

On the structure of positive solutions for a class of quasilinear equations*

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Abstract

This paper studies the existence, nonexistence and uniqueness of positive solutions for a class of quasilinear equations. We also analyze the behavior of this solutions with respect to two parameters κ and λ that appears in the equation. The proof of our main results relies on bifurcation techniques, the sub and supersolution method and a construction of an appropriate large solutions.

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1 Introduction

The main goal of this paper is to study existence, nonexistence, uniqueness and asymptotic behavior of positive solutions for the following class of quasilinear equations

$$\begin{cases} -\Delta u - \kappa \Delta(u^2)u = \lambda u - b(x)|u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathcal{P}_\kappa)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a smooth bounded domain, $p > 1$ is a constant, κ and λ are positive parameters and the weight function $b(x)$ satisfies certain regularity conditions.

Problem (\mathcal{P}_κ) with $\kappa = 0$ becomes the classical semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda u - b(x)|u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (\mathcal{P}_0)$$

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whose positive solutions are *equilibria* or *stationary* solutions of the the following reaction diffusion problem of logistic type

$$\begin{cases} u_t - \Delta u = \lambda u - b(x)u^p & \text{in } \Omega, \quad t > 0 \\ u = 0 & \text{on } \partial\Omega, \quad t > 0 \\ u(0) = u_0 \geq 0, \end{cases} \quad (1.1)$$

see for instance [5] and references therein. Problem (\mathcal{P}_0) has been object of intense study by many authors. If $\lambda \leq \lambda_1$ (λ_1 being the principal eigenvalue of $(-\Delta, H_0^1(\Omega))$) then the problem (\mathcal{P}_0) could have only the trivial solution, see Ambrosetti [1]. In [3] Ambrosetti-Macini prove that if $\lambda > \lambda_1$ then the problem has two nontrivial solutions of constant sign (one positive and the other negative). Soon thereafter Struwe [29] improved the result and proved that if $\lambda > \lambda_2$ the problem (\mathcal{P}_0) has three nontrivial solutions. Subsequently Ambrosetti-Lupo [2] slightly improved the work of Struwe [29] and also presented an approach based on Morse theory. This class of problems involving a more general operator as the p -laplacian (see for instance [27]). In [14] the authors have consider problem (\mathcal{P}_κ) with $b = 0$ and they proved that (\mathcal{P}_κ) has only the trivial solution if $\lambda < \lambda_1$.

The study of quasilinear equations involving the operator $L_\kappa u := \Delta u + \kappa \Delta(u^2)u$ arises in various branches of mathematical physics. It is well-known that nonlinear Schrödinger equations of the form

$$i\partial_t \psi = -\Delta \psi + V(x)\psi - \kappa \Delta[|\psi|^2]\psi - h(|\psi|^2)\psi, \quad (1.2)$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $V = V(x)$ is a given potential, κ is a real constant and h is a real function, have been studied in relation with some mathematical models in physics (see for instance [28]). It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to (1.2) and numerics results on this equation are obtained in [8]. In [24], motivated by the nanotubes and fullerene related structures, it was proposed and shown that a discrete system describing the interaction of a 2-dimensional hexagonal lattice with an excitation caused by an excess electron can be reduced to (1.2) and numerics results have been done on domains of disc type, cylinder type and sphere type.

Setting $\psi(t, x) = \exp(-iFt)u(x)$, $F \in \mathbb{R}$, into the equation (1.2), we obtain a corresponding equation

$$-\Delta u - \kappa \Delta(u^2)u = g(u) - V(x)u \quad \text{in } \Omega, \quad (1.3)$$

where we have renamed $V(x) - F$ to be $V(x)$ and $g(u) = h(u^2)u$.

The quasilinear equation (1.3) in the whole \mathbb{R}^N has received special attention in the past several years, see for instance [11, 13, 19, 28] and references therein. In these papers, we find important results on the existence of nontrivial solutions of (1.3) and a good insight into this quasilinear Schrödinger equation. The main strategies used are the following: the first of them consists in by using a constrained minimization argument, which gives a solution of (1.3) with an unknown Lagrange multiplier λ in front of the nonlinear term, see for example [28]. The other one consists in using a change of variables to get a new semilinear equation and an appropriate Orlicz space framework, for more details see [11, 13, 19]. In general, the existence results for equations of the type (1.3) is obtained by using variational methods. Here, we intend to use bifurcation techniques and the sub and supersolution method in order to analyze (\mathcal{P}_κ) .

In addition to studies involving the operator $L_\kappa u$, another important motivation to study problem (\mathcal{P}_κ) is the fact that many papers have been devoted to study quasilinear and semilinear equations involving logistic terms, which appear naturally in several contexts. For instance, when $\kappa = 0$, problem (\mathcal{P}_κ) becomes the classical logistic equation with linear diffusion and refuge, where $u(x)$ describes the density of the individuals of species at the location $x \in \Omega$, the nonlinearity $g(x, u) := \lambda u - b(x)u^p$ is the well-known logistic reaction term. There are several papers available in the literature dedicated to the analysis of (\mathcal{P}_0) , see for example [9, 21, 25, 26] and references therein.

It is worth mentioning that (\mathcal{P}_κ) can be seen as a quasilinear perturbation of the classical equation (\mathcal{P}_0) , specially when $\kappa \simeq 0$. As we shall see in Theorem 1.1 and 1.3, the presence of this quasilinear term prevents the blow-up (1.6) that occurs with the positive solutions of (\mathcal{P}_0) . Moreover, when $\kappa \downarrow 0$ the positive solutions of (\mathcal{P}_κ) tends to the positive solutions of (\mathcal{P}_0) .

In order to study the behavior of positive solutions of problem (\mathcal{P}_κ) with respect to the parameter $\kappa > 0$, we will assume the following assumptions on $b(x)$:

(b₀) The function $b : \overline{\Omega} \rightarrow [0, \infty)$ belongs to $C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$;

(b₁) The open set $\Omega_+ := \{x \in \Omega; b(x) > 0\}$ satisfies $\overline{\Omega}_+ \subset \Omega$ and there is a finite number of smooth components Ω_+^j , $j = 1, \dots, n$, such that $\overline{\Omega}_+^j \cap \overline{\Omega}_+^i = \emptyset$ if $i \neq j$. Moreover, the open set

$$\Omega_{b,0} := \Omega \setminus \overline{\Omega}_+$$

is connected. It should be noted that $\partial\Omega_+ \subset \Omega$ and $\partial\Omega_{b,0} = \partial\Omega \cup \partial\Omega_+$.

Before to state our main results, in a precise form, let us recall some notations. Throughout, for any function $V \in L^\infty(\Omega)$ called *potential*, by $\lambda_1[-\Delta + V]$ we mean the principal eigenvalue of $-\Delta + V$ in Ω under homogeneous Dirichlet boundary conditions. By simplicity, we also use the convention $\lambda_1 := \lambda_1[-\Delta]$. Moreover, we will denote by $\lambda_{b,0}$ the principal eigenvalue of $-\Delta$ in $\Omega_{b,0}$ under homogeneous Dirichlet boundary conditions.

Now, we are in position to state our main results on the problem (\mathcal{P}_κ) .

Theorem 1.1. *Let $p > 1$, $\kappa > 0$ and assume (b₀). Then problem (\mathcal{P}_κ) has a positive solution if and only if $\lambda > \lambda_1$. Moreover, if $p \geq 3$ or $b(x) \equiv b > 0$ is a constant, it is unique if it exists and it will be denoted by $\Psi_{\lambda,\kappa}$. In addition, the map $\lambda \in (\lambda_1, +\infty) \mapsto \Psi_{\lambda,\kappa} \in \mathcal{C}_0^1(\overline{\Omega})$ is increasing, in the sense that $\Psi_{\lambda,\kappa} > \Psi_{\mu,\kappa}$ if $\lambda > \mu > \lambda_1$. Furthermore,*

$$\lim_{\lambda \downarrow \lambda_1} \|\Psi_{\lambda,\kappa}\|_\infty = 0 \tag{1.4}$$

and for any compact $K \subset \overline{\Omega}_{b,0} \setminus \partial\Omega$,

$$\lim_{\lambda \rightarrow +\infty} \Psi_{\lambda,\kappa} = \infty \quad \text{uniformly in } K. \tag{1.5}$$

Note that we do not assume the hypothesis (b₁) in this theorem. Moreover, we also analyzed the positive solution of (\mathcal{P}_κ) in the particular case $b(x) \equiv b > 0$ constant (see Proposition 3.2). It

should be noted that our assumptions on the weight function $b(x)$ include the case $b \equiv 0$. Thus, Theorem 1.1 still holds for $b \equiv 0$, improving the results of [14, Theorem 1.1].

Concerning the asymptotic behavior of positive solutions $\Psi_{\mu,\kappa}$ in Theorem 1.1, based on the ideas in the paper [12], we will analyze the point-wise behavior of the positive solutions of (\mathcal{P}_κ) with respect to $\kappa \downarrow 0$. To this, let us recall the main result concerning to problem (\mathcal{P}_0) (see for instance Theorem 1.1 in [12] and references therein).

Theorem 1.2. *Assume (b_0) , (b_1) and $p > 1$. Then the following assertions hold:*

- (a) *The problem (\mathcal{P}_0) has a positive solution if and only if $\lambda \in (\lambda_1, \lambda_{b,0})$. Moreover, it is unique if it exists and it will be denoted by Θ_λ . In addition, Θ_λ is a nondegenerate solution of (\mathcal{P}_0) and the map $\lambda \in (\lambda_1, \lambda_{b,0}) \mapsto \Theta_\lambda \in \mathcal{C}_0^1(\overline{\Omega})$ is increasing, in the sense that $\Theta_\lambda > \Theta_\mu$ if $\lambda_{b,0} > \lambda > \mu > \lambda_1$. Furthermore, for each compact $K \subset \overline{\Omega}_{b,0} \setminus \partial\Omega$,*

$$\lim_{\lambda \rightarrow \lambda_{b,0}} \Theta_\lambda = \infty \quad \text{uniformly in } K \quad (1.6)$$

and, for each compact $K \subset \Omega_+$,

$$\lim_{\lambda \rightarrow \lambda_{b,0}} \Theta_\lambda = M_{\lambda_{b,0}} \quad \text{uniformly in } K, \quad (1.7)$$

where $M_{\lambda_{b,0}}$ stands for the minimal positive classical solution of the singular boundary value problem

$$\begin{cases} -\Delta u = \lambda u - b(x)u^p & \text{in } \Omega_+, \\ u = \infty & \text{on } \partial\Omega_+, \end{cases} \quad (1.8)$$

with $\lambda = \lambda_{b,0}$.

- (b) *Problem (1.8) possesses a minimal positive solution for each $\lambda \in \mathbb{R}$ and it will be denoted by M_λ .*

Concerning the asymptotic behavior of positive solutions $\Psi_{\lambda,\kappa}$ given in Theorem 1.1 we will prove the following result.

Theorem 1.3. *Suppose (b_0) and (b_1) . The following assertions are true:*

- (a) *If $\lambda \in (\lambda_1, \lambda_{b,0})$ then $\lim_{\kappa \downarrow 0} \Psi_{\lambda,\kappa} = \Theta_\lambda$ in $\mathcal{C}_0^1(\overline{\Omega})$;*
(b) *If $\lambda \geq \lambda_{b,0}$ then, for any compact $K \subset \overline{\Omega}_{b,0} \setminus \partial\Omega$,*

$$\lim_{\kappa \downarrow 0} \Psi_{\lambda,\kappa} = +\infty \quad \text{uniformly in } K; \quad (1.9)$$

- (c) *Suppose in addition that $p > 3$. If $\lambda \geq \lambda_{b,0}$ then, for any compact $K \subset \Omega_+$*

$$\lim_{\kappa \downarrow 0} \Psi_{\lambda,\kappa} = M_\lambda \quad \text{uniformly in } K, \quad (1.10)$$

where M_λ stands for the minimal positive classical solution of the singular boundary value problem (1.8).

We point out that the stability of the positive solutions of the dual problem associated to (\mathcal{P}_κ) is also analyzed (see Proposition 4.1).

The outline of this paper is as follows: In Section 2, we introduce the dual approach of (\mathcal{P}_κ) and prove the first results which play an important role in our analysis. In Section 3, we show the existence and uniqueness of positive solutions of (\mathcal{P}_κ) . In Section 4, we prove a stability result. Section 4 is devoted to prove a pivotal a priori bounds and in Section 5 we will use these estimates to study the asymptotic behavior of the positive solution of (\mathcal{P}_κ) when $\kappa \downarrow 0$.

2 An auxiliary problem

In this section we introduce the dual approach developed in the papers [11, 19] to deal with (\mathcal{P}_κ) . Specifically, we convert the quasilinear equation (\mathcal{P}_κ) into a semilinear one by using a suitable change of variables. To this end, we argue as follows. For each $\kappa \geq 0$, let $f_\kappa : \mathbb{R} \rightarrow \mathbb{R}$ denotes the solution of the Cauchy problem

$$f'_\kappa(t) = \frac{1}{(1 + 2\kappa f_\kappa^2(t))^{1/2}}, \quad f_\kappa(0) = 0.$$

By the standard theory of ODE, we obtain that f_κ is uniquely determined, invertible and of class $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. Moreover, it is well know that the inverse function of f is given by

$$f_\kappa^{-1}(t) := \int_0^t (1 + 2\kappa s^2)^{1/2} ds, \quad \forall t \geq 0.$$

Thus, by performing the change of variables $u = f_\kappa(v)$ and setting $g(x, s) = \lambda s - b(x)s^{p-1}$ if $s \geq 0$, $x \in \Omega$ and $g(x, s) = 0$ for $s < 0$, $x \in \Omega$, we obtain that problem (\mathcal{P}_κ) is equivalent to the following semilinear elliptic equation

$$\begin{cases} -\Delta v = \lambda f_\kappa(v) f'_\kappa(v) - b(x)(f_\kappa(v))^p f'_\kappa(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.11)$$

Furthermore, we can see that v is a classical positive solution of (2.11) if and only if $u = f_\kappa(v)$ is a positive solution of (\mathcal{P}_κ) (see [11, 19]). Thus, we will analyze the auxiliary problem (2.11).

In order to study problem (2.11), we need to establish some properties of the change of variable $f_\kappa(t)$. Firstly, we recall some useful properties of $f_\kappa(t)$ (see for instance [4, 11]).

Lemma 2.1. *Let $\kappa > 0$ and $t \geq 0$. Then*

- (i) $0 \leq f_\kappa(t) \leq t$;
- (ii) $0 \leq f'_\kappa(t) \leq 1$;
- (iii) $f_\kappa(t) f'_\kappa(t) \leq 1/\sqrt{2\kappa}$;
- (iv) $f''_\kappa(t) = -2\kappa f_\kappa(t) (f'_\kappa(t))^4 = [(f'_\kappa(t))^4 - (f'_\kappa(t))^2]/f_\kappa(t)$;
- (v) $\frac{1}{2} f_\kappa(t) \leq t f'_\kappa(t) \leq f_\kappa(t)$;

$$(vi) \quad \lim_{t \rightarrow 0^+} f_\kappa(t)/t = 1;$$

(vii) The map $t \in (0, \infty) \mapsto f_\kappa(t)/t^{1/2}$ is nondecreasing.

As a consequence of Lemma 2.1, we also have the following properties.

Lemma 2.2. Assume that $\kappa > 0$ and $p > 1$. Then

(i) The map $t \in (0, +\infty) \mapsto f_\kappa(t)f'_\kappa(t)/t$ is of class \mathcal{C}^1 , decreasing and it verifies

$$f_\kappa(t)f'_\kappa(t) \leq t, \quad \forall t \geq 0, \quad (2.12)$$

$$\lim_{t \rightarrow 0^+} \frac{f_\kappa(t)f'_\kappa(t)}{t} = 1 \quad (2.13)$$

and

$$\lim_{t \rightarrow \infty} \frac{f_\kappa(t)f'_\kappa(t)}{t} = 0; \quad (2.14)$$

(ii) For $p \geq 3$, the map $t \in (0, \infty) \mapsto f_\kappa^p(t)f'_\kappa(t)/t$ is of class \mathcal{C}^1 , increasing and it verifies

$$\lim_{t \rightarrow 0^+} \frac{f_\kappa^p(t)f'_\kappa(t)}{t} = 0. \quad (2.15)$$

Proof. To prove that $t \in (0, +\infty) \mapsto f_\kappa(t)f'_\kappa(t)/t$ is decreasing note that, for all $t > 0$,

$$\begin{aligned} \left(\frac{f_\kappa(t)f'_\kappa(t)}{t} \right)' &= \frac{[(f'_\kappa(t))^2 + f_\kappa(t)f''_\kappa(t)]t - f_\kappa(t)f'_\kappa(t)}{t^2} \\ &= \frac{[(f'_\kappa(t))^2 - 2(f_\kappa(t))^2(f'_\kappa(t))^5]t - f_\kappa(t)f'_\kappa(t)}{t^2} < 0, \end{aligned}$$

if and only if, $tf'_\kappa(t) < 2t(f_\kappa(t))^2(f'_\kappa(t))^4 + f_\kappa(t)$, which is true, thanks to Lemma 2.1 (i), (ii) and (v). The inequality (2.12) is a direct consequence of Lemma 2.1 (i) and (ii). The limit (2.13) is obtained by combining Lemma 2.1 (vi) and using that

$$\lim_{t \rightarrow 0^+} f'_\kappa(t) = \lim_{t \rightarrow 0^+} \frac{1}{(1 + 2\kappa f_\kappa^2(t))^{1/2}} = 1.$$

The limit (2.14) follows from Lemma 2.1 (iii).

Now, suppose that $p \geq 3$. To prove that the map $t \in [0, \infty) \mapsto f_\kappa^p(t)f'_\kappa(t)/t$ is increasing, we observe that, by Lemma 2.1 (iv), for all $t > 0$ we have

$$\begin{aligned} \left(\frac{f_\kappa^p(t)f'_\kappa(t)}{t} \right)' &= \frac{[p(f_\kappa(t))^{p-1}(f'_\kappa(t))^2 + f_\kappa^p(t)f''_\kappa(t)]t - f_\kappa^p(t)f'_\kappa(t)}{t^2} \\ &= \frac{[p(f_\kappa(t))^{p-1}(f'_\kappa(t))^2 + (f_\kappa(t))^{p-1}((f'_\kappa(t))^4 - (f'_\kappa(t))^2)]t - f_\kappa^p(t)f'_\kappa(t)}{t^2} > 0, \end{aligned}$$

if and only if, $[p(f'_\kappa(t))^2 + (f'_\kappa(t))^4 - (f'_\kappa(t))^2]t - f_\kappa(t)f'_\kappa(t) > 0$, that is,

$$t(f'_\kappa(t))^4 + (p-1)t(f'_\kappa(t))^2 > f_\kappa(t)f'_\kappa(t). \quad (2.16)$$

On the other hand, since $p \geq 3$, it follows from Lemma 2.1 (v) that

$$tf'_\kappa(t) \geq \frac{f_\kappa(t)}{2} \geq \frac{f_\kappa(t)}{p-1}, \quad \forall t \geq 0.$$

Now, using Lemma 2.1 (ii) in combination with the fact that $t(f'_\kappa(t))^4 > 0$ for all $t > 0$, we conclude that (2.16) is true. Finally, (2.15) is an easy consequence of (2.13) and $\lim_{t \rightarrow 0^+} f^{p-1}(t) = f^{p-1}(0) = 0$. \square

With respect to the map $\kappa \in (0, \infty) \mapsto f_\kappa(t)$ (for each $t \geq 0$), we have the following lemma.

Lemma 2.3. *For each $t > 0$, the function $\kappa \in (0, \infty) \mapsto f_\kappa(t)$ is continuous and decreasing.*

Proof. The continuity of the map $\kappa \in (0, \infty) \mapsto f_\kappa(t)$ follows from the standard theory of ordinary differential equations. To prove that it is decreasing, we argue as follows. Let κ_1, κ_2 be constants such that $0 < \kappa_1 < \kappa_2$. We need prove that $f_{\kappa_2}(t) < f_{\kappa_1}(t)$ for all $t > 0$. Since, for each $t > 0$, the function $\kappa \mapsto f_\kappa^{-1}(t) = \int_0^t (1 + 2\kappa s^2)^{1/2} ds$ is increasing, it suffices to prove that

$$f_{\kappa_2}^{-1}(f_{\kappa_2}(t)) < f_{\kappa_1}^{-1}(f_{\kappa_1}(t)), \quad (2.17)$$

which is equivalent to $t < \int_0^{f_{\kappa_1}(t)} (1 + 2\kappa_2 s^2)^{1/2} ds$. To this, consider the function defined by

$$h(t) = \int_0^{f_{\kappa_1}(t)} (1 + 2\kappa_2 s^2)^{1/2} ds - t, \quad t \geq 0$$

and notice that $h(0) = 0$. We claim that $h'(t) > 0$ for all $t > 0$ which implies that $h(t) > 0$ and hence (2.17) holds. Indeed, observe that $h'(t) = (1 + 2\kappa_2 f_{\kappa_1}^2(t))^{1/2} f'_{\kappa_1}(t) - 1 > 0$ if and only if

$$\frac{1}{(1 + 2\kappa_1 f_{\kappa_1}^2(t))^{1/2}} = f'_{\kappa_1}(t) > \frac{1}{(1 + 2\kappa_2 f_{\kappa_1}^2(t))^{1/2}},$$

which holds if $\kappa_1 < \kappa_2$ and this completes the proof. \square

We finish this section by deriving an *a priori* estimate for positive solutions of (2.11) in the particular case $b(x) \equiv b > 0$. This estimate will be useful to prove a nonexistence result in the next section.

Lemma 2.4. *Let $v \in \mathcal{C}^2(\overline{\Omega})$ be a positive solution of (2.11) with $b(x) \equiv b > 0$ constant. Then*

$$bf_\kappa^{p-1}(v(x)) \leq \lambda, \quad \forall x \in \Omega. \quad (2.18)$$

Proof. Let v be a classical positive solution of (2.11). Since the maximum value of v in $\overline{\Omega}$ is attained in Ω , let $x_0 \in \Omega$ be such that $v(x_0) = \max_{x \in \overline{\Omega}} v(x)$. Thus,

$$0 \leq -(\Delta v)(x_0) = \lambda f_\kappa(v(x_0)) f'_\kappa(v(x_0)) - bf_\kappa^p v(x_0) f'_\kappa(v(x_0))$$

and as $f_\kappa(v(x_0)) f'_\kappa(v(x_0)) > 0$, the previous inequality is equivalent to $bf_\kappa^{p-1}(v(x_0)) \leq \lambda$. Using that $f_\kappa(t)$ is increasing for $t > 0$, we obtain $bf_\kappa^{p-1}(v(x)) \leq bf_\kappa^{p-1}(v(x_0)) \leq \lambda$ for all $x \in \Omega$, and this completes the proof. \square

3 Existence, nonexistence and uniqueness of positive solutions

In this section, we will study the existence, nonexistence and uniqueness of positive solutions of (2.11). We begin by establishing a necessary condition for existence of positive solution for (2.11) (and hence for (\mathcal{P}_κ)).

Lemma 3.1 (Nonexistence). *If (b_0) holds then problem (2.11) does not have positive solutions for $\lambda \leq \lambda_1$. In particular, if $b(x) \equiv 0$ then problem (2.11) does not have positive solutions for $\lambda \leq \lambda_1$.*

Proof. Suppose that $v > 0$ is a solution of (2.11) with $\lambda \leq \lambda_1$. Then, it satisfies

$$\begin{cases} -\Delta v + \tilde{b}(x)v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\tilde{b}(x) := b(x) \frac{f_\kappa^p(v(x))f'_\kappa(v(x))}{v(x)} - \lambda \frac{f_\kappa(v(x))f'_\kappa(v(x))}{v(x)}.$$

Thus, we can infer that $\lambda_1[-\Delta + \tilde{b}(x)] = 0$. Since $b(x)f_\kappa^p(v(x))f'_\kappa(v(x))/v(x) \geq 0$, using the monotonicity properties of the principal eigenvalue combined with (2.12), we conclude that $0 > \lambda_1[-\Delta - \lambda] = \lambda_1 - \lambda$, which is a contradiction and this ends the proof. \square

The next result shows an uniqueness result of positive solutions of (2.11).

Proposition 3.2 (Uniqueness). *Suppose $p \geq 3$ or $b(x) \equiv b > 0$. Then the problem (2.11) admits at most a positive solution.*

Proof. First we will consider the case $p \geq 3$. By the classical Brezis-Oswald result (see [7]), it is sufficient to prove that the function

$$h(x, t) := \lambda \frac{f_\kappa(t)f'_\kappa(t)}{t} - b(x) \frac{f_\kappa^p(t)f'_\kappa(t)}{t}$$

is decreasing in $t > 0$, for each $x \in \Omega$. Thus, the monotonicity follows by Lemma 2.2.

Now, assume that $b(x) \equiv b > 0$ is constant. We will argue by contradiction. Suppose that $v_1 > 0$ and $v_2 > 0$ are solutions of (2.11) with $v_1 \neq v_2$. Denoting, by simplicity, $g_i = f_\kappa(v_i)$ and $g'_i = f'_\kappa(v_i)$ ($i = 1, 2$), we have $-\Delta(v_1 - v_2) = \lambda(g_1g'_1 - g_2g'_2) - b(g_1^p g'_1 - g_2^p g'_2)$. Defining $W : \Omega \rightarrow \mathbb{R}$ by

$$W(x) = \begin{cases} \frac{-\lambda[g_1(x)g'_1(x) - g_2(x)g'_2(x)] + b[g_1^p(x)g'_1(x) - g_2^p(x)g'_2(x)]}{v_1(x) - v_2(x)}, & \text{if } v_1(x) \neq v_2(x), \\ 0 & \text{if } v_1(x) = v_2(x), \end{cases}$$

we have $-\Delta(v_1 - v_2) + W(x)(v_1 - v_2) = 0$ in Ω . Consequently, $\lambda_j[-\Delta + W(x)] = 0$, where $\lambda_j[-\Delta + W(x)]$, $j \geq 1$, stands for an eigenvalue of $-\Delta + W(x)$ in Ω under homogeneous Dirichlet boundary conditions. By the dominance of the principal eigenvalue, we get

$$0 = \lambda_j[-\Delta + W(x)] \geq \lambda_1[-\Delta + W(x)]. \quad (3.19)$$

On the other hand, since v_1 is a positive solution of (2.11), we have

$$\lambda_1 \left[-\Delta - \lambda \frac{g_1 g_1'}{v_1} + b \frac{g_1^p g_1'}{v_1} \right] = 0. \quad (3.20)$$

We claim that

$$-\lambda \frac{g_1 g_1'}{v_1} + b \frac{g_1^p g_1'}{v_1} \leq W \quad \text{in } \Omega, \quad (3.21)$$

with strict inequality in an open subset of Ω . If (3.21) holds, then the proof is completed because we can combine (3.20)-(3.21) and the monotonicity properties of the principal eigenvalue to obtain

$$0 = \lambda_1 \left[-\Delta - \lambda \frac{g_1 g_1'}{v_1} + b \frac{g_1^p g_1'}{v_1} \right] < \lambda_1 [-\Delta + W(x)],$$

which contradicts (3.19). Now, let us prove (3.21). If $v_1(x) = v_2(x)$ then $W(x) = 0$ and (3.21) is equivalent to

$$-\lambda \frac{g_1(x) g_1'(x)}{v_1(x)} + b \frac{g_1^p(x) g_1'(x)}{v_1(x)} \leq 0$$

that is, $bg_1^{p-1} \leq \lambda$ in Ω , which occurs thanks to Lemma 2.4. If $v_1 > v_2$, then $v_1 - v_2 > 0$ and (3.21) is equivalent to

$$-\lambda g_1 g_1' + b g_1^p g_1' + \lambda \frac{g_1 g_1'}{v_1} v_2 - b \frac{g_1^p g_1'}{v_1} v_2 \leq -\lambda g_1 g_1' + \lambda g_2 g_2' + b g_1^p g_1' - b g_2^p g_2' \quad \text{in } \Omega,$$

that is,

$$[\lambda - b g_1^{p-1}] \frac{g_1 g_1'}{v_1} \leq [\lambda - b g_2^{p-1}] \frac{g_2 g_2'}{v_2} \quad \text{in } \Omega. \quad (3.22)$$

Since the map $t \in [0, \infty) \mapsto f_\kappa(t) f_\kappa'(t)/t$ is decreasing, we have

$$0 \leq \frac{g_1 g_1'}{v_1} < \frac{g_2 g_2'}{v_2}. \quad (3.23)$$

Once that $t \in [0, \infty) \mapsto f_\kappa(t) f_\kappa'(t)$ is increasing and by using Lemma 2.4, we can infer that

$$0 \leq \lambda - b g_1^{p-1} \leq \lambda - b g_2^{p-1} \quad \text{in } \Omega. \quad (3.24)$$

Thus, (3.23) and (3.24) imply that (3.22) is true, showing that (3.21) holds for $v_1 > v_2$. The case $v_1 < v_2$ is analogous and this ends the proof. \square

Now, we will show that λ_1 is the unique bifurcation point of positive solutions of (2.11) from the trivial solution. For this, let e_1 be the unique positive solution of

$$\begin{cases} -\Delta v = 1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and let E be the space consisting of all $u \in \mathcal{C}(\overline{\Omega})$ for which there exists $\gamma = \gamma_u > 0$ such that

$$-\gamma e_1(x) \leq u(x) \leq \gamma e_1(x) \quad \forall x \in \Omega,$$

endowed with the norm $\|u\|_E := \inf\{\gamma > 0; -\gamma e_1(x) \leq u(x) \leq \gamma e_1(x), \forall x \in \Omega\}$ and the natural point-wise order. It is not difficult to verify that E is an ordered Banach space whose positive cone, say P , is normal and has nonempty interior. Thus, consider the map $\mathfrak{F} : \mathbb{R} \times E \rightarrow E$ defined by

$$\mathfrak{F}(\lambda, v) = v - (-\Delta)^{-1}[\lambda f_\kappa(v) f'_\kappa(v) - b(x) f_\kappa^p(v) f'_\kappa(v)],$$

where $(-\Delta)^{-1}$ is the inverse of the Laplacian operator under homogeneous Dirichlet boundary conditions. We can see that the application \mathfrak{F} is of \mathcal{C}^1 class and (2.11) can be written in the form

$$\mathfrak{F}(\lambda, v) = 0. \tag{3.25}$$

Moreover, by the Strong Maximum Principle, any nonnegative solution of (3.25) is in fact strictly positive.

Proposition 3.3. *The number λ_1 is a bifurcation point of (2.11) from the trivial solution to a continuum of positive solutions of (2.11). Moreover, it is the unique bifurcation point of positive solutions from $(\lambda, 0)$. If $\Sigma_0 \subset \mathcal{S}$ denotes the component of positive solutions of (2.11) emanating from $(\lambda, 0)$, then Σ_0 is unbounded in $\mathbb{R} \times E$.*

Proof. Observe that (3.25) can be written as $\mathcal{L}(\lambda)v + \mathcal{N}(\lambda, v) = 0$ where $\mathcal{L}(\lambda) = I_E - \lambda(-\Delta)^{-1}$ and

$$\mathcal{N}(\lambda, v) = -(-\Delta)^{-1}[\lambda(f_\kappa(v) f'_\kappa(v) - v) - b(x) f_\kappa^p(v) f'_\kappa(v)].$$

Moreover, thanks to (2.13) and (2.15), we have

$$\lim_{t \rightarrow 0^+} \frac{\lambda(f_\kappa(t) f'_\kappa(t) - t) - b(x) f_\kappa^p(t) f'_\kappa(t)}{t} = 0,$$

and thus $\mathcal{N}(\lambda, v) = o(\|v\|_E)$ as $\|v\|_E \rightarrow 0$. Therefore, we can apply the unilateral bifurcation theorem for positive operators, see [23, Theorem 6.5.5], to conclude the result. \square

Next, we are ready to complete the proof of Theorem 1.1. Actually, it will be a consequence of the following result.

Theorem 3.4. *Let $p > 1$, $\kappa > 0$ and assume (b_0) . Then problem (2.11) possesses a positive solution if and only if $\lambda > \lambda_1$. Moreover, if $p \geq 3$ or $b(x) \equiv b > 0$ is a constant, it is unique if it exists and it will be denoted by $\Theta_{\lambda, \kappa}$. In addition, the map $\lambda \in (\lambda_1, +\infty) \mapsto \Theta_{\lambda, \kappa} \in \mathcal{C}_0^1(\overline{\Omega})$ is increasing, in the sense that $\Theta_{\lambda, \kappa} > \Theta_{\mu, \kappa}$, if $\lambda > \mu > \lambda_1$. Furthermore, $\lim_{\lambda \downarrow \lambda_1} \|\Theta_{\lambda, \kappa}\|_\infty = 0$ and for any compact $K \subset \overline{\Omega}_{b,0} \setminus \partial\Omega$,*

$$\lim_{\lambda \rightarrow +\infty} \Theta_{\lambda, \kappa} = \infty \quad \text{uniformly in } K.$$

Proof. By Proposition 3.3, λ_1 is a bifurcation point of (2.11) from the trivial solution and it is the only one for positive solutions. Moreover, there exists an unbounded continuum Σ_0 of positive solutions emanating from $(\lambda_1, 0)$. In order to prove the existence of a positive solution for every $\lambda > \lambda_1$, it suffices to show that, for every $\lambda_* > \lambda_1$, there exists a constant $C = C(\lambda_*) > 0$ such that

$$\|v\|_\infty \leq C, \quad \forall (\lambda, v) \in \Sigma_0 \text{ and } \lambda \leq \lambda_*. \quad (3.26)$$

Indeed, by the global nature of Σ_0 , this estimate implies that $\text{Proj}_{\mathbb{R}} \Sigma_0 = (\lambda_1, \infty)$, where $\text{Proj}_{\mathbb{R}} \Sigma_0$ is the projection of Σ_0 into \mathbb{R} . To prove (3.26), we will build a family $\overline{W}(\lambda)$ of super solutions of (2.11) and we will apply Theorem 2.2 of [17]. Thus, we consider the continuous map $\overline{W} : [\lambda_1, \lambda_*] \rightarrow C_0^2(\overline{\Omega})$ defined by $\overline{W}(\lambda) = K(\lambda)e$, where $K(\lambda)$ is a positive constant to be chosen later and e is the unique positive solution of

$$\begin{cases} -\Delta v = 1 & \text{in } \widehat{\Omega}, \\ v = 0 & \text{on } \partial\widehat{\Omega}, \end{cases} \quad (3.27)$$

for some regular domain $\Omega \subset\subset \widehat{\Omega}$. Then, $\overline{W}(\lambda) = K(\lambda)e$ is a super solution of (2.11) if

$$1 \geq \lambda \frac{f_\kappa(e)f'_\kappa(e)}{Ke} e - b(x) \frac{f_\kappa^p(e)f'_\kappa(e)}{Ke} e \quad \text{in } \Omega.$$

According to Proposition 2.2, $\lim_{t \rightarrow \infty} f_\kappa(t)f'_\kappa(t)/t = 0$. Consequently, for $K = K(\lambda) > 0$ large enough, $\overline{W}(\lambda) = K(\lambda)e$ is a super solution (but not a solution) of (2.11), for every $\lambda \in [\lambda_1, \lambda_*]$ and $W(\lambda_1) = K(\lambda_1)e > 0$ in Ω . Thus, by Theorem 2.2 of [17], it follows (3.26).

To prove that $\Theta_{\lambda, \kappa} > \Theta_{\mu, \kappa}$ if $\lambda > \mu > \lambda_1$, just note that $\Theta_{\mu, \kappa}$ is a (strict) subsolution of (2.11) if $\mu \in (\lambda_1, \lambda)$. By the uniqueness of positive solution of (2.11) we conclude the result.

The convergence (1.4) is an immediate consequence of Proposition 3.3.

Now, in order to prove (1.5), let $\varphi_{b,0} > 0$ be the eigenfunction associated to $\lambda_{b,0}$ such that $\|\varphi_{b,0}\|_\infty = 1$ and consider

$$\Psi = \begin{cases} \varphi_{b,0} & \text{in } \Omega_{b,0}, \\ 0 & \text{in } \Omega \setminus \overline{\Omega}_{b,0}. \end{cases}$$

It is clear that $\Psi \in H_0^1(\Omega)$. We will show that for $\lambda > \lambda_{b,0}$, $\varepsilon(\lambda)\Psi$ is a subsolution of (2.11) (in the sense of [6]) for a constant $\varepsilon(\lambda) > 0$ to be chosen. Indeed, since $b \equiv 0$ in $\Omega_{b,0}$ and $\Psi = 0$ in $\Omega \setminus \overline{\Omega}_{b,0}$, it suffices to verify that

$$\lambda_{b,0}\varepsilon\varphi_{b,0} = -\Delta(\varepsilon\varphi_{b,0}) \leq \lambda f_\kappa(\varepsilon\varphi_{b,0})f'_\kappa(\varepsilon\varphi_{b,0}) \quad \text{in } \Omega_{b,0},$$

that is,

$$\frac{\lambda_{b,0}}{\lambda} \leq \frac{f_\kappa(\varepsilon\varphi_{b,0})f'_\kappa(\varepsilon\varphi_{b,0})}{\varepsilon\varphi_{b,0}} \quad \text{in } \Omega_{b,0}.$$

According to Lemma 2.2, the map $t \in [0, \infty) \mapsto h_\kappa(t) := f_\kappa(t)f'_\kappa(t)/t$ is decreasing and, hence, is invertible. Then, the above inequality is equivalent to $h_\kappa^{-1}(\lambda_{b,0}/\lambda) \geq \varepsilon\varphi_{b,0}$. Once that $\|\varphi_{b,0}\|_\infty = 1$,

choosing $\varepsilon(\lambda) := h^{-1}(\lambda_{b,0}/\lambda)$, we obtain that $\varepsilon(\lambda)\varphi_{b,0}$ is a subsolution of (2.11). Moreover, it follows from (2.13) that $\lim_{t \rightarrow 0} h^{-1}(t) = +\infty$ and therefore

$$\lim_{\lambda \rightarrow \infty} \varepsilon(\lambda) = \lim_{\lambda \rightarrow \infty} h^{-1}\left(\frac{\lambda_{b,0}}{\lambda}\right) = +\infty. \quad (3.28)$$

Now, the previous arguments establish that $K(\lambda)e$ is a super solution of (2.11) for all K large enough. Thus, since $\min_{x \in \overline{\Omega}} e(x) > 0$, we can choose K such that $\varepsilon(\lambda)\varphi_{b,0} \leq K(\lambda)e$. Therefore, by method of sub and supersolution and the uniqueness of positive solution of (2.11), we can infer that $\varepsilon(\lambda)\varphi_{b,0} \leq \Theta_{\lambda,\kappa}$. Consequently, by (3.28) we obtain (1.5) and this complete the proof. \square

We observe that as a direct consequence of this result, the proof of Theorem 1.1 follows by setting $\Psi_{\lambda,\kappa} := f_{\kappa}(\Theta_{\lambda,\kappa})$.

4 Stability Result

In this section we will provide the stability of the positive solutions of (2.11) with the additional assumption that $p \geq 3$. We recall that the stability of a positive solution (λ_0, u_0) of (2.11) as a steady state of an associated parabolic equation is given by the spectrum of the linearized operator of (2.11), which is

$$\mathcal{L}(\lambda_0, u_0) := -\Delta - \lambda_0[f_{\kappa}(u_0)f'_{\kappa}(u_0)]' + b(x)[f_{\kappa}^p(u_0)f'_{\kappa}(u_0)]',$$

subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$, where $' = d/dt$. Thus, (λ_0, u_0) is said to be linearly asymptotically stable if $\lambda_1[\mathcal{L}(\lambda_0, u_0)] > 0$.

With these considerations, we have the following result:

Proposition 4.1. *Suppose $p \geq 3$ or $b(x) \equiv b > 0$. Then, for each $\lambda > \lambda_1$ and $\kappa > 0$, the unique positive solution $(\lambda, \Theta_{\lambda,\kappa})$ of (2.11) is linearly asymptotically stable, that is,*

$$\lambda_1[\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})] > 0.$$

Proof. Denoting by simplicity $f = f_{\kappa}(\Theta_{\lambda,\kappa})$ and using Lemma 2.1 (iv), we have

$$[ff']' = (f')^2 + ff'' = (f')^2 - 2\kappa f(f')^4 = (f')^4,$$

and

$$[f^p f']' = [f^{p-1}(ff')] = (p-1)f^{p-2}f'(ff') + f^{p-1}(f')^4 = f^{p-1}[(p-1)(f')^2 + (f')^4].$$

Therefore,

$$\mathcal{L}(\lambda, \Theta_{\lambda,\kappa}) = -\Delta - \lambda[(f')^2 - 2\kappa f(f')^4] + b(x)f^{p-1}[(p-1)(f')^2 + (f')^4].$$

By the characterization of the Maximum Principle, see for instance [22, Theorem 2.1] or [20], to prove $\lambda_1[\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})] > 0$ it is sufficient to show that there exists a positive strict super solution of

$\mathcal{L}(\lambda, \Theta_{\lambda, \kappa})$. Let us prove that $\Theta_{\lambda, \kappa}$ is a strict super solution of $\mathcal{L}(\lambda, \Theta_{\lambda, \kappa})$. Indeed, since $\Theta_{\lambda, \kappa}$ is a positive solution of (2.11), we have $-\Delta\Theta_{\lambda, \kappa} = \lambda f f' - b(x) f^p f'$ and therefore

$$\begin{aligned} \mathcal{L}(\lambda, \Theta_{\lambda, \kappa})\Theta_{\lambda, \kappa} &= \lambda f f' - b(x) f^p f' - \lambda (f')^2 \Theta_{\lambda, \kappa} \\ &\quad + 2\lambda \kappa f (f')^4 \Theta_{\lambda, \kappa} + b(x) f^{p-1} [(p-1)(f')^2 + (f')^4] \Theta_{\lambda, \kappa}. \end{aligned} \quad (4.29)$$

Since $p \geq 3$, it follows from Lemma 2.1 (v) that

$$\begin{aligned} f - f' \Theta_{\lambda, \kappa} &= f(\Theta_{\lambda, \kappa}) - f'(\Theta_{\lambda, \kappa}) \Theta_{\lambda, \kappa} > 0 \quad \text{and} \\ (p-1) f' \Theta_{\lambda, \kappa} - f &= (p-1) f'(\Theta_{\lambda, \kappa}) \Theta_{\lambda, \kappa} - f(\Theta_{\lambda, \kappa}) > 0. \end{aligned} \quad (4.30)$$

Moreover, since $b(x) f^{p-1} (f')^3 \Theta_{\lambda, \kappa} \geq 0$ and $2\lambda \kappa f (f')^4 \Theta_{\lambda, \kappa} \geq 0$, we can infer from (4.29) and (4.30) that

$$\mathcal{L}(\lambda, \Theta_{\lambda, \kappa})\Theta_{\lambda, \kappa} > 0,$$

which establishes that $\Theta_{\lambda, \kappa} > 0$ is a strict positive super solution of $\mathcal{L}(\lambda, \Theta_{\lambda, \kappa})$. By characterization of the Maximum Principle, $\lambda_1[\mathcal{L}(\lambda, \Theta_{\lambda, \kappa})] > 0$ and this completes the proof. \square

As a direct consequence we obtain:

Corollary 4.2. *Supposing $p \geq 3$, we have:*

- (i) *For each $\lambda > \lambda_1$, $(\lambda, \Theta_{\lambda, \kappa})$ is a nondegenerate positive solution of (2.11).*
- (ii) *The map $\lambda \in (\lambda_1, +\infty) \mapsto \Theta_{\lambda, \kappa} \in \mathcal{C}_0^1(\Omega)$ is of class \mathcal{C}^∞ .*

Proof. The proof of (i) is standard and, once that $t \in [0, +\infty) \mapsto f_k(t)$ is of class \mathcal{C}^∞ , (ii) follows from implicit function theorem applied to the operator

$$\mathfrak{F}(\lambda, u) := u - (-\Delta)^{-1} [\lambda f_k(u) f'_k(u) - b(x) f_k^p(u) f'_k(u)].$$

\square

A priori bounds in Ω_+ This section is devoted to obtain an *a priori* estimate for positive solutions of (2.11) uniform in $\kappa > 0$, $\kappa \simeq 0$ in any compact subset of Ω_+ . It is a crucial step to prove Theorem 1.3 (c). As we will see below, to obtain these estimates we will assume $p > 3$. To this aim, we need to study the following auxiliary problem

$$\begin{cases} -\Delta v = \lambda v - b_0 g(v) & \text{in } B_r, \\ v = \infty & \text{on } \partial B_r, \end{cases} \quad (4.31)$$

where $b_0 > 0$ is a constant, $B_r := B_r(x_0) = \{x \in \mathbb{R}^N; |x - x_0| < r\}$ is an open ball in \mathbb{R}^N centered in $x_0 \in \mathbb{R}^N$ and

$$g(t) := \frac{f_1^{p+1}(t)}{t}, \quad \forall t > 0. \quad (4.32)$$

First we will prove some important properties of g .

Lemma 4.3. *The map $g : (0, \infty) \rightarrow (0, \infty)$ defined in (4.32) is increasing and it satisfies $g(0) := \lim_{t \rightarrow 0^+} g(t) = 0$. Moreover, there exists a constant $C > 0$ such that*

$$g(t) \geq Ct^{(p-1)/2}, \quad \forall t \geq 1. \quad (4.33)$$

Furthermore,

$$f_\kappa^p(t)f'_\kappa(t) \leq g(t), \quad \forall t > 0 \quad \text{and} \quad 0 < \kappa < 1. \quad (4.34)$$

Proof. In order to prove that g is increasing, note that, by Lemma 2.1 (iii), we have

$$tf_1(t) \geq \frac{f_1(t)}{2} > \frac{f_1(t)}{p+1}, \quad \forall t > 0,$$

since $p > 1$. Thus,

$$g'(t) = \left(\frac{f_1^{p+1}(t)}{t} \right)' = \frac{(p+1)f_1^p(t)t - f_1^{p+1}(t)}{t^2} > 0, \quad \forall t > 0.$$

To conclude the proof of inequality (4.33), observe that for each $t > 0$ one has

$$\frac{g(t)}{t^{(p-1)/2}} = \left(\frac{f_1(t)}{t^{1/2}} \right)^{p+1}.$$

By Lemma 2.1 (vii), $t \mapsto g(t)/t^{(p-1)/2}$ is nondecreasing and thus

$$\frac{g(t)}{t^{(p-1)/2}} \geq g(1), \quad \forall t \geq 1.$$

Choosing $C = g(1)$, we obtain (4.33). Moreover, $\lim_{t \rightarrow 0^+} f_\kappa^p(t)(f_\kappa(t)/t) = g(0) = 0$. Finally, combining the monotonicity of $\kappa \mapsto f_\kappa(\cdot)$ with Lemma 2.1 (v), we get

$$f_\kappa^p(t)f'_\kappa(t) \leq \frac{f_\kappa^{p+1}(t)}{t} < \frac{f_1^{p+1}(t)}{t} = g(t), \quad \forall t > 0,$$

which is the desired result. \square

Now, we will establish an existence result for (1.8). We recall that there are many result about the existence, uniqueness and blow-up rate of large solution of problems related to (4.31), see for instance, [10, 15, 18] and references therein. The following lemma is a consequence of these works.

Lemma 4.4. (i) *Let λ, b_0, M be positive constants and consider the following nonlinear boundary value problem*

$$\begin{cases} -\Delta v = \lambda v - b_0 g(v) & \text{in } B_r, \\ v = M & \text{on } \partial B_r. \end{cases} \quad (4.35)$$

Then (4.35) has an unique positive solution denoted by $\Theta_{[\lambda, b_0, M, B_r]}$.

(ii) Suppose $p > 3$. For each $x \in B_r$, the point-wise limit

$$\Theta_{[\lambda, b_0, \infty, B_r]}(x) := \lim_{M \uparrow \infty} \Theta_{[\lambda, b_0, M, B_r]}(x)$$

is well defined and it is a classical minimal positive solution of (4.31).

Proof. The existence of positive solution for (4.35) can be easily obtained by the method of sub and supersolution and the uniqueness follows from similar arguments used in Section 3.

To prove (ii), we will apply Theorem 1.1 of [10]. Thus, it is sufficient to show that $g \in \mathcal{C}^1([0, \infty))$, $g \geq 0$ and the map $t \in (0, +\infty) \mapsto g(t)/t$ is increasing and the Keller-Osserman condition, i.e.,

$$\int_1^\infty \frac{dt}{\sqrt{G(t)}} < \infty, \quad \text{where} \quad G(t) := \int_0^t g(s) ds. \quad (4.36)$$

Indeed, the regularity and positivity of g is given by Lemma 4.3. To prove that $t \in (0, +\infty) \mapsto g(t)/t$ is increasing, note that

$$\left(\frac{g(t)}{t} \right)' = \left(\frac{f_1^{p+1}(t)}{t^2} \right)' = \frac{(p+1)f_1^p(t)f_1'(t)t^2 - 2tf_1^{p+1}}{t^2} > 0,$$

if and only if $(p+1)tf_1'(t) > 2f_1(t)$. Since $p > 3$, it follows from Lemma 2.1 (v) that

$$(p+1)tf_1'(t) \geq \frac{(p+1)}{2}f_1(t) > 2f_1(t),$$

showing that $g(t)/t$ is increasing. Finally, observing that (4.33) is a sufficient condition for (4.36) to occur, the proof is complete. \square

Now, we are able to prove the main result of this section.

Proposition 4.5. *Suppose $p > 3$. For each compact $K \subset \Omega_+ = \{x \in \Omega; b(x) > 0\}$, there exists a constant $C = C(\lambda, K) > 0$ such that $\|\Theta_{\lambda, \kappa}\|_{\mathcal{C}(K)} \leq C$ for all $\kappa \in (0, 1)$. Recall that $\Theta_{\lambda, \kappa}$ stands for the unique positive solution of (2.11).*

Proof. Let $B_r := B_r(x_0) \subset \subset \Omega_+$. In particular, $b_K := \min_{x \in B_r} b(x) > 0$. By (2.12) and (4.34), for all $0 < \kappa < 1$, $\lambda > \lambda_1$, $\Theta_{\lambda, \kappa}$ satisfies

$$-\Delta \Theta_{\lambda, \kappa} = \lambda f_\kappa(\Theta_{\lambda, \kappa}) f_\kappa'(\Theta_{\lambda, \kappa}) - b(x) f_\kappa^p(\Theta_{\lambda, \kappa}) f_\kappa'(\Theta_{\lambda, \kappa}) \leq \lambda \Theta_{\lambda, \kappa} - b_K g(\Theta_{\lambda, \kappa}) \quad \text{in } B_r.$$

Thus, $\Theta_{\lambda, \kappa}$ is a subsolution of (4.35) for all $M \geq \max_{B_r} \Theta_{\lambda, \kappa}$. Since large constants are positive super solutions of (4.35), by the sub and supersolution method combined with the uniqueness of positive solution of (4.35), we can infer that

$$\Theta_{\lambda, \kappa} \leq \Theta_{[\lambda, M, b_K, B_r]} \quad \text{in } B_r, \quad \forall M \geq \max_{B_r} \Theta_{\lambda, \kappa}, \quad 0 < \kappa < 1.$$

Letting $M \rightarrow \infty$ in the above inequality, we get

$$\Theta_{\lambda, \kappa} \leq \Theta_{[\lambda, \infty, b_K, B_r]} \quad \text{in } B_r; \quad 0 < \kappa < 1.$$

In particular,

$$\Theta_{\lambda,\kappa} \leq \Theta_{[\lambda,\infty,b_K,B_r]} \quad \text{in } B_{r/2}; \quad 0 < \kappa < 1.$$

Consequently, setting $C := \max_{B_{r/2}} \Theta_{[\lambda,\infty,b_K,B_r]}$, we obtain $\|\Theta_{\lambda,\kappa}\|_{C(B_{r/2})} \leq C$. Observe that C depends on $b_K := \min_{x \in B_r} b(x)$, B_r and λ . Finally, since K can be covered by a finite union of such balls, the proof is complete. \square

5 Proof of Theorem 1.3

In this section, we present the proof of Theorem 1.3. Some arguments used here are inspired in [12]. We point out that we will prove the results for the unique positive solution $\Theta_{\lambda,\kappa}$ of (2.11) and therefore we obtain a similar result for the unique positive solution $\Psi_{\lambda,\kappa} = f_\kappa(\Theta_{\lambda,\kappa})$ of (\mathcal{P}_κ) .

Proof of Theorem 1.3. To prove (a), we will apply the Implicit Function Theorem. Suppose $\lambda \in (\lambda_1, \lambda_{b,0})$. Note that, for $\delta > 0$ small enough, $\kappa \in [0, \delta) \mapsto f_\kappa(\cdot)$ is a continuous map and $f'_\kappa = 1/(1+2\kappa f_\kappa^2)^{1/2}$, $\kappa \in [0, \delta) \mapsto f'_\kappa(\cdot)$ is also continuous. Therefore, we can consider a continuous extension of f_κ and f'_κ at $(-\delta, \delta)$. Define $\mathcal{F} : (-\delta, \delta) \times \mathcal{C}_0^1(\overline{\Omega}) \rightarrow \mathcal{C}_0^1(\overline{\Omega})$ by

$$\mathcal{F}(\kappa, v) = v - (-\Delta)^{-1}[\lambda f_\kappa(v) f'_\kappa(v) - b(f_\kappa^p(v) f'_\kappa(v))].$$

Thus, $\mathcal{F}(\kappa, v)$ is continuous in κ and of class \mathcal{C}^1 in v . Moreover, the zeros of \mathcal{F} provide us the positive solution of (2.11) if $\kappa > 0$ and the positive solution of classical logistic equation (\mathcal{P}_0) if $\kappa = 0$, since $f_0(t) = t$, $t \geq 0$. Differentiating with respect to v at $(0, \Theta_\lambda)$, we have

$$D_v \mathcal{F}(0, \Theta_\lambda)v = v - (-\Delta)^{-1}[\lambda v - pb\Theta_\lambda^{p-1}v], \quad \forall v \in \mathcal{C}_0^1(\overline{\Omega}).$$

Since Θ_λ is a nondegenerate positive solution of (\mathcal{P}_0) , the operator $\mathcal{F}(0, \Theta_\lambda)$ is an isomorphism. Thus, it follows from the Implicit Function Theorem that, for $\delta > 0$ small, there exists a continuous map $\kappa \in (-\delta, \delta) \mapsto v(\kappa) \in \mathcal{C}_0^1(\overline{\Omega})$ such that $v(0) = \Theta_\lambda$ and $\mathcal{F}(\kappa, v(\kappa)) = 0$ for each $\kappa \in (-\delta, \delta)$. Observe that $v(\kappa)$ is a positive solution of (2.11) for $\kappa > 0$ and $\kappa \simeq 0$, since Θ_λ lies in the interior of the positive cone of $\mathcal{C}_0^1(\overline{\Omega})$. Consequently, by the uniqueness of positive solution of (2.11), we obtain that $v(\kappa) = \Theta_{\lambda,\kappa}$. In particular, $\lim_{\kappa \downarrow 0} \Theta_{\lambda,\kappa} = \lim_{\kappa \downarrow 0} v(\kappa) = v(0) = \Theta_\lambda$, completing the proof of item (a). Now, we will prove (b). Suppose $\lambda \geq \lambda_{b,0}$. By the monotonicity of $\lambda \mapsto \Theta_{\lambda,\kappa}$, for each $\varepsilon > 0$ small enough, we have $\Theta_{\lambda_{b,0}-\varepsilon,\kappa} < \Theta_{\lambda,\kappa}$. Using the part (a), we can infer that

$$\Theta_{\lambda_{b,0}-\varepsilon} = \lim_{\kappa \downarrow 0} \Theta_{\lambda_{b,0}-\varepsilon,\kappa} < \liminf_{\kappa \downarrow 0} \Theta_{\lambda,\kappa}.$$

Taking into account (1.6), we conclude that

$$+\infty = \lim_{\varepsilon \rightarrow 0^+} \Theta_{\lambda_{b,0}-\varepsilon} \leq \liminf_{\kappa \downarrow 0} \Theta_{\lambda,\kappa} \quad \text{uniformly in compact subsets of } \overline{\Omega}_{b,0} \setminus \partial\Omega.$$

Therefore, $\lim_{\kappa \downarrow 0} \Theta_{\lambda,\kappa} = +\infty$ uniformly in compact subsets of $\overline{\Omega}_{b,0} \setminus \partial\Omega$ which proves (1.9). Conversely, $M_{\lambda_{b,0}} \leq \liminf_{\kappa \downarrow 0} \Theta_{\lambda,\kappa}$ in $\overline{\Omega}_+$, where $M_{\lambda_{b,0}}$ stands for the minimal positive solution of (1.8) with $\lambda = \lambda_{b,0}$, since $\lim_{\varepsilon \rightarrow 0^+} \Theta_{\lambda_{b,0}-\varepsilon} = M_{\lambda_{b,0}}$ in $\overline{\Omega}_+$. In particular, $\lim_{\kappa \downarrow 0} \Theta_{\lambda,\kappa} = \infty$ on

$\partial\Omega_+$. By a rather standard compactness argument combined with Proposition 4.5 (see for instance [21, Proposition 3.3]), we obtain that the point-wise limit

$$M_\lambda(x) := \lim_{\kappa \downarrow 0} \Theta_{\lambda, \kappa}(x)$$

provide us a classical positive solution of (1.8) and this finalizes the proof. \square

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