

New lower bounds on the radius of spatial analyticity for the KdV equation

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Abstract

The radius of spatial analyticity for solutions of the KdV equation is studied. It is shown that the analyticity radius does not decay faster than $t^{-1/4}$ as time t goes to infinity. This improves the works [Selberg, da Silva, Lower bounds on the radius of spatial analyticity for the KdV equation, *Annales Henri Poincaré*, 2017, 18(3): 1009-1023] and [Tefahun, Asymptotic lower bound for the radius of spatial analyticity to solutions of KdV equation, arXiv preprint arXiv:1707.07810, 2017]. Our strategy mainly relies on a higher order almost conservation law in Gevrey spaces, which is inspired by the I -method.

Keywords: KdV equation; Radius of spatial analyticity; I -method.

AMS subject classifications: 35Q53; 35L30.

1 Introduction

In this paper, we are concerned with the Cauchy problem for the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0, \quad t, x \in \mathbb{R}, \quad u(0, x) = u_0(x). \quad (1.1)$$

Here, the unknown function $u(t, x)$ and the datum $u_0(x)$ are real-valued. The KdV equation models the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems. The ill-posedness and well-posedness of the KdV equation in Sobolev spaces H^s have been extensively studied. For instance, Christ, Colliander and Tao [6] showed that the equation (1.1) is ill-posed in $H^s(\mathbb{R})$ for $s < -\frac{3}{4}$. Kenig, Ponce and Vega [15] proved the local well-posedness in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$. With the same range of s , the global well-posedness were obtained by Colliander, Keel, Staffilani, Takaoka and Tao in [7]. In the critical case $s = -\frac{3}{4}$, the KdV equation is globally well-posed. This is shown by Guo[13] and Kishimoto[16] independently.

The linear KdV equation, also called the Airy equation, does not have a global smoothing effect. Precisely, it is only expected that $e^{-t\partial_x^3}u_0(t \neq 0)$ belongs to $H^s(\mathbb{R})$ for a general datum u_0 belonging to $H^s(\mathbb{R})$. Thus, in principle, the solution of (1.1) belongs to at most $H^s(\mathbb{R})$ in general if u_0 belongs to $H^s(\mathbb{R})$. But some interesting things happen if some further restrictions

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are imposed on the datum. In fact, Kato and Ogawa [27] showed that if the datum u_0 belongs to $H^s(\mathbb{R})(s > -\frac{3}{4})$ and satisfies

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k u_0\|_{H^s} < \infty$$

for some positive constant A_0 , then the solution of (1.1) is analytic in both space and time variable. As a direct corollary, if u_0 is the Dirac measure at the origin, then the solution of (1.1) is analytic. Moreover, Tarama [35] proved the following result: If u_0 belongs to $L^2(\mathbb{R})$ and satisfies

$$\int_{-\infty}^{\infty} (1 + |x|)|u_0(x)| dx + \int_0^{\infty} e^{\delta|x|^{\frac{1}{2}}} |u_0(x)| dx < \infty$$

for some positive constant δ , then the solution of (1.1) is analytic in spatial variable x for any $t > 0$. Tarama's result implies that, roughly speaking, the rapid decay of the datum implies the spatial analyticity of the solution for the KdV equation. The phenomenon was investigated by Rubkin [29] in a more general framework. It is proved in [29] that, if $u_0 = \mathcal{O}(e^{-c|x|^\alpha})(x \rightarrow +\infty)$ and satisfies some other slight restrictions, then the solution of (1.1) is

- (a) analytic in x on the whole plane when $\alpha > \frac{1}{2}$,
- (b) analytic in a strip around the real line when $\alpha = \frac{1}{2}$,
- (c) Gevrey-regular if $\alpha < \frac{1}{2}$.

Motivated by these works, it is interesting to study the well-posedness for the KdV equation in analytic function spaces.

A nice choice of the analytic function space is the Gevrey space $G^\sigma(\mathbb{R})(\sigma > 0)$, consisting of functions such that

$$\|f\|_{G^\sigma(\mathbb{R})} := \|e^{\sigma|\xi|} \widehat{f}(\xi)\|_{L^2(\mathbb{R})} < \infty,$$

where $\widehat{f}(\xi)$ denotes the Fourier transform of f . In fact, according to the Paley-Wiener Theorem (see e.g. [14]), a function belongs to G^σ if and only if it can be extended to an analytic function on the strip

$$S_\sigma := \{z \in \mathbb{C} : |\operatorname{Im} z| < \sigma\}.$$

The local well-posedness of the KdV equation in G^σ has been studied by several mathematicians. Grujić and Kalisch [10] showed that, if the datum u_0 belongs to G^{σ_0} for some $\sigma_0 > 0$, then the KdV equation (1.1) has a unique solution $u \in C[-T, T; G^{\sigma_0}]$ with a lifespan T depending on $\|u_0\|_{G^{\sigma_0}}$. Similar results for the periodic KdV equation are proved by Hannah, Himonas and Petronilho [23, 24] and Li [20]. The work by Grujić and Kalisch [10] improved the earlier results of Hayashi [21, 22], where the analyticity radius $\sigma(t)$ of local solution may depend on t . The local well-posedness in [9, 10, 23, 24, 20] shows that, for short times, the KdV equation persists the uniform radius of spatial analyticity as time progresses.

Now we turn to the global well-posedness. In Sobolev spaces H^s , as mentioned above, the study on the global well-posedness of the KdV equation is more or less complete. However, in analytic function spaces, the global well-posedness of the KdV equation is still open, mainly due to the lack of conservation law. In other words, it is not known whether $u(t) \in G^{\sigma_0}$ for all $t > 0$ if $u_0 \in G^{\sigma_0}$, where $u(t)$ is the solution of the KdV equation (1.1). But one can ask the following question instead: For what kind of function $\sigma(t)$ such that $u(t)$ belongs to $G^{\sigma(t)}$ for all $t > 0$ ¹?

In the sequel, we recall some progresses on the problem. With the aid of Liapunov functions with a parameter, Kato and Masuda showed [26, Theorem 2, p. 459] that, for every $T > 0$ fixed,

¹By the embedding $G^\sigma \hookrightarrow G^{\sigma'}$ for $\sigma > \sigma'$, the function $\sigma(t)$ is necessarily less or equal to σ_0 .

there exists $r > 0$ such that $\sigma(t) \geq r$ for $t \in [0, T]$. In particular, the result implies that the solution of the KdV equation is analytic on some strip at any time. Bona and Grujić gave an explicit lower bound of the uniform radius of analyticity by a Gevrey-class approach. In fact, it is shown [1, Theorem 11 and Remark 12, p. 355] that $\sigma(t) \geq e^{-ct^2}$ for large t , where c depends on the Gevrey norm of the datum. Later, Bona, Grujić and Kalisch improved the exponential decay bound to an algebraic lower bound: $\sigma(t) \geq t^{-12}$ for large t , see [2, Corollary 2, p. 795]. More recently, Selberg and Silva [30] obtained a further refinement: $\sigma(t) \geq t^{-\frac{4}{3}-\varepsilon}$ for large t , where ε is an arbitrary positive number. The strategy in [30] is as follows:

- (1) Prove a local well-posedness by contraction mapping principle in G^σ with a lifespan $\delta > 0$;
- (2) Establish an almost conservation law in G^σ , namely²

$$\|u(\delta)\|_{G^\sigma}^2 \leq \|u_0\|_{G^\sigma}^2 + C\sigma^{\frac{3}{4}-\varepsilon}\|u_0\|_{G^\sigma}^2; \quad (1.2)$$

- (3) By shrinking σ gradually, they used repeatedly the local well-posedness and the almost conservation law on the intervals $[0, \delta], [\delta, 2\delta], \dots$, and obtained a global bound of solution on $[0, T]$, with T arbitrarily large.

In a paper [34] on arxiv, Tesfahun removed the ε exponent in the conservation law (1.2), via spacetime dyadic bilinear estimates associated with the KdV equation. This leads to the following improvement: $\sigma(t) \geq t^{-\frac{4}{3}}$ for large t . In this paper, we are able to show that $\sigma(t) \geq t^{-\frac{1}{4}}$ for large t . The precise statement is as follows.

Theorem 1.1. *Let $\sigma_0 > 0$ and $u_0 \in G^{\sigma_0}$. Then the KdV equation (1.1) has a unique smooth solution u such that*

$$u(t) \in G^{\sigma(t)}, \quad t \in \mathbb{R}$$

with the radius of analyticity $\sigma(t)$ satisfying the lower bound

$$\sigma(t) \geq c|t|^{-\frac{1}{4}} \quad \text{as } |t| \rightarrow \infty,$$

where c is a constant depending on $\|u_0\|_{G^{\sigma_0}}$ and σ_0 .

The proof of Theorem 1.1 is based on a simple observation: If one can prove the almost conservation law

$$\|u(\delta)\|_{G^\sigma}^2 \leq \|u_0\|_{G^\sigma}^2 + C(\|u_0\|_{G^\sigma}^2)\sigma^\alpha \quad (1.3)$$

with a larger $\alpha > 0$, then one obtains a better lower bound of $\sigma(t)$ by the strategy in [30]. To this end, inspired by the I -method in [7], we define the modified energies $E_I^2(t), \dots, E_I^4(t)$ (see Section 2 for definitions) in Gevrey space G^σ , and prove that

$$E_I^4(t) \text{ is comparable with } E_I^2(t) \text{ for all } t \in \mathbb{R} \text{ when } E_I^2 \text{ is small,} \quad (1.4)$$

$$|E_I^4(\delta) - E_I^4(0)| \leq C\|u_0\|_{G^\sigma}^5 \sigma^4. \quad (1.5)$$

Combining (1.4) and (1.5) we find that (1.3) holds with $\alpha = 4$ for small $\|u_0\|_{G^\sigma}$. The smallness can be removed by a scaling. In a word, this leads a better lower bound $\sigma(t) \geq c|t|^{-\frac{1}{4}}$.

We do not believe that the lower bound in Theorem 1.1 is optimal. In fact, it is probably can be improved by introducing further modified energies $E_I^5(t), \dots$ in the scheme as in [7].

Finally, we mention some references devoted to the uniform radius of analyticity for other partial differential equations. We refer the readers to [4, 25, 32] for generalized KdV equations, to [3, 5, 33] for Schrödinger equations, to [17, 18, 19] for Euler equations, to [12, 28, 31] for Klein-Gordon equations, and to [8] for the cubic Szegő equation.

²In fact, by letting σ go to 0 in (1.2), one obtained the L^2 conservation law of the KdV equation.

2 Preliminaries

2.1 Local well posedness

First, we introduce some function spaces used in this paper. For $s, b \in \mathbb{R}$, we use $X^{s,b}(\mathbb{R}^2)$ to denote the Bourgain space defined by the norm

$$\|f\|_{X^{s,b}(\mathbb{R}^2)} := \|(1 + |\xi|)^s (1 + |\tau - \xi^3|)^b \widehat{f}(\xi, \tau)\|_{L^2(\mathbb{R}^2)},$$

where $\widehat{f}(\xi, \tau)$ denotes the space-time Fourier transform of $f(x, t)$:

$$\widehat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-i(t\tau + x\xi)} f(x, t) dx dt.$$

Replacing $(1 + |\xi|)^s$ by $e^{\sigma|\xi|}$ in the norm of the Bourgain space, we obtain a Gevrey type Bourgain space $G^{\sigma,b}(\mathbb{R}^2)$ defined by the norm

$$\|f\|_{G^{\sigma,s,b}(\mathbb{R}^2)} := \|e^{\sigma|\xi|} (1 + |\tau - \xi^3|)^b \widehat{f}(\xi, \tau)\|_{L^2(\mathbb{R}^2)}.$$

Also, for $\delta > 0$, we use $X_\delta^{s,b}, G_\delta^{\sigma,b}$ to denote the restrictions of $X^{s,b}$ and $G^{\sigma,b}$ to $\mathbb{R} \times (-\delta, \delta)$, respectively. More precisely, $X_\delta^{s,b}$ and $G_\delta^{\sigma,b}$ are defined by the norms as follows:

$$\|f\|_{X_\delta^{s,b}} = \inf\{\|g\|_{X^{s,b}} : g = f \text{ on } \mathbb{R} \times (-\delta, \delta)\},$$

$$\|f\|_{G_\delta^{\sigma,b}} = \inf\{\|g\|_{G^{\sigma,b}} : g = f \text{ on } \mathbb{R} \times (-\delta, \delta)\}.$$

Next, we give a local well posedness result for the KdV equation.

According to Corollary 2.7 in [15], for $b' \in (\frac{1}{2}, \frac{3}{4}]$ and $b \in (\frac{1}{2}, b']$, there exists a positive constant $c = c(b, b')$ such that

$$\|\partial_x(uv)\|_{X^{0,b'-1}} \leq c \|u\|_{X^{0,b}} \|v\|_{X^{0,b}}. \quad (2.1)$$

Using the obvious inequality $e^{\sigma|\xi|} \leq e^{\sigma|\xi_1|} e^{\sigma|\xi_2|}$, $\xi = \xi_1 + \xi_2$, we deduce from (2.1) that with the same range of b and b'

$$\|\partial_x(uv)\|_{G^{\sigma,b'-1}} \leq c \|u\|_{G^{\sigma,b}} \|v\|_{G^{\sigma,b}}. \quad (2.2)$$

Applying the bilinear estimate (2.2) with $b' = \frac{3}{4}$, and using the contraction mapping principle (or following the proof of [30, Theorem 1]), we obtain the following result.

Proposition 2.1 (local well posedness). *Let $\sigma > 0$ and $b \in (\frac{1}{2}, \frac{3}{4})$. Then, for any $u_0 \in G^\sigma(\mathbb{R})$, there exists a time $\delta > 0$ given by*

$$\delta = \frac{c_0}{(1 + \|u_0\|_{G^\sigma})^{\frac{1}{4-b}}} \quad (2.3)$$

and a unique solution u of (1.1) such that

$$\|u\|_{G_\delta^{\sigma,b}} \leq C \|u_0\|_{G^\sigma}, \quad (2.4)$$

where the constants C, c_0 depend only on b . Moreover, the solution map $u_0 \mapsto u(t)$ is continuous from G^σ to G^σ for every $t \in [-\delta, \delta]$.

Finally, we state a multi-linear estimates will be used later.

Lemma 2.1. *Let $\sigma \geq 0, \delta > 0, -\frac{1}{2} < b' < -\frac{1}{3}$ and $b > \frac{1}{2}$. Let $|D|$ be the Fourier multiplier with symbol $|\xi|$. Then there exists a constant $C = C(b, b')$ such that*

$$\left\| |D| \prod_{i=1}^4 u_i \right\|_{X_\delta^{0, b'}} \leq C \prod_{i=1}^4 \|u_i\|_{X_\delta^{0, b}}. \quad (2.5)$$

Proof. Grünrock [11, Theorem 1] proved the following multi-linear estimates: If $-\frac{1}{2} < b' < -\frac{1}{3}$, $b > \frac{1}{2}$, then for some $C = C(b, b')$

$$\left\| \partial_x \prod_{i=1}^4 u_i \right\|_{X^{0, b'}} \leq C \prod_{i=1}^4 \|u_i\|_{X^{0, b}}. \quad (2.6)$$

By Plancherel's theorem, $\|\partial_x \prod_{i=1}^4 u_i\|_{X^{0, b'}} = \||D| \prod_{i=1}^4 u_i\|_{X^{0, b'}}$. The desired bound (2.5) follows from (2.6) in a standard way, see e.g. [36, Corollary 1]. \square

2.2 Multi-linear forms for KdV

In this section, we borrow some known results from [35] on multi-linear forms for the KdV equation.

Definition 2.1. *A k -multiplier is a function $m : \mathbb{R}^k \mapsto \mathbb{C}$. A k -multiplier is symmetric if $m(\xi_1, \xi_2, \dots, \xi_k) = m(\sigma(\xi_1, \xi_2, \dots, \xi_k))$ for all $\sigma \in S_k$, the group of all permutations on k objects. The symmetrization of a k -multiplier is the multiplier*

$$[m]_{sym}(\xi_1, \xi_2, \dots, \xi_k) = \frac{1}{k!} \sum_{\sigma \in S_k} m(\sigma(\xi_1, \xi_2, \dots, \xi_k)).$$

Definition 2.2. *A k -multiplier generates a k -linear functional or k -form acting on k functions u_1, u_2, \dots, u_k ,*

$$\Lambda_k(m; u_1, u_2, \dots, u_k) = \int_{\xi_1 + \xi_2 + \dots + \xi_k = 0} m(\xi_1, \xi_2, \dots, \xi_k) \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2) \dots \widehat{u}_k(\xi_k).$$

In particular, if $u_1 = u_2 = \dots = u_k = u$ we write $\Lambda_k(m) = \Lambda_k(m; \underbrace{u, u, \dots, u}_{k \text{ times}})$ for brevity.

If m is symmetric, then $\Lambda_k(m)$ is a symmetric k -linear functional. The symmetry is important in the following discussion. To see this, we give a Fourier proof of the fact

$$\int_{\mathbb{R}} u^k u_x dx = 0, \quad k \in \mathbb{N}, u \in \mathcal{S}.$$

Indeed, by the Plancherel's theorem, we write

$$\begin{aligned} \int_{\mathbb{R}} u^k u_x dx &= \int_{\mathbb{R}} i \xi_1 \widehat{u}(\xi_1) \widehat{u}^k(-\xi_1) d\xi_1 \\ &= \int_{\xi_1 + \xi_2 + \dots + \xi_{k+1} = 0} i \xi_1 \widehat{u}(\xi_1) \widehat{u}(\xi_2) \dots \widehat{u}(\xi_k) \\ &= \int_{\xi_1 + \xi_2 + \dots + \xi_{k+1} = 0} i \xi_j \widehat{u}(\xi_1) \widehat{u}(\xi_2) \dots \widehat{u}(\xi_k) \quad (j = 2, \dots, k+1) \\ &= \int_{\xi_1 + \xi_2 + \dots + \xi_{k+1} = 0} i \frac{\xi_1 + \xi_2 + \dots + \xi_{k+1}}{k+1} \widehat{u}(\xi_1) \widehat{u}(\xi_2) \dots \widehat{u}(\xi_k) = 0. \end{aligned}$$

Proposition 2.2. [35, Proposition 1] *Suppose u satisfies the KdV equation (1.1) and that m is a symmetric k -multiplier. Then*

$$\frac{d}{dt}\Lambda_k(m) = \Lambda_k(m\alpha_k) - i\frac{k}{2}\Lambda_k(m(\xi_1, \dots, \xi_{k-1}, \xi_k + \xi_{k+1})\{\xi_k + \xi_{k+1}\}), \quad (2.7)$$

where

$$\alpha_k = i(\xi_1^3 + \dots + \xi_k^3). \quad (2.8)$$

Remark 2.1. *Note that (2.7) still holds if the $k+1$ -multiplier of the second term is symmetrized.*

Let $m : \mathbb{R} \mapsto \mathbb{R}$ be an arbitrary even \mathbb{R} -valued 1-multiplier. Define the associated operator by

$$\widehat{I}f(\xi) = m(\xi)\widehat{f}(\xi).$$

Define the modified energy $E_I^2(t)$ by

$$E_I^2(t) = \|Iu(t)\|_{L^2}^2 = \Lambda_2(m(\xi_1)m(\xi_2)). \quad (2.9)$$

Then using Proposition 2.2 and Remark 2.1 we find

$$\frac{d}{dt}E_I^2(t) = \Lambda_3(M_3), \quad M_3(\xi_1, \xi_2, \xi_3) = -i[m(\xi_1)m(\xi_2 + \xi_3)\{\xi_2 + \xi_3\}]_{sym}. \quad (2.10)$$

Set

$$E_I^3(t) = E_I^2(t) + \Lambda_3(\sigma_3), \quad \beta_3 = -\frac{M_3}{\alpha_3}, \quad (2.11)$$

then by Proposition 2.2 and Remark 2.1 again, we have

$$\frac{d}{dt}E_I^3(t) = \Lambda_4(M_4), \quad M_4(\xi_1, \xi_2, \xi_3, \xi_4) = -i\frac{3}{2}[\beta_3(\xi_1, \xi_2, \xi_3 + \xi_4)\{\xi_3 + \xi_4\}]_{sym}. \quad (2.12)$$

Moreover, defining

$$E_I^4(t) = E_I^3(t) + \Lambda_4(\sigma_4), \quad \beta_4 = -\frac{M_4}{\alpha_4}, \quad (2.13)$$

we have

$$\frac{d}{dt}E_I^4(t) = \Lambda_5(M_5), \quad M_5(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = -2i[\beta_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)\{\xi_4 + \xi_5\}]_{sym}. \quad (2.14)$$

Lemma 2.2. *If m is even and \mathbb{R} -valued and M_4 is given by (2.12), then the following identity holds*

$$\begin{aligned} M_4(\xi_1, \xi_2, \xi_3, \xi_4) &= -\frac{c}{108} \frac{\alpha_4}{\xi_1 \xi_2 \xi_3 \xi_4} [m^2(\xi_1) + m^2(\xi_2) + m^2(\xi_3) + m^2(\xi_4) \\ &\quad - m^2(\xi_1 + \xi_2) - m^2(\xi_1 + \xi_3) - m^2(\xi_1 + \xi_4)] \\ &\quad + \frac{c}{36} \left\{ \frac{m^2(\xi_1)}{\xi_1} + \frac{m^2(\xi_2)}{\xi_2} + \frac{m^2(\xi_3)}{\xi_3} + \frac{m^2(\xi_4)}{\xi_4} \right\}, \end{aligned} \quad (2.15)$$

where c is an absolute constant, α_4 is given by (2.8).

Moreover, it is easy to show that, on the hyperplane $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$,

$$\alpha_4 = \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 \quad (2.16)$$

$$= 3(\xi_1 \xi_2 \xi_3 + \xi_1 \xi_2 \xi_4 + \xi_1 \xi_3 \xi_4 + \xi_2 \xi_3 \xi_4) \quad (2.17)$$

$$= 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4). \quad (2.18)$$

3 Point-wise bounds for M_4 and β_4

The goal of this section is to give some point-wise bounds for multipliers M_4 and β_4 , which will play an important role in the proof of almost conversation law. The multiplier m in M_4 , needed in this paper, is given as follows.

Let $\sigma > 0$. Set

$$m(\xi) = \frac{e^{\sigma\xi} + e^{-\sigma\xi}}{2}, \quad \xi \in \mathbb{R}. \quad (3.1)$$

It is easy to see that

$$e^{\sigma|\xi|}/2 \leq m(\xi) \leq e^{\sigma|\xi|}, \quad \xi \in \mathbb{R},$$

from which, we find

$$\|f\|_{G^\sigma}/2 \leq \|m(D)f\|_{L^2} \leq \|f\|_{G^\sigma}, \quad f \in \mathcal{S}. \quad (3.2)$$

In other words, $\|m(D) \cdot\|_{L^2}$ is an equivalent norm of $\|\cdot\|_{G^\sigma}$.

By Taylor expansion, we have

$$m(\xi) = \sum_{k=0}^{\infty} \frac{(\sigma\xi)^{2k}}{(2k)!}, \quad \xi \in \mathbb{R}. \quad (3.3)$$

Using (3.3), we deduce from Lemma 2.2 that

$$\begin{aligned} M_4(\xi_1, \xi_2, \xi_3, \xi_4) = & -\frac{c}{108} \frac{\alpha_4}{\xi_1 \xi_2 \xi_3 \xi_4} \sum_{k=0}^{\infty} \frac{\sigma^{2k}}{(2k)!} \left[\xi_1^{2k} + \xi_2^{2k} + \xi_3^{2k} + \xi_4^{2k} \right. \\ & \left. - (\xi_1 + \xi_2)^{2k} - (\xi_1 + \xi_3)^{2k} - (\xi_1 + \xi_4)^{2k} \right] \\ & + \frac{c}{36} \sum_{k=0}^{\infty} \frac{\sigma^{2k}}{(2k)!} \left(\xi_1^{2k-1} + \xi_2^{2k-1} + \xi_3^{2k-1} + \xi_4^{2k-1} \right). \end{aligned} \quad (3.4)$$

Note that the terms in the sum of (3.4) are polynomials of $\xi_1, \xi_2, \xi_3, \xi_4$, which allow us to obtain cancelation conveniently on the hyperplane

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0. \quad (3.5)$$

Before giving a detailed analysis of M_4 , we first show that the terms on the right hand side of (3.4) vanishes on the hyperplane (3.5) if $k = 0, 1$. In other words, the sums in (3.4) are only taking for $k \geq 2$. In fact, in the case $k = 0$, using the property (2.17) of α_4 , we find

$$\begin{aligned} RHS(3.4) = & -\frac{c}{108} \frac{\alpha_4}{\xi_1 \xi_2 \xi_3 \xi_4} + \frac{c}{36} \left(\xi_1^{-1} + \xi_2^{-1} + \xi_3^{-1} + \xi_4^{-1} \right) \\ = & \frac{c}{36} \frac{1}{\xi_1 \xi_2 \xi_3 \xi_4} \left(\xi_1 \xi_2 \xi_3 + \xi_1 \xi_2 \xi_4 + \xi_1 \xi_3 \xi_4 + \xi_2 \xi_3 \xi_4 - \frac{\alpha_4}{3} \right) = 0. \end{aligned}$$

In the case $k = 1$, we have

$$\begin{aligned} RHS(3.4) = & -\frac{c}{108} \frac{\alpha_4}{\xi_1 \xi_2 \xi_3 \xi_4} \frac{\sigma^2}{2} \left[\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 - (\xi_1 + \xi_2)^2 - (\xi_1 + \xi_3)^2 - (\xi_1 + \xi_4)^2 \right] \\ & + \frac{c}{36} \frac{\sigma^2}{2} (\xi_1 + \xi_2 + \xi_3 + \xi_4) \\ = & \frac{\sigma^2}{216} (\xi_1 + \xi_2 + \xi_3 + \xi_4) \left(3 + \frac{\alpha_4}{\xi_2 \xi_3 \xi_4} \right) = 0. \end{aligned}$$

Thus, M_4 can be rewritten as

$$\begin{aligned} M_4(\xi_1, \xi_2, \xi_3, \xi_4) = & -\frac{c}{108} \frac{\alpha_4}{\xi_1 \xi_2 \xi_3 \xi_4} \sum_{k=2}^{\infty} \frac{\sigma^{2k}}{(2k)!} \Omega_1(k; \xi_1, \xi_2, \xi_3, \xi_4) \\ & + \frac{c}{36} \sum_{k=1}^{\infty} \frac{\sigma^{2(k+1)}}{(2(k+1))!} \Omega_2(k; \xi_1, \xi_2, \xi_3, \xi_4), \end{aligned} \quad (3.6)$$

where Ω_1 and Ω_2 are given by

$$\Omega_1(k; \xi_1, \xi_2, \xi_3, \xi_4) := \xi_1^{2k} + \xi_2^{2k} + \xi_3^{2k} + \xi_4^{2k} - (\xi_1 + \xi_2)^{2k} - (\xi_1 + \xi_3)^{2k} - (\xi_1 + \xi_4)^{2k}, \quad (3.7)$$

$$\Omega_2(k; \xi_1, \xi_2, \xi_3, \xi_4) := \xi_1^{2k+1} + \xi_2^{2k+1} + \xi_3^{2k+1} + \xi_4^{2k+1}. \quad (3.8)$$

In order to obtain bounds of M_4 and β_4 , according to (3.6) and (2.13), we need to control

$$\frac{\Omega_1}{\xi_1 \xi_2 \xi_3 \xi_4}, \quad \frac{\Omega_2}{\alpha_4}.$$

At a first glance, there are singularities in the two terms. But this is not the case on the hyperplane $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$. We report the fact in the following two subsections.

3.1 Decomposition 1

In this subsection, we get rid of the singularity of $\frac{\Omega_1}{\xi_1 \xi_2 \xi_3 \xi_4}$. To this end, we shall show that for $k \geq 2$

$$\Omega_1 = \xi_1^{2k} + \xi_2^{2k} + \xi_3^{2k} + \xi_4^{2k} - (\xi_1 + \xi_2)^{2k} - (\xi_1 + \xi_3)^{2k} - (\xi_1 + \xi_4)^{2k} = \xi_1 \xi_2 \xi_3 \xi_4 \cdot \text{a polynomial}$$

on the hyperplane $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$. This is contained in the following lemma, in which we give a formula of the polynomial.

Lemma 3.1. *Assume that $\mathbb{N} \ni k \geq 2$ and Ω_1 is given by (3.7). Then*

$$\begin{aligned} \frac{\Omega_1(k)}{\xi_1 \xi_2 \xi_3 \xi_4} = & \sum_{i+j=2k-5} (-1)^i \left((\xi_1^{j+1} + \xi_2^{j+1}) \sum_{m+l=i} \xi_3^m (\xi_3 + \xi_4)^l + (\xi_1^{i+1} - \xi_3^{i+1}) \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_4)^l \right) \\ & + (\xi_2^{i+1} - \xi_3^{i+1}) \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_4)^l + (\xi_1^{i+1} + \xi_2^{i+1}) \sum_{m+l=j} \xi_4^m (\xi_3 + \xi_4)^l \\ & - 2 \sum_{i+j=2k-4} \left(\xi_1^i (\xi_1 + \xi_4)^j + \xi_2^i (\xi_2 + \xi_4)^j + \xi_3^i (\xi_3 + \xi_4)^j + \xi_4^i (\xi_3 + \xi_4)^j \right) \\ & - \sum_{i+j=2k-4} (-1)^i \left(\xi_4^i \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_3)^l + \xi_3^i \sum_{m+l=j} \xi_4^m (\xi_2 + \xi_4)^l \right) \\ & - \sum_{i+j=2k-4} (-1)^i \left(\xi_4^i \sum_{m+l=j} (\xi_2^m + \xi_3^m) (\xi_2 + \xi_3)^l \right). \end{aligned} \quad (3.9)$$

Remark 3.1. *The sums in (3.9) are taking for all nonnegative numbers. For example,*

$$\sum_{m+l=i} \dots = \sum_{m+l=i; m, l \geq 0} \dots$$

Moreover, the sum vanishes if the sum taking over the empty set. For example,

$$\sum_{i+j=-1} \dots = \sum_{i+j=-1; i, j \geq 0} \dots = 0.$$

The sums in the rest of the paper are understood in the same way.

Remark 3.2. In particular, setting $k = 2$ in (3.9), using Remark 3.1, we find

$$\Omega_1(2; \xi_1, \xi_2, \xi_3, \xi_4) = -12\xi_1\xi_2\xi_3\xi_4. \quad (3.10)$$

Proof of Lemma 3.1. The proof is long and the computation is complicated. But the reader can build some intuitions by working out (3.10) following our strategy in the sequel.

We divide the discussion into four steps.

Step 1. Find $\frac{\Omega_1}{\xi_1}$. Rewrite Ω_1 as

$$\Omega_1 = \xi_1^{2k} + \xi_2^{2k} - (\xi_1 + \xi_2)^{2k} + \xi_3^{2k} - (\xi_1 + \xi_3)^{2k} + \xi_4^{2k} - (\xi_1 + \xi_4)^{2k}.$$

Using the elementary identity $x^n - y^n = (x - y) \sum_{i+j=n-1} x^i y^j$ ³ we find

$$\Omega_1 = \xi_1^{2k} - \xi_1 \sum_{i+j=2k-1} \xi_2^i (\xi_1 + \xi_2)^j + \xi_3^i (\xi_1 + \xi_3)^j + \xi_4^i (\xi_1 + \xi_4)^j.$$

Thus, we have

$$\frac{\Omega_1}{\xi_1} = \xi_1^{2k-1} - \sum_{i+j=2k-1} \xi_2^i (\xi_1 + \xi_2)^j + \xi_3^i (\xi_1 + \xi_3)^j + \xi_4^i (\xi_1 + \xi_4)^j. \quad (3.11)$$

Step 2. Find $\frac{\Omega_1}{\xi_1 \xi_2}$. Split

$$\sum_{i+j=2k-1} \xi_2^i (\xi_1 + \xi_2)^j = (\xi_1 + \xi_2)^{2k-1} + \sum_{i+j=2k-1, i \geq 1} \xi_2^i (\xi_1 + \xi_2)^j, \quad (3.12)$$

and use $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ to write

$$\xi_3^i (\xi_1 + \xi_3)^j = \xi_3^i (-\xi_2 - \xi_4)^j, \quad \xi_4^i (\xi_1 + \xi_4)^j = \xi_4^i (-\xi_2 - \xi_3)^j. \quad (3.13)$$

Inserting (3.12) and (3.13) into (3.11) we obtain

$$\begin{aligned} \frac{\Omega_1}{\xi_1} &= \xi_1^{2k-1} - (\xi_1 + \xi_2)^{2k-1} - \sum_{i+j=2k-1, i \geq 1} \xi_2^i (\xi_1 + \xi_2)^j \\ &\quad - \sum_{i+j=2k-1} \left(\xi_3^i (-\xi_2 - \xi_4)^j + \xi_4^i (-\xi_2 - \xi_3)^j \right). \end{aligned} \quad (3.14)$$

On one hand, we have

$$\begin{aligned} &\xi_1^{2k-1} - (\xi_1 + \xi_2)^{2k-1} - \sum_{i+j=2k-1, i \geq 1} \xi_2^i (\xi_1 + \xi_2)^j \\ &= -\xi_2 \sum_{i+j=2k-2} \xi_1^i (\xi_1 + \xi_2)^j - \xi_2 \sum_{i+j=2k-1, i \geq 1} \xi_2^{i-1} (\xi_1 + \xi_2)^j. \end{aligned} \quad (3.15)$$

³We shall use the fact implicitly in the sequel.

Since $\sum_{i+j=2k-1, i \geq 1} \xi_2^{i-1} (\xi_1 + \xi_2)^j = \sum_{i+j=2k-2} \xi_2^i (\xi_1 + \xi_2)^j$,

$$RHS(3.15) = -\xi_2 \sum_{i+j=2k-2} (\xi_1^i + \xi_2^i) (\xi_1 + \xi_2)^j. \quad (3.16)$$

On the other hand,

$$\begin{aligned} - \sum_{i+j=2k-1} \left(\xi_3^i (-\xi_2 - \xi_4)^j + \xi_4^i (-\xi_2 - \xi_3)^j \right) &= - \sum_{i+j=2k-1} \left(\xi_3^i (-\xi_2 - \xi_4)^j + \xi_4^j (-\xi_2 - \xi_3)^i \right) \\ &= \sum_{i+j=2k-1} (-1)^i \left(\xi_3^i (\xi_2 + \xi_4)^j - \xi_4^j (\xi_2 + \xi_3)^i \right). \end{aligned} \quad (3.17)$$

Split the sum in (3.17) into cases: (1) $i, j \geq 1$, (2) $i = 0$, (3) $j = 0$. Then we obtain

$$\begin{aligned} RHS(3.17) &= \sum_{i+j=2k-1, i, j \geq 1} (-1)^i \left(\xi_3^i (\xi_2 + \xi_4)^j - \xi_4^j (\xi_2 + \xi_3)^i \right) \\ &\quad + (\xi_2 + \xi_4)^{2k-1} - \xi_4^{2k-1} + (\xi_2 + \xi_3)^{2k-1} - \xi_3^{2k-1}. \end{aligned} \quad (3.18)$$

Rewrite the term in the sum of (3.18) as

$$\begin{aligned} \xi_3^i (\xi_2 + \xi_4)^j - \xi_4^j (\xi_2 + \xi_3)^i &= \xi_3^i ((\xi_2 + \xi_4)^j - \xi_4^j) - \xi_4^j ((\xi_2 + \xi_3)^i - \xi_3^i) \\ &= \xi_3^i \xi_2 \sum_{m+l=j-1} \xi_4^m (\xi_2 + \xi_4)^l - \xi_4^j \xi_2 \sum_{m+l=i-1} \xi_3^m (\xi_2 + \xi_3)^l. \end{aligned} \quad (3.19)$$

Thanks to (3.19), we deduce from (3.18) that

$$\begin{aligned} RHS(3.17) &= \sum_{i+j=2k-1, i, j \geq 1} (-1)^i \xi_2 \left(\xi_3^i \sum_{m+l=j-1} \xi_4^m (\xi_2 + \xi_4)^l - \xi_4^j \sum_{m+l=i-1} \xi_3^m (\xi_2 + \xi_3)^l \right) \\ &\quad + \xi_2 \left(\sum_{i+j=2k-2} \xi_4^i (\xi_2 + \xi_4)^j + \xi_2 \sum_{i+j=2k-2} \xi_3^i (\xi_2 + \xi_3)^j \right). \end{aligned} \quad (3.20)$$

Combining (3.16) and (3.20) that

$$\begin{aligned} \frac{\Omega_1}{\xi_1 \xi_2} &= - \sum_{i+j=2k-2} (\xi_1^i + \xi_2^i) (\xi_1 + \xi_2)^j + \sum_{i+j=2k-2} \xi_4^i (\xi_2 + \xi_4)^j + \sum_{i+j=2k-2} \xi_3^i (\xi_2 + \xi_3)^j \\ &\quad + \sum_{i+j=2k-1, i, j \geq 1} (-1)^i \left(\xi_3^i \sum_{m+l=j-1} \xi_4^m (\xi_2 + \xi_4)^l - \xi_4^j \sum_{m+l=i-1} \xi_3^m (\xi_2 + \xi_3)^l \right). \end{aligned} \quad (3.21)$$

Step 3. Find $\frac{\Omega_1}{\xi_1 \xi_2 \xi_3}$. Using $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ again, we rewrite (3.21) as

$$\begin{aligned} \frac{\Omega_1}{\xi_1 \xi_2} &= \sum_{i+j=2k-2} \xi_4^i (-\xi_1 - \xi_3)^j - \xi_1^i (-\xi_3 - \xi_4)^j + \sum_{i+j=2k-2} \xi_3^i (\xi_2 + \xi_3)^j - \sum_{i+j=2k-2} \xi_2^i (-\xi_3 - \xi_4)^j \\ &\quad + \sum_{i+j=2k-1, i, j \geq 1} (-1)^i \left(\xi_3^i \sum_{m+l=j-1} \xi_4^m (\xi_2 + \xi_4)^l - \xi_4^j \sum_{m+l=i-1} \xi_3^m (\xi_2 + \xi_3)^l \right) \\ &= A_1 + A_2 + A_3, \end{aligned} \quad (3.22)$$

where A_1, A_2, A_3 are given by

$$A_1 = \sum_{i+j=2k-2} \xi_4^i (-\xi_1 - \xi_3)^j - \xi_1^i (-\xi_3 - \xi_4)^j, \quad (3.23)$$

$$\begin{aligned} A_2 &= \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-1)^i \xi_3^i \sum_{m+l=j-1} \xi_4^m (\xi_2 + \xi_4)^l - \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-1)^i \xi_4^j \sum_{\substack{m+l=i-1, \\ m \geq 1}} \xi_3^m (\xi_2 + \xi_3)^l \\ &+ \sum_{\substack{i+j=2k-2, \\ i \geq 1}} \xi_3^i (\xi_2 + \xi_3)^j, \end{aligned} \quad (3.24)$$

$$A_3 = - \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-1)^i \xi_4^j \sum_{\substack{m+l=i-1, \\ m=0}} \xi_3^m (\xi_2 + \xi_3)^l + (\xi_2 + \xi_3)^{2k-2} - \sum_{i+j=2k-2} \xi_2^i (-\xi_3 - \xi_4)^j.$$

We simplify A_3 as

$$\begin{aligned} A_3 &= - \sum_{i+j=2k-1, i,j \geq 1} (-1)^i \xi_4^j (\xi_2 + \xi_3)^{i-1} + (\xi_2 + \xi_3)^{2k-2} - \sum_{i+j=2k-2} \xi_2^i (-\xi_3 - \xi_4)^j \\ &= \sum_{i+j=2k-1, i,j \geq 1} (-\xi_4)^j (\xi_2 + \xi_3)^{i-1} + (\xi_2 + \xi_3)^{2k-2} - \sum_{i+j=2k-2} \xi_2^i (-\xi_3 - \xi_4)^j \\ &= \sum_{i+j=2k-2} (-\xi_4)^j (\xi_2 + \xi_3)^i - \sum_{i+j=2k-2} \xi_2^i (-\xi_3 - \xi_4)^j \\ &= \sum_{i+j=2k-2} (-\xi_4)^i (\xi_2 + \xi_3)^j - \sum_{i+j=2k-2} \xi_2^j (-\xi_3 - \xi_4)^i \\ &= \sum_{i+j=2k-2} (-1)^i \left(\xi_4^i (\xi_2 + \xi_3)^j - \xi_2^j (\xi_3 + \xi_4)^i \right). \end{aligned} \quad (3.25)$$

To proceed, we rewrite (3.25) as

$$\begin{aligned} RHS(3.25) &= \sum_{i+j=2k-2, i,j \geq 1} (-1)^i \left(\xi_4^i (\xi_2 + \xi_3)^j - \xi_2^j (\xi_3 + \xi_4)^i \right) \\ &+ \xi_4^{2k-2} - (\xi_3 + \xi_4)^{2k-2} + (\xi_2 + \xi_3)^{2k-2} - \xi_2^{2k-2} \\ &= \sum_{i+j=2k-2, i,j \geq 1} (-1)^i \left(\xi_4^i \left((\xi_2 + \xi_3)^j - \xi_2^j \right) - \xi_2^j \left((\xi_3 + \xi_4)^i - \xi_4^i \right) \right) \\ &+ \xi_4^{2k-2} - (\xi_3 + \xi_4)^{2k-2} + (\xi_2 + \xi_3)^{2k-2} - \xi_2^{2k-2} \\ &= \sum_{i+j=2k-2, i,j \geq 1} (-1)^i \xi_3 \left(\xi_4^i \sum_{m+l=j-1} \xi_2^m (\xi_2 + \xi_3)^l - \xi_2^j \sum_{m+l=i-1} \xi_4^m (\xi_3 + \xi_4)^l \right) \\ &+ \xi_3 \sum_{i+j=2k-3} \left(\xi_2^i (\xi_2 + \xi_3)^j - \xi_4^i (\xi_3 + \xi_4)^j \right) \\ &= \sum_{i+j=2k-4} (-1)^i \xi_3 \left(\xi_2^{j+1} \sum_{m+l=i} \xi_4^m (\xi_3 + \xi_4)^l - \xi_4^{i+1} \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_3)^l \right) \\ &+ \xi_3 \sum_{i+j=2k-3} \xi_2^i (\xi_2 + \xi_3)^j - \xi_4^i (\xi_3 + \xi_4)^j. \end{aligned} \quad (3.26)$$

Replacing ξ_2 by ξ_1 in (3.26), we obtain

$$\begin{aligned}
A_1 &= \sum_{i+j=2k-2} \xi_4^i (-\xi_1 - \xi_3)^j - \xi_1^i (-\xi_3 - \xi_4)^j \\
&= \sum_{i+j=2k-2} (-1)^i \left(\xi_4^i (\xi_1 + \xi_3)^j - \xi_1^j (\xi_3 + \xi_4)^i \right) \\
&= \sum_{i+j=2k-4} (-1)^i \xi_3 \left(\xi_1^{j+1} \sum_{m+l=i} \xi_4^m (\xi_3 + \xi_4)^l - \xi_4^{i+1} \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_3)^l \right) \\
&\quad + \xi_3 \sum_{i+j=2k-3} \xi_1^i (\xi_1 + \xi_3)^j - \xi_4^i (\xi_3 + \xi_4)^j. \tag{3.27}
\end{aligned}$$

For A_2 , we have

$$\begin{aligned}
A_2 &= \xi_3 \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-1)^i \xi_3^{i-1} \sum_{m+l=j-1} \xi_4^m (\xi_2 + \xi_4)^l - \xi_3 \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-1)^i \xi_4^j \sum_{\substack{m+l=i-1, \\ m \geq 1}} \xi_3^{m-1} (\xi_2 + \xi_3)^l \\
&\quad + \xi_3 \sum_{\substack{i+j=2k-2, \\ i \geq 1}} \xi_3^{i-1} (\xi_2 + \xi_3)^j \\
&= -\xi_3 \sum_{i+j=2k-3} (-1)^i \xi_3^i \sum_{m+l=j} \xi_4^m (\xi_2 + \xi_4)^l + \xi_3 \sum_{i+j=2k-3} (-1)^i \xi_4^{j+1} \sum_{\substack{m+l=i, \\ m \geq 1}} \xi_3^{m-1} (\xi_2 + \xi_3)^l \\
&\quad + \xi_3 \sum_{i+j=2k-3} \xi_3^i (\xi_2 + \xi_3)^j \tag{3.28}
\end{aligned}$$

It follows from (3.22)-(3.28) that

$$\begin{aligned}
\frac{\Omega_1}{\xi_1 \xi_2 \xi_3} &= \sum_{i+j=2k-4} (-1)^i \left(\xi_2^{j+1} \sum_{m+l=i} \xi_4^m (\xi_3 + \xi_4)^l - \xi_4^{i+1} \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_3)^l \right) \\
&\quad + \sum_{i+j=2k-3} \xi_2^i (\xi_2 + \xi_3)^j - \xi_4^i (\xi_3 + \xi_4)^j \\
&\quad + \sum_{i+j=2k-4} (-1)^i \left(\xi_1^{j+1} \sum_{m+l=i} \xi_4^m (\xi_3 + \xi_4)^l - \xi_4^{i+1} \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_3)^l \right) \\
&\quad + \sum_{i+j=2k-3} \xi_1^i (\xi_1 + \xi_3)^j - \xi_4^i (\xi_3 + \xi_4)^j \tag{3.29} \\
&\quad - \sum_{i+j=2k-3} (-1)^i \xi_3^i \sum_{m+l=j} \xi_4^m (\xi_2 + \xi_4)^l + \sum_{i+j=2k-3} (-1)^i \xi_4^{j+1} \sum_{\substack{m+l=i, \\ m \geq 1}} \xi_3^{m-1} (\xi_2 + \xi_3)^l \\
&\quad + \sum_{i+j=2k-3} \xi_3^i (\xi_2 + \xi_3)^j.
\end{aligned}$$

Step 4. Find $\frac{\Omega_1}{\xi_1 \xi_2 \xi_3 \xi_4}$. For our purpose, we rewrite (3.29) as

$$\frac{\Omega_1}{\xi_1 \xi_2 \xi_3} = B_1 + B_2, \tag{3.30}$$

where B_1 consisting of all terms with a explicit factor ξ_4 , B_2 consisting of the remainder terms.

More precisely, B_1, B_2 are given by

$$\begin{aligned}
B_1 = & \sum_{i+j=2k-4} (-1)^i \left(\xi_2^{j+1} \sum_{\substack{m+l=i, \\ m \geq 1}} \xi_4^m (\xi_3 + \xi_4)^l - \xi_4^{i+1} \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_3)^l \right) + \sum_{\substack{i+j=2k-3, \\ i \geq 1}} -\xi_4^i (\xi_3 + \xi_4)^j \\
& + \sum_{i+j=2k-4} (-1)^i \left(\xi_1^{j+1} \sum_{\substack{m+l=i, \\ m \geq 1}} \xi_4^m (\xi_3 + \xi_4)^l - \xi_4^{i+1} \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_3)^l \right) + \sum_{\substack{i+j=2k-3, \\ i \geq 1}} -\xi_4^i (\xi_3 + \xi_4)^j \\
& - \sum_{i+j=2k-3} (-1)^i \xi_3^i \sum_{\substack{m+l=j, \\ m \geq 1}} \xi_4^m (\xi_2 + \xi_4)^l + \sum_{i+j=2k-3} (-1)^i \xi_4^{j+1} \sum_{\substack{m+l=i, \\ m \geq 1}} \xi_3^{m-1} (\xi_2 + \xi_3)^l, \quad (3.31)
\end{aligned}$$

$$\begin{aligned}
B_2 = & \sum_{i+j=2k-4} (-1)^i \xi_2^{j+1} \sum_{\substack{m+l=i, \\ m=0}} \xi_4^m (\xi_3 + \xi_4)^l + \sum_{i+j=2k-3} \xi_2^i (\xi_2 + \xi_3)^j + \sum_{\substack{i+j=2k-3, \\ i=0}} -\xi_4^i (\xi_3 + \xi_4)^j \\
& + \sum_{i+j=2k-4} (-1)^i \xi_1^{j+1} \sum_{\substack{m+l=i, \\ m=0}} \xi_4^m (\xi_3 + \xi_4)^l + \sum_{i+j=2k-3} \xi_1^i (\xi_1 + \xi_3)^j + \sum_{\substack{i+j=2k-3, \\ i=0}} -\xi_4^i (\xi_3 + \xi_4)^j \\
& - \sum_{i+j=2k-3} (-1)^i \xi_3^i \sum_{\substack{m+l=j, \\ m=0}} \xi_4^m (\xi_2 + \xi_4)^l + \sum_{i+j=2k-3} \xi_3^i (\xi_2 + \xi_3)^j. \quad (3.32)
\end{aligned}$$

Contribution of B_1 . It follows from (3.31) that

$$\begin{aligned}
\frac{B_1}{\xi_4} = & \sum_{i+j=2k-4} (-1)^i \left(\xi_2^{j+1} \sum_{\substack{m+l=i, \\ m \geq 1}} \xi_4^{m-1} (\xi_3 + \xi_4)^l - \xi_4^i \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_3)^l \right) + \sum_{\substack{i+j=2k-3, \\ i \geq 1}} -\xi_4^{i-1} (\xi_3 + \xi_4)^j \\
& + \sum_{i+j=2k-4} (-1)^i \left(\xi_1^{j+1} \sum_{\substack{m+l=i, \\ m \geq 1}} \xi_4^{m-1} (\xi_3 + \xi_4)^l - \xi_4^i \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_3)^l \right) + \sum_{\substack{i+j=2k-3, \\ i \geq 1}} -\xi_4^{i-1} (\xi_3 + \xi_4)^j \\
& - \sum_{i+j=2k-3} (-1)^i \xi_3^i \sum_{\substack{m+l=j, \\ m \geq 1}} \xi_4^{m-1} (\xi_2 + \xi_4)^l + \sum_{i+j=2k-3} (-1)^i \xi_4^j \sum_{\substack{m+l=i, \\ m \geq 1}} \xi_3^{m-1} (\xi_2 + \xi_3)^l \\
= & \sum_{i+j=2k-4} (-1)^i \left(\xi_2^{j+1} \sum_{m+l=i-1} \xi_4^m (\xi_3 + \xi_4)^l - \xi_4^i \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_3)^l \right) - \sum_{i+j=2k-4} \xi_4^i (\xi_3 + \xi_4)^j \\
& + \sum_{i+j=2k-4} (-1)^i \left(\xi_1^{j+1} \sum_{m+l=i-1} \xi_4^m (\xi_3 + \xi_4)^l - \xi_4^i \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_3)^l \right) - \sum_{i+j=2k-4} \xi_4^i (\xi_3 + \xi_4)^j \\
& - \sum_{i+j=2k-3} (-1)^i \xi_3^i \sum_{m+l=j-1} \xi_4^m (\xi_2 + \xi_4)^l + \sum_{i+j=2k-3} (-1)^i \xi_4^j \sum_{m+l=i-1} \xi_3^m (\xi_2 + \xi_3)^l. \quad (3.33)
\end{aligned}$$

Using the fact

$$\begin{aligned}
& - \sum_{i+j=2k-3} (-1)^i \xi_3^i \sum_{m+l=j-1} \xi_4^m (\xi_2 + \xi_4)^l + \sum_{i+j=2k-3} (-1)^i \xi_4^j \sum_{m+l=i-1} \xi_3^m (\xi_2 + \xi_3)^l \\
& = - \sum_{i+j=2k-4} (-1)^i \left(\xi_4^i \sum_{m+l=j} \xi_3^m (\xi_2 + \xi_3)^l + \xi_3^i \sum_{m+l=j} \xi_4^m (\xi_2 + \xi_4)^l \right)
\end{aligned}$$

and rearranging the terms in (3.33), we obtain

$$\begin{aligned}
\frac{B_1}{\xi_4} &= -2 \sum_{i+j=2k-4} \xi_4^i (\xi_3 + \xi_4)^j + \sum_{i+j=2k-4} (-1)^i \left((\xi_1^{i+1} + \xi_2^{i+1}) \sum_{m+l=j-1} \xi_4^m (\xi_3 + \xi_4)^l \right) \\
&- \sum_{i+j=2k-4} (-1)^i \left(\xi_4^i \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_3)^l + \xi_3^i \sum_{m+l=j} \xi_4^m (\xi_2 + \xi_4)^l \right) \\
&- \sum_{i+j=2k-4} (-1)^i \left(\xi_4^i \sum_{m+l=j} (\xi_2^m + \xi_3^m) (\xi_2 + \xi_3)^l \right).
\end{aligned} \tag{3.34}$$

Contribution of B_2 . Using $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$, we rewrite (3.32) as

$$\begin{aligned}
B_2 &= \sum_{i+j=2k-4} (-1)^i \xi_2^{j+1} (\xi_3 + \xi_4)^i + \sum_{i+j=2k-3} \xi_2^i (-\xi_1 - \xi_4)^j - (\xi_3 + \xi_4)^{2k-3} \\
&+ \sum_{i+j=2k-4} (-1)^i \xi_1^{j+1} (\xi_3 + \xi_4)^i + \sum_{i+j=2k-3} \xi_1^i (-\xi_2 - \xi_4)^j - (\xi_3 + \xi_4)^{2k-3} \\
&- \sum_{i+j=2k-3} (-1)^i \xi_3^i (\xi_2 + \xi_4)^j + \sum_{i+j=2k-3} \xi_3^i (-\xi_1 - \xi_4)^j \\
&= \sum_{i+j=2k-4} (-1)^i \xi_2^{j+1} (\xi_3 + \xi_4)^i - \sum_{i+j=2k-3} (-1)^i \xi_3^i (\xi_2 + \xi_4)^j - (\xi_3 + \xi_4)^{2k-3} \\
&+ \sum_{i+j=2k-4} (-1)^i \xi_1^{j+1} (\xi_3 + \xi_4)^i + \sum_{i+j=2k-3} \xi_3^i (-\xi_1 - \xi_4)^j - (\xi_3 + \xi_4)^{2k-3} \\
&+ \sum_{i+j=2k-3} \xi_2^i (-\xi_1 - \xi_4)^j + \sum_{i+j=2k-3} \xi_1^i (-\xi_2 - \xi_4)^j \\
&:= B_{21} + B_{22} + B_{23}.
\end{aligned} \tag{3.35}$$

We deal with B_{21}, B_{22}, B_{23} as follows. For B_{21} ,

$$\begin{aligned}
B_{21} &= \sum_{i+j=2k-4} (-1)^i \xi_2^{j+1} (\xi_3 + \xi_4)^i - \sum_{i+j=2k-3} (-1)^i \xi_3^i (\xi_2 + \xi_4)^j - (\xi_3 + \xi_4)^{2k-3} \\
&= \sum_{i+j=2k-4} (-1)^i \xi_2^{j+1} (\xi_3 + \xi_4)^i - \sum_{\substack{i+j=2k-3, \\ j \geq 1}} (-1)^i \xi_3^i (\xi_2 + \xi_4)^j + \xi_3^{2k-3} - (\xi_3 + \xi_4)^{2k-3} \\
&= \sum_{i+j=2k-4} (-1)^i \left(\xi_2^{j+1} (\xi_3 + \xi_4)^i - \xi_3^i (\xi_2 + \xi_4)^{j+1} \right) + \xi_3^{2k-3} - (\xi_3 + \xi_4)^{2k-3} \\
&= \sum_{i+j=2k-5} (-1)^i \left(\xi_2^{j+1} (\xi_3 + \xi_4)^{i+1} - \xi_3^{i+1} (\xi_2 + \xi_4)^{j+1} \right) \\
&+ \xi_2^{2k-3} - (\xi_2 + \xi_4)^{2k-3} + \xi_3^{2k-3} - (\xi_3 + \xi_4)^{2k-3} \\
&= \xi_4 \sum_{i+j=2k-5} (-1)^i \left(\xi_2^{j+1} \sum_{m+l=i} \xi_3^m (\xi_3 + \xi_4)^l - \xi_3^{i+1} \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_4)^l \right) \\
&- \xi_4 \sum_{i+j=2k-4} \left(\xi_2^i (\xi_2 + \xi_4)^j + \xi_3^i (\xi_3 + \xi_4)^j \right).
\end{aligned} \tag{3.36}$$

Similarly, we have

$$\begin{aligned}
B_{22} = & \xi_4 \sum_{i+j=2k-5} (-1)^i \left(\xi_1^{j+1} \sum_{m+l=i} \xi_3^m (\xi_3 + \xi_4)^l - \xi_3^{i+1} \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_4)^l \right) \\
& - \xi_4 \sum_{i+j=2k-4} \left(\xi_1^i (\xi_1 + \xi_4)^j + \xi_3^i (\xi_3 + \xi_4)^j \right). \tag{3.37}
\end{aligned}$$

For B_{23} , we have

$$\begin{aligned}
B_{23} = & \sum_{i+j=2k-3} \xi_2^i (-\xi_1 - \xi_4)^j + \sum_{i+j=2k-3} \xi_1^i (-\xi_2 - \xi_4)^j \\
= & \sum_{i+j=2k-3} \xi_2^j (-\xi_1 - \xi_4)^i + \sum_{i+j=2k-3} \xi_1^i (-\xi_2 - \xi_4)^j \\
= & \sum_{i+j=2k-3} (-1)^i \left(\xi_2^j (\xi_1 + \xi_4)^i - \xi_1^i (\xi_2 + \xi_4)^j \right) \\
= & \sum_{\substack{i+j=2k-3, \\ i,j \geq 1}} (-1)^i \left(\xi_2^j (\xi_1 + \xi_4)^i - \xi_1^i (\xi_2 + \xi_4)^j \right) + \xi_1^{2k-3} - (\xi_1 + \xi_4)^{2k-3} + \xi_2^{2k-3} - (\xi_2 + \xi_4)^{2k-3} \\
= & \sum_{\substack{i+j=2k-3, \\ i,j \geq 1}} (-1)^i \xi_4 \left(\xi_2^j \sum_{m+l=i-1} \xi_1^m (\xi_1 + \xi_4)^l - \xi_1^i \sum_{m+l=j-1} \xi_2^m (\xi_2 + \xi_4)^l \right) \\
& - \xi_4 \sum_{i+j=2k-4} \left(\xi_1^i (\xi_1 + \xi_4)^j + \xi_2^i (\xi_2 + \xi_4)^j \right) \\
= & \sum_{i+j=2k-5} (-1)^i \xi_4 \left(\xi_1^{i+1} \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_4)^l - \xi_2^{j+1} \sum_{m+l=i} \xi_1^m (\xi_1 + \xi_4)^l \right) \\
& - \xi_4 \sum_{i+j=2k-4} \left(\xi_1^i (\xi_1 + \xi_4)^j + \xi_2^i (\xi_2 + \xi_4)^j \right) \\
= & \sum_{i+j=2k-5} (-1)^i \xi_4 \left(\xi_1^{i+1} \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_4)^l + \xi_2^{i+1} \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_4)^l \right) \\
& - \xi_4 \sum_{i+j=2k-4} \left(\xi_1^i (\xi_1 + \xi_4)^j + \xi_2^i (\xi_2 + \xi_4)^j \right). \tag{3.38}
\end{aligned}$$

It follows from (3.35)-(3.38) that

$$\begin{aligned}
\frac{B_2}{\xi_4} = & \sum_{i+j=2k-5} (-1)^i \left((\xi_1^{j+1} + \xi_2^{j+1}) \sum_{m+l=i} \xi_3^m (\xi_3 + \xi_4)^l + (\xi_1^{i+1} - \xi_3^{i+1}) \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_4)^l \right. \\
& \left. + (\xi_2^{i+1} - \xi_3^{i+1}) \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_4)^l \right) \\
& - 2 \sum_{i+j=2k-4} \left(\xi_1^i (\xi_1 + \xi_4)^j + \xi_2^i (\xi_2 + \xi_4)^j + \xi_3^i (\xi_3 + \xi_4)^j \right). \tag{3.39}
\end{aligned}$$

Combining (3.34) and (3.39) gives the lemma 3.1.

3.2 Decomposition 2

In this subsection, we get rid of the singularity of

$$\frac{\Omega_2}{\alpha_4} = \frac{\Omega_2}{3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)},$$

where we used (2.18). To this end, we shall show that for $k \geq 2$

$$\Omega_2 = \xi_1^{2k+1} + \xi_2^{2k+1} + \xi_3^{2k+1} + \xi_4^{2k+1} = (\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4) \cdot \text{a polynomial}$$

on the hyperplane $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$. This is contained in the following lemma, in which we give a formula of the polynomial.

Lemma 3.2. *Assume that $\mathbb{N} \ni k \geq 1$ and Ω_2 is given by (3.8). Then*

$$\begin{aligned} \frac{\Omega_2(k)}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)} &= \sum_{i+j=2k-2} (-1)^i (2\xi_1^i \xi_4^j + \xi_2^i \xi_3^j) \\ &+ \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} \left((-\xi_3)^i \sum_{m+l=j-1} \xi_1^m (-\xi_4)^l + \xi_4^j \sum_{m+l=i-1} (-\xi_2)^m \xi_3^l \right) \\ &+ \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i \sum_{n+h=i} \xi_1^n (-\xi_4)^h \sum_{m+l=j} \xi_2^m (-\xi_4)^l \\ &+ \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^{i+1} \xi_4^{i+1} \left[\sum_{m+l=j-1} \left(\xi_3^m (-\xi_2)^l + \xi_4^m (-\xi_1)^l \right) \right. \\ &\left. + \sum_{\substack{m+l=j, \\ m,l \geq 1}} \left((-\xi_1)^l \sum_{n+h=m-1} \xi_3^n (-\xi_2)^h + (-\xi_2)^m \sum_{n+h=l-1} \xi_4^n (-\xi_1)^h \right) \right]. \end{aligned} \quad (3.40)$$

Proof of Lemma 3.2. We divide the analysis into three steps.

Step 1. Find $\frac{\Omega_2}{\xi_1 + \xi_2}$. Using $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$, we have

$$\begin{aligned} \Omega_2 &= \xi_1^{2k+1} + \xi_2^{2k+1} + \xi_3^{2k+1} + \xi_4^{2k+1} \\ &= (\xi_1 + \xi_2) \sum_{i+j=2k} (-1)^i \xi_1^i \xi_2^j + (\xi_3 + \xi_4) \sum_{i+j=2k} (-1)^i \xi_3^i \xi_4^j \\ &= (\xi_1 + \xi_2) \sum_{i+j=2k} (-1)^i (\xi_1^i \xi_2^j - \xi_3^i \xi_4^j). \end{aligned} \quad (3.41)$$

From (3.41), we obtain

$$\frac{\Omega_2}{\xi_1 + \xi_2} = \sum_{i+j=2k} (-1)^i (\xi_1^i \xi_2^j - \xi_3^i \xi_4^j). \quad (3.42)$$

Step 2. Find $\frac{\Omega_2}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)}$. Rewrite (3.42) as

$$\frac{\Omega_2}{\xi_1 + \xi_2} = \sum_{\substack{i+j=2k, \\ i,j \geq 1}} (-1)^i (\xi_1^i \xi_2^j - \xi_3^i \xi_4^j) + \xi_1^{2k} - \xi_3^{2k} + \xi_2^{2k} - \xi_4^{2k}. \quad (3.43)$$

On one hand, we have

$$\begin{aligned}\xi_1^{2k} - \xi_3^{2k} + \xi_2^{2k} - \xi_4^{2k} &= (\xi_1 + \xi_3) \sum_{i+j=2k-1} (-1)^i \xi_3^i \xi_1^j + (\xi_2 + \xi_4) \sum_{i+j=2k-1} (-1)^i \xi_4^i \xi_2^j \\ &= (\xi_1 + \xi_3) \sum_{i+j=2k-1} (-1)^i (\xi_3^i \xi_1^j - \xi_4^i \xi_2^j).\end{aligned}\quad (3.44)$$

On the other hand, we have

$$\begin{aligned}\sum_{\substack{i+j=2k, \\ i,j \geq 1}} (-1)^i (\xi_1^i \xi_2^j - \xi_3^i \xi_4^j) &= \sum_{\substack{i+j=2k, \\ i,j \geq 1}} (-1)^i \left(\xi_1^i (\xi_2^j - (-\xi_4)^j) - \xi_4^j (\xi_3^i - (-\xi_1)^i) \right) \\ &= \sum_{\substack{i+j=2k, \\ i,j \geq 1}} (-1)^i \left(\xi_1^i (\xi_2 + \xi_4) \sum_{m+l=j-1} \xi_2^m (-\xi_4)^l - \xi_4^j (\xi_1 + \xi_3) \sum_{m+l=i-1} \xi_3^m (-\xi_1)^l \right) \\ &= \sum_{\substack{i+j=2k, \\ i,j \geq 1}} (-1)^{i+1} (\xi_1 + \xi_3) \left(\xi_1^i \sum_{m+l=j-1} \xi_2^m (-\xi_4)^l + \xi_4^j \sum_{m+l=i-1} \xi_3^m (-\xi_1)^l \right).\end{aligned}\quad (3.45)$$

Combining (3.43)-(3.45) gives that

$$\begin{aligned}\frac{\Omega_2}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)} &= \sum_{\substack{i+j=2k, \\ i,j \geq 1}} (-1)^{i+1} \left(\xi_1^i \sum_{m+l=j-1} \xi_2^m (-\xi_4)^l + \xi_4^j \sum_{m+l=i-1} \xi_3^m (-\xi_1)^l \right) \\ &\quad + \sum_{i+j=2k-1} (-1)^i (\xi_3^i \xi_1^j - \xi_4^i \xi_2^j).\end{aligned}\quad (3.46)$$

Changing variable $i-1 \mapsto i, j-1 \mapsto j$, we find

$$\sum_{\substack{i+j=2k, \\ i,j \geq 1}} (-1)^{i+1} \xi_1^i \sum_{m+l=j-1} \xi_2^m (-\xi_4)^l = \sum_{i+j=2k-2} (-1)^i \xi_1^{i+1} \sum_{m+l=j} \xi_2^m (-\xi_4)^l.\quad (3.47)$$

Similarly,

$$\begin{aligned}\sum_{\substack{i+j=2k, \\ i,j \geq 1}} (-1)^{i+1} \xi_4^j \sum_{m+l=i-1} \xi_3^m (-\xi_1)^l &= \sum_{i+j=2k-2} (-1)^i \xi_4^{j+1} \sum_{m+l=i} \xi_3^m (-\xi_1)^l \\ &= \sum_{i+j=2k-2} (-1)^i \xi_4^{i+1} \sum_{m+l=j} \xi_3^m (-\xi_1)^l.\end{aligned}\quad (3.48)$$

Inserting (3.47)-(3.48) into (3.46), we obtain

$$\begin{aligned}\frac{\Omega_2}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)} &= \sum_{i+j=2k-2} (-1)^i \left(\xi_1^{i+1} \sum_{m+l=j} \xi_2^m (-\xi_4)^l + \xi_4^{i+1} \sum_{m+l=j} \xi_3^m (-\xi_1)^l \right) \\ &\quad + \sum_{i+j=2k-1} (-1)^i (\xi_3^i \xi_1^j - \xi_4^i \xi_2^j).\end{aligned}\quad (3.49)$$

Step 3. Find $\frac{\Omega_2}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}$. It suffices to analyze the two terms on the right hand side

of (3.49). We claim that

$$\begin{aligned} \sum_{i+j=2k-1} (-1)^i (\xi_3^i \xi_1^j - \xi_4^i \xi_2^j) &= (\xi_1 + \xi_4) \sum_{i+j=2k-2} (-1)^i (\xi_1^i \xi_4^j + \xi_2^i \xi_3^j) \\ &\quad + (\xi_1 + \xi_4) \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} \left((-\xi_3)^i \sum_{m+l=j-1} \xi_1^m (-\xi_4)^l + \xi_4^j \sum_{m+l=i-1} (-\xi_2)^m \xi_3^l \right), \end{aligned} \quad (3.50)$$

$$\begin{aligned} &\sum_{i+j=2k-2} (-1)^i \left(\xi_1^{i+1} \sum_{m+l=j} \xi_2^m (-\xi_4)^l + \xi_4^{i+1} \sum_{m+l=j} \xi_3^m (-\xi_1)^l \right) \\ &= (\xi_1 + \xi_4) \left[\sum_{i+j=2k-2} (-1)^i \xi_1^i \xi_4^j + \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i \sum_{n+h=i} \xi_1^n (-\xi_4)^h \sum_{m+l=j} \xi_2^m (-\xi_4)^l \right] \\ &\quad + (\xi_1 + \xi_4) \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^{i+1} \xi_4^{i+1} \left[\sum_{m+l=j-1} \left(\xi_3^m (-\xi_2)^l + \xi_4^m (-\xi_1)^l \right) \right. \\ &\quad \left. + \sum_{\substack{m+l=j, \\ m,l \geq 1}} \left((-\xi_1)^l \sum_{n+h=m-1} \xi_3^n (-\xi_2)^h + (-\xi_2)^m \sum_{n+h=l-1} \xi_4^n (-\xi_1)^h \right) \right]. \end{aligned} \quad (3.51)$$

To prove (3.50), we rewrite

$$\begin{aligned} \sum_{i+j=2k-1} (-1)^i (\xi_3^i \xi_1^j - \xi_4^i \xi_2^j) &= \sum_{i+j=2k-1} \left((-1)^i \xi_3^i \xi_1^j - (-1)^i \xi_4^i \xi_2^j \right) \\ &= \sum_{i+j=2k-1} \left((-1)^i \xi_3^i \xi_1^j + (-1)^j \xi_4^i \xi_2^j \right) = \sum_{i+j=2k-1} \left((-\xi_3)^i \xi_1^j + (-\xi_2)^i \xi_4^j \right) \\ &= \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} \left((-\xi_3)^i \xi_1^j + (-\xi_2)^i \xi_4^j \right) + \xi_1^{2k-1} + \xi_4^{2k-1} - (\xi_2^{2k-1} + \xi_3^{2k-1}). \end{aligned} \quad (3.52)$$

On one hand,

$$\begin{aligned} \xi_1^{2k-1} + \xi_4^{2k-1} - (\xi_2^{2k-1} + \xi_3^{2k-1}) &= (\xi_1 + \xi_4) \sum_{i+j=2k-2} (-1)^i \xi_1^i \xi_4^j - (\xi_2 + \xi_3) \sum_{i+j=2k-2} (-1)^i \xi_2^i \xi_3^j \\ &= (\xi_1 + \xi_4) \sum_{i+j=2k-2} (-1)^i (\xi_1^i \xi_4^j + \xi_2^i \xi_3^j). \end{aligned} \quad (3.53)$$

On the other hand,

$$\begin{aligned}
\sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-\xi_3)^i \xi_1^j + (-\xi_2)^i \xi_4^j &= \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-\xi_3)^i (\xi_1^j - (-\xi_4)^j) + (-\xi_3)^i (-\xi_4)^j + (-\xi_2)^i \xi_4^j \\
&= \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-\xi_3)^i (\xi_1^j - (-\xi_4)^j) + \xi_4^j ((-\xi_2)^i - \xi_3^i) \\
&= \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (-\xi_3)^i (\xi_1 + \xi_4) \sum_{m+l=j-1} \xi_1^m (-\xi_4)^l - \xi_4^j (\xi_2 + \xi_3) \sum_{m+l=i-1} (-\xi_2)^m \xi_3^l \\
&= \sum_{\substack{i+j=2k-1, \\ i,j \geq 1}} (\xi_1 + \xi_4) \left((-\xi_3)^i \sum_{m+l=j-1} \xi_1^m (-\xi_4)^l + \xi_4^j \sum_{m+l=i-1} (-\xi_2)^m \xi_3^l \right). \quad (3.54)
\end{aligned}$$

Then combining (3.52)-(3.54) gives (3.50).

To prove (3.51), we split the sum into two cases: $j = 0, j \geq 1$. We deal with each case as follows. At first,

$$\begin{aligned}
\sum_{\substack{i+j=2k-2, \\ j=0}} (-1)^i \left(\xi_1^{i+1} \sum_{m+l=j} \xi_2^m (-\xi_4)^l + \xi_4^{i+1} \sum_{m+l=j} \xi_3^m (-\xi_1)^l \right) \\
= \xi_1^{2k-1} + \xi_4^{2k-1} = (\xi_1 + \xi_4) \sum_{i+j=2k-2} (-1)^i \xi_1^i \xi_4^j. \quad (3.55)
\end{aligned}$$

Second, for the case $j \geq 1$ we have

$$\begin{aligned}
\sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i \left(\xi_1^{i+1} \sum_{m+l=j} \xi_2^m (-\xi_4)^l + \xi_4^{i+1} \sum_{m+l=j} \xi_3^m (-\xi_1)^l \right) \\
= \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i \left((\xi_1^{i+1} - (-\xi_4)^{i+1}) \sum_{m+l=j} \xi_2^m (-\xi_4)^l + \xi_4^{i+1} \sum_{m+l=j} \xi_3^m (-\xi_1)^l - (-\xi_2)^m \xi_4^l \right). \quad (3.56)
\end{aligned}$$

There are two terms on the right hand side of (3.56). On one hand,

$$\begin{aligned}
\sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i (\xi_1^{i+1} - (-\xi_4)^{i+1}) \sum_{m+l=j} \xi_2^m (-\xi_4)^l \\
= \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i (\xi_1 + \xi_4) \sum_{n+h=i} \xi_1^n (-\xi_4)^h \sum_{m+l=j} \xi_2^m (-\xi_4)^l. \quad (3.57)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i \xi_4^{i+1} \sum_{m+l=j} \left(\xi_3^m (-\xi_1)^l - (-\xi_2)^m \xi_4^l \right) \\
&= \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i \xi_4^{i+1} \left(\xi_3^j - (-\xi_2)^j + (-\xi_1)^j - \xi_4^j \right) \\
&+ \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^i \xi_4^{i+1} \sum_{\substack{m+l=j, \\ m, l \geq 1}} \left((-\xi_1)^l \left(\xi_3^m - (-\xi_2)^m \right) + (-\xi_2)^m \left((-\xi_1)^l - \xi_4^l \right) \right) \\
&= (\xi_1 + \xi_4) \sum_{\substack{i+j=2k-2, \\ j \geq 1}} (-1)^{i+1} \xi_4^{i+1} \left[\sum_{m+l=j-1} \xi_3^m (-\xi_2)^l + \xi_4^m (-\xi_1)^l \right. \\
&+ \left. \sum_{\substack{m+l=j, \\ m, l \geq 1}} \left((-\xi_1)^l \sum_{n+h=m-1} \xi_3^n (-\xi_2)^h + (-\xi_2)^m \sum_{n+h=l-1} \xi_4^n (-\xi_1)^h \right) \right]. \tag{3.58}
\end{aligned}$$

Then combining (3.55)-(3.58) gives (3.51). The desired conclusion follows from (3.50) and (3.51).

3.3 Estimates for β_4

In this subsection, we give two upper bounds for β_4 , based on the analysis of M_4 in subsection 3.1 and 3.2. Since $\beta_4 = -M_4/\alpha_4$ (see (2.13)), using Lemma 3.1 and Lemma 3.2, we obtain

$$\beta_4 = \frac{c}{108} \sum_{k=0}^{\infty} \frac{\sigma^{2(k+2)}}{(2(k+2))!} (\Omega_1(k+2) - \Omega_2(k+1)), \tag{3.59}$$

where $\Omega_1(k+2) := \Omega_1(k+2; \xi_1, \dots, \xi_4)$, $\Omega_2(k+1) := \Omega_2(k+1; \xi_1, \dots, \xi_4)$ are given by

$$\begin{aligned}
\Omega_1(k+2) &= \sum_{i+j=2k-1} (-1)^i \left((\xi_1^{j+1} + \xi_2^{j+1}) \sum_{m+l=i} \xi_3^m (\xi_3 + \xi_4)^l + (\xi_1^{i+1} - \xi_3^{i+1}) \sum_{m+l=j} \xi_2^m (\xi_2 + \xi_4)^l \right. \\
&+ \left. (\xi_2^{i+1} - \xi_3^{i+1}) \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_4)^l + (\xi_1^{i+1} + \xi_2^{i+1}) \sum_{m+l=j} \xi_4^m (\xi_3 + \xi_4)^l \right) \\
&- 2 \sum_{i+j=2k} \left(\xi_1^i (\xi_1 + \xi_4)^j + \xi_2^i (\xi_2 + \xi_4)^j + \xi_3^i (\xi_3 + \xi_4)^j + \xi_4^i (\xi_3 + \xi_4)^j \right) \\
&- \sum_{i+j=2k} (-1)^i \left(\xi_4^i \sum_{m+l=j} \xi_1^m (\xi_1 + \xi_3)^l + \xi_3^i \sum_{m+l=j} \xi_4^m (\xi_2 + \xi_4)^l \right) \\
&- \sum_{i+j=2k} (-1)^i \left(\xi_4^i \sum_{m+l=j} (\xi_2^m + \xi_3^m) (\xi_2 + \xi_3)^l \right), \tag{3.60}
\end{aligned}$$

$$\begin{aligned}
\Omega_2(k+1) &= \sum_{i+j=2k} (-1)^i (2\xi_1^i \xi_4^j + \xi_2^i \xi_3^j) \\
&+ \sum_{\substack{i+j=2k+1, \\ i,j \geq 1}} \left((-\xi_3)^i \sum_{m+l=j-1} \xi_1^m (-\xi_4)^l + \xi_4^j \sum_{m+l=i-1} (-\xi_2)^m \xi_3^l \right) \\
&+ \sum_{\substack{i+j=2k, \\ j \geq 1}} (-1)^i \sum_{n+h=i} \xi_1^n (-\xi_4)^h \sum_{m+l=j} \xi_2^m (-\xi_4)^l \\
&+ \sum_{\substack{i+j=2k, \\ j \geq 1}} (-1)^{i+1} \xi_4^{i+1} \left[\sum_{m+l=j-1} \left(\xi_3^m (-\xi_2)^l + \xi_4^m (-\xi_1)^l \right) \right. \\
&+ \left. \sum_{\substack{m+l=j, \\ m,l \geq 1}} \left((-\xi_1)^l \sum_{n+h=m-1} \xi_3^n (-\xi_2)^h + (-\xi_2)^m \sum_{n+h=l-1} \xi_4^n (-\xi_1)^h \right) \right]. \tag{3.61}
\end{aligned}$$

Taking absolute value on both sides of (3.60), we find

$$\begin{aligned}
|\Omega_1(k+2)| &\leq \sum_{i+j=2k} \left((|\xi_1|^j + |\xi_2|^j) \sum_{m+l=i} |\xi_3|^m |\xi_3 + \xi_4|^l + (|\xi_1|^i + |\xi_3|^i) \sum_{m+l=j} |\xi_2|^m |\xi_2 + \xi_4|^l \right. \\
&+ \left. (|\xi_2|^i + |\xi_3|^i) \sum_{m+l=j} |\xi_1|^m |\xi_1 + \xi_4|^l + (|\xi_1|^i + |\xi_2|^i) \sum_{m+l=j} |\xi_4|^m |\xi_3 + \xi_4|^l \right) \\
&+ 2 \sum_{i+j=2k} \left(|\xi_1|^i |\xi_1 + \xi_4|^j + |\xi_2|^i |\xi_2 + \xi_4|^j + |\xi_3|^i |\xi_3 + \xi_4|^j + |\xi_4|^i |\xi_3 + \xi_4|^j \right) \\
&+ \sum_{i+j=2k} \left(|\xi_4|^i \sum_{m+l=j} |\xi_1|^m |\xi_1 + \xi_3|^l + |\xi_3|^i \sum_{m+l=j} |\xi_4|^m |\xi_2 + \xi_4|^l \right) \\
&+ \sum_{i+j=2k} \left(|\xi_4|^i \sum_{m+l=j} (|\xi_2|^m + |\xi_3|^m) |\xi_2 + \xi_3|^l \right) \\
&\leq 2 \sum_{i+j=2k} \left(|\xi_1|^i |\xi_1 + \xi_4|^j + |\xi_2|^i |\xi_2 + \xi_4|^j + |\xi_3|^i |\xi_3 + \xi_4|^j + |\xi_4|^i |\xi_3 + \xi_4|^j \right) \\
&+ \sum_{i+j=2k} \sum' |\xi_{p_1}|^i \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l. \tag{3.62}
\end{aligned}$$

Here, the sum \sum' is taking over all p_1, p_2, p_3 being different numbers in the set $\{1, 2, 3, 4\}$.

Taking absolute value on both sides of (3.61), we find

$$\begin{aligned}
|\Omega_2(k+1)| &\leq 2 \sum_{i+j=2k} (|\xi_1|^i |\xi_4|^j + |\xi_2|^i |\xi_3|^j) + \sum_{i+j=2k} |\xi_3|^i \sum_{m+l=j} |\xi_1|^m |\xi_4|^l \\
&+ 2 \sum_{i+j=2k} |\xi_4|^j \sum_{m+l=i} |\xi_2|^m |\xi_3|^l + \sum_{i+j=2k} |\xi_4|^i \sum_{m+l=j} |\xi_4|^m |\xi_1|^l \\
&+ \sum_{i+j=2k} \sum_{n+h=i} |\xi_1|^n |\xi_4|^h \sum_{m+l=j} |\xi_2|^m |\xi_4|^l \\
&+ \sum_{i+j=2k} |\xi_4|^i \left[\sum_{m+l=j} \left(|\xi_1|^l \sum_{n+h=m} |\xi_3|^n |\xi_2|^h + |\xi_2|^m \sum_{n+h=l} |\xi_4|^n |\xi_1|^h \right) \right]. \tag{3.63}
\end{aligned}$$

Thanks to (3.59), (3.62), (3.63), we obtain the following lemma.

Lemma 3.3. *We have the following bound for β_4 :*

$$|\beta_4| \leq \frac{|c|}{54} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} (\Theta_1(k) + \Theta_2(k)), \quad (3.64)$$

where c is the constant in Lemma 2.2, and $\Theta_1(k), \Theta_2(k)$ are given by

$$\begin{aligned} \Theta_1(k) = & \sum_{i+j=k} \left(|\xi_1|^i |\xi_1 + \xi_4|^j + |\xi_2|^i |\xi_2 + \xi_4|^j + |\xi_3|^i |\xi_3 + \xi_4|^j + |\xi_4|^i |\xi_3 + \xi_4|^j \right) \\ & + \sum_{i+j=k} \sum_{m+l=j} |\xi_{p_1}|^i \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l, \end{aligned} \quad (3.65)$$

$$\begin{aligned} \Theta_2(k) = & \sum_{i+j=k} (|\xi_1|^i |\xi_4|^j + |\xi_2|^i |\xi_3|^j) + \sum_{i+j=k} |\xi_3|^i \sum_{m+l=j} |\xi_1|^m |\xi_4|^l \\ & + \sum_{i+j=k} |\xi_4|^j \sum_{m+l=i} |\xi_2|^m |\xi_3|^l + \sum_{i+j=k} |\xi_4|^i \sum_{m+l=j} |\xi_4|^m |\xi_1|^l \\ & + \sum_{i+j=k} \sum_{n+h=i} |\xi_1|^n |\xi_4|^h \sum_{m+l=j} |\xi_2|^m |\xi_4|^l \\ & + \sum_{i+j=k} |\xi_4|^i \left[\sum_{m+l=j} \left(|\xi_1|^l \sum_{n+h=m} |\xi_3|^n |\xi_2|^h + |\xi_2|^m \sum_{n+h=l} |\xi_4|^n |\xi_1|^h \right) \right]. \end{aligned} \quad (3.66)$$

To obtain further estimates of β_4 , we need the following lemma.

Lemma 3.4. *Let $p \geq 4$ be an integer. Assume that n_1, n_2, \dots, n_p are integers satisfying*

$$0 \leq n_1 \leq n_2, \dots, n_{p-1} \leq n_p$$

and

$$n_1 + n_p = n_2 + \dots + n_{p-1}.$$

Then we have

$$n_2! n_3! \dots n_{p-1}! \leq n_1! n_p!. \quad (3.67)$$

Proof. Without loss of generality, we assume that $n_2 \leq n_3 \leq \dots \leq n_{p-1}$. We divide the proof into two cases: $p = 4$ and $p \geq 5$.

The case $p = 4$. Using the relation $n! = \Gamma(n+1)$, where $\Gamma: \mathbb{R}^+ \mapsto \mathbb{R}$ is the standard Gamma function given by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0, \quad (3.68)$$

the conclusion (3.67) can be restated as

$$\Gamma(n_2 + 1) \Gamma(n_3 + 1) \leq \Gamma(n_1 + 1) \Gamma(n_4 + 1). \quad (3.69)$$

It remains to prove (3.69). Using Hölder inequality, we deduce from (3.68) that

$$\Gamma(\theta x_1 + (1-\theta)x_2) \leq \Gamma(x_1)^\theta \Gamma(x_2)^{1-\theta}, \quad \theta \in [0, 1], x_1, x_2 > 0. \quad (3.70)$$

Define $f(x) := \ln \Gamma(x)$, $x > 0$. Then (3.70) implies that f is convex. Since Γ is smooth, so is f . Then $f''(x) \geq 0$, $x > 0$. We claim that

$$f(b) + f(c) \leq f(a) + f(d), \quad 0 < a \leq b \leq c \leq d, \quad b + c = a + d. \quad (3.71)$$

In fact, by mean value theorem we have for some $\eta_1 \in [a, b]$, $\eta_2 \in [c, d]$

$$f(b) - f(a) = f'(\eta_1)(b - a), \quad f(d) - f(c) = f'(\eta_2)(d - c).$$

From this, we use mean value theorem again to find for some $\eta_3 \in [\eta_1, \eta_2]$

$$f(a) + f(d) - (f(b) + f(c)) = (f'(\eta_2) - f'(\eta_1))(b - a) = f''(\eta_3)(\eta_2 - \eta_1)(b - a) \geq 0.$$

Thus the claim (3.71) follows. Set

$$a = n_1 + 1, b = n_2 + 1, c = n_3 + 1, d = n_4 + 1$$

in (3.71), we obtain (3.69).

The case $p \geq 5$. Clearly, we have $m!n! \leq (m+n)!$ for all nonnegative integers m, n . Using the fact repeatedly, we find

$$n_3! \cdots n_{p-1}! \leq (n_3 + \cdots + n_{p-1})!.$$

Thus the desired conclusion (3.67) holds if one can show

$$n_2!(n_3 + \cdots + n_{p-1})! \leq n_1!n_p!.$$

But this follows from the proved case $p = 4$. □

We are ready to state our first bound for β_4 .

Lemma 3.5. *Let β_4 be given by (3.59). Then we have for all $\sigma \geq 0$, $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$*

$$|\beta_4| \leq \frac{43|c|}{54} \sigma^4 e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}. \quad (3.72)$$

Proof. Thanks to (3.64), the conclusion (3.72) follows from the following two inequalities:

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \Theta_1(k) \leq 28\sigma^4 e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}, \quad (3.73)$$

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \Theta_2(k) \leq 15\sigma^4 e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}. \quad (3.74)$$

The proof of (3.73). It suffices to show that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \left(|\xi_1|^i |\xi_1 + \xi_4|^j + |\xi_2|^i |\xi_2 + \xi_4|^j + |\xi_3|^i |\xi_3 + \xi_4|^j + |\xi_4|^i |\xi_3 + \xi_4|^j \right) \\ \leq 4\sigma^4 e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}, \end{aligned} \quad (3.75)$$

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \sum_{m+l=j} |\xi_{p_1}|^i |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l \leq 24\sigma^4 e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}. \quad (3.76)$$

To prove (3.75), we first show that

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_1|^i |\xi_1 + \xi_4|^j \leq \sigma^4 e^{\sigma(|\xi_1|+|\xi_4|)}. \quad (3.77)$$

In fact, we expand the left hand side of (3.77) by the binomial theorem to find

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_1|^i |\xi_1 + \xi_4|^j \leq \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \sum_{m+l=j} \frac{j!}{m!l!} |\xi_1|^{i+m} |\xi_4|^l. \quad (3.78)$$

If $i+j=k, m+l=j$, then Lemma 3.4 gives $\frac{j!}{m!l!} \leq \frac{1}{(m+i)!}$. Thus, we deduce from (3.78) that

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_1|^i |\xi_1 + \xi_4|^j \leq \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+1)^4} \sum_{i+j=k} \sum_{m+l=j} \frac{|\xi_1|^{i+m}}{(m+i)!} \frac{|\xi_4|^l}{l!}. \quad (3.79)$$

Since $\sum_{i+j=k} \sum_{m+l=j} \frac{|\xi_1|^{i+m}}{(m+i)!} \frac{|\xi_4|^l}{l!} \leq (k+1) \sum_{i+j=k} \frac{|\xi_1|^i}{i!} \frac{|\xi_4|^j}{j!}$, (3.79) becomes

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_1|^i |\xi_1 + \xi_4|^j \leq \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+1)^3} \sum_{i+j=k} \frac{|\xi_1|^i}{i!} \frac{|\xi_4|^j}{j!} \leq \sigma^4 e^{\sigma(|\xi_1|+|\xi_4|)}. \quad (3.80)$$

This proves (3.77). Similarly, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \left(|\xi_2|^i |\xi_2 + \xi_4|^j + |\xi_3|^i |\xi_3 + \xi_4|^j + |\xi_4|^i |\xi_3 + \xi_4|^j \right) \\ \leq \sigma^4 \left(e^{\sigma(|\xi_2|+|\xi_4|)} + 2e^{\sigma(|\xi_3|+|\xi_4|)} \right). \end{aligned} \quad (3.81)$$

Combining (3.77) and (3.81) implies that (3.75).

To prove (3.76), we first show that

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_1|^i \sum_{m+l=j} |\xi_2|^m |\xi_2 + \xi_3|^l \leq \sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}. \quad (3.82)$$

The idea is similar to that of proving (3.75). In fact, using Lemma 3.4 we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_1|^i \sum_{m+l=j} |\xi_2|^m |\xi_2 + \xi_3|^l \\ \leq \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_1|^i \sum_{m+l=j} |\xi_2|^m \sum_{n+h=l} \frac{l!}{n!h!} |\xi_2|^n |\xi_3|^h \\ \leq \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+1)^4} \sum_{i+j=k} \frac{|\xi_1|^i}{i!} \sum_{m+l=j} \sum_{n+h=l} \frac{|\xi_2|^{m+n}}{(m+n)!} \frac{|\xi_3|^h}{h!} \\ \leq \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+1)^3} \sum_{i+j=k} \frac{|\xi_1|^i}{i!} \sum_{m+l=j} \frac{|\xi_2|^m}{m!} \frac{|\xi_3|^l}{l!} \\ \leq \sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|)}. \end{aligned} \quad (3.83)$$

Clearly, (3.82) follows from (3.83). Similarly, we have for p_1, p_2, p_3 being different numbers in the set $\{1, 2, 3, 4\}$

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_{p_1}|^i \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_3} + \xi_{p_3}|^l \leq \sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}. \quad (3.84)$$

Since the number of different choices of p_1, p_2, p_3 is 24, the conclusion (3.76) follows from (3.84).

The proof of (3.74). It suffices to show that

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} (|\xi_1|^i |\xi_4|^j + |\xi_2|^i |\xi_3|^j) \leq 2\sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}, \quad (3.85)$$

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \left[|\xi_3|^i \sum_{m+l=j} |\xi_1|^m |\xi_4|^l + |\xi_4|^j \sum_{m+l=i} |\xi_2|^m |\xi_3|^l \right] \leq 2\sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}, \quad (3.86)$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_4|^i \left[\sum_{m+l=j} \left(|\xi_1|^l \sum_{n+h=m} |\xi_3|^n |\xi_2|^h + |\xi_2|^m \sum_{n+h=l} |\xi_4|^n |\xi_1|^h \right) \right] \\ \leq 2\sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}, \quad (3.87) \end{aligned}$$

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_4|^i \sum_{m+l=j} |\xi_4|^m |\xi_1|^l \leq \sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}, \quad (3.88)$$

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \sum_{n+h=i} |\xi_1|^n |\xi_4|^h \sum_{m+l=j} |\xi_2|^m |\xi_4|^l \leq 4\sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}. \quad (3.89)$$

Clearly, we have the following inequalities:

$$\frac{1}{k!} \leq \frac{1}{i!} \frac{1}{j!}, \quad i+j=k, \quad (3.90)$$

$$\frac{1}{k!} \leq \frac{1}{i!} \frac{1}{m!} \frac{1}{l!}, \quad i+j=k, m+l=j, \quad (3.91)$$

$$\frac{1}{k!} \leq \frac{1}{i!} \frac{1}{l!} \frac{1}{n!} \frac{1}{h!}, \quad i+j=k, m+l=j, n+h=m. \quad (3.92)$$

Then the equalities (3.85)-(3.87) follows from (3.90)-(3.92). The equality (3.88) follows from (3.77).

It remains to prove (3.89). Indeed, using the elementary inequality $|\xi_1|^n |\xi_4|^h \leq |\xi_1|^i + |\xi_4|^i$ for all $n+h=i$, we find $\sum_{n+h=i} |\xi_1|^n |\xi_4|^h \leq i(|\xi_1|^i + |\xi_4|^i)$. Similarly, $\sum_{m+l=j} |\xi_2|^m |\xi_4|^l \leq j(|\xi_2|^j + |\xi_4|^j)$.

Then we deduce that

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \sum_{n+h=i} |\xi_1|^n |\xi_4|^h \sum_{m+l=j} |\xi_2|^m |\xi_4|^l \\
& \leq \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} ij (|\xi_1|^i + |\xi_4|^i) (|\xi_2|^j + |\xi_4|^j) \\
& \leq \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+2)!} \sum_{i+j=k} (|\xi_1|^i |\xi_4|^j + |\xi_4|^i |\xi_2|^j + |\xi_1|^i |\xi_2|^j + |\xi_4|^k) \\
& \leq \sum_{k=0}^{\infty} \sigma^{k+4} \left(\frac{1}{k!} |\xi_4|^k + \sum_{i+j=k} \frac{1}{i!j!} (|\xi_1|^i |\xi_4|^j + |\xi_4|^i |\xi_2|^j + |\xi_1|^i |\xi_2|^j) \right) \\
& \leq 4\sigma^4 e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}.
\end{aligned}$$

This proves (3.89). \square

The second bound for β_4 is given as follows.

Lemma 3.6. *Let β_4 be given by (3.59). Then we have for all $\sigma \leq 1, \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$*

$$|\beta_4| \leq \frac{|c|}{9} \sum_{p_1 \neq p_2, p_1, p_2 \in \{1,2,3,4\}} \frac{1}{(1+|\xi_{p_1}|)(1+|\xi_{p_2}|)} e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}. \quad (3.93)$$

Proof. Thanks to (3.64), it suffices to show the following two inequalities:

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \Theta_1(k) \leq 3 \sum_{p_1 \neq p_2, p_1, p_2 \in \{1,2,3,4\}} \frac{1}{(1+|\xi_{p_1}|)(1+|\xi_{p_2}|)} e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}, \quad (3.94)$$

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \Theta_2(k) \leq 3 \sum_{p_1 \neq p_2, p_1, p_2 \in \{1,2,3,4\}} \frac{1}{(1+|\xi_{p_1}|)(1+|\xi_{p_2}|)} e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}. \quad (3.95)$$

The proof of (3.94). It suffices to prove that

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \left(|\xi_1|^i |\xi_1 + \xi_4|^j + |\xi_2|^i |\xi_2 + \xi_4|^j + |\xi_3|^i |\xi_3 + \xi_4|^j + |\xi_4|^i |\xi_3 + \xi_4|^j \right) \\
& \leq \sum_{p_1 \neq p_2, p_1, p_2 \in \{1,2,3,4\}} \frac{1}{(1+|\xi_{p_1}|)(1+|\xi_{p_2}|)} e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}, \quad (3.96)
\end{aligned}$$

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \sum_{m+l=j} |\xi_{p_1}|^i \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l \\
& \leq 2 \sum_{p_1 \neq p_2, p_1, p_2 \in \{1,2,3,4\}} \frac{1}{(1+|\xi_{p_1}|)(1+|\xi_{p_2}|)} e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}. \quad (3.97)
\end{aligned}$$

To prove (3.96), inserting the inequality $\sigma(1 + |\xi|) \leq e^{\sigma|\xi|}$ for $0 \leq \sigma \leq 1$ into (3.77), we find

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_1|^i |\xi_1 + \xi_4|^j \leq \frac{1}{(1 + |\xi_2|)(1 + |\xi_3|)} e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}.$$

Similarly, the other terms on the left hand side of (3.96) can be bounded. This proves (3.96).

To prove (3.97), it suffices to establish that if p_1, p_2, p_3, p_4 is a permutation of 1, 2, 3, 4, then

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_{p_1}|^i \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l \leq \frac{1}{(1 + |\xi_{p_1}|)(1 + |\xi_{p_4}|)} e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}. \quad (3.98)$$

(Note that the factor 2 on the right side of (3.97) is needed, if one considers the number of terms for two sums in (3.97).) In fact, on one hand, thanks to (3.83),

$$\sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_{p_1}|^i \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l \leq \sigma^4 e^{\sigma(|\xi_{p_1}| + |\xi_{p_2}| + |\xi_{p_3}|)}. \quad (3.99)$$

On the other hand, similar to the proof of (3.83)

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_{p_1}|^{i+1} \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l &\leq \sum_{k=0}^{\infty} \frac{\sigma^{k+3}}{(k+3)!} \sum_{i+j=k} |\xi_{p_1}|^i \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l \\ &\leq \sigma^3 e^{\sigma(|\xi_{p_1}| + |\xi_{p_2}| + |\xi_{p_3}|)}. \end{aligned} \quad (3.100)$$

Combining (3.99) and (3.100) gives

$$(1 + |\xi_{p_1}|) \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_{p_1}|^i \sum_{m+l=j} |\xi_{p_2}|^m |\xi_{p_2} + \xi_{p_3}|^l \leq \sigma^3 e^{\sigma(|\xi_{p_1}| + |\xi_{p_2}| + |\xi_{p_3}|)}. \quad (3.101)$$

Using $\sigma(1 + |\xi|) \leq e^{\sigma|\xi|}$ for $0 \leq \sigma \leq 1$ again, the inequality (3.98) follows from (3.101).

The proof of (3.95). Using the idea of the proof of (3.94), we obtain the following estimates:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} (|\xi_1|^i |\xi_4|^j + |\xi_2|^i |\xi_3|^j) \\ \leq \left(\frac{1}{(1 + |\xi_1|)(1 + |\xi_4|)} + \frac{1}{(1 + |\xi_2|)(1 + |\xi_3|)} \right) e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}, \end{aligned} \quad (3.102)$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \left[|\xi_3|^i \sum_{m+l=j} |\xi_1|^m |\xi_4|^l + |\xi_4|^j \sum_{m+l=i} |\xi_2|^m |\xi_3|^l + |\xi_4|^i \sum_{m+l=j} |\xi_4|^m |\xi_1|^l \right] \\ \leq \left(\frac{2}{(1 + |\xi_1|)(1 + |\xi_4|)} + \frac{1}{(1 + |\xi_2|)(1 + |\xi_3|)} \right) e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}, \end{aligned} \quad (3.103)$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} |\xi_4|^i \left[\sum_{m+l=j} \left(|\xi_1|^l \sum_{n+h=m} |\xi_3|^n |\xi_2|^h + |\xi_2|^m \sum_{n+h=l} |\xi_4|^n |\xi_1|^h \right) \right] \\ \leq \left(\frac{1}{(1 + |\xi_1|)(1 + |\xi_4|)} + \frac{1}{(1 + |\xi_2|)(1 + |\xi_3|)} \right) e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)}, \end{aligned} \quad (3.104)$$

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{\sigma^{k+4}}{(k+4)!} \sum_{i+j=k} \sum_{n+h=i} |\xi_1|^n |\xi_4|^h \sum_{m+l=j} |\xi_2|^m |\xi_4|^l \\ & \leq \left(\frac{2}{(1+|\xi_1|)(1+|\xi_2|)} + \left(\frac{1}{1+|\xi_1|} + \frac{1}{1+|\xi_2|} \right) \frac{1}{1+|\xi_4|} \right) e^{\sigma(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}. \end{aligned} \quad (3.105)$$

Combining (3.102)-(3.105) implies (3.95). \square

4 The analyticity radius for KdV

In this section, we first shall prove an almost conservation law for the KdV equation 1.1 in Gevrey class spaces, based on the upper bounds in the subsection 3.3. Then we using the almost conservation law and an iteration argument to prove Theorem 1.1.

4.1 Almost conservation law

Recall that the energy $E_I^2(t) = \|Iu\|_{L^2}^2$, see (2.9). The following lemma shows that, for every $t \in \mathbb{R}$, the energy $E_I^4(t)$ is comparable to $E_I^2(t)$ if $\|Iu\|_{L^2}$ is small.

Lemma 4.1. *Let I be the operator defined with the Fourier symbol m given by (3.1), $0 < \sigma \leq 1$. Then there exists an absolute constant C such that for $t \in \mathbb{R}$*

$$|E_I^4(t) - E_I^2(t)| \leq C(\|Iu\|_{L^2}^3 + \|Iu\|_{L^2}^4). \quad (4.1)$$

Proof. Since $E_I^4(t) = E_I^2(t) + \Lambda_3(\beta_3; u, u, u) + \Lambda_4(\beta_4; u, u, u, u)$, it suffices to show

$$|\Lambda_3(\beta_3; u, u, u)| \leq \|Iu\|_{L^2}^3, \quad (4.2)$$

$$|\Lambda_4(\beta_4; u, u, u, u)| \leq \|Iu\|_{L^2}^4. \quad (4.3)$$

Without loss of generality, we assume that \widehat{u} is nonnegative.

Proof of (4.3). According to Lemma 3.6, using the property of Fourier transform, we find

$$\begin{aligned} |\Lambda_4(\beta_4; u, u, u, u)| & \leq \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \frac{|c|}{9} \sum_{\substack{p_1 \neq p_2, p_1, p_2 \in \{1,2,3,4\}}} \frac{1}{(1+|\xi_{p_1}|)(1+|\xi_{p_2}|)} \prod_{i=1}^4 e^{\sigma|\xi_i|} u(\xi_i) \\ & \leq \frac{4|c|}{3} \int_{\mathbb{R}} |\mathcal{F}^{-1}\left(\frac{1}{1+|\xi|}\right) e^{\sigma|\xi|} \widehat{u}(\xi)|^2 |\mathcal{F}^{-1} e^{\sigma|\xi|} \widehat{u}(\xi)|^2 dx, \end{aligned} \quad (4.4)$$

where \mathcal{F}^{-1} denotes the inverse Fourier transform. Using the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we derive from (4.4) that

$$\begin{aligned} |\Lambda_4(\beta_4; u, u, u, u)| & \leq \frac{4|c|}{3} \|\mathcal{F}^{-1}\left(\frac{1}{1+|\xi|}\right) e^{\sigma|\xi|} \widehat{u}(\xi)\|_{L^\infty}^2 \|\mathcal{F}^{-1} e^{\sigma|\xi|} \widehat{u}(\xi)\|_{L^2}^2 \\ & \leq C \|e^{\sigma|\xi|} \widehat{u}(\xi)\|_{L^2}^4 \leq 2^4 C \|Iu\|_{L^2}^4. \end{aligned}$$

This proves (4.3).

Proof of (4.2). The idea is similar to (4.3). We only give a sketch. If $1 \leq k \in \mathbb{N}$ and $\xi_1 + \xi_2 + \xi_3 = 0$, then one can show that

$$\xi_1^{2k+1} + \xi_2^{2k+1} + \xi_3^{2k+1} = \xi_1 \xi_2 \xi_3 \sum_{i+j=2k-2} \left(\xi_3^j \left((-\xi_1)^i + (-\xi_2)^i \right) + \xi_1^i (-\xi_2)^j \right). \quad (4.5)$$

In particular, this gives $\alpha_3 = i(\xi_1^3 + \xi_2^3 + \xi_3^3) = 3i\xi_1\xi_2\xi_3$ for $\xi_1 + \xi_2 + \xi_3 = 0$. Recall that $\beta_3 = i[m(\xi_1)m(\xi_2 + \xi_3)\{\xi_2 + \xi_3\}]_{sym}/\alpha_3$, by (3.3) we find

$$\beta_3 = -\frac{1}{9\xi_1\xi_2\xi_3} \sum_{k=1}^{\infty} \frac{\sigma^{2k}}{(2k)!} (\xi_1^{2k+1} + \xi_2^{2k+1} + \xi_3^{2k+1}). \quad (4.6)$$

Combining (4.5) and (4.6) gives

$$\beta_3 = -\frac{1}{9} \sum_{k=1}^{\infty} \frac{\sigma^{2k}}{(2k)!} \sum_{i+j=2k-2} \left(\xi_3^j ((-\xi_1)^i + (-\xi_2)^i) + \xi_1^i (-\xi_2)^j \right).$$

From this, one can show that for $0 < \sigma \leq 1$

$$|\beta_3| \leq \sum_{i=1}^3 \frac{1}{1 + |\xi_i|} e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3|)}. \quad (4.7)$$

Then (4.2) follows from (4.7). \square

Lemma 4.2. *Let I be the operator defined with the Fourier symbol m given by (3.1), $0 < \sigma \leq 1$. Then for $b \in (\frac{1}{2}, \frac{2}{3})$ there exists a constant $C = C(b)$ such that*

$$\left| \int_0^\delta \Lambda_5(M_5; u, u, u, u, u) dt \right| \leq C\sigma^4 \|Iu\|_{X_\delta^{0,b}}^5. \quad (4.8)$$

Proof. Without loss of generality, we assume that \widehat{u} is nonnegative again. Recall that (see (2.14))

$$M_5(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = -2i[\beta_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)\{\xi_4 + \xi_5\}]_{sym},$$

using Lemma 3.5, we find

$$|M_5| \leq \frac{86|c|}{27} \sigma^4 e^{\sigma(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4| + |\xi_5|)} \max_{i=1,2,\dots,5} |\xi_i|. \quad (4.9)$$

We first use the bound (4.9), and then Parseval identity to obtain that

$$\begin{aligned} \left| \int_0^\delta \Lambda_5(M_5; u, u, u, u, u) dt \right| &\leq \frac{86|c|}{27} \sigma^4 \int_0^\delta \int_{\xi_1 + \xi_2 + \dots + \xi_5 = 0} \max_{i=1,2,\dots,5} |\xi_i| \prod_{i=1}^5 e^{\sigma|\xi_i|} \widehat{u}(\xi_i) dt \\ &= \frac{86|c|}{27} \sigma^4 \int_0^\delta \int_{\mathbb{R}} |D| e^{\sigma|D|} u \cdot (e^{\sigma|D|} u)^4 dx dt, \end{aligned} \quad (4.10)$$

where $|D|$ and $e^{\sigma|D|}$ are the Fourier multiplier with symbol $|\xi|$ and $e^{\sigma|\xi|}$, respectively. The integral on right hand side of (4.10) can be bounded by

$$\int_0^\delta \int_{\mathbb{R}} e^{\sigma|D|} u \cdot |D| (e^{\sigma|D|} u)^4 dx dt \leq \|e^{\sigma|D|} u\|_{X_\delta^{0,1-b}} \| |D| (e^{\sigma|D|} u)^4 \|_{X_\delta^{0,b-1}} \quad (4.11)$$

for all $b \in (\frac{1}{2}, 1)$. Applying Lemma 2.1 with $u_i = e^{\sigma|D|} u, i = 1, 2, 3, 4, b' = b - 1$, we find for $b \in (\frac{1}{2}, \frac{2}{3})$

$$\| |D| (e^{\sigma|D|} u)^4 \|_{X_\delta^{0,b-1}} \leq C \|e^{\sigma|D|} u\|_{X_\delta^{0,b}}^4, \quad (4.12)$$

where C is a constant depends only on b . Note that $1-b < b$ (since $b > \frac{1}{2}$), we have $\|e^{\sigma|D|}u\|_{X_\delta^{0,1-b}} \leq \|e^{\sigma|D|}u\|_{X_\delta^{0,b}}$. Inserting (4.12) into (4.11), we obtain

$$\int_0^\delta \int_{\mathbb{R}} e^{\sigma|D|}u \cdot |D|(e^{\sigma|D|}u)^4 dx dt \leq C \|e^{\sigma|D|}u\|_{X_\delta^{0,b}}^5. \quad (4.13)$$

Combining (4.10) and (4.13) we get

$$\left| \int_0^\delta \Lambda_5(M_5; u, u, u, u, u) dt \right| \leq \frac{86|c|}{27} C \sigma^4 \|e^{\sigma|D|}u\|_{X_\delta^{0,b}}^5 \leq C' \sigma^4 \|Iu\|_{X_\delta^{0,b}}^5$$

with $C' = \frac{86|c|}{27} C 2^5$. This completes the proof. \square

Lemma 4.2 implies an almost conservation of $E_I^4(t)$ for $t \in [0, \delta]$ when σ goes to zero. This together with Lemma 4.1 will show that the energy $E_I^2(t)$ is almost conserved.

Corollary 4.1. *Let $u \in G_\delta^{\sigma,b}$ be the solution of (1.1) obtained in Proposition 2.1, $b > \frac{1}{2}$. Assume that $0 < \sigma \leq 1$ and $\|Iu_0\|_{L^2} = \varepsilon_0 < 1$, where I is defined by the Fourier symbol m given by (3.1). Then for all $t \in [0, \delta]$*

$$\|Iu(t)\|_{L^2}^2 \leq \varepsilon_0^2 + \mathcal{O}(\varepsilon_0^3) + C\varepsilon_0^5\sigma^4. \quad (4.14)$$

Proof. Since $\|Iu\|_{X_\delta^{0,b}}$ is comparable with $\|u\|_{G_\delta^{\sigma,b}}$, the bound (2.4) implies that $\|Iu\|_{X_\delta^{0,b}} \leq C\|Iu_0\|_{L^2}$ for some constant $C > 0$. Using the embedding $X_\delta^{0,b} \hookrightarrow L_t^\infty L_x^2$ when $b > \frac{1}{2}$ and $\|Iu_0\|_{L^2} = \varepsilon_0 < 1$, we deduce from Lemma 4.1 that

$$E_I^4(0) = E_I^2(0) + \mathcal{O}(\varepsilon_0^3), \quad (4.15)$$

and, moreover, for all $t \in (0, \delta]$

$$E_I^4(t) = E_I^2(t) + \mathcal{O}(\varepsilon_0^3). \quad (4.16)$$

Thanks to Lemma 4.2, we find for all $t \in (0, \delta]$

$$|E_I^4(t) - E_I^4(0)| \leq C\varepsilon_0^5\sigma^4. \quad (4.17)$$

Combining (4.15)-(4.17) implies the desired inequality (4.14). \square

4.2 The proof of Theorem 1.1

Let $u_0 \in G^{\sigma_0}$ with some $\sigma_0 > 0$. We can not use the almost conservation law above directly, since the norm $\|u_0\|_{G^{\sigma_0}}$ may be large. To over the difficulty, we need to make a scaling on the solution. Precisely, for every $\lambda > 0$, set

$$u_\lambda(t, x) := \lambda^{-2} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right).$$

Clearly, $u_\lambda(t, x)$ is also a solution of the KdV equation (1.1) on $[0, \lambda^3 T] \times \mathbb{R}$ if $u(t, x)$ is a solution on $[0, T] \times \mathbb{R}$. The spatial Fourier transform has the relation

$$\widehat{u}_\lambda(t, \xi) = \lambda^{-1} \widehat{u}\left(\frac{t}{\lambda^3}, \lambda\xi\right). \quad (4.18)$$

In particular, we have $\widehat{u_\lambda}(0, \xi) = \lambda^{-1} \widehat{u_0}(\lambda \xi)$. This implies that for all $\sigma > 0$

$$\|u_\lambda(0, \cdot)\|_{G^\sigma} = \lambda^{-\frac{3}{2}} \|u_0\|_{G^{\frac{\sigma}{\lambda}}}. \quad (4.19)$$

For every $\varepsilon_0 \in (0, 1)$, set

$$\lambda := \left(1 + \frac{\|u_0\|_{G^{\sigma_0}}}{\varepsilon_0}\right)^{\frac{2}{3}}. \quad (4.20)$$

Using the embedding $G^\sigma \hookrightarrow G^{\frac{\sigma}{\lambda}}$ since $\lambda \geq 1$, and by (4.19) we obtain

$$\|u_\lambda(0, \cdot)\|_{G^{\sigma_0}} \leq \varepsilon_0. \quad (4.21)$$

According to Proposition 2.1, problem (1.1) has a unique rescaled solution $u_\lambda(t, x)$ with datum $u_\lambda(0, x)$ on the interval $t \in [0, \delta]$, where

$$\delta = \frac{c_0}{(1 + \|u_0\|_{G^{\sigma_0}})^{\frac{1}{\frac{3}{4}-b}}}. \quad (4.22)$$

Since $\|Iu_\lambda(0, \cdot)\|_{L^2} \leq \|u_\lambda(0, \cdot)\|_{G^{\sigma_0}} \leq \varepsilon_0$, thanks to Corollary 4.1, we obtain for $t \in [0, \delta]$

$$\|u(t)\|_{G^\sigma} \leq 2\|Iu(t)\|_{L^2} \leq 2\sqrt{\varepsilon_0^2 + \mathcal{O}(\varepsilon_0^3)} + C\varepsilon_0^5\sigma^4 \leq 4\varepsilon_0 \quad (4.23)$$

where $\sigma = \min\{1, \sigma_0\}$, ε_0 is chosen small enough. Thus $\|u_\lambda(\delta)\|_{G^\sigma} \leq 4\varepsilon_0$. This allows us to take $u_\lambda(\delta)$ as a new data, by virtue of (4.22), to obtain a solution on the interval $[\delta, 2\delta]$. Follow this line, by using the local well posedness result and almost conservation law repeatedly, we shall prove that, for arbitrarily large T ,

$$\sup_{t \in [0, T]} \|u_\lambda(t)\|_{G^{\sigma(t)}} \leq 4\varepsilon_0, \quad (4.24)$$

with for large t

$$\sigma(t) \geq c|t|^{-\frac{1}{4}}. \quad (4.25)$$

Now arbitrarily fixed T large. With a little abuse using of notations, we still denote $E_I^j(t)$ ($j = 2, 3, 4$) the energies defined in Subsection 2.2 with u_λ in place of u . Choose $m \in \mathbb{N}$ such that $T \in [m\delta, (m+1)\delta)$. We shall use induction to show for $k = \{1, 2, \dots, m+1\}$ that

$$\sup_{t \in [0, k\delta]} |E_I^4(t) - E_I^4(0)| \leq Ck\varepsilon_0^5\sigma^4, \quad (4.26)$$

$$\sup_{t \in [0, k\delta]} \|u_\lambda(t)\|_{G^\sigma} \leq 4\varepsilon_0. \quad (4.27)$$

In fact, for $k = 1$, (4.26) and (4.27) follows from Corollary 4.1 and (4.23), respectively. Now assume that (4.26) and (4.27) hold for some $k \in \{1, 2, \dots, m\}$. Take $u_\lambda(k\delta)$ as a new data, by Proposition 2.1, we obtain a solution u_λ on the interval $[k\delta, (k+1)\delta]$, and

$$\sup_{t \in [k\delta, (k+1)\delta]} \|u_\lambda(t)\|_{G^\sigma} \leq 4C\varepsilon_0. \quad (4.28)$$

Moreover, we apply Corollary 4.1 with u_λ on the interval $[k\delta, (k+1)\delta]$ to find

$$\sup_{t \in [k\delta, (k+1)\delta]} |E_I^4(t) - E_I^4(k\delta)| \leq C\varepsilon_0^5 \sigma^4. \quad (4.29)$$

Combining (4.29) and the induction hypothesis (4.26), we obtain

$$\sup_{t \in [0, (k+1)\delta]} |E_I^4(t) - E_I^4(0)| \leq C(k+1)\varepsilon_0^5 \sigma^4. \quad (4.30)$$

This proves (4.26) with k replaced by $k+1$. Using Lemma 4.1, we deduce from (4.30) that

$$\sup_{t \in [0, (k+1)\delta]} E_I^2(t) \leq \varepsilon_0^2 + \mathcal{O}(\varepsilon_0^3) + C(k+1)\varepsilon_0^5 \sigma^4 = \varepsilon_0^2 + \mathcal{O}(\varepsilon_0^3) + C\varepsilon_0^5 \quad (4.31)$$

provided that

$$(k+1)\sigma^4 = 1. \quad (4.32)$$

By (4.31), we can choose ε_0 small enough such that

$$\sup_{t \in [0, (k+1)\delta]} E_I^2(t) \leq 4\varepsilon_0^2. \quad (4.33)$$

It follows from (4.33) that

$$\sup_{t \in [0, (k+1)\delta]} \|u_\lambda(t)\|_{G^\sigma} \leq 2 \sup_{t \in [0, (k+1)\delta]} \sqrt{E_I^2(t)} \leq 4\varepsilon_0.$$

This proves (4.27) with k replaced by $k+1$.

Since $\varepsilon_0 \in (0, 1)$, we find the lifespan, of local solution, $\delta \sim 1$. Then it follows from (4.32) that

$$\sigma = (k+1)^{-\frac{1}{4}} \geq \left(\frac{T}{\delta} + 1\right)^{-\frac{1}{4}} \geq cT^{-\frac{1}{4}}, \quad (4.34)$$

where c is an absolute constant. Thus, we have proved (4.24) and (4.25).

Now we pass the result of u_λ to that of u . Thanks to (4.18), we have

$$\widehat{u}(t, \xi) = \lambda \widehat{u_\lambda}(\lambda^3 t, \frac{\xi}{\lambda}). \quad (4.35)$$

Fixed T arbitrarily large. It follows from (4.24), (4.25) and (4.35) that

$$\sup_{t \in [0, T]} \|u(t)\|_{G^\sigma} = \lambda^{\frac{3}{2}} \sup_{t \in [0, T]} \|u_\lambda(t)\|_{G^{\lambda^3 \sigma}} \leq 4\lambda^{\frac{3}{2}} \varepsilon_0 \quad (4.36)$$

with

$$\sigma \geq \frac{cT^{-\frac{1}{4}}}{\lambda^3}. \quad (4.37)$$

By virtue of (4.20), we deduce from (4.36)-(4.37) that

$$\sup_{t \in [0, T]} \|u(t)\|_{G^\sigma} \leq 4(1 + \|u_0\|_{G^{\sigma_0}})$$

with

$$\sigma \geq c'T^{-\frac{1}{4}},$$

where $c' = \frac{c}{(1 + \frac{\|u_0\|_{G^{\sigma_0}}}{\varepsilon_0})^2}$. This completes of the proof.

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