{2\epsilon}^+} t^{1-2\sigma}|\nabla\bar{\eta}_\epsilon|^{n+2-2\sigma}\Big)^{\frac{2}{n+2-2\sigma}} \Big(\int_{\mathcal{B}_{2\epsilon}^+} t^{1-2\sigma}F^{2\frac{n+2-2\sigma}{n-2\sigma}}\Big)^{\frac{n-2\sigma}{n+2-2\sigma}} \\ &\le \frac{C}{\epsilon^2}\Big(\int_{\mathcal{B}_{2\epsilon}^+} t^{1-2\sigma}\Big)^{\frac{2}{n+2-2\sigma}} \Big(\int_{\mathcal{B}_{2\epsilon}^+} t^{1-2\sigma}F^{2\frac{n+2-2\sigma}{n-2\sigma}}\Big)^{\frac{n-2\sigma}{n+2-2\sigma}} \\ &\le C\Big(\int_{\mathcal{B}_{2\epsilon}^+} t^{1-2\sigma}F^{2\frac{n+2-2\sigma}{n-2\sigma}}\Big)^{\frac{n-2\sigma}{n+2-2\sigma}}, \end{aligned} \tag{3.28}$$

where C is a positive constant independent of ϵ. Since, by (3.19), the Harnack inequality (3.3) and (3.1), we have

$$\begin{aligned} \int_{\mathcal{B}_{2\epsilon}^+} t^{1-2\sigma}F^{2\frac{n+2-2\sigma}{n-2\sigma}} &\le C(l,q)\int_{\mathcal{B}_{2\epsilon}^+} t^{1-2\sigma}U^{2q_0\frac{n+2-2\sigma}{n-2\sigma}} \\ &\le C(l,q)\int_0^{2\epsilon}\Big(\sup_{\partial^+\mathcal{B}_s^+} U\Big)^{2q_0\frac{n+2-2\sigma}{n-2\sigma}} s^{n+1-2\sigma}ds \\ &\le C(l,q)\int_0^{2\epsilon}\Big(\inf_{\partial^+\mathcal{B}_s^+} U\Big)^{2q_0\frac{n}{n-2\sigma}} s^{-2q_0\frac{4\sigma(1-\sigma)}{(p-1)(n-2\sigma)}} s^{n+1-2\sigma}ds \\ &\le C(l,q)\int_0^{2\epsilon}\Big(\inf_{\partial B_s} u\Big)^{2q_0\frac{n}{n-2\sigma}} s^{n-1}ds \\ &\le C(l,q)\int_0^{2\epsilon}\Big(\frac{1}{|\partial B_s|}\int_{\partial B_s} u^{2q_0\frac{n}{n-2\sigma}}\Big) s^{n-1}ds \\ &\le C(l,q)\int_{B_{2\epsilon}} u^{2q_0\frac{n}{n-2\sigma}} = C(l,q)\int_{B_{2\epsilon}} u^{\frac{(p-1)n}{2\sigma}}. \end{aligned} \tag{3.29}$$

Here we have used the fact $2q_0\frac{n}{n-2\sigma}=\frac{(p-1)n}{2\sigma}>1$ because $p-1>\frac{2\sigma}{n-2\sigma}$. Now, we have from (3.18), (3.28) and (3.29)

$$\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\bar{\eta}_\epsilon|^2F^2\to 0 \quad \text{as } \epsilon\to 0.$$

This together with (3.25) and (3.26), we obtain

$$\int_{\mathcal{B}_R^+} t^{1-2\sigma}\eta^2|\nabla(F(U))|^2\le C(q)\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\eta|^2F^2 \tag{3.30}$$

and

$$\left(\int_{B_R}(\eta F)^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{n}}\le C(q)\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\eta|^2F^2. \tag{3.31}$$

Let $l\to+\infty$, since $F(u)\to u^q$, we have

$$\int_{\mathcal{B}_R^+} t^{1-2\sigma}\eta^2|\nabla U^q|^2\le C(q)\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\eta|^2U^{2q} \tag{3.32}$$

and

$$\left(\int_{B_R}(\eta u^q)^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{n}}\le C(q)\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\eta|^2U^{2q}. \tag{3.33}$$

By a similar estimate as in (3.29), we can obtain

$$\begin{aligned}\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla\eta|^2U^{2q} &\le C(q)\int_0^R\left(\inf_{\partial^+\mathcal{B}_s^+}U^{2q-2q_0\frac{2(1-\sigma)}{n-2\sigma}}\right)s^{n-1}ds\\ &\le C(q)\left(\int_{B_R}u^{2q}\right)^{1-\frac{q_0}{q}\frac{2(1-\sigma)}{n-2\sigma}}\end{aligned} \tag{3.34}$$

Inequality (3.32), (3.33) and (3.34) can be iterated a *finite* number of times to show that

$$U\in W^{1,2}(t^{1-2\sigma},\mathcal{B}_R^+) \quad \text{and} \quad u\in L^q(B_R) \quad \text{for all } q>0.$$

Furthermore, U satisfies

$$\begin{cases}-\mathrm{div}(t^{1-2\sigma}\nabla U)=0 & \text{in } \mathcal{B}_R^+,\\ \frac{\partial U}{\partial\nu^\sigma}(x,0)=U^p(x,0) & \text{on } \partial^0\mathcal{B}_R^+.\end{cases}$$

Indeed, for $\epsilon>0$ small, let $\bar{\eta}_\epsilon$ be a smooth cut-off function as in (3.27). Let $\psi\in C_c^\infty(\mathcal{B}_R^+\cup\partial^0\mathcal{B}_R^+)$. It follows from (1.4) that

$$\int_{\mathcal{B}_R^+} t^{1-2\sigma}\nabla U\nabla(\psi\bar{\eta}_\epsilon)=\int_{\partial^0\mathcal{B}_R^+}U^p\psi\bar{\eta}_\epsilon. \tag{3.35}$$

Since

$$\left|\int_{\mathcal{B}_R^+} t^{1-2\sigma}\nabla U\nabla\bar{\eta}_\epsilon\psi\right|\le C\epsilon^{\frac{n-2\sigma}{2}}\int_{\mathcal{B}_R^+} t^{1-2\sigma}|\nabla U|^2\to 0 \quad \text{as } \epsilon\to 0,$$

by the dominated convergence theorem, let $\epsilon \to 0$ in (3.35), we obtain

$$\int_{\mathcal{B}_R^+} t^{1-2\sigma} \nabla U \nabla \psi = \int_{\partial^0 \mathcal{B}_R^+} U^p \psi.$$

Since $u \in L^q(B_R)$ for some $q > \frac{n}{2\sigma}$, it follows from Proposition 2.10 in [23] that U is Hölder continuous in $\overline{\mathcal{B}_{R/2}^+}$. The proof of the proposition is completed. □

Corollary 3.1. *Let U be a nonnegative solution of* (1.3) *with* $\frac{n}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$. *Then either the origin 0 is a removable singularity or* $\lim_{|X|\to 0} U(x,t) = +\infty$.

Proof. By Proposition 3.4, if the origin is not a removable singularity, then there exists a sequence of points $\{x_j\}$ such that

$$r_j = |x_j| \to 0 \quad \text{and} \quad U(x_j, 0) \to +\infty \quad \text{as } j \to \infty.$$

By the Harnack inequality (3.2), we have

$$\inf_{|X|=r_j} U(X) \geq C^{-1} U(x_j, 0).$$

By the maximum principle,

$$U(X) \geq \inf_{|X|=r_j, r_{j+1}} U(X) \geq C^{-1} \min(U(x_j,0), U(x_{j+1},0)) \quad \text{in } r_{j+1} \leq |X| \leq r_j.$$

Hence, we have $U(X) \to +\infty$ as $|X| \to 0$. □

Proof of Theorem 1.1. By Proposition 3.1,

$$u(x) \leq c|x|^{-\frac{2\sigma}{p-1}}.$$

If (1.5) does not hold, then

$$\liminf_{x\to 0} |x|^{\frac{2\sigma}{p-1}} u(x) = 0.$$

It follows Proposition 3.3 that

$$\lim_{x\to 0} |x|^{\frac{2\sigma}{p-1}} u(x) = 0 \tag{3.36}$$

We will prove that the origin is a removable singularity. It suffices to establish (3.18) by Proposition 3.4.

Let $\tau = \frac{n-2\sigma}{p-1}(p - \frac{n}{n-2\sigma})$ and $0 < \delta < 1$. Define

$$\Phi = |X|^{-\tau} - \delta t^{2\sigma} |X|^{-(\tau+2\sigma)}.$$

Then we can choose δ small (depending only on n , σ and p) such that

$$\begin{cases} -\text{div}(t^{1-2\sigma}\nabla\Phi) \geq 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial \Phi}{\partial \nu^\sigma}(x,0) = 2\delta\sigma |x|^{-2\sigma}\phi(x) & \text{on } \mathbb{R}^n \backslash \{0\}, \end{cases} \tag{3.37}$$

where $\phi(x) = \Phi(x,0) = |x|^{-\tau}$. Take $\xi_1(s) \in C_c^\infty(\mathbb{R})$ satisfying $0 \le \xi_1 \le 1$ in $\mathbb{R}$ and

$$\xi_1(s) = \begin{cases} 0 & \text{if } |s| \le 1, \\ 1 & \text{if } |s| \ge 2. \end{cases}$$

Take $\xi_2(s) \in C_c^\infty(\mathbb{R})$ satisfying $0 \le \xi_2 \le 1$ in $\mathbb{R}$ and

$$\xi_2(s) = \begin{cases} 1 & \text{if } |s| \le \frac{1}{2}, \\ 0 & \text{if } |s| \ge \frac{3}{4}. \end{cases}$$

For any $\epsilon > 0$ small, we choose $\zeta(X) \in C_c^\infty(\epsilon < |X| < 1)$ as follow

$$\zeta(X) = \begin{cases} \xi_1(\frac{|X|}{\epsilon}) & \text{if } |X| \le \frac{1}{2}, \\ \xi_2(|X|) & \text{if } |X| \ge \frac{1}{2}. \end{cases}$$

Using $\zeta\Phi$ as a test function in (1.4), and the divergence theorem we obtain

$$\begin{aligned} \int_{B_1} u\zeta\phi\left(\frac{1}{\phi}\frac{\partial\Phi}{\partial\nu^\sigma} - u^{p-1}\right) = \int_{\mathcal{B}_1^+} U\Big\{&\text{div}(t^{1-2\sigma}\nabla\Phi)\zeta + 2t^{1-2\sigma}\nabla\zeta\cdot\nabla\Phi \\ &+ \text{div}(t^{1-2\sigma}\nabla\zeta)\Phi\Big\} - \int_{B_1} u\phi\frac{\partial\zeta}{\partial\nu^\sigma}. \end{aligned} \tag{3.38}$$

Since ζ is radial, we have

$$\frac{\partial\zeta}{\partial\nu^\sigma}(x,0) = -\lim_{t\to 0^+} t^{1-2\sigma}\partial_t\zeta = 0 \quad \text{in } B_1. \tag{3.39}$$

By (3.37), we obtain

$$\int_{\mathcal{B}_1^+} U\text{div}(t^{1-2\sigma}\nabla\Phi)\zeta \le 0. \tag{3.40}$$

Using (3.1) and the Harnack inequality (3.2), we have

$$\begin{aligned} &\int_{\mathcal{B}_1^+} U\Big\{2t^{1-2\sigma}\nabla\zeta\cdot\nabla\Phi + \text{div}(t^{1-2\sigma}\nabla\zeta)\Phi\Big\} \\ &\le C_1 + \int_{\mathcal{B}_{2\epsilon}^+\setminus\mathcal{B}_\epsilon^+} U\Big\{2t^{1-2\sigma}|\nabla\zeta||\nabla\Phi| + |\text{div}(t^{1-2\sigma}\nabla\zeta)|\Phi\Big\} \\ &\le C_1 + C\epsilon^{-\frac{2\sigma}{p-1}-1}\int_{\mathcal{B}_{2\epsilon}^+\setminus\mathcal{B}_\epsilon^+}\left(t^{1-2\sigma}|X|^{-(\tau+1)} + |X|^{-(\tau+2\sigma)}\right) \\ &\qquad + C\epsilon^{-\frac{2\sigma}{p-1}-\tau}\int_{\mathcal{B}_{2\epsilon}^+\setminus\mathcal{B}_\epsilon^+} t^{1-2\sigma}\left(|\Delta\zeta| + \epsilon^{-1}\left|\xi_1'(\frac{|X|}{\epsilon})\right|\frac{1}{|X|}\right) \\ &\le C_1 + C\epsilon^{n-(\tau+2\sigma)-\frac{2\sigma}{p-1}} \le C < +\infty \end{aligned} \tag{3.41}$$

with C independent of ϵ. Hence, we obtain

$$\int_{B_1} u\zeta\phi\left(\frac{1}{\phi}\frac{\partial\Phi}{\partial\nu^\sigma} - u^{p-1}\right) \le C < +\infty.$$

By (3.36), we have $u^{p-1} = o(|x|^{-2\sigma})$, and while

$$\frac{1}{\phi}\frac{\partial \Phi}{\partial \nu^{\sigma}} = 2\delta\sigma|x|^{-2\sigma} \qquad \text{in } B_1\backslash\{0\}.$$

Therefore

$$\int_{B_1} u\zeta|x|^{-(n-\frac{2\sigma}{p-1})} = \int_{B_1} u\zeta|x|^{-(\tau+2\sigma)} \leq C < +\infty \tag{3.42}$$

with C independent of ϵ. Again by (3.1), we have

$$\begin{aligned}\int_{2\epsilon\leq|x|\leq 1/2} u^{\frac{(p-1)n}{2\sigma}} &\leq C\int_{2\epsilon\leq|x|\leq 1/2} u|x|^{-\frac{2\sigma}{p-1}[\frac{(p-1)n}{2\sigma}-1]}\\ &\leq C\int_{B_1} u\zeta|x|^{-(n-\frac{2\sigma}{p-1})} \leq C < +\infty\end{aligned}$$

with C independent of ϵ. Thus, we establish (3.18) and complete the proof of Theorem 1.1. □

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