

# INFINITELY MANY NON-RADIAL SOLUTIONS TO A CRITICAL EQUATION ON ANNULUS

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ABSTRACT. In this paper, we build infinitely many non-radial sign-changing solutions to the critical problem:

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (P)$$

on the annulus  $\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$ ,  $N \geq 3$ . In particular, for any integer  $k$  large enough, we build a non-radial solution which look like the unique positive solution  $u_0$  to (P) crowned by  $k$  negative bubbles arranged on a regular polygon with radius  $r_0$  such that  $r_0^{\frac{N-2}{2}} u_0(r_0) =: \max_{a \leq r \leq b} r^{\frac{N-2}{2}} u_0(r)$ .

## 1. INTRODUCTION

This paper deals with the existence of solutions to the critical elliptic problem:

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  and  $N \geq 3$ .

It is well known that the geometry of the domain  $\Omega$  plays a crucial role in the solvability of the problem (1.1). Indeed, if  $\Omega$  is a star-shaped domain, the classical Pohozaev identity [30] implies that (1.1) does not have any solutions. While if  $\Omega = \{x \in \mathbb{R}^N : a < |x| < b\}$  is an annulus, Kazdan and Warner [21] found a positive solution and infinitely many radial sign-changing solutions. Without any symmetry assumptions, the existence of solutions is a delicate issue. The first existence result is due to Coron in [10] who proved that problem (1.1) has a positive solution in domain  $\Omega$  with a small hole. Later, Bahri and Coron in [2] proved that actually a positive solution always exists as long as the domain has non-trivial homology with  $\mathbb{Z}_2$ -coefficients. However, this last condition is not necessary since solutions to problem (1.1) in contractible domains have been found by Dancer [11], Ding [17], Passaseo [28, 29] and Clapp and Weth [6]. The existence of sign-changing solutions is an even more delicate issue and it is known only for domains which have some symmetries or a small hole. The first existence result is due to Marchi and Pacella [24] for symmetric domains with thin channels. Successively, Clapp and Weth [6] found sign-changing solutions in a symmetric domain with a small hole. A first attempt to remove the symmetry assumption is due to Clapp and Weth [7], who found a second solution to (1.1) in a domain with a small hole but they were not able to say if it changes sign or not. Sign-changing solutions in a domain with a small hole have been found by Clapp, Musso and Pistoia in [8]. Recently, Musso and Pistoia [25] and Ge, Musso

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1991 *Mathematics Subject Classification.* Primary 35J91; Secondary 35B33.

*Key words and phrases.* Critical exponent, Non-radial solutions, Annulus domain.

and Pistoia [18] (see also [19]) proved that in a domain (not necessarily symmetric) with a small hole the number of sign-changing solutions to problem (1.1) becomes arbitrary large as the size of the hole decreases. The existence of a large number of sign-changing solutions in a domain with a hole of arbitrary size is due to Clapp and Pacella in [5], provided the domain has enough symmetry.

It is largely open for the problem of the existence of infinitely many sign-changing solutions in a general domain with non-trivial homology in the spirit of the famous Bahri and Coron's result.

Here, we will focus on the existence of infinitely many sign-changing solutions to problem (1.1) when  $\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$  is an annulus. The existence of infinitely many radial solutions was established by Kazdan and Warner in [21]. On the other hand, an annulus is invariant under many group actions and then it is natural to expect non-radial solutions which are invariant under these group actions. Indeed, Y.Y. Li in [22] improved a previous result by Coffman [9] and he found for any integer  $k \geq 1$  in a sufficiently thin annulus some non-radial solutions which are invariant under the action of the group  $\mathfrak{G}_k \times \mathfrak{O}(N-2)$ , when  $N \geq 4$ . Here  $\mathfrak{O}(N-2)$  denotes the group of orthogonal  $(N-2) \times (N-2)$  matrices and  $\mathfrak{G}_k$  is the subgroup of matrices which rotates  $\mathbb{R}^2$  with angles equal to integer multiple of  $\frac{2\pi}{k}$ . Recently, Clapp in [4] found infinitely many non-radial solutions which are invariant under the action of a suitable group whose orbits are infinite, provided  $N = 4$  or  $N \geq 6$ .

In this paper we prove the existence of infinitely many new non-radial solutions which are invariant under the action of a group whose orbits are finite and they are not invariant under the action of the group  $\mathfrak{G}_k \times \mathfrak{O}(N-2)$ . Moreover, as far as we know, this is the first example of non-radial solutions in the 3-dimensional annulus.

Let us state our main result. Let  $\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$  be an annulus. Assume that

$$\text{the unique positive radial solution } u_0 \text{ to (1.1) is non-degenerate.} \quad (1.2)$$

The uniqueness has been proved by Ni and Nussbaum [26]. The non-degeneracy will be studied in Appendix A and it is true for most radii  $a$  and  $b$ . Let us introduce the functions:

$$U_{\xi,\lambda}(y) = C_N \lambda^{\frac{N-2}{2}} \left( \frac{1}{1 + \lambda^2 |y - \xi|^2} \right)^{\frac{N-2}{2}}, \quad \xi, y \in \mathbb{R}^N, \lambda > 0$$

which are all the positive solutions of the following critical problem on the whole space:

$$-\Delta U = U^{\frac{N+2}{N-2}} \quad \text{in } \mathbb{R}^N, \quad (1.3)$$

where  $C_N$  is a constant dependent on  $N$  (see [1, 27, 31]). We call  $U_{\xi,\lambda}(y)$  the bubble centered at the point  $\xi$  with scaling parameter  $\lambda$ . Let us introduce its projection  $PU_{\xi,\lambda}$  onto  $H_0^1(\Omega)$ , namely the solution of the Dirichlet problem:

$$\begin{cases} -\Delta PU_{\xi,\lambda} = U_{\xi,\lambda}^{\frac{N+2}{N-2}}, & \text{in } \Omega, \\ PU_{\xi,\lambda} = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Let  $k \geq 1$  be an integer. Let us choose the centers of the bubbles as the  $k$  vertices of a regular  $k$ -polygon with radius  $r$  inside  $\Omega$  as:

$$\xi_j = r \xi_j^*, \quad \xi_j^* := (e^{i\frac{2\pi}{k}j}, \mathbf{0}), \quad \mathbf{0} \in \mathbb{R}^{N-2}, \quad j = 1, 2, \dots, k, \quad r \in (a, b) \quad (1.5)$$

and the concentration parameter as:

$$\lambda = \ell k^2, \ell \in [\eta, \eta^{-1}] \text{ for some } \eta > 0 \text{ small enough.} \quad (1.6)$$

Finally, we introduce the space

$$\begin{aligned} H_s := \{ & u \in H_0^1(\Omega) : \\ & u(x_1, x_2, \dots, x_i, \dots, x_N) = u(x_1, x_2, \dots, -x_i, \dots, x_N), \quad i = 2, \dots, N, \\ & u(re^{i\theta}, x_3, \dots, x_N) = u\left(re^{i(\theta + \frac{2\pi}{k}j)}, x_3, \dots, x_N\right), \quad j = 1, \dots, k \} \end{aligned}$$

Now, we can state our main result.

**Theorem 1.1.** *Let  $\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$  be an annulus. Assume (1.2). Then there exists an integer  $k_0 > 0$ , such that for any integer  $k \geq k_0$ , problem (1.1) has a solution*

$$u_k(x) = u_0(x) - \sum_{j=1}^k PU_{r_k \xi_j^*, \lambda_k}(x) + \varphi_k(x).$$

Where as  $k \rightarrow \infty$

- (i)  $r_k \rightarrow r_0 \in (a, b)$  and  $r_0^{\frac{N-2}{2}} u_0(r_0) := \max_{a \leq r \leq b} r^{\frac{N-2}{2}} u_0(r)$
- (ii)  $\lambda_k/k^2 \rightarrow \ell_0 > 0$
- (iii)  $\varphi_k \in H_s$  and  $\|\varphi_k\|_{H_0^1(\Omega)} \rightarrow 0$

The paper is inspired by recent results obtained by Del Pino, Musso, Pacard and Pistoia [15, 16], where the authors constructed for any  $N \geq 3$  infinitely many sign-changing solutions to (1.3) which look like the solution  $U_{0,1}$  crowned with  $k$  negative bubbles arranged on a regular polygon with radius near 1.

For the proof of our theorem, it relies on a Ljapunov-Schmidt procedure which allows us to reduce the problem of finding a solution to (1.1) whose profile at main order is  $u_0 - \sum_{j=1}^k PU_{r \xi_j^*, \lambda}$  to a 2-dimensional problem, namely finding the concentration parameter  $\lambda > 0$  in (1.6) and the radius  $r \in (a, b)$  of the  $k$ -regular polygon whose vertices are the concentration points as in (1.5). The basic outline is similar to that in [15], but we carry out the reduction argument in a different way. Indeed, the invariance by Kelvin's transform which is one of the main ingredient in the proof of [15], does not hold for problem (1.1). In particular, all our estimates are more straightforward than those used in [15].

This paper is organized as follows. In Section 2 we study the linearized equation around the approximate solution and we reduce the problem to a finite dimensional one. In Section 3 we study the reduced problem and we complete the proof of Theorem 1.1. Appendix A is devoted to the study of the non-degeneracy of the positive radial solution  $u_0$ .

## 2. FINITE-DIMENSIONAL REDUCTION

Let us introduce the norms:

$$\|u\|_* = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right)^{-1} \lambda^{-\frac{N-2}{2}} |u(y)| \quad (2.1)$$

and

$$\|f\|_{**} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N+2}{2} + \tau}} \right)^{-1} \lambda^{-\frac{N+2}{2}} |f(y)|, \quad (2.2)$$

where  $\tau = \frac{1}{2}$ . Since we assume that  $\lambda \sim k^2$ , it holds

$$\sum_{j=2}^k \frac{1}{|\lambda\xi_j - \lambda\xi_1|^\tau} \leq \frac{Ck}{\lambda^\tau} \leq C.$$

Set  $U_j =: U_{\xi_j, \lambda}(y)$ ,  $P_j =: PU_{\xi_j, \lambda}(y)$  and  $U_* = u_0 - \sum_j^k P_j$ .

Denote

$$Z_{j,1} = \frac{\partial P_j}{\partial \lambda}, \quad Z_{j,2} = \frac{\partial P_j}{\partial r}, \quad j = 1, 2, \dots, k.$$

We consider the following linearized problem:

$$\begin{cases} L_k \varphi := -\Delta \varphi - (2^* - 1)|U_*|^{2^*-2} \varphi = h + \sum_{l=1}^2 c_l \sum_{j=1}^k U_j^{2^*-2} Z_{j,l}, & \text{in } \Omega, \\ \varphi \in H_s, \quad \sum_{j=1}^k \int_{\Omega} U_j^{2^*-2} Z_{j,l} \varphi = 0, & l = 1, 2, \end{cases} \quad (2.3)$$

for some real numbers  $c_l$ .

**Lemma 2.1.** *Suppose that  $\varphi_k$  solves (2.3) for  $h = h_k$ . If  $\|h_k\|_{**}$  goes to zero as  $k \rightarrow +\infty$ , so does  $\|\varphi_k\|_*$ .*

*Proof.* We argue by contradiction. Suppose that there exist  $k \rightarrow +\infty$ ,  $r_k \rightarrow r_0$ ,  $\lambda_k \in [L_0 k^2, L_1 k^2]$  and  $\varphi_k$  solving (2.3) for  $h = h_k$ ,  $\lambda = \lambda_k$ ,  $r = r_k$  with  $\|h_k\|_{**} \rightarrow 0$  and  $\|\varphi_k\|_* \geq c > 0$ . Without loss of generality, we may assume that  $\|\varphi_k\|_* = 1$ . In the following, for simplicity reason, we drop the subscript  $k$ .

Since we assume  $u_0$  is non-degenerate, the following linear operator:

$$\tilde{L}_0 \varphi := -\Delta \varphi - (2^* - 1)u_0^{2^*-2} \varphi, \quad \varphi \in H_0^1(\Omega),$$

is invertible. Let  $G(y, x)$  be the corresponding Green's function. It is easy to prove that there exists a constant  $C > 0$ , such that

$$|G(y, x)| \leq \frac{C}{|y - x|^{N-2}}. \quad (2.4)$$

We rewrite (2.3) as:

$$\begin{cases} L_0 \varphi = (2^* - 1)(|U_*|^{2^*-2} - u_0^{2^*-2}) \varphi + h + \sum_{l=1}^2 c_l \sum_{j=1}^k U_j^{2^*-2} Z_{j,l}, & \text{in } \Omega, \\ \varphi \in H_s, \quad \sum_{j=1}^k \int_{\Omega} U_j^{2^*-2} Z_{j,l} \varphi = 0, & l = 1, 2. \end{cases} \quad (2.5)$$

Then

$$\varphi(y) = \int_{\Omega} G(z, y) \left[ (2^* - 1)(|U_*|^{2^*-2} - u_0^{2^*-2})\varphi + h + \sum_{l=1}^2 c_l \sum_{j=1}^k U_j^{2^*-2} Z_{j,l} \right].$$

Using (2.4), we obtain

$$|\varphi(y)| \leq C \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left| \left[ (2^* - 1)(|U_*|^{2^*-2} - u_0^{2^*-2})\varphi + h + \sum_{l=1}^2 c_l \sum_{j=1}^k U_j^{2^*-2} Z_{j,l} \right] \right|.$$

As in [32], we have

$$\begin{aligned} & \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left| |U_*|^{2^*-2} - u_0^{2^*-2} \right| |\varphi| \\ & \leq C \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left( \sum_{j=1}^k P_j \right)^{2^*-2} |\varphi| \\ & \leq C \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \left( \sum_{j=1}^k U_j \right)^{2^*-2} |\varphi| \tag{2.6} \\ & \leq C \|\varphi\|_* \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \left( \sum_{j=1}^k U_j \right)^{2^*-2} \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda|z - \xi_j|)^{\frac{N-2}{2} + \tau}} \\ & \leq C \|\varphi\|_* \lambda^{\frac{N-2}{2}} \sum_{j=1}^m \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau + \theta}}. \end{aligned}$$

$$\begin{aligned} & \int_{\Omega} \frac{1}{|z-y|^{N-2}} |h(z)| dz \\ & \leq C \|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^k \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|z - \xi_j|)^{\frac{N+2}{2} + \tau}} dz \tag{2.7} \\ & \leq C \|h\|_{\alpha, **} \lambda^{\frac{N-2}{2}} \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}}, \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left| \sum_{j=1}^k U_j^{2^*-2} Z_{j,l} \right| dz \\ & \leq C \lambda^{\frac{N+2}{2} + n_l} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^k \frac{1}{(1 + \lambda|z - \xi_j|)^{N+2}} \\ & \leq C \lambda^{\frac{N-2}{2} + n_l} \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{N-2}} \tag{2.8} \\ & \leq C \lambda^{\frac{N-2}{2} + n_l} \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}}, \end{aligned}$$

where  $n_2 = 1$ ,  $n_1 = -1$ .

To estimate  $c_l$ ,  $l = 1, 2$ , multiplying the both sides of (2.3) by the function  $Z_{1,l}$ , ( $l = 1, 2$ ) and integrating on  $\Omega$ , we see that  $c_l$  satisfies:

$$\begin{aligned} & \sum_{h=1}^2 c_h \sum_{j=1}^k \int_{\Omega} U_j^{2^*-2} Z_{j,h} Z_{1,l} \\ &= \int_{\Omega} \left( -\Delta \varphi - (2^* - 1) |U_*|^{2^*-2} \varphi \right) Z_{1,l} - \int_{\Omega} h Z_{1,l}. \end{aligned} \quad (2.9)$$

We have

$$\begin{aligned} & \left| \int_{\Omega} h Z_{1,l} \right| \\ & \leq C \|h\|_{**} \int_{\mathbb{R}^N} \frac{\lambda^{\frac{N-2}{2}+n_l}}{(1+\lambda|z-\xi_1|)^{N-2}} \sum_{j=1}^k \frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|z-\xi_j|)^{\frac{N-2}{2}+\tau}} \\ & \leq C \lambda^{n_l} \|h\|_{**} \left( C + C \sum_{j=2}^k \frac{1}{(\lambda|\xi_j - \xi_1|)^{\tau}} \right) \leq C \lambda^{n_l} \|h\|_{**}. \end{aligned} \quad (2.10)$$

On the other hand, direct calculation gives

$$\begin{aligned} & \left| \int_{\Omega} \left( -\Delta \varphi - (2^* - 1) |U_*|^{2^*-2} \varphi \right) Z_{1,l} \right| \\ &= \left| \int_{\Omega} \left( -\Delta Z_{1,l} - (2^* - 1) |U_*|^{2^*-2} Z_{1,l} \right) \varphi \right| \\ &= (2^* - 1) \left| \int_{\Omega} (U_1^{2^*-2} - |U_*|^{2^*-2}) Z_{1,l} \right| \varphi \\ & \leq C \lambda^{n_l} \|\varphi\|_* \int_{\Omega} \left( u_0^{2^*-1} + \left( \sum_{j=2}^k U_j \right)^{2^*-2} \right) U_1 \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|z-\xi_j|)^{\frac{N-2}{2}+\tau}} \\ & \leq O \left( \lambda^{n_l} \|\varphi\|_* \left( \frac{1}{\lambda^2} + \frac{1}{\lambda^{\frac{N-2}{2}}} \right) \right). \end{aligned} \quad (2.11)$$

And it is easy to check that

$$\sum_{j=1}^k \int_{\Omega} U_j^{2^*-2} Z_{j,h} Z_{1,l} = (\bar{c} + o(1)) \delta_{hl} \lambda^{2n_l}, \quad (2.12)$$

for some constant  $\bar{c} > 0$ .

Now inserting (2.12) into (2.9), we find

$$c_l = \frac{1}{\lambda^{n_l}} \left( o(\|\varphi\|_{\alpha,*}) + O(\|h\|_{\alpha,**}) \right). \quad (2.13)$$

So,

$$\|\varphi\|_* \leq \left( o(1) + \|h\|_{**} + \frac{\sum_{j=1}^k \frac{1}{(1+\lambda|y-\xi_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^k \frac{1}{(1+\lambda|y-\xi_j|)^{\frac{N-2}{2}+\tau}} \right). \quad (2.14)$$

Since  $\|\varphi\|_* = 1$ , we obtain from (2.14) that there is  $R > 0$  such that

$$\|\lambda^{\frac{N-2}{2}} \varphi\|_{L^\infty(B_{R/\lambda}(\xi_j))} \geq a > 0, \quad (2.15)$$

for some  $j$ . But  $\tilde{\varphi}(y) = \lambda^{-\frac{N-2}{2}} \varphi(\lambda(y - x_j))$  converges uniformly in any compact set to a solution  $u$  of

$$-\Delta u - (2^* - 1)U_{0,\Lambda}^{2^*-2}u = 0, \quad \text{in } \mathbb{R}^N, \quad (2.16)$$

for some  $\Lambda \in [\Lambda_1, \Lambda_2]$ , where  $\Lambda_1, \Lambda_2$  are two constants, and  $u$  is perpendicular to the kernel of (2.16). So  $u = 0$ . This is a contradiction to (2.15).  $\square$

From Lemma 2.1, applying the same argument as in the proof of Proposition 4.1 in [13], we can prove the following result:

**Lemma 2.2.** *There exist  $k_0 > 0$  and a constant  $C > 0$  independent of  $k$ , such that for  $k \geq k_0$  and all  $h \in L^\infty(\mathbb{R}^N)$ , problem (2.3) has a unique solution  $\varphi_k \equiv L_k(h)$ . Moreover,*

$$\|\varphi_k\|_* \leq C\|h\|_{**}, \quad |c_l| \leq \frac{C}{\lambda^{n_l}} \|h\|_{**}. \quad (2.17)$$

Now we consider the following non-linear problem:

$$\begin{cases} -\Delta(U_* + \varphi) = |U_* + \varphi|^{2^*-2}(U_* + \varphi) + \sum_{l=1}^2 c_l \sum_{j=1}^k U_j^{2^*-2} Z_{j,l}, & \text{in } \Omega, \\ \varphi \in H_s, \quad \int_{\Omega} \sum_{j=1}^k U_j^{2^*-2} Z_{j,l} \varphi = 0, \quad l = 1, 2. \end{cases} \quad (2.18)$$

The main result of this section is:

**Proposition 2.3.** *There exists a positive integer  $k_0$  such that for each  $k \geq k_0, \lambda \in [\eta k^2, \eta^{-1} k^2], r \in [a + \tau, b - \tau]$ , where  $\tau$  and  $\eta$  are positive and small, (2.18) has a unique solution  $\varphi = \varphi_{r,\lambda} \in H_s$  satisfying*

$$\|\varphi\|_* \leq C\lambda^{-\frac{N-2}{4}-\sigma}, \quad |c_l| \leq C\lambda^{-\frac{N-2}{4}-\sigma-n_l}, \quad (2.19)$$

where  $\sigma > 0$  is a small constant.

Rewrite (2.18) as:

$$\begin{cases} -\Delta\varphi - (2^* - 1)|U_*|^{2^*-2}\varphi = N(\varphi) + l_k + \sum_{l=1}^2 c_l \sum_{j=1}^k U_j^{2^*-2} Z_{j,l}, & \text{in } \Omega, \\ \varphi \in H_s, \quad \int_{\Omega} \sum_{j=1}^k U_j^{2^*-2} Z_{j,l} \varphi = 0, \quad l = 1, 2, \end{cases} \quad (2.20)$$

where

$$N(\varphi) = |U_* + \varphi|^{2^*-2}(U_* + \varphi) - |U_*|^{2^*-2}U_* - (2^* - 1)|U_*|^{2^*-2}\varphi,$$

and

$$l_k = |U_*|^{2^*-2}U_* - u_0^{2^*-1} + \sum_{j=1}^k U_j^{2^*-1}.$$

In order to apply the contraction mapping principle to prove that (2.20) is uniquely solvable, we have to estimate  $N(\varphi)$  and  $l_k$  respectively.

**Lemma 2.4.** *We have*

$$\|N(\varphi)\|_{**} \leq C\|\varphi\|_*^{\min(2^*-1, 2)}.$$

*Proof.* If  $N \geq 6$ , then  $2^* - 2 \leq 1$ . So we have

$$|N(\varphi)| \leq C|\varphi|^{2^*-1},$$

which gives

$$\begin{aligned} |N(\varphi)| &\leq C\|\varphi\|_*^{2^*-1} \left( \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^*-1} \\ &\leq C\|\varphi\|_*^{2^*-1} \lambda^{\frac{N+2}{2}} \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N+2}{2} + \tau}} \left( \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^\tau} \right)^{\frac{4}{N-2}} \\ &\leq C\|\varphi\|_*^{2^*-1} \lambda^{\frac{N+2}{2}} \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N+2}{2} + \tau}}. \end{aligned}$$

Therefore,

$$\|N(\varphi)\|_{**} \leq C\|\varphi\|_*^{2^*-1}.$$

Similarly, if  $3 \leq N \leq 5$ , then  $2^* - 3 > 0$ . In view of  $U_j \geq c_0 > 0$  in  $\Omega$ , we find

$$\begin{aligned} |N(\varphi)| &\leq C(|u_0|^{2^*-3} + \sum_{j=1}^k U_j^{2^*-3})\varphi^2 + C|\varphi|^{2^*-1} \\ &\leq C(\|\varphi\|_*^2 + \|\varphi\|_*^{2^*-1})\lambda^{\frac{N+2}{2}} \left( \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^*-1} \\ &\leq C\|\varphi\|_*^2 \sum_{j=1}^k \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - \xi_j|)^{\frac{N+2}{2} + \tau}}. \end{aligned}$$

□

Next, we estimate  $l_k$ .

**Lemma 2.5.** *There is a constant  $\sigma > 0$ , such that*

$$\|l_k\|_{**} \leq C\lambda^{-\frac{N-2}{4} - \sigma}.$$

*Proof.* Write

$$\begin{aligned} l_k &= \left[ |U_*|^{2^*-2}U_* - u_0^{2^*-1} + \sum_{j=1}^k P_j^{2^*-1} \right] + \sum_{j=1}^k (U_j^{2^*-1} - P_j^{2^*-1}) \\ &=: J_1 + J_2. \end{aligned}$$

First, we estimate  $\|J_2\|_{**}$ . We have

$$0 \leq U_j^{2^*-1} - P_j^{2^*-1} \leq \frac{CU_j^{2^*-2}}{\lambda^{\frac{N-2}{2}}}.$$

Let us determine the number  $\alpha > 0$ , such that



$$\frac{CU_j^{2^*-2}}{\lambda^{\frac{N-2}{2}}} \leq \frac{C\lambda^{-\alpha}\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-\xi_j|)^{\frac{N+2}{2}+\tau}}.$$

The above inequality is equivalent to

$$(1+\lambda|y-\xi_j|)^{\frac{N+2}{2}+\tau-4} \leq C\lambda^{-\alpha}\lambda^{N-2}.$$

Note that  $\tau = \frac{1}{2}$ . We find that  $\frac{N+2}{2} + \tau - 4 \geq 0$  if  $N \geq 5$ . In view of  $1 + \lambda|y - x_j| \leq C\lambda$  in  $\Omega$ . We have

$$(1+\lambda|y-\xi_j|)^{\frac{N+2}{2}+\tau-4} \leq C\lambda^{\frac{N+2}{2}+\tau-4} = C\lambda^{-\frac{N-1}{2}}\lambda^{N-2}.$$

As an result,  $\alpha = \frac{N-1}{2}$ . Thus, we get

$$\|J_2\|_{**} \leq C\lambda^{-\frac{N-1}{2}}, \quad \text{if } N \geq 5. \quad (2.21)$$

If  $N \leq 5$ , it holds  $\frac{N+2}{2} + \tau - 4 < 0$ . Thus

$$(1+\lambda|y-\xi_j|)^{\frac{N+2}{2}+\tau-4} \leq C = C\lambda^{2-N}\lambda^{N-2}.$$

So  $\alpha = N - 2$ . Hence, we obtain

$$\|J_2\|_{**} \leq C\lambda^{2-N}, \quad \text{if } N \leq 5. \quad (2.22)$$

In order to estimate  $\|J_1\|_{**}$ . We define

$$\Omega_j = \left\{ y : y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, \left\langle \frac{y'}{|y'|}, \frac{\xi_j'}{|\xi_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

Using the assumed symmetry, we just need to estimate  $J_1$  in  $\Omega_1$ . Let  $S = \Omega_1 \cap B_{1/\sqrt{\lambda}}(\xi_1)$ .

Note that, it holds  $P_1 \geq c_0 > 0$  in  $S$ , and

$$|U_*|^{2^*-2}U_* = |u_0 - \sum_{j=2}^k P_j - P_1|^{2^*-2}(u_0 - \sum_{j=2}^k P_j - P_1),$$

we have

$$|J_1| \leq P_1^{2^*-2}(u_0 + \sum_{j=2}^k P_j) + J_3,$$

where  $|J_3| \leq C$  in  $S$ .

Since

$$\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-\xi_1|)^{\frac{N+2}{2}+\tau}} \geq \frac{\lambda^{\frac{N+2}{2}}}{(1+\sqrt{\lambda})^{\frac{N+2}{2}+\tau}} \geq a_0\lambda^{\frac{N+2}{4}-\frac{\tau}{2}}, \quad y \in S,$$

it holds

$$|J_3| \leq C\lambda^{-\frac{N+2}{4}+\frac{\tau}{2}} \frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-\xi_1|)^{\frac{N+2}{2}+\tau}}, \quad y \in S.$$

On the other hand

$$|P_1^{2^*-2}(u_0 + \sum_{j=2}^k P_j)| \leq CU_1^{2^*-2},$$

and if  $N \geq 5$ ,

$$(1 + \lambda|y - \xi_1|)^{\frac{N+2}{2} + \tau - 4} \leq C\lambda^{\frac{1}{2}(\frac{N+2}{2} + \tau - 4)},$$

which gives

$$|P_1^{2^*-2}(u_0 + \sum_{j=2}^k P_j)| \leq CU_1^{2^*-2} \leq \lambda^{-\frac{N+2}{4} + \frac{\tau}{2}} \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - \xi_1|)^{\frac{N+2}{2} + \tau}}, \quad y \in S.$$

If  $N = 3, 4$ ,

$$(1 + \lambda|y - \xi_1|)^{\frac{N+2}{2} + \tau - 4} \leq C,$$

which gives

$$|P_1^{2^*-2}(u_0 + \sum_{j=2}^k P_j)| \leq CU_1^{2^*-2} \leq \lambda^{-\frac{N-2}{2}} \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - \xi_1|)^{\frac{N+2}{2} + \tau}}, \quad y \in S.$$

Therefore, we have proved

$$|J_1| \leq C\lambda^{-\frac{N-2}{4} - \sigma} \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - \xi_1|)^{\frac{N+2}{2} + \tau}}, \quad y \in S. \quad (2.23)$$

On the other hand, we note that, in  $\Omega_1 \setminus S$ , it holds  $P_1 \leq C$ . Thus

$$\begin{aligned} |J_1| &\leq C \sum_{j=1}^k U_j \\ &\leq \frac{C}{\lambda^{\frac{N-2}{2}} |y - \xi_1|^{N-2}} + \frac{C}{\lambda^{\frac{N-2}{2}} |y - \xi_1|^{N-2-\tau}} \sum_{j=2}^k \frac{1}{|\xi_j - \xi_1|^\tau} \\ &\leq \frac{C}{\lambda^{\frac{N-2}{2} - \tau} |y - \xi_1|^{N-2-\tau}}. \end{aligned}$$

Now we determine  $\beta > 0$ , such that

$$\frac{1}{\lambda^{\frac{N-2}{2} - \tau} |y - \xi_1|^{N-2-\tau}} \leq C\lambda^{-\beta} \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - \xi_1|)^{\frac{N+2}{2} + \tau}}, \quad y \in \Omega_1 \setminus S. \quad (2.24)$$

It holds

$$\frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - \xi_1|)^{\frac{N+2}{2} + \tau}} \geq \frac{c'}{\lambda^\tau |y - \xi_1|^{\frac{N+2}{2} + \tau}}, \quad y \in \Omega_1 \setminus S.$$

So (2.24) holds if

$$\frac{1}{\lambda^{\frac{N-2}{2} - \tau} |y - \xi_1|^{N-2-\tau}} \leq \frac{C\lambda^{-\beta}}{\lambda^\tau |y - \xi_1|^{\frac{N+2}{2} + \tau}}, \quad y \in \Omega_1 \setminus S.$$

which is equivalent to

$$C|y - \xi_1|^{N-2-2\tau-\frac{N+2}{2}} \geq \lambda^{\beta+2\tau-\frac{N-2}{2}}, \quad y \in \Omega_1 \setminus S. \quad (2.25)$$

Since  $|y - \xi_1| \geq \frac{1}{\sqrt{\lambda}}$ , we can take

$$\beta = \frac{N-2}{2} - 2\tau - \frac{1}{2}\left(N-2-2\tau - \frac{N+2}{2}\right) = \frac{N+2}{4} - \tau,$$

if  $N-2-2\tau - \frac{N+2}{2} \geq 0$ . That is  $N \geq 8$ . If  $N \leq 8$ , we can take  $\beta = \frac{N-2}{2} - 2\tau$ . So, we have proved

$$|J_1| \leq C\lambda^{-\frac{N-2}{4}-\sigma} \frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-\xi_1|)^{\frac{N+2}{2}+\tau}}, \quad y \in \Omega_1 \setminus S. \quad (2.26)$$

Combining (2.23) and (2.26), we find that there exists  $\sigma > 0$ , such that

$$|J_1| \leq C\lambda^{-\frac{N-2}{4}-\sigma} \frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-\xi_1|)^{\frac{N+2}{2}+\tau}}, \quad y \in \Omega_1. \quad (2.27)$$

This gives

$$\|J_1\|_{**} \leq C\lambda^{-\frac{N-2}{4}-\sigma}.$$

□

Now we are ready to the proof of Proposition 2.3.

*Proof of Proposition 2.3.* First we recall that  $\lambda \in [\eta k^2, \eta^{-1} k^2]$  for some  $\eta > 0$ . Set

$$\mathcal{N} = \left\{ w : w \in C(\mathbb{R}^N) \cap H_s, \|w\|_* \leq \frac{1}{\lambda^{\frac{N-2}{4}}}, \int_{\Omega} \sum_{j=1}^k U_j^{2^*-2} Z_{j,l} w = 0 \right\},$$

where  $l = 1, 2$ . Then (2.20) is equivalent to

$$\varphi = \mathcal{A}(\varphi) =: L_k(N(\varphi)) + L_k(l_k), \quad (2.28)$$

here  $L_k$  is defined in Lemma 2.2. We will prove that  $\mathcal{A}$  is a contraction map from  $\mathcal{N}$  to  $\mathcal{N}$ .

First, we have

$$\begin{aligned} \|\mathcal{A}(\varphi)\|_* &\leq C\|N(\varphi)\|_{**} + C\|l_k\|_{**} \\ &\leq C\|\varphi\|_*^{\min\{2^*-1, 2\}} + C\frac{1}{\lambda^{\frac{N-2}{4}+\sigma}} \\ &\leq \frac{1}{\lambda^{\frac{N-2}{4}}}. \end{aligned}$$

Hence,  $\mathcal{A}$  maps  $\mathcal{N}$  to  $\mathcal{N}$ .

On the other hand, we see

$$\begin{aligned} \|\mathcal{A}(\varphi_1) - \mathcal{A}(\varphi_2)\|_* &= \|L_k(N(\varphi_1)) - L_k(N(\varphi_2))\|_* \\ &\leq C\|N(\varphi_1) - N(\varphi_2)\|_{**}. \end{aligned}$$

It is easy to check that if  $N \geq 6$ , then

$$\begin{aligned} &|N(\varphi_1) - N(\varphi_2)| \\ &\leq |N'(\varphi_1 + \theta\varphi_2)| |\varphi_1 - \varphi_2| \\ &\leq C(|\varphi_1|^{2^*-2} + |\varphi_2|^{2^*-2}) |\varphi_1 - \varphi_2| \\ &\leq C(\|\varphi_1\|_*^{2^*-2} + \|\varphi_2\|_*^{2^*-2}) \|\varphi_1 - \varphi_2\|_* \left( \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-\xi_j|)^{\frac{N-2}{2}+\tau}} \right)^{2^*-1}. \end{aligned}$$

As before, we have

$$\left( \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^* - 1} \leq C \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N+2}{2} + \tau}}.$$

Hence,

$$\begin{aligned} \|\mathcal{A}(\varphi_1) - \mathcal{A}(\varphi_2)\|_* &\leq C(\|\varphi_1\|_*^{2^* - 2} + \|\varphi_2\|_*^{2^* - 2})\|\varphi_1 - \varphi_2\|_* \\ &\leq \frac{1}{2}\|\varphi_1 - \varphi_2\|_*. \end{aligned}$$

Therefore,  $\mathcal{A}$  is a contraction map.

The case  $N \leq 5$  can be proved in a similar way.

Now by using the contraction mapping theorem, there exists a unique  $\varphi = \varphi_{r,\lambda} \in \mathcal{N}$  such that (2.28) holds. Moreover, by Lemmas 2.2, 2.4 and 2.5, we deduce

$$\begin{aligned} \|\varphi\|_* &\leq \|L_k(N(\varphi))\|_* + \|L_k(l_k)\|_* \\ &\leq C\|N(\varphi)\|_{**} + C\|l_k\|_{**} \\ &\leq C\left(\frac{1}{\lambda}\right)^{\frac{N-2}{4} + \sigma}. \end{aligned}$$

Moreover, we get the estimate of  $c_l$  from (2.17).  $\square$

### 3. THE PROOF OF THE MAIN THEOREM

We look for a solution to (1.1) as  $u = U_* + \varphi$ , where  $\varphi = \varphi_k$  is the function obtained in Proposition 2.3. Let us introduce the energy functional whose critical points are solutions to (1.1)

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{2^*} \int_{\Omega} |u|^{2^*}. \quad (3.1)$$

and the reduced energy

$$I_k(\ell, r) := I(U_* + \varphi). \quad (3.2)$$

Where

$$\lambda = \ell k^2, \ell \in [\eta, \eta^{-1}] \text{ for some } \eta > 0 \text{ small enough.}$$

We have the following result

**Proposition 3.1.** (i)  $U_* + \varphi$  is a critical point of  $I$  if and only if  $(\ell, r)$  is a critical point of the reduced energy  $I_k$   
(ii) We have

$$I_k(\ell, r) = I(u_0) + kA + \frac{1}{k^{N-2}} F(\ell, r) + o\left(\frac{1}{k^{N-2}}\right)$$

uniformly in compact sets of  $(0, +\infty) \times (a, b)$ , where

$$F(\ell, r) := B \frac{u_0(r)}{\ell^{\frac{N-2}{2}}} - C \frac{1}{r^{N-2} \ell^{N-2}}$$

for some positive constants  $A, B$  and  $C$ .

*Proof.* The proof of (i) is quite standard. We only prove (ii). First of all we prove that

$$I(U_* + \varphi) = I(U_*) + kO\left(\lambda^{-\frac{N-2}{2} - 2\sigma}\right), \text{ for some } \sigma < 0. \quad (3.3)$$

First of all, we have

$$\begin{aligned}
I(U_* + \varphi) &= I(U_*) + \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} (u_0^{2^*-1} - \sum_{j=1}^k U_j^{2^*-1}) \varphi \\
&\quad - \frac{1}{2^*} \int_{\Omega} (|U_* + \varphi|^{2^*} - |U_*|^{2^*}).
\end{aligned} \tag{3.4}$$

It follows from (2.18) that

$$\int_{\Omega} |\nabla \varphi|^2 = \int_{\Omega} |U_* + \varphi|^{2^*-2} (U_* + \varphi) \varphi - \int_{\Omega} (u_0^{2^*-1} - \sum_{j=1}^k U_j^{2^*-1}) \varphi. \tag{3.5}$$

Thus, we obtain

$$\begin{aligned}
&I(U_* + \varphi) \\
&= I(U_*) + \frac{1}{2} \int_{\Omega} |U_* + \varphi|^{2^*-2} (U_* + \varphi) \varphi + \frac{1}{2} \int_{\Omega} (u_0^{2^*-1} - \sum_{j=1}^k U_j^{2^*-1}) \varphi \\
&\quad - \frac{1}{2^*} \int_{\Omega} (|U_* + \varphi|^{2^*} - |U_*|^{2^*}) \\
&= I(U_*) + \frac{1}{2} \int_{\Omega} (u_0^{2^*-1} - \sum_{j=1}^k U_j^{2^*-1} - |U_*|^{2^*-2} U_*) \varphi \\
&\quad + \frac{1}{2} \int_{\Omega} (|U_* + \varphi|^{2^*-2} (U_* + \varphi) - |U_*|^{2^*-2} U_*) \varphi \\
&\quad - \frac{1}{2^*} \int_{\Omega} (|U_* + \varphi|^{2^*} - |U_*|^{2^*} - 2^* |U_*|^{2^*-2} U_* \varphi).
\end{aligned} \tag{3.6}$$

Write

$$l_k = \left[ |U_*|^{2^*-2} U_* - u_0^{2^*-1} + \sum_{j=1}^k P_j^{2^*-1} \right] + \sum_{j=1}^k (U_j^{2^*-1} - P_j^{2^*-1}).$$

It follows from Lemma 2.5, there is a constant  $\sigma > 0$ , such that

$$\|l_k\|_{**} \leq C \lambda^{-\frac{N-2}{4}-\sigma}.$$

By Proposition 2.3, we can obtain from (3.6) that if  $N \geq 6$ ,

$$\begin{aligned}
&I(U_* + \varphi) \\
&= I(U_*) + O\left(\|l_k\|_{**} \|\varphi\|_*\right) \sum_{j=1}^k \int_{\Omega} \left( \sum_{j=1}^k \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N+2}{2} + \tau}} \right) \left( \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2}{2} + \tau}} \right) \\
&\quad + O\left(\|\varphi\|_*^{2^*}\right) \int_{\Omega} \left( \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^*} \\
&= I(U_*) + k O\left(\lambda^{-\frac{N-2}{2}-2\sigma}\right).
\end{aligned} \tag{3.7}$$

While if  $N \leq 5$ , then

$$\begin{aligned}
& I(U_* + \varphi) \\
&= I(U_*) + O\left(\|l_k\|_{**} \|\varphi\|_*\right) \sum_{j=1}^k \int_{\Omega} \left( \sum_{j=1}^k \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N+2}{2} + \tau}} \right) \left( \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2}{2} + \tau}} \right) \\
&\quad + O\left(\|\varphi\|_*^{2^*}\right) \int_{\Omega} \left( \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^*} \\
&\quad + O\left(\|\varphi\|_*^2\right) \int_{\Omega} |U_*|^{2^* - 3} \left( \sum_{j=1}^k \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^2 \\
&= I(U_*) + kO\left(\lambda^{-\frac{N-2}{2} - 2\sigma}\right).
\end{aligned} \tag{3.8}$$

That concludes the proof of (3.3).

Next, we prove that

$$I(U_*) = I(u_0) + k \left[ A - \frac{B_1 k^{N-2}}{r^{N-2} \lambda^{N-2}} + \frac{B_2 u_0(r)}{\lambda^{\frac{N-2}{2}}} + O\left(\frac{1}{\lambda^{\frac{N-2}{2}(1+\delta)}}\right) \right] \tag{3.9}$$

where  $A$ ,  $B_1$ ,  $B_2$  and are positive constants,  $\delta > 0$  is small.

Recall that  $P_j$  satisfies (1.4) and set  $V = \sum_{j=1}^k P_j$ . We have

$$\begin{aligned}
& \int_{\Omega} |\nabla U_*|^2 \\
&= \int_{\Omega} |\nabla u_0|^2 + \int_{\Omega} |\nabla V|^2 - 2 \int_{\Omega} \nabla V \nabla u_0 \\
&= \int_{\Omega} |\nabla u_0|^2 + \int_{\Omega} |\nabla V|^2 - 2 \int_{\Omega} u_0^{2^* - 1} V.
\end{aligned} \tag{3.10}$$

Let  $\Omega_j =: \{(r \cos \theta, r \sin \theta, x') \mid \frac{2\pi(j-1)}{k} - \frac{\pi}{k} \leq \theta \leq \frac{2\pi(j)}{k} + \frac{\pi}{k}, x' \in \mathbb{R}^{N-2}\} \cap \Omega, j = 1, \dots, k$ . Then by the symmetry, we have

$$\int_{\Omega} u_0^{2^* - 1} V = k \int_{\Omega_1} u_0^{2^* - 1} V,$$

and

$$\int_{\Omega} |U_*|^{2^*} = k \int_{\Omega_1} |U_*|^{2^*}.$$

Let  $S =: \Omega_1 \cap B_{\lambda^{-\frac{1}{2}}}(\xi_1)$ , by suing the following inequality:

$$|1 - t|^p = 1 - pt + O(t^2) = 1 - pt + O(t^\alpha), 1 < \alpha \leq 2, \forall 0 \leq t \leq c,$$

where  $c$  is some constant, we obtain

$$\begin{aligned}
& \int_S |U_*|^{2^*} \\
&= \int_S V^{2^*} - 2^* \int_S V^{2^*-1} u_0 + O\left(\int_S v^{2^*-1-\delta} u_0^\delta\right) \\
&= \int_S V^{2^*} - 2^* \int_S V^{2^*-1} u_0 + O\left(\lambda^{-\frac{(1+\delta)(N-2)}{2}}\right),
\end{aligned} \tag{3.11}$$

where  $\delta > 0$  is small.

On the other hand, we have

$$\begin{aligned}
& \int_{\Omega_1 \setminus S} |U_*|^{2^*} \\
&= \int_{\Omega_1 \setminus S} u_0^{2^*} - 2^* \int_{\Omega_1 \setminus S} u_0^{2^*-1} V + O\left(\int_{\Omega_1 \setminus S} u_0^{2^*-2} V^2\right) \\
&= \int_{\Omega} u_0^{2^*} - 2^* \int_{\Omega_1 \setminus S} u_0^{2^*-1} V + O\left(\int_{\Omega_1 \setminus S} u_0^{2^*-2} V^2\right) + O\left(\lambda^{-\frac{N}{2}}\right),
\end{aligned} \tag{3.12}$$

since

$$\int_{\Omega_1 \setminus S} u_0^{2^*} = \int_{\Omega_1} u_0^{2^*} + O\left(\lambda^{-\frac{N}{2}}\right). \tag{3.13}$$

Note that, for any  $y \in \Omega_0$ , we have

$$\sum_{j=2}^k \frac{1}{|y - \xi_j|^{N-2}} \leq \frac{C}{|y - \xi_1|^{N-2-\tau}} \sum_{j=2}^k \frac{1}{|\xi_j - \xi_1|^\tau} \leq \frac{Ck}{|y - \xi_1|^{N-2-\tau}}$$

for  $\tau \in (0, 1)$ . So we obtain

$$\begin{aligned}
\int_{\Omega_1 \setminus S} u_0^{2^*-2} V^2 &\leq C \int_{\Omega_1 \setminus S} V^2 \\
&\leq \int_{\Omega_1 \setminus S} \left( \sum_{j=1}^k \frac{1}{\lambda^{\frac{N-2}{2}} |y - \xi_j|^{N-2}} \right)^2 \\
&\leq \frac{C}{\lambda^{N-2}} \int_{\Omega_1 \setminus S} \left( \frac{1}{|y - \xi_1|^{N-2}} + \frac{k}{|y - \xi_1|^{N-2-\tau}} \right)^2 \\
&\leq \frac{C}{\lambda^{N-2}} \int_{\Omega_1 \setminus S} \left( \frac{1}{|y - \xi_1|^{2(N-2)}} + \frac{k}{|y - \xi_1|^{2(N-2-\tau)}} \right) \\
&\leq \frac{C}{\lambda^{N-2}} \left( \lambda^{\frac{1}{2}(N-4)} + k^2 \lambda^{\frac{1}{2}(N-4-2\tau)} \right) \\
&\leq \frac{C}{\lambda^{\frac{N}{2}-1+2\tau}},
\end{aligned} \tag{3.14}$$

since  $k^2 \sim \lambda$ . As a consequence,

$$\begin{aligned}
& \int_{\Omega_1 \setminus S} |U_*|^{2^*} \\
&= \int_{\Omega_1} u_0^{2^*} - 2^* \int_{\Omega_1 \setminus S} u_0^{2^*-1} V + O\left(\frac{1}{\mu^{\frac{N-2}{2}(1+\delta)}}\right).
\end{aligned} \tag{3.15}$$

Combining the above obtained results, we get

$$\begin{aligned} I(U_*) &= I(u_0) + \frac{1}{2} \int_{\Omega} |DV|^2 - \frac{k}{2^*} \int_S V^{2^*} + k \int_S V^{2^*-1} u_0 \\ &\quad - \int_{\Omega} u_0^{2^*-1} V + k \int_{\Omega_1 \setminus S} u_0^{2^*-1} V + O\left(\frac{k}{\lambda^{\frac{N-2}{2}(1+\delta)}}\right). \end{aligned} \quad (3.16)$$

Now we compute those integrals in (3.16) one by one:

$$\begin{aligned} & - \int_{\Omega} u_0^{2^*-1} V + k \int_{\Omega_1 \setminus S} u_0^{2^*-1} V \\ &= -k \int_{\Omega_1} u_0^{2^*-1} V + k \int_{\Omega_1 \setminus S} u_0^{2^*-1} V \\ &= -k \int_S u_0^{2^*-1} V \\ &= O\left(k \int_S \frac{1}{\lambda^{\frac{N-2}{2}}} \sum_{j=1}^k \frac{1}{|y - \xi_j|^{N-2}}\right) \\ &= O\left(k \int_S \frac{1}{\lambda^{\frac{N-2}{2}}} \left(\frac{1}{|y - \xi_1|^{N-2}} + \frac{k}{|y - \xi_1|^{N-2-\tau}}\right)\right) \\ &= O\left(\frac{k}{\lambda^{\frac{N-2}{2}}} \left(\frac{1}{\lambda} + \frac{k}{\lambda^{1+\frac{\tau}{2}}}\right)\right) = O\left(\frac{k}{\lambda^{\frac{(1+\delta)(N-2)}{2}}}\right). \end{aligned} \quad (3.17)$$

We have that for any  $y \in S$

$$\begin{aligned} \sum_{j=2}^k \frac{P_j(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} &\leq C \sum_{j=2}^k \frac{1}{(1 + |y - \lambda(\xi_j - \xi_1)|)^{N-2}} \\ &\leq C \sum_{j=2}^k \frac{1}{|\lambda(\xi_j - \xi_1)|^{N-2}} \\ &\leq \frac{C |\ln k|^{\sigma_N} k^{N-2}}{\lambda^{N-2}} \leq \frac{C |\ln k|^{\sigma_N}}{\lambda^{\frac{N-2}{2}}}, \end{aligned}$$

where  $\sigma_N = 0$  if  $N \geq 4$  and  $\sigma_3 = 1$ , if  $N = 3$ . So

$$\begin{aligned} & \left| \left(\frac{V(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}}\right)^{2^*-1} - \left(\frac{P_1(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}}\right)^{2^*-1} \right| \\ &\leq C \left( \frac{1}{(1 + |y|)^4} \frac{|\ln k|^{\sigma_N}}{\lambda^{\frac{N-2}{2}}} + \frac{|\ln k|^{(2^*-1)\sigma_N}}{\lambda^{\frac{N+2}{2}}} \right). \end{aligned}$$

Thus, we have



$$\begin{aligned}
& \int_S V^{2^*-1} u_0 \\
&= \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_{\sqrt{\lambda}}(0)} \left( \frac{V(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right)^{2^*-1} u_0(\xi_1 + \lambda^{-1}y) \\
&= \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_{\sqrt{\lambda}}(0)} \left( \frac{P_1(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right)^{2^*-1} u_0(\xi_1 + \lambda^{-1}y) \\
&\quad + \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_{\sqrt{\lambda}}(0)} \left[ \left( \frac{V(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right)^{2^*-1} - \left( \frac{P_1(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right)^{2^*-1} \right] u_0(\xi_1 + \lambda^{-1}y) \\
&= \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_{\sqrt{\lambda}}(0)} \left( \frac{P_1(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right)^{2^*-1} u_0(\xi_1 + \lambda^{-1}y) \\
&\quad + \frac{1}{\lambda^{\frac{N-2}{2}}} O\left( \lambda^{\frac{N-4}{2}} \frac{|\ln k|^{\sigma_N}}{\lambda^{\frac{N-2}{2}}} + \lambda^{\frac{N}{2}} \frac{|\ln k|^{(2^*-1)\sigma_N}}{\lambda^{\frac{N+2}{2}}} \right) \\
&= \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_{\sqrt{\lambda}}(0)} \left( U(y) + O\left( \frac{1}{\lambda^{N-2}} \right) \right)^{2^*-1} \left( u_0(\xi_1) + O\left( \frac{1}{\lambda} \right) \right) \\
&\quad + \frac{1}{\lambda^{\frac{N-2}{2}}} O\left( \lambda^{\frac{N-4}{2}} \frac{|\ln k|^{\sigma_N}}{\lambda^{\frac{N-2}{2}}} + \lambda^{\frac{N}{2}} \frac{|\ln k|^{(2^*-1)\sigma_N}}{\lambda^{\frac{N+2}{2}}} \right) \\
&= \frac{u_0(\xi_1)}{\lambda^{\frac{N-2}{2}}} \left( \int_{\mathbb{R}^N} U^{2^*-1} + O\left( \frac{1}{\lambda^\delta} \right) \right).
\end{aligned} \tag{3.18}$$

Finally, it is standard to prove

$$\begin{aligned}
& \frac{1}{2} \int_\Omega |\nabla V|^2 - \frac{k}{2^*} \int_S V^{2^*} = k \left( \frac{1}{2} \int_{\Omega_0} |\nabla V|^2 - \frac{k}{2^*} \int_S V^{2^*} \right) \\
&= k \left[ \int_{\mathbb{R}^N} |\nabla U_{0,1}|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} U_{0,1}^{2^*} - \sum_{j=2}^k \frac{B_0}{\lambda^{N-2} |x_j - x_1|} + O\left( \frac{1}{\lambda^{\frac{N-2}{2}(1+\delta)}} \right) \right] \\
&= k \left[ A - \frac{B_1 k^{N-2}}{\lambda^{N-2} r_0^{N-2}} + O\left( \frac{1}{\lambda^{\frac{N-2}{2}(1+\delta)}} \right) \right]
\end{aligned} \tag{3.19}$$

Combining the above obtained results, we get (3.9).

Finally, the claim follows by the choice of  $\lambda$  in (1.6).  $\square$

We are now ready to prove the main theorem.

*Proof of Theorem 1.1: completed.* We apply Proposition 3.1. It is easy to check that  $F$  has a maximum point at the point  $(\ell_0, r_0)$  where  $r_0$  maximizes the function  $r \rightarrow r^{\frac{N-2}{2}} u_0(r)$  and  $\ell_0 := \left( \frac{2B}{C u_0(r_0) r_0^{N-2}} \right)^{\frac{2}{N-2}}$ , which is stable under  $C^0$ -perturbation. Therefore, the reduced energy  $I_k$  has a critical point  $(\ell_k, r_k)$ , which produces the solution  $U_* + \phi$  to the problem (1.1).  $\square$

## APPENDIX A. NON-DEGENERACY OF THE POSITIVE RADIAL SOLUTION

Without loss of generality we can assume that the annulus is  $\mathcal{A}_R := \{x \in \mathbb{R}^n : R \leq |x| \leq 1\}$  (i.e.  $a = R$  and  $b = 1$ ).

Let  $u_R$  be the unique positive radial solution to the following problem:

$$\begin{cases} -\Delta u = u^p & \text{in } \mathcal{A}_R, \\ u = 0 & \text{on } \partial\mathcal{A}_R. \end{cases} \quad (\text{A.1})$$

Here we set  $p := \frac{N+2}{N-2}$ ,  $N \geq 3$ .

**Proposition A.1.** *There exists a sequence of radii  $(R_k)_{k \in \mathbb{N}}$ , such that  $u_R$  is non-degenerate for any  $R \neq R_k$ .*

*Proof.* (i) Let us consider the following linear problem:

$$\begin{cases} -\Delta v = pu_R^{p-1}v & \text{in } \mathcal{A}_R, \\ v = 0 & \text{on } \partial\mathcal{A}_R. \end{cases} \quad (\text{A.2})$$

We denote by  $\lambda_k = k(k+n-2)$  for  $k = 0, 1, 2, \dots$  the eigenvalues of  $-\Delta$  on the sphere  $\mathbb{S}^{n-1}$ . Let  $\{\Phi_i^k : 1 \leq i \leq m_k\}$  denote a basis for the  $k^{\text{th}}$  eigenspace of  $-\Delta$ . Then for any function  $v = v(r, \theta)$  on the annulus  $\mathcal{A}_R$  we may write

$$v(r, \theta) = \sum_{k \geq 0} a_k(r) \tilde{\Phi}_i^k(\theta), \quad r \in (1, R), \theta \in \mathbb{S}^{n-1}, \quad (\text{A.3})$$

where each  $a_k$  is a radial solution to

$$\begin{cases} a_k'' + \frac{n-1}{r}a_k' + \left(pu_R^{p-1}(r) - \frac{\lambda_k}{r^2}\right)a_k(r) = 0 & \text{in } (R, 1), \\ a_k(R) = a_k(1) = 0, \end{cases} \quad (\text{A.4})$$

and

$$\tilde{\Phi}_i^k(\theta) = \sum_{i=1}^{m_k} c_i \Phi_i^k(\theta), \quad \text{for some } c_i \in \mathbb{R}.$$

(ii) Argue as in Proposition 2.1 in [3], we have that

$$a_0(r) = 0 \text{ for any } r \in (R, 1). \quad (\text{A.5})$$

It means that  $u_0$  is non-degenerate in the space of radial functions.

(iii) For any integer  $k \geq 1$ , let  $\mu_{ki} = \mu_{ki}(R)$ ,  $i \geq 1$  be the sequence of the eigenvalues of the problem:

$$\begin{cases} \phi'' + \frac{n-1}{r}\phi' + \left(pu_R^{p-1}(r) - \frac{\lambda_k}{r^2}\right)\phi = -\mu_{ki}\phi & \text{in } (R, 1), \\ \phi(R) = \phi(1) = 0. \end{cases} \quad (\text{A.6})$$

We point out that if

$$\mu_{ki}(R) \neq 0 \text{ for any } k \geq 1 \text{ and } i \geq 1, \quad (\text{A.7})$$

then any solutions to (A.4)  $a_k \equiv 0$ .

So by (A.3) (together with (A.5)) we deduce that any solutions to (A.2)  $v \equiv 0$ , i.e.  $u_R$  is non-degenerate.

(iv) By Corollary 2.4 in [23] we get

$$\mu_{11}(R) < 0 \text{ and } \mu_{ki}(R) > 0 \text{ for } k \geq 1 \text{ and } i \geq 2, \text{ for any } R \in (0, 1). \quad (\text{A.8})$$

It only remains to check the behavior of the first eigenvalue  $\mu_{k1}(R)$  for any  $k \geq 2$ . We know by Lemma 3.1 in [23] that

$$\lim_{R \rightarrow 1} \mu_{k1}(R) = -\infty, \text{ for any } k \geq 1.$$

(v) If  $\phi$  solves (A.6) then  $\psi(t) = \phi(t(1-R) + 2R-1)$  solves the following problem:

$$\begin{cases} \psi'' + \frac{(n-1)(1-R)}{t(1-R)+2R-1} \psi' + (1-R)^2 \left( pw_R^{p-1}(t) - \frac{\lambda_k}{(t(1-R)+2R-1)^2} \right) \psi = \lambda_k \psi, & \text{in } (1, 2), \\ \psi(1) = \psi(2) = 0. \end{cases} \quad (\text{A.9})$$

Where

$$\lambda_k = \lambda_k(R) := -(1-R)^2 \mu_{k1}(R).$$

On the other hand, we see that  $w_R(t) = u_R(t(1-R) + 2R-1)$  solves the following problem:

$$\begin{cases} w_R'' + \frac{(n-1)(1-R)}{t(1-R)+2R-1} w_R' + (1-R)^2 w_R^p(t) = 0 & \text{in } (1, 2), \\ w_R(1) = w_R(2) = 0. \end{cases} \quad (\text{A.10})$$

(vi) We claim that

*for any  $k \geq 2$  there exists a finite number of radii  $R_{k1}, \dots, R_{k\ell(k)}$  such that  $\lambda_k(R_{ki}) = 0$  for  $i = 1, \dots, \ell(k)$ .*

The proof for the claim could follow the same arguments as in Lemma 2.2 (c) of [14]. Indeed, using a result due to Kato (see Example 2.12, page 380 in [20]), we could prove that each function  $R \rightarrow \lambda_k(R)$  is analytic so it can only vanish at a finite number of points. We can prove that the function  $W : (0, 1) \rightarrow C^2(I)$ ,  $I = [1, 2]$ , defined by  $W(R)(t) = w_R(t)$  is analytic using the same arguments developed by Dancer in [12]. □

## REFERENCES

- [1] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geom. 11 (1976), 573–598.
- [2] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, Comm. Pure Appl. Math. 41 (1988), 253–294.
- [3] T. Bartsch, M. Clapp, M. Grossi and F. Pacella, Asymptotically radial solutions in expanding annular domains, Math. Ann. 352 (2012), 485–515.
- [4] M. Clapp, Entire nodal solutions to the pure critical exponent problem arising from concentration. J. Differential Equations. 261 (2016), 3042–3060.
- [5] M. Clapp, F. Pacella, Multiple solutions to the pure critical exponent problem in domains with a hole of arbitrary size. Math. Z. 259 (2008), 575–589.
- [6] M. Clapp, T. Weth, Minimal nodal solutions of the pure critical exponent problem on a symmetric domain. Calc. Var. 21 (2004), 1–14.
- [7] M. Clapp, T. Weth, Two solutions of the Bahri–Coron problems in punctured domains via the fixed point transfer. Commun. Contemp. Math. 10 (2008), 81–101.
- [8] M. Clapp, M. Musso, A. Pistoia, Multipeak solutions to the pure critical exponent problem in punctured domains. J. Funct. Anal. 256 (2009), 275–306.
- [9] C. Coffman, A nonlinear boundary value problem with many positive solutions, J. Differ. Equ. 54 (1984), 429–437.
- [10] J. M. Coron, Topologie et cas limite des injections de Sobolev. C.R. Acad. Sc. Paris. 299 (1984), 209–212.
- [11] E. N. Dancer, A note on an equation with critical exponent. Bull. London Math. Soc. 20 (1988), 600–602.
- [12] E. N. Dancer, Real analyticity and non-degeneracy. Math. Ann. 325 (2003), 369–392.

- [13] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem, *Calc. Var. Partial Differential Equations.* 16 (2003), 113–145.
- [14] M. del Pino, J. Wei, Supercritical elliptic problems in domains with small holes. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24(2007), 507–520.
- [15] M. del Pino, M. Musso, F. Pacard and A. Pistoia, Large energy entire solutions for the Yamabe equation, *J. Diff. Eqns*, 251 (2011), 2568–2597.
- [16] M. del Pino, M. Musso, F. Pacard, A. Pistoia, Torus action on  $S^n$  and sign changing solutions for conformally invariant equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 12 (2013), 209–237.
- [17] W. Ding, Positive solutions of  $\Delta u + u^{2^*-1} = 0$  on contractible domains. *J. Partial Diff. Eq.* 2 (1989), 83–88.
- [18] Y. Ge, M. Musso and A. Pistoia, Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains, *Communications in Partial Differential Equations*, 35 (2010), 1419–1457.
- [19] Y. Ge, M. Musso, A. Pistoia, D. Pollack, A refined result on sign changing solutions for a critical elliptic problem. *Commun. Pure Appl. Anal.* 12 (2013), 125–155.
- [20] T. Kato, *Perturbation Theory for Linear Operators.* Classics in Mathematics. Springer-Verlag, Berlin (1995).
- [21] J. Kazdan, F. W. Warner, Remarks on some quasilinear elliptic equations. *Comm. Pure Appl. Math.* 28 (1975), 567–597.
- [22] Y. Y. Li, Existence of many positive solutions of semilinear elliptic equations on annulus, *J. Differ. Equ.*, 83 (1990), 348–367.
- [23] Lin, Song-Sun, Existence of positive nonradial solutions for nonlinear elliptic equations in annular domains. *Trans. Amer. Math. Soc.* 332 (1992), 775–791.
- [24] M. V. Marchi, F. Pacella, On the existence of nodal solutions of the equation  $\Delta u + u^{2^*-2}u = 0$  with Dirichlet boundary conditions. *Diff. Int. Eq.* 6 (1993), 849–862.
- [25] M. Musso and A. Pistoia, Sign changing solutions to a Bahri-Coron's problem in pierced domains, *Discrete Contin. Dyn. Syst.* 21 (2008), 295–306.
- [26] W. M. Ni and R. D. Nussbaum, Uniqueness and non-uniqueness for positive radial solutions of  $\Delta u + f(u, r) = 0$ , *Comm. Pure Appl. Math.* 38 (1985), 67–108.
- [27] M. Obata, Conformal changes of Riemannian metrics on a Euclidean sphere, in: *Differential Geometry (in Honor of Kentaro Yano)*, Kinokuniya, Tokyo. 1972, 347–353.
- [28] D. Passaseo, Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains. *Manuscripta Math.* 65 (1989), 147–165.
- [29] D. Passaseo, The effect of the domain shape on the existence of positive solutions of the equation  $\Delta u + u^{2^*-1} = 0$  *Top. Meth. Nonl. Anal.* 3 (1994), 27–54.
- [30] S. Pohozaev, Eigenfunctions of the equation  $-\Delta u + \lambda f(u) = 0$ , *Sov. Math. Dokl.* 6 (1965), 1408–1411.
- [31] G. Talenti, Best constants in Sobolev inequality, *Ann. Math.* 10 (1976), 353–372.
- [32] J. Wei and S. Yan, Infinitely many solutions for the prescribed scalar curvature problem on  $S^n$ , *Journal of Funct. Anal.* (2010), 3048–3081.

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