

Finite time blow up for wave equations with strong damping in an exterior domain

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Abstract

We consider the initial boundary value problem in exterior domain for semilinear wave equations with power-type nonlinearity $|u|^p$. We will establish blow-up results when p is less than or equal to Strauss' exponent which is the same one for the whole space case \mathbb{R}^n .

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1. Introduction

This paper concerns the initial boundary value problem of the strongly damped wave equation in an exterior domain. Let $\Omega \subset \mathbb{R}^n$ be an exterior domain whose obstacle $O \subset \mathbb{R}^n$ is bounded with smooth compact boundary $\partial\Omega$. We consider the initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = |u|^p & t > 0, x \in \Omega, \\ u(0, x) = \varepsilon u_0(x), \quad u_t(0, x) = \varepsilon u_1(x) & x \in \Omega, \\ u = 0, & t \geq 0, x \in \partial\Omega, \end{cases} \quad (1.1)$$

where the unknown function u is real-valued, $n \geq 1$, $\varepsilon > 0$, and $p > 1$. Throughout this paper, we assume that

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega), \quad \text{and} \quad u_0, u_1 \geq 0. \quad (1.2)$$

Without loss of generality, we assume that $0 \in O \subset\subset B(R)$, where $B(R) := \{x \in \mathbb{R}^n : |x| < R\}$ is a ball of radius R centred at the origin, and that

$$\text{supp} u_i \subset B(R), \quad i = 0, 1. \quad (1.3)$$

For the simplicity of notations, $\|\cdot\|_q$ and $\|\cdot\|_{H^1}$ ($1 \leq q \leq \infty$) stand for the usual $L^q(\Omega)$ -norm and $H_0^1(\Omega)$ -norm, respectively.

First, the following local well-posedness result is needed.

Proposition 1. [3, see Proposition 2.1]

Let $1 < p < \infty$. Under the assumptions (1.2)-(1.3), there exists a maximal existence time $T_{\max} > 0$ such that the problem (1.1) possesses a unique weak solution

$$u \in C([0, T_{\max}), H_0^1(\Omega)) \cap C^1([0, T_{\max}), L^2(\Omega)),$$

where $0 < T_{\max} \leq \infty$. Moreover, $u(t, \cdot)$ is supported in the ball $B(t + R)$. In addition:

$$\text{either } T_{\max} = \infty \quad \text{or else} \quad T_{\max} < \infty \quad \text{and} \quad \|u(t, \cdot)\|_{H_0^1} + \|u_t(t, \cdot)\|_2 \rightarrow \infty \quad \text{as } t \rightarrow T_{\max}. \quad (1.4)$$

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Remark 1. We say that u is a global solution of (1.1) if $T_{\max} = \infty$, while in the case of $T_{\max} < \infty$, we say that u blows up in finite time.

Let $p_c(n) = +\infty$, for $n = 1$, and let $p_c(n)$, for $n \geq 2$, be the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

The number $p_c(n)$ is known as the critical exponent (Strauss exponent) of the semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u = |u|^p & t > 0, x \in \Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \end{cases} \quad (1.5)$$

since it divides $(1, \infty)$ into two subintervals such that the following description holds: If $p \in (1, p_c(n))$, then solutions with nonnegative initial values blow-up in finite time; if $p \in (p_c(n), \infty)$, then solutions with small (and sufficiently regular) initial values exist for all time (see e.g. [7]). The proof has an interesting and exciting history that spans three decades. We only give a brief summary here and refer the reader to [7, 4] and the references therein for details. The problem as regards the existence or nonexistence of global solutions is sometimes referred to as the Conjecture of Strauss [8]. The same problem was also posed by Glassey [5].

Our main result is the following

Theorem 1. *Assume that the initial data satisfy (1.2)-(1.3). If*

$$\begin{cases} 1 < p < p_c(1) = \infty, \\ 1 < p \leq p_c(2), \\ 1 < p < p_c(n), \quad n \geq 3, \end{cases}$$

then the solution of the problem (1.1) blows up in finite time.

Remark 2. It still an open problem to prove that the solution of the problem (1.1) blows up in finite time for $p = p_c(n)$, $n \geq 3$.

This paper is organized as follows: in Section 2, we present several preliminaries. Section 3 contains the proofs of the blow-up theorem (Theorem 1).

2. Preliminaries

In this section, we give some preliminary properties that will be used in the proof of Theorem 1. In [6, p. 386], Sideris obtained the following well-known ODE blow-up result:

Lemma 1. *Let $p > 1$, $a \geq 1$, and $(p-1)a > q-2$. If $F \in C^2([0, T])$ satisfies*

1. $F(t) \geq \delta(t+R)^a$, and
2. $\frac{d^2 F(t)}{dt^2} \geq k(t+R)^{-q}[F(t)]^p$,

with some positive constants δ, k , and R , then $T < \infty$.

To prove the main results in this paper when $n = 2$, we will concentrate on the improvement of the above well-known Sideris ODE blow-up result, for when the differential inequality involves a logarithmic term.

Lemma 2. [1, Lemma 2.3] *Let $p > 1$, $a \geq 1$, and $(p-1)a > q-2$. If $F \in C^2([0, T])$ satisfies*

1. $F(t) \geq \delta(t+R)^a$, and
2. $\frac{d^2 F(t)}{dt^2} \geq k[\ln(t+R)]^{-q/2}(t+R)^{-q}[F(t)]^p$,

with some positive constants δ, k , and R , then $T < \infty$.

Lemma 3. [1, Lemma 2.4] Let $p > 1$, $a \geq 1$, and $(p-1)a = q-2$. If $F \in C^2([0, T])$ satisfies, when $t \geq T_0 > 0$,

1. $F(t) \geq K_0(t+R)^a$, and
2. $\frac{d^2 F(t)}{dt^2} \geq K_1[\ln(t+R)]^{-q/2}(t+R)^{-q}[F(t)]^p$,

with some positive constants K_0, K_1, T_0 and R . Fixing K_1 , there exists a positive constant c_0 , independent of R and T_0 , such that if $K_0 \geq c_0$, then $T < \infty$.

We also need of the following special functions.

Lemma 4. [9, Lemma 2.2] There exists a function $\phi_0(x) \in C^2(\Omega)$ satisfying the following boundary value problem

$$\begin{cases} \Delta \phi_0(x) = 0, & \text{in } \Omega, \quad n \geq 3, \\ \phi_0|_{\partial\Omega} = 0, \\ |x| \rightarrow \infty, & \phi_0(x) \rightarrow 1. \end{cases} \quad (2.1)$$

Moreover, $\phi_0(x)$ satisfies: for all $x \in \Omega$, $0 < \phi_0(x) < 1$.

Lemma 5. [1, Lemma 2.5] There exists a function $\phi_0(x) \in C^2(\Omega)$ satisfying the following boundary value problem

$$\begin{cases} \Delta \phi_0(x) = 0, & \text{in } \Omega, \quad n = 2, \\ \phi_0|_{\partial\Omega} = 0, \\ |x| \rightarrow \infty, & \phi_0(x) \rightarrow +\infty, \quad \text{and } \phi_0(x) \text{ increase at the rate of } \ln(|x|). \end{cases} \quad (2.2)$$

Moreover, $\phi_0(x)$ satisfies: for all $x \in \Omega$, $0 < \phi_0(x) \leq C \ln r$, where $r = |x|$ and $C > 0$ is a positive constant.

Lemma 6. [2, Lemma 2.2] There exists a function $\phi_0(x) \in C^2([0, \infty))$ satisfying the following boundary value problem

$$\begin{cases} \Delta \phi_0(x) = 0, & x > 0, \\ \phi_0|_{x=0} = 0, \\ x \rightarrow \infty, & \phi_0(x) \rightarrow +\infty, \quad \text{and } \phi_0(x) \text{ increase at the rate of linear function } x. \end{cases} \quad (2.3)$$

Moreover, $\phi_0(x)$ satisfies: there exist two positive constants C_1 and C_2 such that, for all $x > 0$, we have $C_1 x \leq \phi_0(x) \leq C_2 x$.

Similarly, we have the following

Lemma 7. There exists a function $\varphi_1(x) \in C^2(\Omega)$ satisfying the following boundary value problem

$$\begin{cases} \Delta \varphi_1(x) = \frac{1}{2}\varphi_1(x), & \text{in } \Omega, \quad n \geq 1, \\ \varphi_1|_{\partial\Omega} = 0. \end{cases} \quad (2.4)$$

Moreover, $\varphi_1(x)$ satisfies: there exists positive constant C_1 , for all $x \in \Omega$, $0 < \varphi_1(x) \leq C_1(1 + |x|)^{-(n-1)/2}e^{|x|}$.

Proof. It is sufficient to take $\varphi_1(x) = \phi_1(\frac{x}{\sqrt{2}})$ where ϕ_1 is the function defined by [9, Lemma 2.3] on $\frac{1}{\sqrt{2}}\Omega$ instead of Ω . \square

In order to continue the description of the following lemmas, we define the following test function

$$\psi_1(x, t) = \varphi_1(x)e^{-t}, \quad \forall x \in \Omega, t \geq 0.$$

It is easy to check that

$$(\psi_1)_t(x, t) = -\psi_1(x, t), \quad (\psi_1)_{tt}(x, t) = \psi_1(x, t), \quad \text{and} \quad \Delta \psi_1(x, t) = \frac{1}{2}\psi_1(x, t).$$

Lemma 8. [9, Lemma 2.4]

Let $p > 1$, $n \geq 1$. Then, for all $t \geq 0$, we have

$$\int_{\Omega \cap \{|x| \leq t+R\}} [\psi_1(x, t)]^{p'} dx \leq C(t+R)^{n-1-(n-1)p'/2},$$

where $p' = p/(p-1)$ and C is a positive constant.

Lemma 9. Let $p > 1$, $n \geq 1$. Then, for all $t \geq 0$, we have

$$\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} [\psi_1(x, t)]^{p'} dx \leq C(t+R)^{n-1-(n-1)p'/2},$$

where $p' = p/(p-1)$ and C is a positive constant.

For the case $n = 2$, we can improve the last inequality, more precisely, there exists $R_1 \gg 1$ such that, for all $t \geq R_1$, we have

$$\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} [\psi_1(x, t)]^{p'} dx \leq C(t+R)^{1-p'/2} (\ln(t+R))^{-1/(p-1)}.$$

Proof. For the case $n \geq 3$, see [9, Lemma 2.5]. For the case of $n = 2$ see [1, Lemma 2.8]. Finally, for the one dimensional case see [2, Lemma 2.5]. \square

3. Proof of Theorem 1

Theorem 1 is a consequence of the lower bound and the blowup result about nonlinear differential inequalities in Lemmas 1 and ??.

To outline the method, we will introduce the following functions:

$$\begin{cases} F_0(t) = \int_{\Omega} u(x, t) \phi_0(x) dx, \\ F_1(t) = \int_{\Omega} u(x, t) \psi_1(x, t) dx. \end{cases}$$

By density we can assume that the solution u is sufficiently smooth, which implies that $F_0(t)$ and $F_1(t)$ are well-defined C^2 -functions for all $t \geq 0$. The following lemma is dedicated to obtain a lower bound on $F_1(t)$.

Lemma 10. Let $n \geq 1$. Under the assumptions (1.2)-(1.3), let (u, u_t) be the solution of the problem (1.1) such that

$$(u, u_t) \in C([0, T], H_0^1(\Omega)) \times C([0, T], L^2(\Omega)),$$

and

$$\text{supp}(u, u_t) \subset B(t+R) := \{x \in \Omega : |x| < t+R\}.$$

Then, for all $t \geq 0$, we have

$$F_1(t) \geq \left(\frac{\varepsilon}{3} (1 - e^{-\frac{3}{2}t}) + \varepsilon e^{-\frac{3}{2}t} \right) \int_{\Omega} \varphi_1(x) u_0(x) dx + \frac{2\varepsilon}{3} (1 - e^{-\frac{3}{2}t}) \int_{\Omega} \varphi_1(x) u_1(x) dx \geq \varepsilon c_0 > 0.$$

Proof. We multiply (1.1) by the test function $\psi_1 \in C^2(\Omega \times \mathbb{R})$ and integrate over $\Omega \times [0, t]$, we get

$$\int_0^t \int_{\Omega} u_{ss} \psi_1 dx ds - \int_0^t \int_{\Omega} \Delta u \psi_1 dx ds - \int_0^t \int_{\Omega} \Delta u_s \psi_1 dx ds = \int_0^t \int_{\Omega} |u|^p \psi_1 dx ds. \quad (3.1)$$

Use integration by parts, we have

$$\begin{aligned} \int_0^t \int_{\Omega} u_{ss} \psi_1 dx ds &= - \int_0^t \int_{\Omega} u_s (\psi_1)_s dx ds + \int_{\Omega} u_t(x, t) \psi_1(x, t) dx - \varepsilon \int_{\Omega} u_1(x) \varphi_1(x) dx \\ &= \int_0^t \int_{\Omega} u(x, s) \psi_1(x, s) dx ds + \int_{\Omega} [u(x, t) + u_t(x, t)] \psi_1(x, t) dx - \varepsilon \int_{\Omega} [u_0(x) + u_1(x)] \varphi_1(x) dx, \end{aligned}$$

where we have used the fact that $(\psi_1)_s(x, s) = -\psi_1(x, s)$ and $(\psi_1)_{ss}(x, s) = \psi_1(x, s)$, for all $x \in \Omega$, $s \geq 0$. Moreover

$$\begin{aligned} \int_0^t \int_{\Omega} \Delta u \psi_1 dx ds &= - \int_0^t \int_{\Omega} \nabla u \nabla \psi_1 dx ds + \int_0^t \int_{\partial\Omega} \psi_1 \nabla u \cdot \mathbf{n} d\sigma ds \\ &= \int_0^t \int_{\Omega} u \Delta \psi_1 dx ds - \int_0^t \int_{\partial\Omega} u \nabla \psi_1 \cdot \mathbf{n} d\sigma ds + \int_0^t \int_{\partial\Omega} \psi_1 \nabla u \cdot \mathbf{n} d\sigma ds, \\ &= \frac{1}{2} \int_0^t \int_{\Omega} u \psi_1 dx ds, \end{aligned}$$

where we have used the boundary conditions and the fact that $\Delta \psi_1(x, s) = \frac{1}{2} \psi_1(x, s)$, for all $x \in \Omega$, $s \geq 0$. Similarly

$$\begin{aligned} \int_0^t \int_{\Omega} \Delta u_s \psi_1 dx ds &= - \int_0^t \int_{\Omega} \Delta u (\psi_1)_s dx ds + \int_{\Omega} \psi_1(x, t) \Delta u(x, t) dx - \int_{\Omega} \varphi_1(x) \Delta u(x, 0) dx \\ &= \int_0^t \int_{\Omega} \Delta u \psi_1 dx ds + \int_{\Omega} \psi_1(x, t) \Delta u(x, t) dx - \int_{\Omega} \varphi_1(x) \Delta u(x, 0) dx \\ &= \frac{1}{2} \int_0^t \int_{\Omega} u \psi_1 dx ds + \frac{1}{2} \int_{\Omega} \psi_1(x, t) u(x, t) dx - \frac{1}{2} \varepsilon \int_{\Omega} \varphi_1(x) u_0(x) dx, \end{aligned}$$

where a similar calculation as above was applied. Combining the above equalities, we conclude from (3.1) that

$$\begin{aligned} \int_0^t \int_{\Omega} |u|^p \psi_1 dx ds &= \int_0^t \int_{\Omega} u \psi_1 dx ds + \int_{\Omega} [u(x, t) + u_t(x, t)] \psi_1(x, t) dx - \varepsilon \int_{\Omega} [u_0(x) + u_1(x)] \varphi_1(x) dx \\ &\quad - \frac{1}{2} \int_0^t \int_{\Omega} u \psi_1 dx ds \\ &\quad - \frac{1}{2} \int_0^t \int_{\Omega} u \psi_1 dx ds - \frac{1}{2} \int_{\Omega} \psi_1(x, t) u(x, t) dx + \frac{1}{2} \varepsilon \int_{\Omega} \varphi_1(x) u_0(x) dx \\ &= \frac{d}{dt} \int_{\Omega} u(x, t) \psi_1(x, t) dx + \frac{3}{2} \int_{\Omega} u(x, t) \psi_1(x, t) dx - \frac{1}{2} \varepsilon \int_{\Omega} \varphi_1(x) u_0(x) dx - \varepsilon \int_{\Omega} \varphi_1(x) u_1(x) dx \\ &= \frac{d}{dt} F_1(t) + \frac{3}{2} F_1(t) - \frac{1}{2} \varepsilon \int_{\Omega} \varphi_1(x) u_0(x) dx - \varepsilon \int_{\Omega} \varphi_1(x) u_1(x) dx. \end{aligned}$$

So by $\psi_1 > 0$, we have

$$\begin{aligned} \frac{d}{dt} F_1(t) + \frac{3}{2} F_1(t) &= \int_0^t \int_{\Omega} |u|^p \psi_1 dx ds + \frac{1}{2} \varepsilon \int_{\Omega} \varphi_1(x) u_0(x) dx + \varepsilon \int_{\Omega} \varphi_1(x) u_1(x) dx \\ &\geq \frac{1}{2} \varepsilon \int_{\Omega} \varphi_1(x) u_0(x) dx + \varepsilon \int_{\Omega} \varphi_1(x) u_1(x) dx. \end{aligned}$$

Multiply the above expression by $e^{\frac{3}{2}t}$, we obtain

$$\frac{d}{dt} (e^{\frac{3}{2}t} F_1(t)) \geq \frac{\varepsilon}{2} e^{\frac{3}{2}t} \int_{\Omega} \varphi_1(x) u_0(x) dx + \varepsilon e^{\frac{3}{2}t} \int_{\Omega} \varphi_1(x) u_1(x) dx,$$

and integrating the last differential inequality over $[0, t]$, we get

$$e^{\frac{3}{2}t} F_1(t) - F_1(0) \geq \frac{\varepsilon}{3} (e^{\frac{3}{2}t} - 1) \int_{\Omega} \varphi_1(x) u_0(x) dx + \frac{2\varepsilon}{3} (e^{\frac{3}{2}t} - 1) \int_{\Omega} \varphi_1(x) u_1(x) dx.$$

As $F_1(0) = \varepsilon \int_{\Omega} \varphi_1(x) u_0(x) dx$, we arrive at

$$F_1(t) \geq \left(\frac{\varepsilon}{3} (1 - e^{-\frac{3}{2}t}) + \varepsilon e^{-\frac{3}{2}t} \right) \int_{\Omega} \varphi_1(x) u_0(x) dx + \frac{2\varepsilon}{3} (1 - e^{-\frac{3}{2}t}) \int_{\Omega} \varphi_1(x) u_1(x) dx \geq \varepsilon c_0 > 0.$$

□

Next, in order to apply Lemma 1 on $F_0(t)$, we multiply (1.1) by ϕ_0 and integrate over Ω

$$\int_{\Omega} u_{tt} \phi_0 dx - \int_{\Omega} \Delta u \phi_0 dx - \int_{\Omega} \Delta u_t \phi_0 dx = \int_{\Omega} |u|^p \phi_0 dx.$$

By using integration by parts, boundary conditions and Lemma 4, we can easily check that

$$\int_{\Omega} \Delta u \phi_0 dx = \int_{\Omega} u \Delta \phi_0 dx = 0,$$

and

$$\int_{\Omega} \Delta u_t \phi_0 dx = \int_{\Omega} u_t \Delta \phi_0 dx = 0.$$

Therefore

$$\frac{d^2}{dt^2} F_0(t) = \int_{\Omega} u_{tt} \phi_0 dx = \int_{\Omega} |u|^p \phi_0 dx. \quad (3.2)$$

To estimate the right-hand side of the last equality, we use Hölder's inequality

$$\begin{aligned} \left| \int_{\Omega} u(x, t) \phi_0(x) dx \right| &= \left| \int_{\Omega \cap \{|x| \leq t+R\}} u(x, t) [\phi_0(x)]^{1/p} [\phi_0(x)]^{(p-1)/p} dx \right| \\ &\leq \left(\int_{\Omega \cap \{|x| \leq t+R\}} |u(x, t) [\phi_0(x)]^{1/p}|^p dx \right)^{1/p} \left(\int_{\Omega \cap \{|x| \leq t+R\}} |[\phi_0(x)]^{(p-1)/p}|^{p'} dx \right)^{1/p'}, \end{aligned}$$

where $p' = p/(p-1)$, and then

$$\begin{aligned} |F_0(t)|^p &= \left| \int_{\Omega} u(x, t) \phi_0(x) dx \right|^p \leq \left(\int_{\Omega \cap \{|x| \leq t+R\}} |u(x, t)|^p \phi_0(x) dx \right) \left(\int_{\Omega \cap \{|x| \leq t+R\}} \phi_0(x) dx \right)^{p-1} \\ &\leq \left(\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \right) \left(\int_{\Omega \cap \{|x| \leq t+R\}} \phi_0(x) dx \right)^{p-1}. \end{aligned}$$

So

$$\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \geq \frac{|F_0(t)|^p}{\left(\int_{\Omega \cap \{|x| \leq t+R\}} \phi_0(x) dx \right)^{p-1}}. \quad (3.3)$$

At this stage, we distinguish the following four cases.

Case $n \geq 3$: By lemma 4, we have $0 < \phi_0(x) < 1$, then (3.3) implies

$$\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \geq \frac{|F_0(t)|^p}{\left(\int_{\{|x| \leq t+R\}} 1 dx \right)^{p-1}} = |F_0(t)|^p [\text{Vol}(\mathbf{B}^n)]^{-(p-1)} (t+R)^{-n(p-1)},$$

where \mathbf{B}^n stands for the unit closed ball in \mathbb{R}^n . Combining that above inequalities, we infer that

$$\frac{d^2}{dt^2} F_0(t) \geq k(t+R)^{-n(p-1)} |F_0(t)|^p, \quad (3.4)$$

where $k = [\text{Vol}(\mathbf{B}^n)]^{-(p-1)} > 0$. So F_0 satisfies the second inequality in Lemma 1. To provide that F_0 is also verifies the first inequality in Lemma 1, we relate $\frac{d^2}{dt^2}F_0(t)$ to F_1 using again Hölder's inequality

$$\begin{aligned} \left| \int_{\Omega} u(x, t) \psi_1(x, t) dx \right| &= \left| \int_{\Omega \cap \{|x| \leq t+R\}} u(x, t) [\phi_0(x)]^{1/p} [\phi_0(x)]^{-1/p} \psi_1(x, t) dx \right| \\ &\leq \left(\int_{\Omega \cap \{|x| \leq t+R\}} |u(x, t) [\phi_0(x)]^{1/p}|^p dx \right)^{1/p} \left(\int_{\Omega \cap \{|x| \leq t+R\}} [|\phi_0(x)]^{-1/p} \psi_1(x, t)|^{p'} dx \right)^{1/p'} \\ &\leq \left(\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \right)^{1/p} \left(\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} [\psi_1(x, t)]^{p'} dx \right)^{1/p'}, \end{aligned}$$

then

$$|F_1(t)|^p = \left| \int_{\Omega} u(x, t) \psi_1(x, t) dx \right|^p \leq \left(\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \right) \left(\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} [\psi_1(x, t)]^{p'} dx \right)^{p-1}.$$

So, by using Lemmas 9 and 10, we get

$$\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \geq \frac{|F_1(t)|^p}{\left(\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} [\psi_1(x, t)]^{p'} dx \right)^{p-1}} \geq \frac{(\varepsilon c_0)^p}{(C(t+R)^{n-1-(n-1)p'/2})^{p-1}} = L(t+R)^{-(n-1)(p/2-1)},$$

where $L > 0$ is a positive constant independent of t . Therefore, (3.2) implies

$$\frac{d^2}{dt^2}F_0(t) \geq L(t+R)^{-(n-1)(p/2-1)}.$$

Integrate twice, we have

$$F_0(t) \geq \delta(t+R)^{n+1-(n-1)p/2} + \frac{dF_0(0)}{dt}t + F_0(0),$$

for a positive constant $\delta > 0$. As $1 < p < p_c(n)$, it is easy to check that $n+1-(n-1)p/2 > 1$. Hence the following estimate is valid when t is sufficiently large:

$$F_0(t) \geq \frac{1}{2}\delta(t+R)^{n+1-(n-1)p/2}. \quad (3.5)$$

Estimates (3.4)-(3.9) and Lemma 1 with parameters

$$a \equiv n+1-(n-1)p/2, \quad \text{and} \quad q \equiv n(p-1)$$

imply Theorem 1 for all exponents p such that

$$(p-1)(n+1-(n-1)p/2) > n(p-1)-2, \quad \text{and} \quad p > 1.$$

Note that the last condition on p is equivalent to $p \in (1, p_c(n))$.

Case $n = 1$: In one dimensional case, the exterior domain is reduced on the semi-infinite interval $[0, \infty)$. By lemma 6, we have $0 < \phi_0(x) < C_2x$, then (3.3) implies

$$\int_0^{\infty} |u(x, t)|^p \phi_0(x) dx \geq \frac{|F_0(t)|^p}{\left(\int_0^{t+R} C_2x dx \right)^{p-1}} = C(t+R)^{-2(p-1)}|F_0(t)|^p,$$

and then by (3.2) we get

$$\frac{d^2}{dt^2}F_0(t) \geq k(t+R)^{-2(p-1)}|F_0(t)|^p, \quad (3.6)$$

where $k = C^{-1} > 0$. So F_0 satisfies the second inequality in Lemma 1. To provide that F_0 is also verifies the first inequality in Lemma 1, we relate $\frac{d^2}{dt^2}F_0(t)$ to F_1 using again Hölder's inequality

$$\begin{aligned} \left| \int_0^\infty u(x, t)\psi_1(x, t) dx \right| &= \left| \int_0^{t+R} u(x, t)[\phi_0(x)]^{1/p}[\phi_0(x)]^{-1/p}\psi_1(x, t) dx \right| \\ &\leq \left(\int_0^{t+R} |u(x, t)[\phi_0(x)]^{1/p}|^p dx \right)^{1/p} \left(\int_0^{t+R} |[\phi_0(x)]^{-1/p}\psi_1(x, t)|^{p'} dx \right)^{1/p'} \\ &\leq \left(\int_0^\infty |u(x, t)|^p \phi_0(x) dx \right)^{1/p} \left(\int_0^{t+R} [\phi_0(x)]^{-1/(p-1)}[\psi_1(x, t)]^{p'} dx \right)^{1/p'}, \end{aligned}$$

then

$$|F_1(t)|^p = \left| \int_0^\infty u(x, t)\psi_1(x, t) dx \right|^p \leq \left(\int_0^\infty |u(x, t)|^p \phi_0(x) dx \right) \left(\int_0^{t+R} [\phi_0(x)]^{-1/(p-1)}[\psi_1(x, t)]^{p'} dx \right)^{p-1}.$$

So, by using Lemmas 9 and 10, we get

$$\int_0^\infty |u(x, t)|^p \phi_0(x) dx \geq \frac{|F_1(t)|^p}{\left(\int_0^{t+R} [\phi_0(x)]^{-1/(p-1)}[\psi_1(x, t)]^{p'} dx \right)^{p-1}} \geq \frac{(\varepsilon c_0)^p}{C^{p-1}} = L,$$

where $L > 0$ is a positive constant independent of t . Therefore, (3.2) implies

$$\frac{d^2}{dt^2}F_0(t) \geq L.$$

Integrate twice on $[0, t + R]$, we have

$$F_0(t) \geq \frac{1}{2}L(t + R)^2 + \frac{dF_0(0)}{dt}t + F_0(0).$$

Hence the following estimate is valid when t is sufficiently large:

$$F_0(t) \geq \frac{1}{4}L(t + R)^2. \quad (3.7)$$

Estimates (3.7) together with (3.6) and Lemma 1 with parameters

$$a \equiv 2, \quad \text{and} \quad q \equiv 2(p - 1)$$

imply Theorem 1 for all exponents p such that

$$1 < p < p_c(1) = \infty.$$

Case $n = 2$ and $p < p_c(2)$: As $0 \notin \bar{\Omega}$, then without loss of generality we can assume that $B_2(0) \cap \Omega = \emptyset$, ($B_2(0)$ stands for the closed ball of center 0 and radius 2). By lemma 5, we have

$$\begin{aligned} \int_{\Omega \cap \{|x| \leq t+R\}} \phi_0(x) dx &\leq \int_{\{|x| \leq t+R\} \setminus B_2(0)} C \ln |x| dx \\ &= \int_0^{2\pi} \int_2^{t+R} C \ln r \cdot r dr d\theta \\ &= \pi C \left[(t + R)^2 \ln(t + R) - 4 \ln(2) - \frac{1}{2}((t + R)^2 - 4) \right] \\ &\leq \pi C (t + R)^2 \ln(t + R). \end{aligned}$$

Therefore, (3.2) and (3.3) imply

$$\frac{d^2}{dt^2}F_0(t) = \int_{\Omega} |u(x, t)|^p \phi_0(x) dx \geq \frac{|F_0(t)|^p}{(\pi C(t+R)^2 \ln(t+R))^{p-1}} = k[\ln(t+R)]^{-(p-1)}(t+R)^{-2(p-1)}|F_0(t)|^p \quad (3.8)$$

where $k > 0$. So F_0 satisfies the second inequality in Lemma 2. To provide that F_0 is also verifies the first inequality in Lemma 2, we relate $\frac{d^2}{dt^2}F_0(t)$ to F_1 using again Hölder's inequality

$$\begin{aligned} \left| \int_{\Omega} u(x, t)\psi_1(x, t) dx \right| &= \left| \int_{\Omega \cap \{|x| \leq t+R\}} u(x, t)[\phi_0(x)]^{1/p}[\phi_0(x)]^{-1/p}\psi_1(x, t) dx \right| \\ &\leq \left(\int_{\Omega \cap \{|x| \leq t+R\}} |u(x, t)[\phi_0(x)]^{1/p}|^p dx \right)^{1/p} \left(\int_{\Omega \cap \{|x| \leq t+R\}} |[\phi_0(x)]^{-1/p}\psi_1(x, t)|^{p'} dx \right)^{1/p'} \\ &\leq \left(\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \right)^{1/p} \left(\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)}[\psi_1(x, t)]^{p'} dx \right)^{1/p'}, \end{aligned}$$

then

$$|F_1(t)|^p = \left| \int_{\Omega} u(x, t)\psi_1(x, t) dx \right|^p \leq \left(\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \right) \left(\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)}[\psi_1(x, t)]^{p'} dx \right)^{p-1}.$$

So, by using Lemmas 9 and 10, we get

$$\int_{\Omega} |u(x, t)|^p \phi_0(x) dx \geq \frac{|F_1(t)|^p}{\left(\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)}[\psi_1(x, t)]^{p'} dx \right)^{p-1}} \geq \frac{(\varepsilon C_0)^p}{(C(t+R)^{1-p'/2})^{p-1}} = L(t+R)^{-(p/2-1)},$$

where $L > 0$ is a positive constant independent of t . Therefore, (3.2) implies

$$\frac{d^2}{dt^2}F_0(t) \geq L(t+R)^{-(p/2-1)}.$$

Integrate twice, we have

$$F_0(t) \geq \delta(t+R)^{3-p/2} + \frac{dF_0(0)}{dt}t + F_0(0),$$

for a positive constant $\delta > 0$. As $1 < p < p_c(2)$, it is easy to check that $3 - p/2 > 1$. Hence the following estimate is valid when t is sufficiently large:

$$F_0(t) \geq \frac{1}{2}\delta(t+R)^{3-p/2}. \quad (3.9)$$

Estimates (3.8), (3.9) and Lemma 2 with parameters

$$a \equiv 3 - p/2, \quad \text{and} \quad q \equiv 2(p-1)$$

imply Theorem 1 for all exponents p such that

$$(p-1)(3-p/2) > 2(p-1) - 2, \quad \text{and} \quad p > 1.$$

Case $n = 2$ and $p = p_c(2)$: As the subcritical case ($p < p_c(2)$), we have

$$\frac{d^2}{dt^2}F_0(t) \geq K_1[\ln(t+R)]^{-(p-1)}(t+R)^{-2(p-1)}|F_0(t)|^p, \quad (3.10)$$

where $K_1 > 0$, and

$$\frac{d^2}{dt^2}F_0(t) = \int_{\Omega} |u(x, t)|^p \phi_0(x) dx \geq \frac{|F_1(t)|^p}{\left(\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)}[\psi_1(x, t)]^{p'} dx \right)^{p-1}}.$$

Next, we use Lemma 10 and the fact that (see Lemma 9)

$$\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} [\psi_1(x, t)]^{p'} dx \leq C(t+R)^{1-p'/2} (\ln(t+R))^{-1/(p-1)}.$$

we conclude that

$$\frac{d^2}{dt^2} F_0(t) \geq \frac{\varepsilon^p c_0^p}{\left(C(t+R)^{1-p'/2} (\ln(t+R))^{-1/(p-1)}\right)^{p-1}} \geq L(t+R)^{-(p/2-1)} (\ln(t+R)) \quad (3.11)$$

where $L > 0$ is a positive constant independent of t . Integrate twice, we have when t is sufficiently large:

$$F_0(t) \geq C(t+R)^{3-p/2} \ln t.$$

As $\lim_{t \rightarrow \infty} \ln t = \infty$, we infer that

$$F_0(t) \geq K_0(t+R)^{3-p/2}, \quad (3.12)$$

with $K_0 > 0$ being arbitrarily large when t is sufficiently large. Estimates (3.12) together with (3.10) and Lemma 3 with parameters

$$a \equiv 3 - p/2, \quad \text{and} \quad q \equiv 2(p-1)$$

imply Theorem 1, since exponent $p = p_c(2)$ satisfies

$$(p-1)(3-p/2) = 2(p-1) - 2, \quad \text{and} \quad p > 1.$$

□

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