

BOUNDARY REGULARITY FOR NONLOCAL OPERATORS WITH KERNELS OF VARIABLE ORDERS

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ABSTRACT. We study the boundary regularity of solutions of the Dirichlet problem for the nonlocal operator with a kernel of variable orders. Since the order of differentiability of the kernel is not represented by a single number, we consider the generalized Hölder space. We prove that there exists a unique viscosity solution of $Lu = f$ in D , $u = 0$ in $\mathbb{R}^n \setminus D$, where D is a bounded $C^{1,1}$ open set, and that the solution u satisfies $u \in C^V(D)$ and $u/V(d_D) \in C^\alpha(D)$ with the uniform estimates, where V is the renewal function and $d_D(x) = \text{dist}(x, \partial D)$.

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1. INTRODUCTION

In this paper, we will consider the viscosity solutions for the following Dirichlet (exterior) problem

$$(1.1) \quad \begin{cases} -\phi(-\Delta)u = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^n \setminus D, \end{cases}$$

where ϕ is in the class of functions called *Bernstein function*, which contains $\phi(\lambda) = \lambda^\alpha$ with $0 < \alpha < 1$, and D is a bounded $C^{1,1}$ open set in \mathbb{R}^n . For example, if $\phi(\lambda) = \lambda^\alpha$, then $-\phi(-\Delta) = -(-\Delta)^\alpha$ is a fractional Laplacian.

We will focus on the boundary behavior of the viscosity solutions of the Dirichlet problem (1.1) under assumptions (1.6) and (1.7) below.

1.1. Probabilistic point of view. The operator $-\phi(-\Delta)$ can be understood as the infinitesimal generator of subordinate Brownian motions, thus we can use probabilistic tools to study the behavior of solutions of (1.1).

Let $S = (S_t)_{t \geq 0}$ be a subordinator, that is, an increasing Lévy process in \mathbb{R} . It is known that its Laplace exponent is given by

$$\mathbb{E}[e^{-\lambda S_t}] = \exp(-t\phi(\lambda)), \quad \lambda > 0,$$

where the function $\phi : (0, \infty) \rightarrow (0, \infty)$ satisfies $\lim_{\lambda \downarrow 0} \phi(\lambda) = 0$ and

$$(1.2) \quad \phi(\lambda) = b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda x}) \mu(dx)$$

with a drift $b \geq 0$ and a measure μ on $(0, \infty)$ satisfying $\int_{(0, \infty)} (1 \wedge x) \mu(dx) < \infty$. It is known that the function ϕ of the form (1.2) is a *Bernstein function*, it means, $\phi : (0, \infty) \rightarrow (0, \infty)$ is a C^∞ -function satisfying

$$(-1)^{n+1} \phi^{(n)}(\lambda) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

Here $\phi^{(n)}$ is the n -th derivative of ϕ . Also, it is known that every Bernstein function can be uniquely represented by (1.2).

Subordinate Brownian motion $Y = (Y_t)_{t \geq 0} = (B_{S_t})_{t \geq 0}$ in \mathbb{R}^n is a Lévy process obtained by replacing the time of Brownian motion in \mathbb{R}^n by an independent subordinator. Then, the characteristic exponent of Y is given by $z \mapsto \phi(|z|^2)$. Also, the Lévy measure of the process has a density $y \mapsto j(|y|)$ where $j : (0, \infty) \rightarrow (0, \infty)$ is the function given by

$$(1.3) \quad j(r) = j_n(r) = \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{r^2}{4t}} \mu(dt),$$

and we have

$$(1.4) \quad \phi(|z|^2) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(z \cdot y)) j(|y|) dy.$$

Let A be the infinitesimal generator of Y . Then, by [34, Section 4.1] we have

$$(1.5) \quad Au(x) = -\phi(-\Delta)u(x) = \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x) - \mathbf{1}_{\{|y| \leq 1\}} y \cdot \nabla u(x)) j(|y|) dy.$$

for any $u \in C^2(\mathbb{R}^n)$. See Section 1.4 for the definition of function spaces and Section 2 for the definition of infinitesimal generator.

Note that when $\phi(\lambda) = \lambda^\alpha$ with $0 < \alpha < 1$, the corresponding subordinate Brownian motion in \mathbb{R}^n is a rotationally symmetric 2α -stable process. We also have $j(|y|) = c(n, \alpha) |y|^{-n-2\alpha}$. Thus the corresponding infinitesimal generator is the fractional Laplacian $-(\Delta)^\alpha$.

Now we introduce some conditions which we will impose in this paper. The first condition is *weak scaling condition at the infinity* for ϕ , that is, there exist constants $0 < \alpha_1 \leq \alpha_2 < 1$ and $b_1 \geq 1$ such that

$$(1.6) \quad b_1^{-1} \left(\frac{R}{r}\right)^{\alpha_1} \leq \frac{\phi(R)}{\phi(r)} \leq b_1 \left(\frac{R}{r}\right)^{\alpha_2} \quad \text{for all } 1 \leq r \leq R < \infty.$$

The constant 1 in above condition can be changed into other positive constant without loss of generality. Note that (1.2) and (1.6) imply that $b = 0$ and that μ is an infinite measure. The second one is that the Lévy density of process satisfies

$$(1.7) \quad j(r+1) \leq b_2 j(r) \quad \text{for all } r \geq 1$$

for some constant $b_2 > 0$. (1.7) is valid for any complete Bernstein function satisfying (1.6). See [35, Definition 6.1] and [27, Theorem 13.3.5] for details. Moreover, we also have (1.7) when (1.6) holds for any $0 < r \leq R < \infty$ (See [3, Corollary 22]).

We will see that the *renewal function* V with respect to one dimensional Lévy process is related to the boundary behavior of solutions. This function plays an important role throughout this paper. For the definition of the *renewal function*, see Section 2.2.

1.2. Analytic point of view. In analytic point of view, nonlocal operators can be defined via the Fourier transformation. For example, the fractional Laplacian is defined by

$$\begin{aligned} -(-\Delta)^{\sigma/2}f(x) &:= -(|\xi|^\sigma \hat{f})^\vee(x) \\ &= P.V. \int_{\mathbb{R}^n} \frac{f(y) - f(x)}{|y - x|^{n+\sigma}} dy \\ &= \int_{\mathbb{R}^n} \frac{f(y) - f(x) - \nabla f(x) \cdot (y - x) \mathbf{1}_{\{|y-x|<k\}}}{|y - x|^{n+\sigma}} dy \end{aligned}$$

for $f \in C_c^\infty(\mathbb{R}^n)$ and it is well-known that

$$\lim_{\sigma \rightarrow 2} (2 - \sigma)c(n, \sigma)(-\Delta)^{\sigma/2}f(x) = -\Delta f(x).$$

Moreover, Caffarelli and Silvestre [14] provided Harnack inequality and interior $C^{1,\alpha}$ regularity for fully nonlinear integro-differential equations associated with kernels comparable to that of fractional Laplacian, which remain uniform as $\sigma \rightarrow 2$. These results were generalized in [23] and [19] to more general integro-differential equations. These results make the theory of integro-differential operators and elliptic differential operators become unified.

The fractional Laplacian $(-\Delta)^{\sigma/2}f$ can be also thought as the normal derivative of some extension of f (the Dirichlet to Neumann operator of f). Consider the extension problem

$$\begin{cases} -\nabla(y^{1-\sigma}\nabla u) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = f(x) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

It is known in [13] that the following holds:

$$(-\Delta)^{\sigma/2}f(x) = \partial_\nu u(x, 0) = -\lim_{y \rightarrow 0} y^{1-\sigma} u_y(x, y),$$

where $\partial_\nu u$ is the outward normal derivative of u on the boundary $\{y = 0\}$.

We are interested in the operator of the form

$$(1.8) \quad Lu(x) = P.V. \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x)) j(|y|) dy$$

where $j : (0, \infty) \rightarrow (0, \infty)$ is a non-increasing function satisfying (1.4), (1.6) and (1.7), or satisfying (2.3) and (2.4) in Section 2.1. Let us call the function $j(|y|)$ be the *kernel* of operator L . Note that $Lu(x)$ is well-defined if $u \in C^2(x) \cap B(\mathbb{R}^n)$, where $C^2(x)$ denotes the family of all functions which are C^2 in some neighborhood of x and $B(\mathbb{R}^n)$ denotes the family of all bounded functions defined on \mathbb{R}^n , and this is why we needed the assumption $0 < \alpha_1 \leq \alpha_2 < 1$. Due to the symmetry of the kernel $j(|y|)dy$, the operator can be rewritten without the principal value as

$$(1.9) \quad \begin{aligned} Lu(x) &= \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x) - \mathbf{1}_{\{|y|\leq 1\}} y \cdot \nabla u(x)) j(|y|) dy \\ &= \frac{1}{2} \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) + u(x-y) - 2u(x)) j(|y|) dy \end{aligned}$$

when $u \in C^2(x) \cap B(\mathbb{R}^n)$. The important point to note here is that $Lu = Au$ for $u \in C^2(\mathbb{R}^n)$ when $j(|y|)$ in (1.5) and (1.8) are the same. In Section 3.2 we discuss the connection between two operators in (1.5) and (1.8).

We will consider the *viscosity solution* of $Lu = f$ in D . A function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ which is upper (resp. lower) semicontinuous on \overline{D} is said to be a *viscosity subsolution* (resp. *viscosity supersolution*) to $Lu = f$, and we write $Lu \geq f$ (resp. $Lu \leq f$) in *viscosity sense*, if for any $x \in D$ and a test function $v \in C^2(x)$ satisfying $v(x) = u(x)$ and

$$v(y) > u(y) \quad (\text{resp. } <), \quad y \in \mathbb{R}^n \setminus \{x\},$$

it holds that

$$Lv(x) \geq f(x) \quad (\text{resp. } \leq).$$

A function u is said to be a *viscosity solution* if u is both sub and supersolution.

We are going to prove the Hölder regularity of viscosity solutions of nonlocal Dirichlet problem

$$(1.10) \quad \begin{cases} Lu = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^n \setminus D, \end{cases}$$

up to the boundary using the gradient heat kernel estimates and prove higher boundary regularity using PDE tools: barriers, comparison principle, and Harnack inequality. It is important that the boundary condition in (1.10) is given not only on ∂D but on the whole complement of D because of the nonlocal character of the operator L . See Section 3.2 for details.

The PDE approach can be applied to nonlinear integro-differential equations. There are many literatures dealing with regularity results with PDE approach. See [1, 7, 14, 21, 23, 29] and [19]. We expect that similar results such as Harnack inequality and Hölder regularity hold for nonlinear equations with our L .

1.3. History. Over the last few decades there have been a lot of studies for the nonlocal operators, and regularity theory for nonlocal operators is one of the main areas as the one for local operators. In [8] Bass and Levin proved Hölder regularity of harmonic functions with respect to a class of pure jump Markov processes in \mathbb{R}^n , whose kernels are comparable to those of symmetric stable processes. Bass and Kassmann generalized this result to kernels with variable order in [5, 6]. Bass also established in [2] the Schauder estimates for stable-like operators in \mathbb{R}^n . All these works were done by probabilistic methods.

On the other hand, in [33] Silvestre provided a purely analytic proof of Hölder estimates for solutions to integro-differential equation. His assumptions include the case of an operator with variable orders. In [14] Caffarelli and Silvestre generalized this result to fully nonlinear integro-differential equations associated with symmetric kernels comparable to fractional Laplacian by PDE methods. Kim and Lee, in [21] and [23], extended this result to fully nonlinear integro-differential equations associated with nonsymmetric kernels. A singular regularity theory for parabolic nonlocal nonlinear equations was also established at [22]. In [1], Bae proved Hölder regularity for solutions of fully nonlinear integro-differential equations with kernels of variable orders in [1]. Bae and Kassmann in [7] established Schauder estimates for integro-differential equation with kernels of variable orders. In [19], they extended the regularity results for the integro-differential operators of the fractional Laplacian type by Caffarelli and Silvestre [14] to those for the integro-differential operators associated with symmetric, regularly varying kernels at zero.

There are relatively fewer results concerning boundary regularity of solutions of Dirichlet problem. For the boundary regularity for local operators, see [15]. Kim and Lee proved regularity up to the boundary for the fractional heat flow in [20]. The boundary regularity up to the boundary is well-known for the fractional Laplacian, and for fully nonlinear integro-differential equations, when D is a bounded $C^{1,1}$ domain. See [28, 29]. Ros-Oton and Serra also proved the similar result when D is a bounded $C^{1,\alpha}$ or C^1 domain in [30]. However, there is no boundary regularity result for the operators with kernels having variable orders.

1.4. Notation. In this paper, we denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For any nonnegative functions f and g , $f(r) \asymp g(r)$ for $r > 0$ (resp. $0 < r \leq r_0$) means that there is a constant $c \geq 1$ such that $c^{-1}f(r) \leq g(r) \leq cf(r)$ for $r > 0$ (resp. $0 < r \leq r_0$). We call c the *comparison constant* of f and g . We also denote $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ for the open ball and $d_D(x) := \text{dist}(x, D^c)$ for the distance between $x \in D$ and D^c . For $n \geq 1$, let $\omega_n = \int_{\mathbb{R}^n} \mathbf{1}_{\{|y| \leq 1\}} dy$ be the volume of n -dimensional ball.

We denote by $C(D)$ the Banach space of bounded and continuous functions on D , equipped with the supremum norm $\|f\|_{C(D)} := \sup_{x \in D} |f(x)|$, and denote by $C^k(D)$, $k \geq 1$, the Banach space of k -times continuously differentiable functions on D , equipped with the norm $\|f\|_{C^k(D)} := \sum_{|\gamma| \leq k} \sup_{x \in D} |D^\gamma f(x)|$. Also, denote $C_0(D) := \{u \in C(D) : u \text{ vanishes at the boundary of } D\}$. For $x \in \mathbb{R}^n$, define $C^1(x)$ as the collection of functions which are C^1 in some neighborhood of x . Similarly, we define $C^2(x)$, $C^{1,1}(x)$, etc. For $0 < \alpha < 1$, the Hölder space $C^\alpha(\mathbb{R}^n)$ is defined as

$$(1.11) \quad C^\alpha(\mathbb{R}^n) := \{f \in C(\mathbb{R}^n) \mid \|f\|_{C^\alpha(\mathbb{R}^n)} < \infty\},$$

equipped with the C^α -norm

$$\|f\|_{C^\alpha(\mathbb{R}^n)} := \|f\|_{C(\mathbb{R}^n)} + \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Also, for given open set $D \subset \mathbb{R}^n$ we define $C^\alpha(D)$ by

$$C^\alpha(D) := \{f \in C(D) \mid \|f\|_{C^\alpha(D)} < \infty\}$$

with the norm

$$\|f\|_{C^\alpha(D)} := \|f\|_{C(D)} + \sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For given function $h : (0, \infty) \rightarrow (0, \infty)$, we define Generalized Hölder space $C^h(D)$ for bounded open set D by

$$(1.12) \quad C^h(D) := \{f \in C(D) \mid \|f\|_{C^h(D)} < \infty\},$$

equipped with the norm

$$\|f\|_{C^h(D)} := \|f\|_{C(D)} + \sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{h(|x - y|)}.$$

We define seminorm $[\cdot]_{C^h(D)}$ by

$$[f]_{C^h(D)} := \sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{h(|x - y|)}.$$

We denote the diameter of D by $\text{diam}(D)$. Note that if $h_1 \asymp h_2$ in $0 < r \leq \text{diam}(D)$, $\|\cdot\|_{C^{h_1}(D)}$ and $\|\cdot\|_{C^{h_2}(D)}$ are equivalent and $C^{h_1}(D) = C^{h_2}(D)$.

We say that $D \subset \mathbb{R}^n$ (when $n \geq 2$) is a $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$ there exist a $C^{1,1}$ -function $\varphi = \varphi_z : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \leq \Lambda$, $|\nabla\varphi(x) - \nabla\varphi(w)| \leq \Lambda|x - w|$ and an orthonormal coordinate system CS_z of $z = (z_1, \dots, z_{n-1}, z_n) := (\tilde{z}, z_n)$ with origin at z such that $D \cap B(z, R_0) = \{y = (\tilde{y}, y_n) \in B(0, R_0) \text{ in } CS_z : y_n > \varphi(\tilde{y})\}$. The pair (R_0, Λ) will be called the $C^{1,1}$ characteristics of the open set D . Note that a $C^{1,1}$ open set D with characteristics (R_0, Λ) can be unbounded and disconnected, and the distance between two distinct components of D is at least R_0 . By a $C^{1,1}$ open set in \mathbb{R} with a characteristic $R_0 > 0$, we mean an open set that can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least R_0 and the infimum of the distances between these intervals is at least R_0 .

1.5. Main theorems. The main results of this paper are the existence and the uniqueness of the viscosity solution u of (1.1), the generalized Hölder regularity estimates of such solution u and the regularity of the quotient $u\phi(d_D^{-2})$ up to the boundary.

The boundary estimate for nonlinear PDE has been studied for a long time, where the solution behaves as a linear function. See [11] and references therein. For the degenerate or singular PDE, [22], it has been proved that the solution behaves in various ways just as that of the fractional Laplace equation. In [28], Ros-Oton and Serra applied the known techniques for local operators to fractional Laplacian, which has a nice scaling invariance and a simple barrier of the form x_n^α . On the other hand, our ϕ has only a weak scaling condition at infinity and it has a general form which allows nontrivial boundary behavior different from x_n^α . In this paper, we track down u in every scale to find scaling invariant uniform estimates only with the weak scaling condition at infinity. We also construct the renewal function, $V(\cdot)$, of the ladder height process defined at (2.7) to overcome the lack of a simple barrier. In addition, we provide the existence and uniqueness theory for given Dirichlet problem by utilizing the concept of viscosity solution.

The first result is the Hölder estimates up to the boundary of solutions of the Dirichlet problem (1.1). Unlike the case of the fractional Laplacian, it is inappropriate to represent Hölder regularity as a single number since kernel in (1.8) has variable orders. Therefore it is natural to consider a generalized Hölder space.

Theorem 1.1 (Hölder estimates up to the boundary). *Assume that D is a bounded $C^{1,1}$ open set in \mathbb{R}^n , and ϕ is a Bernstein function satisfying (1.6) and (1.7). If $f \in C(D)$, then there exists a unique viscosity solution u of (1.10) and $u \in C^{\bar{\phi}}(D)$. Moreover, we have*

$$\|u\|_{C^{\bar{\phi}}(D)} \leq C \|f\|_{C(D)},$$

where $\bar{\phi}(r) := \phi(r^{-2})^{-1/2}$, for some constant $C > 0$ depending only on n, D , and ϕ .

We will prove Theorem 1.1 using the potential operator, which is the inverse of the operator L , and the estimates on the transition density and its spatial derivatives, see Section 3 for details. In whole space \mathbb{R}^n , estimates on any order of spatial derivatives of the transition density are known. Based on these estimates, Bae and Kassmann established Schauder estimates for the integro-differential operators with kernels of variable orders in [7]. However, in a bounded $C^{1,1}$ open set, estimates on the first order derivative of the transition density are only known. Higher order regularities up to the boundary require further research in future.

It is well known that $\bar{\phi}$ is comparable to renewal function V (see Section 2.2.) Thus any solution u of Dirichlet problem (1.1) is in C^V up to the boundary by Theorem 1.1. Hence it is of importance to study the regularity of $u/V(d_D)$ up to the boundary. The following is our second main result.

Theorem 1.2 (Boundary estimates). *Assume that D is a bounded $C^{1,1}$ open set in \mathbb{R}^n , and ϕ is a Bernstein function satisfying (1.6) and (1.7). If $f \in C(D)$ and u is the viscosity solution of (1.10), then $u/V(d_D) \in C^\alpha(D)$ and*

$$\left\| \frac{u}{V(d_D)} \right\|_{C^\alpha(D)} \leq C \|f\|_{C(D)}$$

for some constants $\alpha > 0$ and $C > 0$ depending only on n, D , and ϕ .

One of the methods proving the above result follows the standard argument of Krylov in [26]. In the other words, we are going to control the oscillation of the function $u\phi(d_D^{-2})^{1/2}$ near the boundary using barriers, comparison principle, and the Harnack inequality. However, the construction of barriers are highly nontrivial. The difficulty mainly comes from the fact that the operator (1.8) is not scale-invariant.

In fact, we will prove Theorems 1.1 and 1.2 for a little more general operators including $-\phi(-\Delta)$. In section 2 we will state the generalization of these theorems, and we collect some known results about the renewal function V . We will prove Theorem 1.1 in Section 3, and Theorem 1.2 in section 4.

2. PRELIMINARIES

The operators we consider in this paper coincides with infinitesimal generators of isotropic unimodal Lévy processes for $C^2(\mathbb{R}^n)$ functions. Thus, in Section 2.1 we first explain the definitions and properties of Lévy processes, and some related concepts. Then we introduce some additional conditions that will be needed in this paper. With these concepts, we state Theorems 2.1 and 2.2, which are generalized version of Theorems 1.1 and 1.2. Throughout this paper, we prove Theorems 2.1 and 2.2.

Next, in Section 2.2 we will define the renewal function V , which will be act as a barrier, and record some properties of renewal function.

2.1. Lévy processes. Let $X = (X_t, \mathbb{P}^x, t \geq 0, x \in \mathbb{R}^n)$ be a Lévy process in \mathbb{R}^n defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^x)$ with $\mathbb{P}^x(X_0 = x) = 1$. For the precise definition of Lévy process, see [32, Definition 1.5]. Note that $\mathbb{P}^x(X_t \in A) = \mathbb{P}^0(X_t + x \in A)$. By Lévy-Khintchine formula, the characteristic exponent of Lévy process is given by

$$\mathbb{E}^0[e^{iz \cdot X_t}] = e^{t\Phi(z)}, \quad z \in \mathbb{R}^n,$$

where

$$\Phi(z) = -\frac{1}{2}z \cdot Uz + i\gamma \cdot z + \int_{\mathbb{R}^n} (e^{iz \cdot x} - 1 - iz \cdot x \mathbf{1}_{\{|x| \leq 1\}}) J(dx)$$

with an $n \times n$ symmetric nonnegative-definite matrix $U = (U_{ij})$, $\gamma \in \mathbb{R}^n$ and a measure $J(dx)$ on $\mathbb{R}^n \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^n \setminus \{0\}} (1 \wedge |x|^2) J(dx) < \infty.$$

Let $(P_t)_{t \geq 0}$ be a transition semigroup for X , it means that

$$P_t f(x) := \mathbb{E}^x[f(X_t)] = \mathbb{E}^0[f(x + X_t)].$$

Now, define the *infinitesimal generator* A of X by

$$Au(x) := \lim_{t \downarrow 0} \frac{P_t u(x) - u(x)}{t}$$

if the limit exists. By [34, Section 4.1], Au is well-defined for $u \in C^2(\mathbb{R}^n)$ and represented by

$$Au(x) = \frac{1}{2} \sum_{i,j=1}^n U_{ij} \partial_{ij} u(x) + \sum_{i=1}^n \gamma_i \partial_i u(x) + \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) - u(x) - \mathbf{1}_{\{|y| \leq 1\}} y \cdot \nabla u(x)) J(dy).$$

Throughout this paper, we will assume that X is an isotropic unimodal pure jump Lévy process with an infinite Lévy measure, that is, $U = 0$, $\gamma = 0$ and $J(dy)$ is an infinite measure with an isotropic density $J(|y|)dy$, where $r \mapsto J(r)$ is non-increasing. Under these assumptions, X possesses transition density $p : (0, \infty) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying

$$P_t f(x) = \mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^n} f(y) p(t, |x-y|) dy$$

and characteristic exponent $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is an isotropic function. From now on, we regard isotropic functions J and Φ as functions on \mathbb{R}_+ .

For every open subset $D \subset \mathbb{R}^n$, let $\tau_D := \inf\{t > 0 : X_t \notin D\}$ be the first exit time of D by X . We define subprocess $X^D = (X_t^D)_{t \geq 0}$, which is called *the killed process of X upon D* , by $X_t^D = X_t$ when $t < \tau_D$ and $X_t^D = \partial$ when $t \leq \tau_D$ where ∂ is a cemetery point. Since X has the transition density, X^D also possesses the transition density $p_D(t, x, y)$ with

$$p_D(t, x, y) = p(t, |x-y|) - \mathbb{E}^x[p(t - \tau_D, |X_{\tau_D} - y|); \tau_D < t],$$

and its transition semigroup $(P_t^D)_{t \geq 0}$ is represented by

$$P_t^D f(x) := \mathbb{E}^x[f(X_t^D)] = \int_D f(y) p_D(t, x, y) dy.$$

Now we are ready to introduce main assumptions in this paper. Note that, under settings above, the infinitesimal generator can be rewritten as

$$(2.1) \quad Au(x) = \frac{1}{2} \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) + u(x-y) - 2u(x)) J(|y|) dy$$

for $u \in C^2(\mathbb{R}^n)$. Moreover, it is known in [9, Lemma 2.6] that (2.1) still holds for $u \in C^2(x) \cap C_0(\mathbb{R}^n)$. Recall that the operator L in (1.8) with kernel $J(|y|)$ is represented as

$$(2.2) \quad Lu(x) = \frac{1}{2} \int_{\mathbb{R}^n \setminus \{0\}} (u(x+y) + u(x-y) - 2u(x)) J(|y|) dy$$

for $u \in C^2(x) \cap B(\mathbb{R}^n)$ since J is symmetric. We record that $Au(x) = Lu(x)$ for any $u \in C^2(x) \cap C_0(\mathbb{R}^n)$ for the next use.

We first assume that the characteristic exponent Φ satisfies weak scaling condition with constants $a_1 \geq 1$ and $0 < \alpha_1 \leq \alpha_2 < 1$ so that

$$(2.3) \quad a_1^{-1} \left(\frac{R}{r}\right)^{2\alpha_1} \leq \frac{\Phi(R)}{\Phi(r)} \leq a_1 \left(\frac{R}{r}\right)^{2\alpha_2} \quad \text{for all } 1 < r \leq R \leq \infty.$$

We also assume that The Lévy measure of the isotropic unimodal pure jump Lévy process X has the density $y \rightarrow J(|y|)$ and it satisfies that there exists a constant $a_2 > 0$ such that

$$(2.4) \quad J(r+1) \leq a_2 J(r) \quad \text{for all } r > 0, \quad \text{and} \quad r \mapsto -\frac{J'(r)}{r} \quad \text{is non-increasing.}$$

Let

$$\varphi(r) := \frac{J(1)}{J(r)r^n}.$$

By [3], for any $c > 0$ we have $\Phi(r^{-1})^{-1} \asymp \varphi(r)$ in $0 < r \leq c$ with comparison constant depending only on c and n . Thus, there exists a constant $a_3 = a_3(n, a_1) \geq 1$ such that

$$(2.5) \quad a_3^{-1} \left(\frac{R}{r}\right)^{2\alpha_1} \leq \frac{\varphi(R)}{\varphi(r)} \leq a_3 \left(\frac{R}{r}\right)^{2\alpha_2} \quad \text{for all } 0 < r \leq R \leq 1,$$

where α_1 and α_2 are constants in (2.3). Note that (2.5) implies that $\varphi(r) \leq cr^{2\alpha_1}$ for $r \leq 1$, so by definition of φ we see that $J(|y|)dy$ is an infinite measure.

We say that $D \subset \mathbb{R}^d$ (when $d \geq 2$) is a $C^{1,1}$ open set with $C^{1,1}$ characteristics (R_0, Λ) if there exist a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$ there exist a $C^{1,1}$ -function $\varphi = \varphi_z : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \leq \Lambda$, $|\nabla\varphi(x) - \nabla\varphi(w)| \leq \Lambda|x-w|$ and an orthonormal coordinate system CS_z of $z = (z_1, \dots, z_{d-1}, z_d) := (\tilde{z}, z_d)$ with origin at z such that $D \cap B(z, R_0) = \{y = (\tilde{y}, y_d) \in B(0, R_0) \text{ in } CS_z : y_d > \varphi(\tilde{y})\}$. The pair (R_0, Λ) will be called the $C^{1,1}$ characteristics of the open set D . Note that a bounded $C^{1,1}$ open set D with characteristics (R_0, Λ) can be disconnected, and the distance between two distinct components of D is at least R_0 . By a $C^{1,1}$ open set in \mathbb{R} with a characteristic $R_0 > 0$, we mean an open set that can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least R_0 and the infimum of the distances between these intervals is at least R_0 .

Now, consider the following Dirichlet (exterior) problem on a bounded $C^{1,1}$ open set $D \subset \mathbb{R}^n$:

$$(2.6) \quad \begin{cases} Lu = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^n \setminus D, \end{cases}$$

where L is the operator in (2.2), which coincides with (1.10) when the process X is a subordinate Brownian motion. We will prove the following theorems, which contain Theorem 1.1 and 1.2 (See Remark 2.3 below), in Sections 3 and 4, respectively.

Theorem 2.1 (Hölder estimates up to the boundary). *Assume that D is a bounded $C^{1,1}$ open set in \mathbb{R}^n , and X is an isotropic pure jump Lévy process satisfying (2.3) and (2.4). If $f \in C(D)$, then there exists a unique viscosity solution u of (2.6) and $u \in C^{\bar{\phi}}(D)$. Moreover, we have*

$$\|u\|_{C^{\bar{\phi}}(D)} \leq C\|f\|_{C(D)},$$

where $\bar{\phi}(r) := \varphi(r)^{1/2}$, for some constant $C > 0$ depending only on n, D , and Φ .

Theorem 2.2 (Boundary estimates). *Assume that D is a bounded $C^{1,1}$ open set in \mathbb{R}^n , and X is an isotropic pure jump Lévy process satisfying (2.3) and (2.4). If $f \in C(D)$ and u is the viscosity solution of (2.6), then $u/V(d_D) \in C^\alpha(D)$ and*

$$\left\| \frac{u}{V(d_D)} \right\|_{C^\alpha(D)} \leq C\|f\|_{C(D)}$$

for some constants $\alpha > 0$ and $C > 0$ depending only on n, D , and Φ .

In the next remark, we explain that assumptions in Theorem 1.1 and Theorem 1.2 imply assumptions in Theorem 2.1 and Theorem 2.2.

Remark 2.3. *When X is a subordinate Brownian motion satisfying (1.6) and (1.7), we have (2.3) by using $\Phi(r) = \phi(r^2)$ and (1.6). We also have that by (1.3)*

$$J(r) = J_n(r) = \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{r^2}{4t}} \mu(dt).$$

Thus $J(r)$ is decreasing. Also, differentiating above equation we obtain

$$-\frac{J'_n(r)}{r} = 2\pi J_{n+2}(r), \quad r > 0,$$

so $-\frac{J(r)}{r}$ is decreasing. Therefore, (2.4) holds.

Note that by [3, Corollary 23] and (2.3) we have $\varphi(r) \asymp \Phi(r^{-1})^{-1}$. Using this and $\Phi(r) = \phi(r^2)$, both $\bar{\phi}$'s in Theorem 1.1 and Theorem 2.1 are comparable. Therefore, two $C^{\bar{\phi}}(D)$ norms are equivalent.

2.2. Renewal function. Let $Z = (Z_t)_{t \geq 0}$ be an one-dimensional Lévy process with characteristic exponent $\Phi(|z|)$ and $M_t := \sup\{Z_s : 0 \leq s \leq t\}$ be the supremum of Z . Let $L = (L_t)_{t \geq 0}$ be a local time of $M_t - Z_t$ at 0, which satisfies

$$L_t = \int_0^t \mathbf{1}_{\{M_t = Z_t\}}(s) ds.$$

Note that since $t \mapsto L_t$ is non-decreasing and continuous with probability 1, we can define the right-continuous inverse of L by

$$L^{-1}(t) := \inf\{s > 0 : L(s) > t\}.$$

The mapping $t \mapsto L^{-1}(t)$ is non-decreasing and right-continuous a.s. The process $L^{-1} = (L_t^{-1})_{t \geq 0}$ with $L_t^{-1} = L^{-1}(t)$ is called *the ascending ladder time process* of Z . *The ascending ladder height process* $H = (H_t)_{t \geq 0}$ is defined as

$$H_t := \begin{cases} M_{L_t^{-1}} (= Z_{L_t^{-1}}) & \text{if } L_t^{-1} < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

(See [17] for details.) Define the renewal function of the ladder height process H with respect to Φ by

$$(2.7) \quad V(x) = \int_0^\infty \mathbb{P}(H_s \leq x) ds, \quad x \in \mathbb{R}.$$

It is known that $V(x) = 0$ if $x \leq 0$, $V(\infty) = \infty$ and V is strictly increasing, differentiable on $[0, \infty)$. So, there exists the inverse function $V^{-1} : [0, \infty) \rightarrow [0, \infty)$.

In the following lemma we collect some basic scaling properties of renewal function in [3] and [4].

Lemma 2.4. *For any $c > 0$, There exist constants $C_i(c) = C_i(c, n, a_1, \alpha_1, \alpha_2) > 0$ for $i = 1, 2, 3$ such that*

$$(2.8) \quad C_1^{-1} \varphi(r) \leq V(r)^2 \leq C_1 \varphi(r), \quad 0 < r \leq c,$$

$$(2.9) \quad C_2^{-1} \left(\frac{R}{r}\right)^{\alpha_1} \leq \frac{V(R)}{V(r)} \leq C_2 \left(\frac{R}{r}\right)^{\alpha_2}, \quad 0 < r \leq R \leq c \quad \text{and}$$

$$(2.10) \quad C_3^{-1} \left(\frac{T}{t}\right)^{1/\alpha_2} \leq \frac{V^{-1}(T)}{V^{-1}(t)} \leq C_3 \left(\frac{T}{t}\right)^{1/\alpha_1}, \quad 0 < t \leq T < V(c).$$

Proof. By [3, Corollary 3] and [4, Proposition 2.4], we have

$$(V(r))^{-2} \asymp \Phi(r^{-1}), \quad r > 0.$$

with comparison constant depending only on n . Combining this with $\Phi(r^{-1})^{-1} \asymp \varphi(r)$ in $0 < r \leq c$, we conclude (2.8).

By (2.8) and (2.5) we have (2.9). Using [3, Remark 4], we also obtain the weak scaling property of the inverse function in (2.10). \square

The most important property of renewal function in this paper is the following: $w(x) := V(x_n)$ is a solution of the following Dirichlet problem :

$$(2.11) \quad \begin{cases} Lw = 0 & \text{in } \mathbb{R}_+^n, \\ w = 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}_+^n, \end{cases}$$

where L is of the form (2.2) and $\mathbb{R}_+^n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ is upper half plane (see [18, Theorem 3.3]).

The following estimates for derivatives of V are in [18, Proposition 3.1] and [24, Theorem 1.2].

Lemma 2.5. *Assume X is an isotropic pure jump Lévy process satisfying (2.3) and (2.4). Then $r \mapsto V(r)$ is twice-differentiable for any $r > 0$. Moreover, for any $c > 0$ there exists a constant $C(c) = C(c, n, a_1, \alpha_1, \alpha_2) > 0$ such that*

$$(2.12) \quad |V''(r)| \leq C \frac{V'(r)}{r \wedge c}, \quad V'(r) \leq C \frac{V(r)}{r \wedge c}.$$

We are going to utilize the space $C^V(D)$ in Section 3 and adopt $V(d_D)$ as a barrier in Section 4.

3. HÖLDER REGULARITY UP TO THE BOUNDARY

In this section, we give the proof of Theorem 2.1. First we introduce the following Dirichlet heat kernel estimates from [12, Corollary 1.6] and [25, Theorem 1.1 and 1.2]. We reformulate here for the usage of our proofs.

Theorem 3.1. *Let X be an isotropic unimodal Lévy process satisfying (2.3) and (2.4). Let $D \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ open set satisfying $\text{diam}(D) \leq 1$ and $p_D(t, x, y)$ be the Dirichlet heat kernel for X on D . Then $x \mapsto p_D(t, x, y)$ is differentiable for any $y \in D, t > 0$, and there exist constants $C_i = C_i(n, D, a_1, a_2, \alpha_1, \alpha_2, \Phi(1)) > 0, i = 1, \dots, 4$ satisfying the following estimates:*

(a) *For any $(t, x, y) \in (0, 1] \times D \times D$,*

$$p_D(t, x, y) \leq C_1 \left(1 \wedge \frac{V(d_D(x))}{t^{1/2}} \right) \left(1 \wedge \frac{V(d_D(y))}{t^{1/2}} \right) p(t, |x - y|/4)$$

and

$$|\nabla_x p_D(t, x, y)| \leq C_2 \left[\frac{1}{d_D(x) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{t})} \right] p_D(t, x, y).$$

(b) *For any $(t, x, y) \in [1, \infty) \times D \times D$,*

$$p_D(t, x, y) \leq C_3 e^{-\lambda_1 t} V(d_D(x)) V(d_D(y))$$

and

$$|\nabla_x p_D(t, x, y)| \leq C_4 \left[\frac{1}{d_D(x) \wedge 1} \vee \frac{1}{V^{-1}(1)} \right] p_D(t, x, y),$$

where $-\lambda_1 = -\lambda_1(n, a_1, a_2, \alpha_1, \alpha_2, \Phi(1)) < 0$ is the largest eigenvalue of the generator of $X^{B(0,1)}$.

In the estimates of Theorem 3.1, we used $d_D(x) \vee d_D(y) \leq \text{diam}(D) \leq 1, V(r) \asymp \varphi(r)^{1/2}$ in $0 < r \leq 1$ and $\frac{1}{V^{-1}(\sqrt{t})} \asymp \varphi^{-1}(t)$ to reformulate theorems in our references. In addition, estimates in [12, Corollary 1.6] are of the form

$$p_D(t, x, y) \leq c e^{-\lambda(D)t} V(d_D(x)) V(d_D(y))$$

where $-\lambda(D) < 0$ is the largest eigenvalue of the generator of X^D . Using [16, (6.4.14) and Lemma 6.4.5], we have $\lambda(D) = \inf\{\int_{\mathbb{R}^n} -Lu(x)u(x)dx \mid \|u\|_2 = 1, \text{supp}(u) \subset D\}$, thus we can obtain $\lambda_1 \leq \lambda(D)$. This implies heat kernel estimates in Theorem 3.1(b).

Without loss of generality, we will always assume $\text{diam}(D) \leq 1$ in this paper.

3.1. Potential operator for the killed process of subordinate Brownian motion. In this subsection, we assume that $D \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ open set with $\text{diam}(D) \leq 1$ and X is a Lévy process satisfying (2.3) and (2.4), which are conditions in Theorem 3.1. We define the *Green function of X^D* by

$$G^D(x, y) = \int_0^\infty p_D(t, x, y) dt$$

for $x, y \in D$ with $x \neq y$. Note that by Theorem 3.1(b), $G^D(x, y)$ is finite for any $x \neq y$.

We define a potential operator R^D for X^D as

$$(3.1) \quad R^D f(x) := \int_0^\infty \int_D p_D(t, x, y) f(y) dy dt.$$

Using definitions of P_t^D and G^D , we also have

$$(3.2) \quad R^D f(x) = \int_{D \setminus \{x\}} G^D(x, y) f(y) dy = \int_0^\infty P_t^D f(x) dt.$$

In the next subsection, we will see that R^D acts as the inverse of $-A$.

First we will prove interior Hölder estimate of $R^D f$. For the next usage, we prove the following proposition for the functions in $L^\infty(D)$.

Proposition 3.2. *For any $f \in L^\infty(D)$ and any ball $B(x_0, r) \subset D$ satisfying $d_D(x_0) \leq 2r$, we have $R^D f \in C^V(B/2)$ and there is a constant $C = C(n, a_1, a_2, \alpha_1, \alpha_2, D, \Phi(1)) > 0$ satisfying*

$$(3.3) \quad \|R^D f\|_{C^V(B/2)} \leq C (\|f\|_{L^\infty(D)} + \|R^D f\|_{C(B)})$$

Here we have denoted $B = B(x_0, r)$ and $B/2 = B(x_0, r/2)$.

Proof. We have $|x - y| < r$ for any $x, y \in B/2$. Thus, we have

$$\begin{aligned} [R^D f]_{C^V(B/2)} &\leq \sup_{|h| \leq r} \sup_{x \in B/2} \frac{|R^D f(x+h) - R^D f(x)|}{V(|h|)} \\ &\leq \sup_{|h| \leq r} \int_0^\infty \sup_{x \in B/2} \frac{|P_s^D f(x+h) - P_s^D f(x)|}{V(|h|)} ds \\ &\leq \sup_{|h| \leq r} \left(\int_0^{V(|h|)V(r)} + \int_{V(|h|)V(r)}^{V(r)^2} + \int_{V(r)^2}^\infty \right) \sup_{x \in B/2} \frac{|P_s^D f(x+h) - P_s^D f(x)|}{V(|h|)} ds \\ &=: \sup_{|h| \leq r} (\text{I} + \text{II} + \text{III}). \end{aligned}$$

To estimate I, we use $|P_s^D f(x)| \leq \|f\|_{L^\infty(D)}$ so that

$$(3.4) \quad \begin{aligned} \text{I} &= \int_0^{V(|h|)V(r)} \sup_{x \in B/2} \frac{|P_s^D f(x+h) - P_s^D f(x)|}{V(|h|)} ds \\ &\leq \int_0^{V(|h|)V(r)} \frac{2\|f\|_{L^\infty(D)}}{V(|h|)} ds \leq c_1 V(r) \|f\|_{L^\infty(D)}. \end{aligned}$$

To estimate II, we will use Theorem 3.1(a). Since $s \leq V(r)^2$ and $x \in B/2$, we obtain

$$\frac{1}{d_D(x) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{s})} \leq \frac{c_2}{V^{-1}(\sqrt{s})}.$$

Therefore, for $s \leq V(r)^2$ we have

$$|\nabla_x P_s^D f(x)| \leq c_3 \left(\frac{1}{d_D(x) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{s})} \right) \|P_s^D f\|_{L^\infty(D)} \leq \frac{c_2 c_3}{V^{-1}(\sqrt{s})} \|f\|_{L^\infty(D)}$$

for every $x \in D$. Here we used Theorem 3.1(a) for the first inequality. Using above inequality we conclude

$$(3.5) \quad \begin{aligned} \text{II} &= \int_{V(|h|)V(r)}^{V(r)^2} \sup_{x \in B/2} \frac{|P_s^D f(x+h) - P_s^D f(x)|}{V(|h|)} ds \\ &\leq \frac{|h|}{V(|h|)} \int_{V(|h|)V(r)}^{V(r)^2} \sup_{x \in B/2} |\nabla_x P_s^D f(x^*)| ds \\ &\leq c_2 c_3 \|f\|_{L^\infty(D)} \frac{|h|}{V(|h|)} \int_{V(|h|)V(r)}^{V(r)^2} \frac{1}{V^{-1}(\sqrt{s})} ds, \end{aligned}$$

where x^* is a point on the segment between x and $x+h$. Using change of variables with $s = V^2(t)$ in the first equality and Lemma 2.5 for the second inequality, we get

$$(3.6) \quad \int_{V(r)V(|h|)}^{V(r)^2} \frac{1}{V^{-1}(\sqrt{s})} ds = 2 \int_{V^{-1}(V(r)^{1/2}V(|h|)^{1/2})}^r \frac{V(t)V'(t)}{t} dt \leq c_4 \int_\varepsilon^r \frac{V(t)}{t} \frac{V(t)}{t} dt,$$

where $\varepsilon := V^{-1}(V(|h|)^{1/2}V(r)^{1/2})$. Also, by (2.9) we have

$$\frac{V(t)}{V(\varepsilon)} \leq c_5 \left(\frac{t}{\varepsilon}\right)^{\alpha_2} \leq c_5 \frac{t}{\varepsilon}, \quad t \geq \varepsilon$$

and

$$\int_0^r \frac{V(t)}{t} dt = \int_0^r \frac{V(r)}{t} \frac{V(t)}{V(r)} dt \leq c_6 V(r) \int_0^r \frac{1}{t} \left(\frac{t}{r}\right)^{\alpha_1} dt \leq c_7 V(r).$$

Using above two inequalities, we deduce from (3.6) that

$$(3.7) \quad \int_{V(r)V(|h|)}^{V(r)^2} \frac{1}{V^{-1}(\sqrt{s})} ds \leq c_4 \int_\varepsilon^r \frac{V(t)}{t} \frac{V(t)}{t} dt \leq c_8 \frac{V(\varepsilon)}{\varepsilon} \int_0^r \frac{V(t)}{t} dt \\ \leq c_9 V(r) \frac{V(\varepsilon)}{\varepsilon} = c_9 V(r) \frac{V(|h|)^{1/2}V(r)^{1/2}}{V^{-1}(V(|h|)^{1/2}V(r)^{1/2})}.$$

Combining (3.5) and (3.7), we conclude that

$$\begin{aligned} \text{II} &\leq c_{10} \|f\|_{L^\infty(D)} \frac{|h|}{V(|h|)} \cdot V(r) \frac{V(|h|)^{1/2}V(r)^{1/2}}{V^{-1}(V(|h|)^{1/2}V(r)^{1/2})} \\ &= c_{10} \|f\|_{L^\infty(D)} V(r) \frac{V(r)}{u} \frac{V^{-1}(u^2/V(r))}{V^{-1}(u)} \leq c_{11} \|f\|_{L^\infty(D)} V(r) \left(\frac{u}{V(r)}\right)^{\frac{1}{\alpha_2}-1} \leq c_{11} V(r) \|f\|_{L^\infty(D)}, \end{aligned}$$

where $u := V(h)^{1/2}V(r)^{1/2} \leq V(r)$. Here we used (2.10) and $\alpha_2 < 1$ for the second line.

For III, first note that for any $V(r)^2 \leq s \leq 1$,

$$\frac{1}{d_D(x) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{s})} \vee \frac{1}{V^{-1}(1)} \leq \frac{1}{r} \vee \frac{1}{V^{-1}(\sqrt{s})} \vee \frac{1}{V^{-1}(1)} \leq \frac{1}{r}.$$

So, by Theorem 3.1(a) we have for $V(r)^2 \leq s \leq 1$,

$$(3.8) \quad |\nabla_x p_D(s, x, y)| \leq \frac{c_{12}}{r} p_D(s, x, y) \leq \frac{c_{13}}{r} \left(1 \wedge \frac{V(d_D(x))}{s^{1/2}}\right) \left(1 \wedge \frac{V(d_D(y))}{s^{1/2}}\right) p(s, |x-y|/4) \\ \leq \frac{c_{14}}{r} \frac{V(r)}{\sqrt{s}} p(s, |x-y|/4).$$

Here in the second line we used $V(d_D(x)) \leq c_{15}V(r)$, which follows from (2.9) and $d_D(x) \leq 2r$. Thus, we obtain

$$(3.9) \quad |P_s^D f(x+h) - P_s^D f(x)| = |h| |\nabla_x P_s^D f(x_*)| \leq |h| \|f\|_{L^\infty(D)} \int_D |\nabla_x p_D(s, x^*, y)| dy \\ \leq c_{16} |h| \|f\|_{L^\infty(D)} \frac{V(r)}{r\sqrt{s}} \int_D p\left(s, \frac{|x^* - y|}{4}\right) dy \leq c_{17} |h| \|f\|_{L^\infty(D)} \frac{V(r)}{r\sqrt{s}},$$

where x_* is a point on the line segment between x and $x+h$. Here we used $\int_{\mathbb{R}^n} p(s, y/4) dy = 4^n$ for the last inequality.

For $s \geq 1$, using Theorem 3.1(b) we have

$$(3.10) \quad |\nabla_x p_D(s, x, y)| \leq \frac{c_{18}}{r} p_D(s, x, y) \leq \frac{c_{19}}{r} e^{-\lambda_1 s} V(d_D(x)) V(d_D(y)) \leq \frac{c_{20} V(r)}{r} e^{-\lambda_1 s},$$

Here we used $d_D(x) \leq 2r$, $d_D(y) \leq 1$ and (2.9) in the last inequality. Thus we arrive

$$(3.11) \quad |P_s^D f(x+h) - P_s^D f(x)| = |h| |\nabla_x P_s^D f(x_*)| \leq |h| \|f\|_{L^\infty(D)} \int_D |\nabla_x p_D(s, x^*, y)| dy \\ \leq c_{21} |h| \|f\|_{L^\infty(D)} \frac{V(r)}{r} \int_D e^{-\lambda_1 s} dy \leq c_{22} |h| \|f\|_{L^\infty(D)} \frac{V(r)}{r} e^{-\lambda_1 s},$$

where x_* is a point on the line segment between x and $x+h$.

Now combining (3.9) and (3.11), we obtain

$$\begin{aligned}
 \text{III} &= \int_{V(r)^2}^{\infty} \frac{|P_s^D f(x+h) - P_s^D f(x)|}{V(|h|)} ds = \left(\int_{V(r)^2}^1 + \int_1^{\infty} \right) \frac{|P_s^D f(x+h) - P_s^D f(x)|}{V(|h|)} ds \\
 (3.12) \quad &\leq c_{23} \frac{V(r)}{r} \frac{|h|}{V(|h|)} \|f\|_{L^\infty(D)} \left(\int_{V(r)^2}^1 \frac{1}{\sqrt{s}} ds + \int_1^{\infty} e^{-\lambda_1 s} ds \right) \\
 &\leq c_{24} \|f\|_{L^\infty(D)} (2 - 2V(r) + \lambda_1^{-1}).
 \end{aligned}$$

The last inequality follows from $\frac{V(r)}{V(|h|)} \leq c_{25} \left(\frac{r}{|h|}\right)^{\alpha_2} \leq c_{25} \left(\frac{r}{|h|}\right)$ since $|h| \leq r$.

Combining (3.4), (3.7) and (3.12), we conclude

$$[Rf]_{C^V(B/2)} \leq c_{26} (1 + V(r)) \|f\|_{L^\infty(D)} \leq c_{26} (1 + V(1)) \|f\|_{L^\infty(D)}.$$

Above inequality and that $\|Rf\|_{C^V(B/2)} = [Rf]_{C^V(B/2)} + \|Rf\|_{C(B/2)}$ finish the proof. \square

We next provide an upper bound of $R^D f$ near the boundary. In the proof we apply the estimates on the Green function in [18, Theorem 1.6].

Lemma 3.3. *There exists a constant $C = C(n, a_1, a_2, \alpha_1, \alpha_2, D, \Phi(1)) > 0$ such that*

$$|R^D f(x)| \leq C \|f\|_{L^\infty(D)} V(\text{diam}(D)) V(d_D(x))$$

for any $f \in L^\infty(D)$ and $x \in D$.

Proof. The estimate on the Green function in [18, Theorem 1.6] and (2.8) give that for any $x, y \in D$,

$$\begin{aligned}
 (3.13) \quad G^D(x, y) &\leq c_1 \frac{\varphi(|x-y|)}{|x-y|^n} \left(1 \wedge \frac{\varphi(d_D(x))}{\varphi(|x-y|)} \right)^{1/2} \left(1 \wedge \frac{\varphi(d_D(y))}{\varphi(|x-y|)} \right)^{1/2} \\
 &\leq c_1 \frac{\varphi(|x-y|)^{1/2}}{|x-y|^n} \varphi(d_D(x))^{1/2} \leq c_2 \frac{V(|x-y|)}{|x-y|^n} V(d_D(x)).
 \end{aligned}$$

Substituting (3.13) to (3.2) we obtain

$$(3.14) \quad |R^D f(x)| \leq c_3 \|f\|_{L^\infty(D)} V(d_D(x)) \int_D \frac{V(|x-y|)}{|x-y|^n} dy.$$

Also, using (2.9) we have

$$\begin{aligned}
 (3.15) \quad \int_D \frac{V(|x-y|)}{|x-y|^n} dy &\leq \int_{B(x, \text{diam}(D))} \frac{V(|x-y|)}{|x-y|^n} dy \leq c_4 \int_0^{\text{diam}(D)} \frac{V(r)}{r} dr \\
 &\leq c_5 \frac{V(\text{diam}(D))}{\text{diam}(D)^{\alpha_1}} \int_0^{\text{diam}(D)} r^{\alpha_1-1} dr \leq c_6 V(\text{diam}(D)).
 \end{aligned}$$

Combining above two inequalities we have proved the lemma. \square

Remark 3.4. *As a corollary of Lemma 3.3, we have*

$$\|R^D f\|_{L^\infty(D)} \leq C \|f\|_{L^\infty(D)}.$$

Hence we can simplify (3.3) to

$$(3.16) \quad \|R^D f\|_{C^V(B/2)} \leq \tilde{C} \|f\|_{L^\infty(D)}$$

for some constant $\tilde{C} = \tilde{C}(n, a_1, a_2, \alpha_1, \alpha_2, D, \Phi(1)) > 0$.

Now we are ready to prove Theorem 2.1 for the function $R^D f$.

Proposition 3.5. *Assume $f \in L^\infty(D)$. Then, $R^D f \in C^V(D)$ and there exists a constant $C > 0$ such that*

$$(3.17) \quad \|R^D f\|_{C^V(D)} \leq C \|f\|_{L^\infty(D)}.$$

The constant $C > 0$ depends only on $n, a_1, a_2, \alpha_1, \alpha_2, D$ and $\Phi(1)$.

Proof. By (3.16) we have

$$(3.18) \quad |R^D f(x) - R^D f(y)| \leq c_1 \|f\|_{L^\infty(D)} V(|x - y|)$$

for all x, y satisfying $|x - y| < d_D(x)/2$. We want to show that (3.18) holds, perhaps with a bigger constant, for all $x, y \in D$.

Let (R_0, Λ) be the $C^{1,1}$ characteristics of D . Then D can be covered by finitely many balls of the form $B(z_i, d_D(z_i)/2)$ with $z_i \in D$ and finitely many sets of the form $B(z_j^*, R_0) \cap D$ with $z_j^* \in \partial D$. Thus, it is enough to show that (3.18) holds for all $x, y \in B(z_j^*, R_0) \cap D$ possibly with a larger constant.

Fix $B(z_0^*, R_0) \cap D$ and assume that the outward normal vector at z_0 is $(0, \dots, 0, -1)$. This is possible because the operator is invariant under the rotation. Now let $x = (x', x_n)$ and $y = (y', y_n)$ be two points in $B(z_0^*, R_0) \cap D$, and let $r = |x - y|$. Let us define for $k \geq 0$

$$x^k = (x', x_n + \lambda^k r) \quad \text{and} \quad y^k = (y', y_n + \lambda^k r),$$

for some $1 - 2^{-1}(1 + \Lambda^2)^{-1/2} \leq \lambda < 1$. Since $(1 + \Lambda^2)^{-1/2}(x^k)_n \leq d_D(x^k)$, we have

$$|x^k - x^{k+1}| = \lambda^k(1 - \lambda)r \leq \frac{1}{2\sqrt{1 + \Lambda^2}}(x^k)_n \leq \frac{1}{2}d_D(x^k).$$

Thus, we have from (3.18) that

$$|R^D f(x^k) - R^D f(x^{k+1})| \leq c_1 \|f\|_{L^\infty(D)} V(|x^k - x^{k+1}|) = c_1 \|f\|_{L^\infty(D)} V(\lambda^k(1 - \lambda)r)$$

and similarly that $|R^D f(y^k) - R^D f(y^{k+1})| \leq c_1 \|f\|_{L^\infty(D)} V(\lambda^k(1 - \lambda)r)$. Moreover, note that the distance from the line segment joining x^0 and y^0 to the boundary ∂D is more than $r(1 - \Lambda/2)$. Thus, this line can be split into finitely many line segments of length less than $r(1 - \Lambda/2)/2$. The number of small line segments depends only on Λ . Therefore, we have $|R^D f(x^0) - R^D f(y^0)| \leq c_2 \|f\|_{L^\infty(D)} V(r)$ and hence

$$\begin{aligned} & |R^D f(x) - R^D f(y)| \\ & \leq |R^D f(x^0) - R^D f(y^0)| + \sum_{k \geq 0} (|R^D f(x^k) - R^D f(x^{k+1})| + |R^D f(y^k) - R^D f(y^{k+1})|) \\ & \leq c_3 \|f\|_{L^\infty(D)} (V(r) + \sum_{k \geq 0} V(\lambda^k(1 - \lambda)r)) \\ & \leq c_4 \|f\|_{L^\infty(D)} V(r) \left(1 + c_5 \sum_{k \geq 0} (\lambda^k(1 - \lambda))^{\alpha_1} \right) \\ & \leq c_6 \|f\|_{L^\infty(D)} V(r). \end{aligned}$$

Recall that $r = |x - y|$. This finishes the proof. \square

In the next subsection, we will prove that the function $u = -R^D f$ is the unique viscosity solution for (2.6) when $f \in C(D)$.

3.2. Nonlocal operator and infinitesimal generator. In this section we establish the relation between viscosity solutions of (2.6) and solutions of the following:

$$(3.19) \quad \begin{cases} Au = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^n \setminus D. \end{cases}$$

In [9], the authors discussed the relation between operators A and L , for instance, domain or values of the operators; see [9] for the application to heat equations.

At the beginning of this section we apply the strategies in [9] to our settings and obtain some related properties. After then, we obtain comparison principle for the viscosity solution. Combining these results, we finally obtain the existence and uniqueness for Dirichlet problems (2.6) and (3.19). Moreover, these two solutions coincide under some conditions. Also, in Section 4.2 we obtain Harnack inequality, which is one of the key ingredients for the standard argument of Krylov in [26]. In Section 4.3 we will make use of Harnack inequality and the comparison principle

to prove Theorem 2.2.

Let $D \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ open set and let

$$\mathcal{D} = \mathcal{D}(D) := \{u \in C_0(D) : Au \in C(D)\}$$

be the domain of operator A . Recall that by [9, Lemma 2.6] we have

$$(3.20) \quad Au(x) = Lu(x)$$

for any $u \in C^2(x) \cap C_0(\mathbb{R}^n)$, $x \in D$. We first show that $u = -R^D f$ satisfies (3.19) when f is continuous.

Lemma 3.6. *Let $f \in C(D)$ and define $u = -R^D f$. Then, u is a solution for (3.19).*

Proof. First we claim that for any $u \in C_0(D)$ and $x \in D$,

$$(3.21) \quad Au(x) = \lim_{t \downarrow 0} \frac{P_t^D u(x) - u(x)}{t}.$$

To show (3.21), we follow the proof in [9, Theorem 2.3]. Note that our domain of operator is slightly different from it in [9, (2.8)].

We first observe that for any $u \in \mathcal{D}$ and $x \in D$,

$$\begin{aligned} P_t^D u(x) - P_t u(x) &= \mathbb{E}^x u(X_t^D) - \mathbb{E}^x u(X_t) \\ &= \mathbb{E}^x [u(X_t^D) \mathbf{1}_{\{\tau_D \geq t\}}] - \mathbb{E}^x [u(X_t) \mathbf{1}_{\{\tau_D \geq t\}}] - \mathbb{E}^x [u(X_t) \mathbf{1}_{\{\tau_D < t\}}] \\ &= -\mathbb{E}^x [u(X_t) \mathbf{1}_{\{\tau_D < t\}}]. \end{aligned}$$

Indeed, the first and the third term in the second line cancel. Hence

$$(3.22) \quad \frac{P_t^D u(x) - u(x)}{t} - \frac{P_t u(x) - u(x)}{t} = -\frac{\mathbb{E}^x [u(X_t) \mathbf{1}_{\{\tau_D < t\}}]}{t} = \frac{\mathbb{E}^x [(u(X_{\tau_D}) - u(X_t)) \mathbf{1}_{\{\tau_D < t\}}]}{t}.$$

Meanwhile, by the strong Markov property we obtain

$$|\mathbb{E}^x [(u(X_{\tau_D}) - u(X_t)) \mathbf{1}_{\{\tau_D < t\}}]| \leq \mathbb{E}^x [|\mathbb{E}^{X_{\tau_D}} [u(X_0) - u(X_{t-\tau_D})]| \mathbf{1}_{\{\tau_D < t\}}].$$

Since $u \in C_0(D)$ is uniformly continuous, with stochastic continuity of Lévy process we have that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$|\mathbb{E}^z [u(X_s)] - u(z)| < \varepsilon$$

for any $z \in D$ and $0 < s \leq \delta$. Combining above two equations we conclude

$$|\mathbb{E}^x [(u(X_{\tau_D}) - u(X_t)) \mathbf{1}_{\{\tau_D < t\}}]| \leq \varepsilon \mathbb{P}^x(\tau_D < t)$$

for $0 < t \leq \delta$. Since D is open, for any $x \in D$ we have a constant $r_x > 0$ such that $B(x, r_x) \subset D$. Using [10, Theorem 5.1 and Proposition 2.27(d)] there exists some $M > 0$ such that

$$\frac{\mathbb{P}^x(\tau_D < t)}{t} \leq \frac{\mathbb{P}^x(\tau_{B(x, r_x)} < t)}{t} \leq M \quad \text{for all } t > 0.$$

Combining above inequalities we obtain that

$$\begin{aligned} \lim_{t \downarrow 0} \left| \frac{P_t^D u(x) - u(x)}{t} - Au(x) \right| &= \lim_{t \downarrow 0} \left| \frac{P_t^D u(x) - u(x)}{t} - \frac{P_t u(x) - u(x)}{t} \right| \\ &\leq \varepsilon \lim_{t \downarrow 0} \frac{\mathbb{P}^x[\tau_D < t]}{t} \leq \varepsilon M. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily, this concludes the claim.

Now we prove the lemma. Note that $u = 0$ in D^c immediately follows from the definition of R^D . Then, by (3.21) and (3.2) we have that for $x \in D$,

$$\begin{aligned}
(3.23) \quad Au(x) &= A(-R^D f)(x) = -\lim_{t \downarrow 0} \frac{P_t^D(R^D f)(x) - R^D f(x)}{t} \\
&= -\lim_{t \downarrow 0} \frac{1}{t} \left[P_t^D \left(\int_0^\infty P_s^D f(\cdot) ds \right) (x) - \int_0^\infty P_s^D f(x) ds \right] \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left(-\int_0^\infty P_{t+s}^D f(x) ds + \int_0^\infty P_s^D f(x) ds \right) \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left(-\int_t^\infty P_s^D f(x) ds + \int_0^\infty P_s^D f(x) ds \right) \\
&= \lim_{t \downarrow 0} \frac{\int_0^t P_s^D f(x) ds}{t} = f(x).
\end{aligned}$$

Indeed, the third line follows from the semigroup property $P_s^D P_t^D = P_{s+t}^D$ and that $R^D f \in C_0(D)$ which follows from Proposition 3.2. This finishes the proof. \square

The next lemma shows that every solution of (3.19) is a viscosity solution of (2.6).

Lemma 3.7. *Assume that $f \in C(D)$ and $u \in \mathcal{D}$ satisfies $Au = f$ in D . Then, u is a viscosity solution of $Lu = f$.*

Proof. For any $x_0 \in D$ and test function $v \in C^2(\mathbb{R}^n)$ with $v(x_0) = u(x_0)$ and $v(y) > u(y)$ for $y \in \mathbb{R}^n \setminus \{x_0\}$, we have

$$Av(x_0) = Lv(x_0).$$

Since $v(x_0) = u(x_0)$ and $P_t^D v(x_0) \geq P_t^D u(x_0)$ for every $t > 0$, we have

$$Av(x_0) = \lim_{t \downarrow 0} \frac{P_t^D v(x_0) - v(x_0)}{t} \geq \lim_{t \downarrow 0} \frac{P_t^D u(x_0) - u(x_0)}{t} = Au(x_0).$$

Thus, we arrive

$$Lv(x_0) \geq Au(x_0),$$

which concludes that u is a viscosity solution of (2.6). \square

Now we see comparison principle in [14]. This implies the uniqueness of viscosity solution for (2.6).

Theorem 3.8 (Comparison principle). *Let D be a bounded open set in \mathbb{R}^n . Let u and v be bounded functions satisfying $Lu \geq f$ and $Lv \leq f$ in D in viscosity sense for some continuous function f , and let $u \leq v$ in $\mathbb{R}^n \setminus D$. Then $u \leq v$ in D .*

Proof. We first claim that L satisfies [14, Assumption 5.1]. More precisely, there exists constant $r_0 \geq 1$ such that for every $r \geq r_0$, there exists a constant $\delta = \delta(r) > 0$ satisfying $Lw > \delta$ in B_r , where $w(x) = 1 \wedge \frac{|x|^2}{r^3}$.

Let $r_0 = 4$, $r \geq 4$ and $x \in B_r$. Note that by $r \geq 4$ we have

$$\frac{|y|^2}{r^3} \leq \frac{4r^2}{r^3} \leq 1, \quad y \in B_{2r}.$$

Thus, for $y \in B_r$ we obtain

$$w(x+y) + w(x-y) - 2w(x) = \frac{|x+y|^2 + |x-y|^2 - 2|x|^2}{r^3} = \frac{2|y|^2}{r^3}.$$

On the other hand, for $y \in B_r^c$ we have

$$w(x+y) + w(x-y) - 2w(x) \geq \frac{2|y|^2}{r^3} \wedge (1 - 2w(x)) > 0.$$

Therefore, since $w \in C^2(\mathbb{R}^d)$ we have

$$\begin{aligned} Lw(x) &:= \frac{1}{2} \int_{\mathbb{R}^n} (w(x+y) + w(x-y) - 2w(x)) J(y) dy \\ &= \frac{1}{2} \int_{B_r} (w(x+y) + w(x-y) - 2w(x)) J(y) dy + \frac{1}{2} \int_{B_r^c} (w(x+y) + w(x-y) - 2w(x)) J(y) dy \\ &\geq \frac{1}{r^3} \int_{B_r} |y|^2 J(y) dy =: \delta(r) > 0 \end{aligned}$$

for every $r \geq r_0 = 4$ and $x \in B_r$. Since L satisfies [14, Assumption 5.1], we can apply Theorem 5.2 therein, which proves the theorem. \square

The following uniqueness of viscosity solution is immediate.

Corollary 3.9. *Let D be a bounded open set in \mathbb{R}^n and let $f \in C(D)$. Then there is at most one viscosity solution of (2.6).*

Here is the main result in this section.

Theorem 3.10. *Assume that $f \in C(D)$. Then, $u = -R^D f \in \mathcal{D}$ is the unique solution of (3.19). Also, u is the unique viscosity solution of (2.6).*

Proof. By Lemma 3.6, we have that $u = -R^D f \in \mathcal{D}$ is solution of (3.19). Now, Lemma 3.7 and Corollary 3.9 conclude the proof. \square

Proof of Theorem 2.1 By Theorem 3.10, the unique viscosity solution for (2.6) is given by $u = -R^D f$. Therefore, Proposition 3.5 yields the Hölder regularity of viscosity solution with respect to C^V -norm. By (2.8), we have $V \asymp \bar{\phi}$ and this concludes the proof. \square

4. BOUNDARY REGULARITY

4.1. Barriers. Throughout this section, $D \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ open set. Without loss of generality, we assume that $\text{diam}(D) \leq 1$. Since d_D is only $C^{1,1}$ near ∂D , we need to consider the following “regularized version” of d_D .

Definition 4.1. *We call $\psi : D \rightarrow (0, \infty)$ the regularized version of d_D if $\psi \in C^{1,1}(D)$ and it satisfies*

$$(4.1) \quad \tilde{C}^{-1} d_D(x) \leq \psi(x) \leq \tilde{C} d_D(x), \quad \|\nabla \psi(x)\| \leq \tilde{C} \quad \text{and} \quad \|\nabla \psi(x) - \nabla \psi(y)\| \leq \tilde{C}|x - y|$$

for any $x, y \in D$, where the constant $\tilde{C} > 0$ depends only on D .

For $D = B(0, 1)$, there exists a regularized version of $d_{B(0,1)}$ which is C^2 and isotropic. Denote this function by Ψ and let $C = C(n)$ be the constant in (4.1) for the function Ψ . For any open ball $B_r := B(x_0, r)$, we will take the regularized version of d_{B_r} which is defined by $\Psi_r(x) := \Psi(\frac{x-x_0}{r})$. Then, Ψ_r satisfies

$$(4.2) \quad C^{-1} d_{B_r}(x) \leq \Psi_r(x) \leq C d_{B_r}(x), \quad \|\nabla \Psi_r\| \leq C \quad \text{and} \quad \|\nabla^2 \Psi_r(x)\| \leq \frac{C}{r}$$

for any $x, y \in B(x_0, r)$. The last estimate follows from the fact that $\Psi \in C^2(B_r)$.

We first introduce the following three lemmas which will be used to construct a barrier for L .

Lemma 4.2. *Assume that D is a bounded $C^{1,1}$ open set and let ψ be a regularized version of d_D . Then, for every $x \in \mathbb{R}^n$ and $x_0 \in D$ we have*

$$(4.3) \quad |\psi(x) - (\psi(x_0) + \nabla \psi(x_0) \cdot (x - x_0))_+| \leq \tilde{C}|x - x_0|^2$$

where \tilde{C} is the constant in (4.1). In addition, when $D = B(0, r)$ and $\psi = \Psi_r$ we have (4.3) with $\tilde{C} = \frac{C}{r}$ where C is the constant in (4.2).

Proof. Let $\tilde{\psi}$ be a $C^{1,1}$ extension of $\psi|_D$ satisfying $\tilde{\psi} \leq 0$ in $\mathbb{R}^n \setminus D$. Then, since $\tilde{\psi} \in C^{1,1}(\mathbb{R}^n)$ we clearly have

$$(4.4) \quad |\tilde{\psi}(x) - \psi(x_0) - \nabla\psi(x_0) \cdot (x - x_0)| = |\tilde{\psi}(x) - \tilde{\psi}(x_0) - \nabla\tilde{\psi} \cdot (x - x_0)| \leq \tilde{C}|x - x_0|^2$$

in all of $x \in \mathbb{R}^n$. Using $|a_+ - b_+| \leq |a - b|$ and $(\tilde{\psi})_+ = \psi$, we have

$$|\psi(x) - (\psi(x_0) + \nabla\psi(x_0) \cdot (x - x_0))_+| \leq |\tilde{\psi}(x) - \psi(x_0) - \nabla\psi(x_0) \cdot (x - x_0)| \leq \tilde{C}|x - x_0|^2$$

for all $x \in \mathbb{R}^n$. If $D = B(0, r)$ and $\psi = \Psi_r$, the constant \tilde{C} in (4.4) become $\frac{C}{r}$. Thus, the conclusion of lemma follows. \square

Next lemma is a collection of inequalities which will be used for this section. Note that we can easily check these inequalities when $\varphi(r) = r^{2\alpha}$ and $V(r) = r^\alpha$ with $0 < \alpha < 1$. The inequalities (4.6) and (4.8) are in [4, Lemma 3.5]. We provide the proof for the completeness.

Lemma 4.3. *There exists a constant $C_1 = C_1(n, a_1, \alpha_1, \alpha_2) > 0$ such that for any $0 < r \leq 1$,*

$$(4.5) \quad \int_0^r \frac{s}{\varphi(s)} ds \leq \frac{C_1 r^2}{\varphi(r)},$$

$$(4.6) \quad \int_r^\infty \frac{1}{s\varphi(s)} ds \leq \frac{C_1}{\varphi(r)},$$

$$(4.7) \quad \int_0^r \frac{1}{V(s)} ds \leq \frac{C_1 r}{V(r)}, \quad \int_0^r \frac{V(s)}{s} ds \leq C_1 V(r)$$

and

$$(4.8) \quad \int_r^\infty \frac{V(s)}{s\varphi(s)} ds \leq \frac{C_1}{V(r)}.$$

Proof. The inequalities (4.5) and (4.7) can be proved using weak scaling conditions (2.5) and (2.9): by (2.5), we have

$$\int_0^r \frac{s}{\varphi(s)} ds = \int_0^r \frac{s}{\varphi(r)} \frac{\varphi(r)}{\varphi(s)} ds \leq c_1 \int_0^r \frac{s}{\varphi(r)} \left(\frac{r}{s}\right)^{2\alpha_2} ds = \frac{c_1}{2 - 2\alpha_2} \frac{r^2}{\varphi(r)},$$

and by (2.9) we have

$$\int_0^r \frac{1}{V(s)} ds = \int_0^r \frac{1}{V(r)} \frac{V(r)}{V(s)} ds \leq \int_0^r c_2 \left(\frac{r}{s}\right)^{\alpha_2} ds = \frac{c_2}{1 - \alpha_2} \frac{r}{V(r)}$$

and

$$\int_0^r \frac{V(s)}{s} ds = \int_0^r \frac{V(r)}{s} \frac{V(s)}{V(r)} ds \leq \int_0^r \frac{V(r)}{s} c_2 \left(\frac{s}{r}\right)^{\alpha_1} ds = \frac{c_2}{\alpha_1} V(r).$$

Let $\mathcal{P}(r) := \int_{\mathbb{R}} (1 \wedge \frac{|x|^2}{r^2}) J(x) dx$ be the Pruitt function of X . By [3, (6) and Lemma 1] and (2.8), we have a constant $c_3 > 0$ satisfying

$$(4.9) \quad \mathcal{P}(r) \leq c\varphi(r)^{-1} \leq c_3 V(r)^{-2}, \quad r > 0.$$

Let $\mathcal{P}_1(r) := \int_r^\infty \frac{1}{s\varphi(s)} ds$. Note that we have

$$(4.10) \quad \mathcal{P}_1(r) = \omega_n^{-1} \int_{B(0,r)^c} \left(1 \wedge \frac{|x|^2}{r^2}\right) J(|x|) dx \leq \omega_n^{-1} \mathcal{P}(r) \leq c_4 V(r)^{-2}, \quad r > 0.$$

Thus, (4.10) and (2.8) imply (4.6). Also, using integration by parts and (4.10) we have

$$\begin{aligned} \int_r^\infty \frac{V(s)}{s\varphi(s)} ds &= \int_r^\infty V(s) d(-\mathcal{P}_1)(s) \\ &= V(r)\mathcal{P}_1(r) - \lim_{s \rightarrow \infty} V(s)\mathcal{P}_1(s) + \int_r^\infty V'(s)\mathcal{P}_1(s) ds \\ &\leq c_5 \left(\frac{1}{V(r)} - \lim_{s \rightarrow \infty} \frac{1}{V(s)} + \int_r^\infty \frac{V'(s)}{V(s)^2} ds \right) = \frac{2c_5}{V(r)}, \end{aligned}$$

which concludes (4.8). \square

Lemma 4.4. *Let $U \subset \mathbb{R}^n$ be a $C^{1,1}$ open set, which can be unbounded. Then there exists a constant $C_2 = C_2(n, U, a_1, a_2, \alpha_1, \alpha_2) > 0$ such that for any $x \in U$ and $0 < r \leq 1$,*

$$(4.11) \quad \int_{U \cap (B(x,r) \setminus B(x, d_U(x)/2))} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \leq \frac{C_2 r}{V(r)}.$$

Proof. Fix $x \in U$ and denote $\rho := d_U(x) < 2r$, $B_r := B(x, r)$ for $r > 0$ and $B_r = \emptyset$ for $r \leq 0$. First note that there is a constant $\kappa = \kappa(U) > 0$ such that the level set $\{d_U \geq t\} = \{x \in U \mid d_U(x) \geq t\}$ is $C^{1,1}$ for any $t \in (0, \kappa]$ since U is $C^{1,1}$. Without loss of generality we can assume $\kappa \leq r$ because κ can be arbitrarily small.

Since $B_R \cap \{d_U \geq \kappa\} = \emptyset$ for every $R \leq \kappa - \rho$, we have

$$\begin{aligned} & \int_{(B_r \setminus B_{\rho/2}) \cap \{d_U \geq \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \\ &= \int_{(B_r \setminus B_{\max\{\rho/2, \kappa-\rho\}}) \cap \{d_U \geq \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \\ &\leq \int_{(B_r \setminus B_{2\kappa/3}) \cap \{d_U \geq \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)}, \end{aligned}$$

where the last line follows from $\rho/2 \vee (\kappa - \rho) \geq \frac{2\kappa}{3}$. Using

$$\kappa \leq d_U(y) \leq r + \kappa \leq 2r \quad \text{and} \quad \frac{2\kappa}{3} \leq |x-y| \leq r$$

for every $y \in (B_r \setminus B_{2\kappa/3}) \cap \{d_U \geq \kappa\}$, we arrive that for any $x \in U$,

$$(4.12) \quad \begin{aligned} & \int_{(B_r \setminus B_{2\kappa/3}) \cap \{d_U \geq \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \\ &\leq \int_{(B_r \setminus B_{2\kappa/3}) \cap \{d_U \geq \kappa\}} \frac{V(2r)}{\kappa} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \\ &\leq c_1 \frac{V(r)}{\kappa} \int_0^r \frac{s}{\varphi(s)} ds \leq c_2(\kappa) \frac{r^2}{V(r)} \leq c_2(\kappa) \frac{r}{V(r)}, \end{aligned}$$

where we used (2.8) and (4.5) for the second last inequality. Thus, it suffices to estimate the integrand (4.11) in the set $(B_r \setminus B_{\rho/2}) \cap \{0 < d_U < \kappa\}$.

We will utilize the following estimates on Hausdorff measure in [RV15], that is, there exists a constant $c_3(U) > 0$ such that that for every $x \in U$ and $t \in (0, \kappa)$,

$$(4.13) \quad \mathcal{H}^{n-1}(\{d_U = t\} \cap (B_{2^{-k+1}r} \setminus B_{2^{-k}r})) \leq c_3(2^{-k}r)^{n-1}$$

which follows from the fact that the level set $\{d_U = t\}$ is $C^{1,1}$ for $t \in (0, \kappa)$.

Let us denote $C_n := B_{r2^{-n}}$ for $n \geq 0$ and let $M \in \mathbb{N}$ be the natural number satisfying $2^{-M}r \leq \rho/2 \leq 2^{-M+1}r$. Using $|x-y| \geq 2^{-k}r$ for every $y \in C_{k-1} \setminus C_k$ and φ is increasing for the third line, we have

$$\begin{aligned} & \int_{(B_r \setminus B_{\rho/2}) \cap \{0 < d_U < \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \\ &\leq \sum_{k=1}^M \int_{(C_{k-1} \setminus C_k) \cap \{0 < d_U < \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \\ &\leq \sum_{k=1}^M \frac{1}{(2^{-k}r)^{n-2} \varphi(2^{-k}r)} \int_{(C_{k-1} \setminus C_k) \cap \{0 < d_U < \kappa\}} \frac{V(d_U(y))}{d_U(y)} dy \\ &= \sum_{k=1}^M \frac{1}{(2^{-k}r)^{n-2} \varphi(2^{-k}r)} \int_{(C_{k-1} \setminus C_k) \cap \{0 < d_U < \kappa\}} \frac{V(d_U(y))}{d_U(y)} |\nabla d_U(y)| dy. \end{aligned}$$

Here we used $|\nabla d_U(y)| = 1$ for $y \in \{0 < d_U < \kappa\}$ for the last line. (See [31].) For any $1 \leq k \leq M$ and $y \in C_{k-1}$ we have $d_U(y) \leq 2^{-k+1}r + \rho \leq (2^{-k+1} + 2^{-M+2})r \leq 6 \cdot 2^{-k}r$, which implies $C_{k-1} \subset \{d_U < 6 \cdot 2^{-k}r\}$. Thus, combining this with above inequality we have

$$(4.14) \quad \begin{aligned} & \int_{(B_r \setminus B_{\rho/2}) \cap \{0 < d_U < \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \\ & \leq \sum_{k=1}^M \frac{1}{(2^{-k}r)^{n-2} \varphi(2^{-k}r)} \int_{(C_{k-1} \setminus C_k) \cap \{0 < d_U < 6 \cdot 2^{-k}r\}} \frac{V(d_U(y))}{d_U(y)} |\nabla d_U(y)| dy. \end{aligned}$$

Plugging $u(y) = d_U(y)$ and $g(y) = \frac{V(d_U(y))}{d_U(y)}$ into the following coarea formula

$$\int_D g(y) |\nabla u(y)| dy = \int_{-\infty}^{\infty} \left(\int_{u^{-1}(t)} g(y) d\mathcal{H}_{n-1}(y) \right) dt,$$

we obtain

$$(4.15) \quad \begin{aligned} & \sum_{k=1}^M \frac{1}{(2^{-k}r)^{n-2} \varphi(2^{-k}r)} \int_{(C_{k-1} \setminus C_k) \cap \{0 < d_U < 6 \cdot 2^{-k}r\}} \frac{V(d_U(y))}{d_U(y)} |\nabla d_U(y)| dy \\ & = \sum_{k=1}^M \frac{1}{(2^{-k}r)^{n-2} \varphi(2^{-k}r)} \int_0^{6 \cdot 2^{-k}r} \int_{(C_{k-1} \setminus C_k) \cap \{d=t\}} \frac{V(t)}{t} d\mathcal{H}^{n-1}(y) dt \\ & \leq \sum_{k=1}^M \frac{1}{(2^{-k}r)^{n-2} \varphi(2^{-k}r)} \int_0^{6 \cdot 2^{-k}r} c_3 (2^{-k}r)^{n-1} \frac{V(t)}{t} dt \\ & = c_3 \sum_{k=1}^M \frac{2^{-k}r}{\varphi(2^{-k}r)} \int_0^{6 \cdot 2^{-k}r} \frac{V(t)}{t} dt \leq c_4 \sum_{k=1}^M \frac{2^{-k}r}{\varphi(2^{-k}r)} V(6 \cdot 2^{-k}r), \end{aligned}$$

where we used (4.13) for the third line and (4.7) for the last line. Also, by (2.9) and (2.8),

$$(4.16) \quad \begin{aligned} \sum_{k=1}^M \frac{2^{-k}r}{\varphi(2^{-k}r)} V(6 \cdot 2^{-k}r) & \leq \sum_{k=1}^M \frac{2^{-k}r}{V(2^{-k}r)} = \sum_{k=1}^M \int_{2^{-k}r}^{2^{-k+1}r} \frac{1}{V(2^{-k}r)} ds \\ & \leq \int_0^r \frac{1}{V(s)} ds \leq c_5 \frac{r}{V(r)}, \end{aligned}$$

where in the last two inequalities we have used that V is increasing and (4.7).

Using (4.14), (4.15), and (4.16), we conclude

$$\int_{(B_r \setminus B_{\rho/2}) \cap \{d < \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{dy}{|x-y|^{n-2} \varphi(|x-y|)} \leq \frac{c_4 c_5 r}{V(r)}.$$

This and (4.12) finish the proof. \square

Now we are ready to show that $V(\psi)$ acts as a barrier of L on D .

Proposition 4.5. *Let L be given by (2.2) and ψ be a regularized version of d_D . Then there exists a constant $\tilde{C}_3 = \tilde{C}_3(n, a_1, a_2, \alpha_1, \alpha_2, D) > 0$ such that*

$$(4.17) \quad |L(V(\psi))| \leq \tilde{C}_3 \quad \text{in } D.$$

where V is the renewal function with respect to Φ . In addition, if $D = B(0, r)$ is a ball with radius r , there exists a constant $C_3 = C_3(n, a_1, a_2, \alpha_1, \alpha_2) > 0$ such that

$$(4.18) \quad |L(V(\psi))| \leq \frac{C_3}{V(r)} \quad \text{in } B(0, r),$$

where $\psi = \Psi_r$ is a regularized version of $d_{B(0,r)}$ defined in (4.2). Note that C_3 is independent of r .

Proof. We prove (4.18) only. The proof of (4.17) is similar.

Let $x_0 \in B_r := B(0, r)$ and $\rho := d_{B_r}(x_0)$. First we prove (4.18) for the case $\rho \geq \kappa r > 0$ with $\kappa = 1/(8C^2)$. In this case, we have

$$(4.19) \quad \begin{aligned} |L(V(\psi))(x_0)| &= \left| \int_{\mathbb{R}^n} \left(\frac{V(\psi(x_0 + y)) + V(\psi(x_0 - y))}{2} - V(\psi(x_0)) \right) \frac{J(1)}{|y|^n \varphi(|y|)} dy \right| \\ &\leq \int_{B_{\kappa r/2}} \|\nabla^2[V(\psi(x_*))]\| \frac{J(1)}{|y|^{n-2} \varphi(|y|)} dy \\ &\quad + \int_{B_{\kappa r/2}^c} \left| \frac{V(\psi(x_0 + y)) + V(\psi(x_0 - y))}{2} - V(\psi(x_0)) \right| \frac{J(1)}{|y|^n \varphi(|y|)} dy, \end{aligned}$$

where x^* is a point on the segment between $x_0 - y$ and $x_0 + y$, so that $d_{B_r}(x_*) \geq \kappa r/2$ when $y \in B_{\kappa r/2}$. Using (2.9), (4.2), and Lemma 2.5, we have

$$\|\nabla^2[V(\psi(x_*))]\| \leq |V''(\psi(x))| \|\nabla \psi(x)\|^2 + |V'(\psi(x))| \|\nabla^2 \psi(x)\| \leq \frac{c_1(\kappa)V(r)}{r^2},$$

which yields to estimate the first term of (4.19) by

$$\begin{aligned} \int_{B_{\kappa r/2}} \|\nabla^2[V(\psi(x_*))]\| \frac{J(1)}{|y|^{n-2} \varphi(|y|)} dy &\leq c_1 \frac{V(r)}{r^2} \int_{B_{\kappa r/2}} \frac{1}{|y|^{n-2} \varphi(|y|)} dy \\ &= c_2 \frac{V(r)}{r^2} \int_0^{\kappa r/2} \frac{s}{\varphi(s)} ds \leq \frac{c_3}{V(r)}. \end{aligned}$$

In the last inequality above, we have used (4.5), (2.5), and (2.8). For the second term, using $\psi(x) \leq Cd_{B_r}(x) \leq Cr$ for any $x \in B_r$, we have

$$\left| \frac{V(\psi(x_0 + y)) + V(\psi(x_0 - y))}{2} - V(\psi(x_0)) \right| \leq 2V(Cr) \leq c_4 V(r).$$

Therefore,

$$\int_{B_{\kappa r/2}^c} \left| \frac{V(\psi(x_0 + y)) + V(\psi(x_0 - y))}{2} - V(\psi(x_0)) \right| \frac{J(1)}{|y|^n \varphi(|y|)} dy \leq c_5 V(r) \int_{\kappa r/2}^{\infty} \frac{1}{s \varphi(s)} ds \leq \frac{c_6(\kappa)}{V(r)}.$$

In the last inequality we have used (4.6), (2.5), and (2.8). Therefore, (4.18) for the case $\rho \geq \kappa r$ holds with $C_3 = c_3 + c_6$.

Now it suffices to consider the case $\rho < \kappa r$. Denote

$$l(x) := (\psi(x_0) + \nabla \psi(x_0) \cdot (x - x_0))_+,$$

which satisfies

$$L(V(l)) = 0 \quad \text{on} \quad \{l > 0\}$$

by (2.11). Note that $\psi(x_0) = l(x_0)$ and $\nabla \psi(x_0) = \nabla l(x_0)$. Moreover, by (4.3) we have

$$(4.20) \quad |\psi(x) - l(x)| \leq \frac{C}{r} |x - x_0|^2.$$

For any $0 < a \leq b \leq C$, there exists $a_* \in [a, b]$ satisfying $|V(a) - V(b)| = |a - b|V'(a_*)$. Using Lemma 2.5 in the first inequality we have

$$|V(a) - V(b)| = |a - b|V'(a_*) \leq c_7 |a - b| \frac{V(a_*)}{a_*} \leq c_8 |a - b| \frac{V(a)}{a}.$$

Here we used (2.9) with $c = C$ for the second inequality. Therefore, for any $a, b \in (0, C]$ we have

$$|V(a) - V(b)| \leq c_8 |a - b| \left(\frac{V(a)}{a} + \frac{V(b)}{b} \right).$$

Also, one can easily see the following inequality

$$(4.21) \quad |V(a) - V(b)| \leq c_8 |a - b| \left(\frac{V(a)}{a} \mathbf{1}_{\{a>0\}} + \frac{V(b)}{b} \cdot \mathbf{1}_{\{b>0\}} \right)$$

for any $0 \leq a, b \leq C$ by using Lemma 2.5.

By (4.20) and (4.21) we have that for any $x \in B_r(x_0)$,

$$(4.22) \quad |V(\psi(x)) - V(\ell(x))| \leq \frac{c_8}{r} |x - x_0|^2 \left(\frac{V(\psi(x))}{\psi(x)} \mathbf{1}_{\{\psi(x) > 0\}} + \frac{V(\ell(x))}{\ell(x)} \mathbf{1}_{\{\ell(x) > 0\}} \right) \\ \leq \frac{c_9}{r} |x - x_0|^2 \left(\frac{V(d_{B_r}(x))}{d_{B_r}(x)} \mathbf{1}_{\{d_{B_r}(x) > 0\}} + \frac{V(\ell(x))}{\ell(x)} \mathbf{1}_{\{\ell(x) > 0\}} \right),$$

where we used $\psi(x) \leq Cd_{B_r}(x) \leq C$ and $\ell(x) = (\psi(x_0) + \nabla\psi(x_0) \cdot (x - x_0))_+ \leq Cd_{B_r}(x_0) + Cr \leq C$ for the first inequality and (2.9) for the second.

On the other hand, for any $x \in B_{\rho/2}(x_0)$ with $\rho \leq \kappa r$ we have

$$|\ell(x) - \psi(x)| \leq \frac{C}{r} |x - x_0|^2 \leq \frac{C}{r} \rho^2 \leq C\kappa\rho$$

and

$$C^{-1} \frac{\rho}{2} \leq C^{-1} d_{B_r}(x) \leq \psi(x).$$

Thus, using $\kappa = 1/(8C^2)$ we obtain

$$\frac{1}{2} \psi(x) \leq \ell(x) \leq 2\psi(x) \quad \text{for any } x \in B_{\rho/2}(x_0).$$

Using $\frac{\rho}{2} \leq d_{B_r}(x) \leq 2\rho$, we arrive at

$$\psi(x), \ell(x) \in [(4C)^{-1}\rho, 4C\rho].$$

Therefore, there exists $y \in ((4C)^{-1}\rho, 4C\rho)$ satisfying

$$\frac{V(\psi(x)) - V(\ell(x))}{\psi(x) - \ell(x)} = V'(y),$$

so using (4.20) and (2.12), we have

$$(4.23) \quad |V(\psi(x)) - V(\ell(x))| = |\psi(x) - \ell(x)| V'(y) \leq \frac{c_{10}}{r} |x - x_0|^2 \frac{V(y)}{y} \\ \leq \frac{c_{11}}{r} |x - x_0|^2 \frac{V((4C)^{-1}\rho)}{(4C)^{-1}\rho} \leq \frac{c_{12}}{r} |x - x_0|^2 \frac{V(\rho)}{\rho}$$

for $x \in B_{\rho/2}(x_0)$. Here we used (2.12) and (2.9) for the second line. Also, for any $x \in B_r^c(x_0)$ we have

$$V(\ell(x)) = V(\psi(x_0) + (x - x_0)\nabla\psi(x_0)) \leq V(C\rho + C|x - x_0|) \leq V(2C|x - x_0|) \leq c_{13}V(|x - x_0|)$$

and

$$V(\psi(x)) \leq V(Cr) \leq V(C|x - x_0|) \leq c_{13}V(|x - x_0|),$$

where we have used (2.9) and $\rho \leq r \leq |x - x_0|$. Thus we obtain

$$(4.24) \quad |V(\psi) - V(\ell)|(x) \leq c_{14}V(|x - x_0|)$$

for $x \in B_r^c(x_0)$. Therefore, by taking $x = y + x_0$ for (4.22), (4.23), and (4.24) we have

$$|V(\psi) - V(\ell)|(y + x_0) \leq c \begin{cases} \frac{1}{r} \frac{V(\rho)}{\rho} |y|^2 & \text{for } y \in B_{\rho/2} \\ \frac{|y|^2}{r} \left(\frac{V(d_{B_r}(x_0+y))}{d_{B_r}(x_0+y)} \mathbf{1}_{\{d_{B_r}(x_0+y) > 0\}} + \frac{V(\ell(x_0+y))}{\ell(x_0+y)} \mathbf{1}_{\{\ell(x_0+y) > 0\}} \right) & \text{for } y \in B_r \setminus B_{\rho/2} \\ V(|y|) & \text{for } y \in B_r^c \end{cases}$$

where $c = c_9 \vee c_{12} \vee c_{14}$. Hence, recalling that $L(V(\ell))(x_0) = 0$ and $\psi(x_0) = \ell(x_0)$, we find that

$$\begin{aligned}
 |L(V(\psi))(x_0)| &= |L(V(\psi) - V(\ell))(x_0)| \\
 &= \int_{\mathbb{R}^n} |V(\psi) - V(\ell)|(x_0 + y) \frac{J(1)}{|y|^{n\varphi(|y|)}} dy \\
 &\leq \frac{c}{r} \frac{V(\rho)}{\rho} \int_{B_{\rho/2}} |y|^2 \frac{J(1)}{|y|^{n\varphi(|y|)}} dy + c \int_{B_r^c} V(|y|) \frac{J(1)}{|y|^{n\varphi(|y|)}} dy \\
 &\quad + c \int_{B_r \setminus B_{\rho/2}} \frac{|y|^2}{r} \left(\frac{V(d_{B_r}(x_0 + y))}{d_{B_r}(x_0 + y)} \mathbf{1}_{\{d_{B_r}(x_0 + y) > 0\}} + \frac{V(\ell(x_0 + y))}{\ell(x_0 + y)} \mathbf{1}_{\{\ell(x_0 + y) > 0\}} \right) \frac{J(1)}{|y|^{n\varphi(|y|)}} dy \\
 &=: \text{I} + \text{II} + \text{III}.
 \end{aligned}$$

For I, using (4.5) we have

$$\begin{aligned}
 \text{I} &= \frac{c}{r} \frac{V(\rho)}{\rho} \int_{B_{\rho/2}} |y|^2 \frac{J(1)}{|y|^{n\varphi(|y|)}} dy = \frac{c_{15}}{r} \frac{V(\rho)}{\rho} \int_0^{\rho/2} \frac{s}{\varphi(s)} ds \\
 &\leq \frac{c_{16}}{r} \frac{V(\rho)}{\rho} \frac{(\rho/2)^2}{\varphi(\rho/2)} \leq \frac{c_{17}}{V(r)} \left(\frac{\rho}{r} \frac{V(r)}{V(\rho)} \right) \leq \frac{c_{18}}{V(r)},
 \end{aligned}$$

where we used (2.8) and (2.9) for the last two inequalities. Also, using (4.8) we obtain

$$\text{II} = c \int_{B_r^c} V(|y|) \frac{J(1)}{|y|^{n\varphi(|y|)}} dy = c_{19} \int_r^\infty \frac{V(s)}{s\varphi(s)} ds \leq \frac{c_{20}}{V(r)}.$$

For the estimate of III, we first observe that for any $y \in \{\ell > 0\} := H$,

$$\left| \frac{\ell(y)}{d_H(y)} \right| = \|\nabla\psi(x_0)\| \leq C.$$

Thus, by (2.9) we have

$$\frac{V(\ell(y))}{\ell(y)} \leq c_{21} \frac{V(Cd_H(y))}{d_H(y)} \leq c_{22} \frac{V(d_H(y))}{d_H(y)}.$$

Therefore, using Lemma 4.4 for B_r and the half plane $H := \{\ell > 0\}$ for each line, we conclude

$$\begin{aligned}
 \text{III} &= \frac{c}{r} \int_{B_r \cap (B_1(x_0) \setminus B_{\rho/2}(x_0))} \frac{V(d_{B_r}(y))}{d_{B_r}(y)} \frac{J(1)}{|y - x_0|^{n-2}\varphi(|y - x_0|)} dy \\
 &\quad + \frac{c}{r} \int_{H \cap (B_1(x_0) \setminus B_{\rho/2}(x_0))} \frac{V(\ell(y))}{\ell(y)} \frac{J(1)}{|x - y|^{n-2}\varphi(|x_0 - y|)} dy \\
 &\leq \frac{c_{23}}{r} \frac{r}{V(r)} + \frac{c_{24}}{r} \int_{H \cap (B_1(x_0) \setminus B_{\rho/2}(x_0))} \frac{V(d_H(y))}{d_H(y)} \frac{1}{|x - y|^{n-2}\varphi(|x_0 - y|)} dy \leq \frac{c_{25}}{V(r)}.
 \end{aligned}$$

Combining estimates of I,II and III we arrive

$$|L(V(\psi))(x_0)| \leq \text{I} + \text{II} + \text{III} \leq (c_{18} + c_{20} + c_{25}) \frac{1}{V(r)}$$

and (4.18) follows. \square

4.2. Subsolution and Harnack inequality. In this section we construct a subsolution from the barrier we have obtained in Proposition 4.5. Recall that we defined the domain of infinitesimal generator A by

$$\mathcal{D} = \mathcal{D}(D) = \{u \in C_0(D) : Au \in C(D)\}$$

in Section 3.2. It is uncertain whether $V(\psi) \in \mathcal{D}(D)$ since $A(V(\psi))$ is not continuous in general. To make our barrier included in the domain of operator, we construct a new domain of generator which contains $V(\psi)$. For given $C^{1,1}$ bounded open set D and open subset U in D , define

$$\mathcal{F} = \mathcal{F}(D, U) := \{u \in C_0(D) : Au \in L^\infty(U)\}.$$

for the usage of proof. Denote $\mathcal{F}(D) = \mathcal{F}(D, D)$. Clearly $\mathcal{F}(D, U_2) \subset \mathcal{F}(D, U_1)$ for any $U_1 \subset U_2$. We first prove that $V(\psi) \in \mathcal{F}(D)$.

Lemma 4.6. *Let ψ be the regularized version of d_D . Then, $A(V(\psi)) = L(V(\psi))$ in D . Moreover, $V(\psi) \in \mathcal{F}(D)$.*

Proof. Let $u \in C_0(D)$ be a twice-differentiable function in D . Assume that $\nabla^2 u$ is bounded in some $U \subset\subset D$. We first claim that

$$(4.25) \quad Lu(x) = Au(x) \quad \text{for any } x \in U.$$

Indeed, fix $x \in U$ and let $r_x > 0$ be a constant satisfying $B = B(x, r_x) \subset U$. Without loss of generality we can assume $r_x \leq 1$. Note that there exists a constant $c_1 > 0$ such that $2|u| + r_x^2 \|\nabla^2 u\| \leq c_1$ in U . Then we have

$$(4.26) \quad \begin{aligned} Au(x) &= \lim_{t \downarrow 0} \frac{P_t u(x) - u(x)}{t} = \lim_{t \downarrow 0} \frac{1}{t} \left(\int_{\mathbb{R}^n} u(x+y) p(t, |y|) dy - u(x) \right) \\ &= \lim_{t \downarrow 0} \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{p(t, |y|)}{t} dy. \end{aligned}$$

Since there is a constant $c_2 > 0$ such that $\frac{p(t, r)}{t} \leq c_2 J(r)$ for any $t > 0$ and $r > 0$, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \left| \frac{u(x+y) + u(x-y)}{2} - u(x) \right| \frac{p(t, |y|)}{t} dy \\ & \leq \int_B \left| \frac{u(x+y) + u(x-y)}{2} - u(x) \right| \frac{p(t, |y|)}{t} dy + \int_{B^c} \left| \frac{u(x+y) + u(x-y)}{2} - u(x) \right| \frac{p(t, |y|)}{t} dy \\ & \leq c_1 \int_B \frac{|y|^2}{r_x^2} \frac{p(t, |y|)}{t} dy + c_1 \int_{B^c} \frac{p(t, |y|)}{t} dy \leq c_1 \int_{\mathbb{R}^n} \left(\frac{|y|^2}{r_x^2} \wedge 1 \right) (c_3 J(|y|)) dy < \infty \end{aligned}$$

for any $t > 0$ so that we can apply dominated convergence theorem in the right-handed side of (4.26). Thus, using $\lim_{t \downarrow 0} \frac{p(t, r)}{t} = J(r)$ we obtain

$$\begin{aligned} Au(x) &= \lim_{t \downarrow 0} \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{p(t, |y|)}{t} dy \\ &= \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) J(|y|) dy = Lu(x). \end{aligned}$$

This concludes the claim. Now, by Lemma 2.5 we have that $V(\psi) \in C_0(D)$ is twice-differentiable and $\nabla^2 V(\psi)$ is locally bounded on D . Therefore, we arrive $L(V(\psi)) = A(V(\psi))$ in D . It immediately follows from (4.17) that $V(\psi) \in \mathcal{F}(D)$. \square

Now we are ready to construct a subsolution with respect to the generator A .

Lemma 4.7 (subsolution). *There exist a constant $C_4 = C_4(n, a_1, a_2, \alpha_1, \alpha_2) > 0$ independent of r and a radial function $w = w_r \in \mathcal{F}(B_{4r})$ satisfying*

$$\begin{cases} Aw \geq 0 & \text{in } B_{4r} \setminus B_r, \\ w \leq V(r) & \text{in } B_r, \\ w \geq C_4 V(4r - |x|) & \text{in } B_{4r} \setminus B_r, \\ w \equiv 0 & \text{in } \mathbb{R}^n \setminus B_{4r}, \end{cases}$$

where $B_r := B(0, r)$.

Proof. Let $\Psi = \Psi_{4r}$ be the regularized version of $d_{B_{4r}}$ in (4.2) and choose a function $\eta \in C_c^\infty(B_1)$ satisfying $\|\eta\|_{C(B_1)} = 1$ and $\eta \equiv 1$ on $B_{1/2}$. Define $\eta_r(x) := V(r)\eta(x/r) \in C_c^\infty(B_r)$. Then, we

have

$$\begin{aligned} |A\eta_r(x)| &= |L\eta_r(x)| \leq \int_{\mathbb{R}^n} \left| \frac{\eta_r(x+y) + \eta_r(x-y)}{2} - \eta_r(x) \right| J(|y|) dy \\ &\leq (\|\nabla^2 \eta_r\|_{L^\infty(B_r)} + \|\eta_r\|_{L^\infty(B_r)}) \int_{\mathbb{R}^n} (|y|^2 \wedge 1) J(|y|) dy < \infty \end{aligned}$$

for any $x \in \mathbb{R}^n$, which implies $\eta_r \in \mathcal{F}(B_{4r})$. Also, for $x \in B_{4r} \setminus B_r$,

$$\begin{aligned} A\eta_r(x) &= L\eta_r(x) = \int_{\mathbb{R}^n} \frac{\eta_r(x+y) + \eta_r(x-y)}{2} \frac{J(1)}{|y|^n \varphi(|y|)} dy \\ &= \int_{\mathbb{R}^n} \eta_r(x+y) \frac{J(1)}{|y|^n \varphi(|y|)} dy \geq \int_{B(-x, r/2)} \frac{V(r)J(1)}{|y|^n \varphi(|y|)} dy \geq \frac{c_1(r/2)^n V(r)}{(9r/2)^n \varphi(9r/2)} \geq \frac{c_2}{V(4r)}. \end{aligned}$$

Here we used (2.8) and (2.9) for the last inequality.

Define a function \tilde{w}_r by

$$\tilde{w}_r = \frac{c_2}{C_3} V(\Psi) + \eta_r,$$

where C_3 is the constant in Proposition 4.5. We have $\tilde{w}_r \in \mathcal{F}(B_{4r})$ by Lemma 4.6. Also, for $x \in B_{4r} \setminus B_r$, using Proposition 4.5 and Lemma 4.6 again, we have

$$A\tilde{w}_r(x) = \frac{c_2}{C_3} AV(\Psi)(x) + A\eta_r(x) \geq -\frac{c_2}{C_3} |LV(\Psi)(x)| + A\eta_r(x) \geq -\frac{c_2}{V(4r)} + \frac{c_2}{V(4r)} = 0$$

and

$$\tilde{w}_r(x) = \frac{c_2}{C_3} V(\Psi(x)) \geq c_3 V(d_D(x)) = c_3 V(4r - |x|).$$

For $x \in B_r$,

$$\tilde{w}_r(x) \leq \frac{c_2}{C_3} V(4Cr) + V(r) \leq c_4 V(r)$$

by (4.2) and (2.9). Define $w_r(x) := \frac{1}{c_4} \tilde{w}_r(x)$. Then w_r satisfies all assertions in Lemma 4.7 with constant $C_4 = \frac{c_3}{c_4}$, which is independent of r . \square

We end this section with the Harnack inequality and the maximum principle of probabilistic version. For local operators, the Harnack inequality implies Hölder regularity of solutions of differential equations. However for nonlocal operators, as Silvestre mentioned in [33], this is not true because the nonnegativity of the function u is required in the whole space \mathbb{R}^n . The Harnack inequality, maximum principle, and the subsolution constructed in Lemma 4.7 will play a key role in the proof of Theorem 2.2. We emphasize that the following theorem is the Harnack inequality for harmonic function with respect to A , and it does not imply the Harnack inequality for the viscosity solution with respect to L . See [14] for the statement of Harnack inequality for viscosity solution.

Theorem 4.8 (Harnack inequality). *[36, Theorem 2.2] Let D be a bounded $C^{1,1}$ open set. Then, there exists a constant $C > 0$ such that for any ball $B(x_0, r) \subset D$, and any nonnegative function $u \in \mathcal{F}(D)$ satisfying $Au = 0$ a.e. in $B(x_0, r)$, we have*

$$\sup_{B(x_0, r/2)} u \leq C \inf_{B(x_0, r/2)} u.$$

Also, we have the following maximum principle.

Lemma 4.9 (Maximum principle). *Let D be a bounded $C^{1,1}$ open set and U be an open subset of D . If the function $u \in \mathcal{F}(D, U)$ satisfies $Au = 0$ a.e. in U and $u \geq 0$ in U^c , then $u \geq 0$ in \mathbb{R}^n .*

Proof. Suppose that there exists $x \in U$ satisfying $u(x) < 0$. Since $u \in C_0(D)$, the set $U_- := \{x \in \mathbb{R}^d : u(x) < 0\}$ is bounded and open set with positive Lebesgue measure. For any $t > 0$ we have

$$\begin{aligned} \int_{U_-} P_t u(x) - u(x) dx &= \int_{U_-} \int_{\mathbb{R}^d} u(y) p(t, |x-y|) dy dx - \int_{U_-} u(x) dx \\ &= \int_{\mathbb{R}^d} u(y) \int_{U_-} p(t, |x-y|) dx dy - \int_{U_-} u(y) dy \\ &= \int_{U_-} u(y) \int_{U_-} p(t, |x-y|) dx dy + \int_{U_-} u(y) \left(\int_{U_-} p(t, |x-y|) dx - 1 \right) dy \\ &\geq \int_{U_-} u(y) \left(\int_{U_-} p(t, |x-y|) dx - 1 \right) dy. \end{aligned}$$

Since U_- is bounded, $\text{diam}(U_-) =: R < \infty$. Thus, for any $y \in U_- \subset B(y, R)$,

$$\frac{1 - \int_{U_-} p(t, |x-y|) dx}{t} \geq \frac{1 - \int_{B(y,R)} p(t, |x-y|) dx}{t} = \frac{1}{t} (1 - \mathbb{P}^y(X_t \in B(y, R))) = \frac{\mathbb{P}^0(|X_t| \geq R)}{t}.$$

Using heat kernel estimates in [3, Theorem 21], we have $p(t, r) \asymp \left(\varphi^{-1}(t)^{-n} \wedge \frac{t}{r^n \varphi(r)} \right)$ for $(t, r) \in (0, 1] \times \mathbb{R}_+$. Note that $\frac{t}{r^n \varphi(r)} \leq \varphi^{-1}(t)^{-n}$ for $t \leq \varphi(r)$. Thus, there exists $\varepsilon = \varepsilon(R) > 0$ satisfying

$$\frac{\mathbb{P}^0(|X_t| \geq R)}{t} \geq \frac{1}{t} \int_{R \leq |z| \leq 2R} p(t, |z|) dz \geq c \int_R^{2R} \frac{1}{r \varphi(r)} dr \geq \varepsilon \quad \text{for all } t \in (0, \varphi(R)].$$

Combining above estimates we obtain

$$\int_{U_-} \frac{P_t u(x) - u(x)}{t} dx \geq -\varepsilon \int_{U_-} u(y) dy \quad \text{for all } t \in (0, \varphi(R)].$$

Letting $t \rightarrow 0$, we conclude

$$0 = \int_{U_-} A u(x) dx = \lim_{t \rightarrow 0} \int_{U_-} \frac{P_t u(x) - u(x)}{t} dx \geq -\varepsilon \int_{U_-} u(y) dy > 0,$$

which is contradiction. Therefore, $u \geq 0$ in \mathbb{R}^n . \square

4.3. Proof of Theorem 2.2. In this section we will prove Theorem 2.2. More precisely, we prove the Hölder regularity for the function $u/V(d_D)$ up to the boundary of D . We will control the oscillation of this function using the Harnack inequality, the maximum principle and the subsolution constructed in Lemma 4.7.

Let us adopt notations in [28, Definition 3.3]. Let $\kappa > 0$ be a fixed small constant and let $\kappa' = 1/2 + 2\kappa$. Given $x_0 \in \partial D$ and $r > 0$, define

$$D_r = D_r(x_0) = B(x_0, r) \cap D$$

and

$$D_{\kappa' r}^+ = D_{\kappa' r}^+(x_0) = B(x_0, \kappa' r) \cap \{x \in D : -x \cdot \nu(x_0) \geq 2\kappa r\},$$

where $\nu(x_0)$ is the unit outward normal at x_0 . Since D is a bounded $C^{1,1}$ open set, there exists $\rho_0 > 0$ such that for each $x_0 \in \partial D$ and $r \leq \rho_0$, there exists an orthonormal system CS_{x_0} with its origin at x_0 and a $C^{1,1}$ -function $\Psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ satisfying $\Psi(\vec{0}) = 0$, $\nabla_{CS_{x_0}} \Psi(\vec{0}) = 0$, $\|\Psi\|_{C^{1,1}} \leq \kappa$, and

$$\{y = (\tilde{y}, y_n) \text{ in } CS_{x_0} : |\tilde{y}| < 2r, \Psi(\tilde{y}) < y_n < 2r\} \subset D.$$

Then we have

$$(4.27) \quad B(y, \kappa r) \subset D_r(x_0) \text{ for all } y \in D_{\kappa' r}^+(x_0),$$

and we can take a $C^{1,1}$ subdomain $D_r^{1,1}$ satisfying $D_r \subset D_r^{1,1} \subset D_{2r}$ and

$$(4.28) \quad \text{dist}(y, \partial D_r^{1,1}) = d_D(y)$$

for all $y \in D_r$. Since D_r is not $C^{1,1}$ in general, we will use this subdomain instead of D_r .

Since D is bounded and $C^{1,1}$ again, we can assume that for each $x_0 \in \partial D$ and $r \leq \rho_0$,

$$(4.29) \quad B(y^* - 4\kappa r\nu(y^*), 4\kappa r) \subset D_r(x_0) \text{ and } B(y^* - 4\kappa r\nu(y^*), \kappa r) \subset D_{\kappa' r}^+(x_0)$$

for all $y \in D_{r/2}(x_0)$, where $y^* \in \partial D$ is the unique boundary point satisfying $|y - y^*| = d_D(y)$.

The following oscillation lemma is the key lemma to prove Theorem 2.2.

Lemma 4.10 (Oscillation lemma). *Assume $f \in C(D)$ and let $u \in \mathcal{D}$ be the viscosity solution of (2.6). Then there exist constants $\gamma \in (0, 1)$ and $C_1 > 0$, depending only on $n, a_1, a_2, \alpha_1, \alpha_2$ and D , such that*

$$(4.30) \quad \sup_{D_r(x_0)} \frac{u}{V(d_D)} - \inf_{D_r(x_0)} \frac{u}{V(d_D)} \leq C_1 V(r)^\gamma \|f\|_{L^\infty(D)}$$

for any $x_0 \in \partial D$ and $r > 0$.

To prove the oscillation lemma, we need some preparation. Note that in the following two lemmas we aim to verify inequalities for every function $u \in \mathcal{F}$, since we want to utilize the subsolution constructed in Lemma 4.7. The first one is a generalized version of Harnack inequality.

Lemma 4.11 (Harnack inequality). *There exists a constant $C_2 = C_2(n, a_1, a_2, \alpha_1, \alpha_2, D) > 0$ such that for any $r \leq \rho_0, x_0 \in \partial D$ and nonnegative function $u \in \mathcal{F}(D, D_r^{1,1})$,*

$$(4.31) \quad \sup_{D_{\kappa' r}^+(x_0)} \frac{u}{V(d_D)} \leq C_2 \left(\inf_{D_{\kappa' r}^+(x_0)} \frac{u}{V(d_D)} + \|Au\|_{L^\infty(D_r^{1,1})} V(r) \right).$$

Proof. We first prove that if a nonnegative function v satisfies $Av = 0$ a.e. in $D_r^{1,1}$, then

$$(4.32) \quad \sup_{D_{\kappa' r}^+(x_0)} \frac{v}{V(d_D)} \leq c \inf_{D_{\kappa' r}^+(x_0)} \frac{v}{V(d_D)}$$

for a constant $c > 0$ which is independent of r and v . Indeed, for each $y \in D_{\kappa' r}^+$, we have $B(y, \kappa r) \subset D_r^{1,1}$ by (4.27) hence $Av = 0$ a.e. in $B(y, \kappa r)$. We may cover $D_{\kappa' r}^+$ by finitely many balls $B(y_i, \kappa r/2)$. Here the number of balls is independent of r . By the Theorem 4.8, we have for each i ,

$$\sup_{B(y_i, \kappa r/2)} v \leq c_1 \inf_{B(y_i, \kappa r/2)} v.$$

If $x \in B(y_i, \kappa r/2)$, we have $\kappa r/2 \leq d_D(x) \leq r/2 + 5\kappa r/2$. Thus, using (2.9) we obtain

$$\sup_{B(y_i, \kappa r/2)} \frac{v}{V(d_D)} \leq \sup_{B(y_i, \kappa r/2)} \frac{v}{V(\kappa r/2)} \leq c_2 \inf_{B(y_i, \kappa r/2)} \frac{v}{V(r/2 + 5\kappa r/2)} \leq c_2 \inf_{B(y_i, \kappa r/2)} \frac{v}{V(d_D)}.$$

Now (4.32) follows from the standard covering argument, possibly with a larger constant.

We next prove (4.31). Let us write $u = u_1 + u_2$, where $u_1 := u + R^{D_r^{1,1}} Au$ and $u_2 := -R^{D_r^{1,1}} Au$. We claim that $u_1 \geq 0$ in \mathbb{R}^n and $Au_1 = 0$ a.e. in $D_r^{1,1}$.

Following the calculations of (3.21) we obtain that for any open subset $U \subset D$, $x \in U$ and $u \in \mathcal{F}(D, U)$,

$$(4.33) \quad Au(x) = \lim_{t \downarrow 0} \frac{P_t u(x) - u(x)}{t} = \lim_{t \downarrow 0} \frac{P_t^U u(x) - u(x)}{t}.$$

Let us emphasize that we only have used $u \in C_0(D)$ in (3.21) so we can repeat the same argument for $u \in \mathcal{F}(D, U)$.

Let $g \in L^\infty(U)$. Deducing $R^U g \in C_0(U)$ from Proposition 3.2 and (3.1), we obtain the following counterpart of (3.23): For any $x \in U$,

$$\begin{aligned}
AR^U g(x) &= A \left(\int_0^\infty P_s^U g(\cdot) ds \right) (x) = \lim_{t \downarrow 0} \frac{1}{t} \left(P_t^U \left(\int_0^\infty P_s^U g(\cdot) ds \right) (x) - \int_0^\infty P_s^U g(x) ds \right) \\
(4.34) \quad &= \lim_{t \downarrow 0} \frac{1}{t} \left(\int_0^\infty P_{s+t}^U g(x) ds - \int_0^\infty P_s^U g(x) ds \right) \\
&= - \lim_{t \downarrow 0} \frac{\int_0^t P_s^U g(x) ds}{t} = - \lim_{t \downarrow 0} \frac{\int_0^t P_s g(x) ds}{t}.
\end{aligned}$$

Here we used (4.33) for the first line. Let

$$U_g := \{x \in U : \lim_{r \downarrow 0} \frac{1}{r^n} \int_{B(x,r)} |g(x) - g(y)| dy = 0\}.$$

Then, we have $|U \setminus U_g| = 0$ since $g \in L^\infty(U) \subset L^1(U)$. For $x \in U_g$, we have

$$|P_t g(x) - g(x)| = \left| \int_{\mathbb{R}^n} p(t, |x-y|) (g(y) - g(x)) dy \right| \leq \int_{\mathbb{R}^n} p(t, |x-y|) |g(y) - g(x)| dy.$$

Let $\varepsilon > 0$. Using $p(t, r) \asymp \left(\varphi^{-1}(t)^{-n} \wedge \frac{t}{r^n \varphi(r)} \right)$ for $t \in (0, 1] \times \mathbb{R}_+$ in [3, Theorem 21] again, there exist constants $c_3(\varepsilon), c_4(\varepsilon) > 0$ such that for any $t \in (0, 1]$ and $r > 0$,

$$p(t, r) \leq c_4 \varphi^{-1}(t)^{-n}$$

and

$$\mathbb{P}^x(|X_t| > c_3 \varphi^{-1}(t)) \leq \varepsilon.$$

Indeed, using (4.6) and (2.5) we have

$$\mathbb{P}^x(|X_t| > c_3 \varphi^{-1}(t)) = \int_{|z| > c_3 \varphi^{-1}(t)} p(t, |z|) dz \leq c_4 t \int_{c_3 \varphi^{-1}(t)}^\infty \frac{dr}{r \varphi(r)} \leq \frac{c_5 t}{\varphi(c_3 \varphi^{-1}(t))} \leq c_6 c_3^{-2\alpha_1}.$$

Thus, we obtain

$$\begin{aligned}
|P_t g(x) - g(x)| &\leq \int_{B(x, c_3 \varphi^{-1}(t))} p(t, |x-y|) |g(y) - g(x)| dy + \int_{B(x, c_3 \varphi^{-1}(t))^c} p(t, |x-y|) |g(y) - g(x)| dy \\
&\leq c_4 \varphi^{-1}(t)^{-n} \int_{B(x, c_3 \varphi^{-1}(t))} |g(y) - g(x)| dy + 2 \|g\|_\infty \int_{B(x, c_3 \varphi^{-1}(t))} p(t, |x-y|) dy \\
&\leq c_4 \varphi^{-1}(t)^{-n} \int_{B(x, c_3 \varphi^{-1}(t))} |g(y) - g(x)| dy + 2 \|g\|_\infty \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $x \in U_g$, we conclude

$$\lim_{t \downarrow 0} |P_t g(x) - g(x)| = 0.$$

Combining this with (4.34) we arrive that for any open subset $U \subset D$ and $g \in L^\infty(D)$,

$$(4.35) \quad AR^U g = -g \quad \text{a.e. in } U.$$

Since $u \in \mathcal{F}(D, U)$, we have $Au \in L^\infty(U)$. Thus, taking $U = D_r^{1,1}$ and $g = Au$ in (4.35) we conclude

$$Au_1 = Au + AR^{D_r^{1,1}} Au = 0 \quad \text{a.e. in } D_r^{1,1}.$$

Also, $u_1 \geq 0$ follows from applying Lemma 4.9 with above equation and $u_1 = u \geq 0$ in $\mathbb{R}^n \setminus D_r^{1,1}$.

Applying (4.32) to u_1 , we get

$$\sup_{D_{\kappa' r}^+} \frac{u_1}{V(d_D)} \leq c_7 \inf_{D_{\kappa' r}^+} \frac{u_1}{V(d_D)}.$$

Meanwhile, using (4.28) and Lemma 3.3 we have

$$|u_2(x)| \leq c_8 \|Au\|_{L^\infty(D_r^{1,1})} V(\text{diam}(D_r^{1,1})) V(\text{dist}(x, \partial D_r^{1,1})) \leq c_9 \|Au\|_{L^\infty(D_r^{1,1})} V(r) V(d_D(x))$$

for all $x \in D_r^{1,1}$. Therefore, combining above two inequalities we conclude that

$$\begin{aligned} \sup_{D_{\kappa' r}^+} \frac{u}{V(d_D)} &\leq \sup_{D_{\kappa' r}^+} \frac{u_1}{V(d_D)} + \sup_{D_{\kappa' r}^+} \frac{u_2}{V(d_D)} \leq c_5 \inf_{D_{\kappa' r}^+} \frac{u_1}{V(d_D)} + \sup_{D_{\kappa' r}^+} \frac{u_2}{V(d_D)} \\ &\leq c_5 \inf_{D_{\kappa' r}^+} \frac{u}{V(d_D)} + (c_5 + 1) \sup_{D_{\kappa' r}^+} \frac{|u_2|}{V(d_D)} \leq C_2 \left(\inf_{D_{\kappa' r}^+} \frac{u}{V(d_D)} + \|Au\|_{L^\infty(D_r^{1,1})} V(r) \right). \end{aligned}$$

□

The next lemma gives the link between $D_{\kappa' r}^+$ and $D_{r/2}$. Here we are going to use the subsolution w in Lemma 4.7.

Lemma 4.12. *Let $r \leq \rho_0, x_0 \in \partial D$. If $u \in \mathcal{F}(D, D_r^{1,1})$ is nonnegative, then there exists a constant $C_3 = C_3(n, a_1, a_2, \alpha_1, \alpha_2, D) > 0$ such that*

$$\inf_{D_{\kappa' r}^+(x_0)} \frac{u}{V(d_D)} \leq C_3 \left(\inf_{D_{r/2}(x_0)} \frac{u}{V(d_D)} + \|Au\|_{L^\infty(D_r^{1,1})} V(r) \right).$$

Proof. First assume that Au is nonnegative. As in the proof of Lemma 4.11, we write $u = u_1 + u_2$, where $u_1 = u + R^{D_r^{1,1}} Au$ and $u_2 = -R^{D_r^{1,1}} Au$. Then u_1 is a nonnegative solution for

$$\begin{cases} Au_1 = 0 & \text{a.e. in } D_r^{1,1}, \\ u_1 = u & \text{in } \mathbb{R}^n \setminus D_r^{1,1}. \end{cases}$$

Let

$$m := \inf_{D_{\kappa' r}^+} \frac{u_1}{V(d_D)} \geq 0.$$

For $y \in D_{r/2}$, we have either $y \in D_{\kappa' r}^+$ or $d_D(y) < 4\kappa r$ by (4.29).

If $y \in D_{\kappa' r}^+$, then clearly

$$(4.36) \quad m \leq \frac{u_1(y)}{V(d_D(y))}.$$

If $d_D(y) < 4\kappa r$, let y^* be the closest point to y on $\partial D_r^{1,1}$ and let $\tilde{y} = y^* - 4\kappa r \nu(y^*)$. By (4.29), we have $B_{4\kappa r}(\tilde{y}) \subset D_r$ and $B_{\kappa r}(\tilde{y}) \subset D_{\kappa' r}^+$.

Now consider $w \in \mathcal{F}(B_{4\kappa r}(\tilde{y})) \subset \mathcal{F}(D, B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}))$ satisfying

$$\begin{cases} Aw \geq 0 & \text{in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ w \leq V(\kappa r) & \text{in } B_{\kappa r}(\tilde{y}), \\ w \geq c_1 V(4\kappa r - |x - \tilde{y}|) & \text{in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ w \equiv 0 & \text{in } \mathbb{R}^n \setminus B_{4\kappa r}(\tilde{y}), \end{cases}$$

which can be obtained by translating the subsolution in Lemma 4.7. Since $Au_1 = 0$ a.e. in $B_{4\kappa r}(\tilde{y})$, we have

$$\begin{cases} Au_1 = 0 \leq A(mw) & \text{a.e. in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ u_1 \geq mV(d_D) \geq mw & \text{in } B_{\kappa r}(\tilde{y}), \\ u_1 \geq 0 = mw & \text{in } \mathbb{R}^n \setminus B_{4\kappa r}(\tilde{y}). \end{cases}$$

Now by the maximum principle in Lemma 4.9 with the function $u_1 - mw$ and $U = B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y})$, we obtain $u_1 \geq mw$ in \mathbb{R}^n . In particular, for $y \in B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y})$,

$$u_1(y) \geq c_1 m V(4\kappa r - |y - \tilde{y}|) = c_1 m V(d_D(y)).$$

Therefore, we obtain

$$\inf_{D_{\kappa' r}^+} \frac{u_1}{V(d_D)} \leq c_2 \inf_{D_{r/2}} \frac{u_1}{V(d_D)}.$$

On the other hand, u_2 satisfies

$$|u_2(x)| \leq c_3 \|Au\|_{L^\infty(D_r^{1,1})} V(r)V(d_D(x))$$

for all $x \in D_r^{1,1}$, which gives the desired result. \square

We prove the oscillation lemma (4.30) by using Lemmas 4.11 and 4.12.

Proof of Lemma 4.10 As a consequence of Remark 3.4, by dividing $\|f\|_{L^\infty(D)}$ on both sides of (2.6) if necessary, we may assume $\|f\|_{L^\infty(D)} \leq 1$ and $\|u\|_{C(D)} = \|R^D f\|_{C(D)} \leq c_1$ without loss of generality. Fix $x_0 \in \partial D$. We will prove that there exist constants $c_2 > 0, \rho_1 \in (0, \rho_0/16]$, and $\gamma \in (0, 1)$ and monotone sequences $(m_k)_{k \geq 0}$ and $(M_k)_{k \geq 0}$ such that $M_k - m_k = V(r_{k+1}/2)^\gamma$,

$$-V(\rho_1/16) \leq m_k \leq m_{k+1} < M_{k+1} \leq M_k \leq V(\rho_1/16),$$

and

$$m_k \leq \frac{u}{c_2 V(d_D)} \leq M_k \text{ in } D_{r_k} = D_{r_k}(x_0)$$

for all $k \geq 0$, where $r_k = \rho_1 8^{-k}$. If we have such constants and sequences, then for any $0 < r \leq \rho_1$ we have $k \geq 0$ satisfying $r \in (r_{k+1}, r_k]$ and

$$\sup_{D_r} \frac{u}{V(d_D)} - \inf_{D_r} \frac{u}{V(d_D)} \leq \sup_{D_{r_k}} \frac{u}{V(d_D)} - \inf_{D_{r_k}} \frac{u}{V(d_D)} \leq c_2(M_k - m_k) = c_2 V(r_{k+1}/2)^\gamma \leq c_2 V(r)^\gamma.$$

Also, for any $r > \rho_1$ we have

$$\sup_{D_r} \frac{u}{V(d_D)} - \inf_{D_r} \frac{u}{V(d_D)} \leq c_3 \leq c_4 V(\rho_1)^\gamma \leq c_4 V(r)^\gamma$$

by Lemma 3.3. Above two inequalities conclude the lemma so it suffices to construct such constants and sequences.

Let us use the induction on k . The case $k = 0$ follows from Lemma 3.3 provided we take c_2 large enough. The constants ρ_1 and γ will be chosen later. Assume that we have sequences up to m_k and M_k . Let ψ be the regularized version of d_D . We may assume that $\psi = d_D$ in $\{d_D(x) \leq \rho_1\}$. Define

$$u_k = V(\psi) \left(\frac{u}{c_2 V(\psi)} - m_k \right) = \frac{1}{c_2} u - m_k V(\psi)$$

in \mathbb{R}^n . Note that $u_k \in \mathcal{F}(D)$ since $Au = f$ by the consequence of Theorem 3.10. Moreover, for $x \in D_{r_k/4}^{1,1}$ we have $u_k^- \in C^2(x)$ since we know that $u_k^- \equiv 0$ in $B(x_0, r_k)$ by the induction hypothesis. Thus, we have $Au_k^-(x) = Lu_k^-(x)$ by (3.20), which implies that Au_k^- is well-defined in $D_{r_k/4}^{1,1}$, and so is Au_k^+ . We will apply Lemmas 4.11 and 4.12 for the function u_k^+ and $r = r_k/4$ to find m_{k+1} and M_{k+1} . By (4.17) and Lemma 4.6, we have

$$\begin{aligned} |Au_k^+| &\leq |Au_k| + |Au_k^-| \leq \left| \frac{1}{c_2} Au - m_k AV(\psi) \right| + |Au_k^-| \\ (4.37) \quad &\leq \left(\frac{1}{c_2} |f| + V(\rho_1/16) |L(V(\psi))| \right) + |Au_k^-| \leq c_3 + |Au_k^-| \end{aligned}$$

in D . Thus, we need to estimate $|Au_k^-|$ in $D_{r_k/4}^{1,1}$ for the usage of Lemmas 4.11 and 4.12.

Let $x \in D_{r_k/4}^{1,1}$. By the induction hypothesis, we have $u_k^- \equiv 0$ in $B(x_0, r_k)$, which implies that $u_k^- \in C^2(x)$. Thus, we compute the value $Au_k^-(x)$ using the operator L as follows:

$$\begin{aligned} (4.38) \quad 0 \leq Au_k^-(x) &= Lu_k^-(x) = \frac{1}{2} \int_{\mathbb{R}^n} (u_k^-(x+h) + u_k^-(x-h)) \frac{J(1)}{|h|^n \varphi(|h|)} dh \\ &= \int_{x+h \notin B_{r_k}} u_k^-(x+h) \frac{J(1)}{|h|^n \varphi(|h|)} dh. \end{aligned}$$

For any $y \in B_{r_0} \setminus B_{r_k}$, there is $0 \leq j < k$ such that $y \in B_{r_j} \setminus B_{r_{j+1}}$. Since $c_2^{-1}u \geq m_j V(\psi)$ and $d_D = \psi$ in B_{r_j} , we have

$$\begin{aligned} u_k(y) &= c_2^{-1}u(y) - m_k V(\psi(y)) \geq (m_j - m_k)V(\psi(y)) \\ &\geq (m_j - M_j + M_k - m_k)V(d_D(y)) \geq -(V(r_{j+1}/2)^\gamma - V(r_{k+1}/2)^\gamma)V(r_j). \end{aligned}$$

It follows from $r_{j+1} \leq |y - x_0| < r_j \leq 8|y - x_0| \leq 1$ that

$$(4.39) \quad \begin{aligned} u_k^-(y) &\leq c_4 (V(|y - x_0|/2)^\gamma - V(r_k/16)^\gamma) V(8|y - x_0|) \\ &\leq c_5 (V(|y - x_0|/2)^\gamma - V(r_k/16)^\gamma) V(|y - x_0|/2). \end{aligned}$$

Note that (4.39) possibly with a larger constant also holds for $y \in \mathbb{R}^n \setminus B_{r_0}$ because $\|u_k\|_{C(\mathbb{R}^n)} \leq c_1 c_2^{-1} + V(1/16)V(\tilde{C})$ for any k and

$$(V(|y - x_0|/2)^\gamma - V(r_k/16)^\gamma) V(|y - x_0|/2) \geq (V(\rho_1/2)^\gamma - V(\rho_1/16)^\gamma) V(\rho_1/2) > 0$$

for any $y \in \mathbb{R}^n \setminus B_{r_0}$. Thus, by (4.38) and (4.39), we have

$$|Au_k^-(x)| \leq c_6 \int_{x+y \notin B_{r_k}} (V(|x+h-x_0|/2)^\gamma - V(r_k/16)^\gamma) \frac{V(|x+y-x_0|/2)}{|h|^n \varphi(|h|)} dh.$$

If $x+y \notin B_{r_k}$, then $|h| \geq |x+h-x_0| - |x-x_0| \geq r_k - r_k/2 = r_k/2$ and $|x+h-x_0| \leq r_k/2 + |h| \leq 2|h|$. Thus, recalling that $\mathcal{P}_1(r) = \int_r^\infty \frac{ds}{s\varphi(s)}$, we obtain

$$\begin{aligned} |Au_k^-(x)| &\leq c_6 \int_{|h| \geq r_k/2} (V(|h|)^\gamma - V(r_k/16)^\gamma) \frac{V(|h|)}{|h|^n \varphi(|h|)} dh \\ &\leq c_7 \int_{r_k/2}^\infty (V(s)^\gamma - V(r_k/16)^\gamma) V(s) d(-\mathcal{P}_1)(s) \\ &= c_7 \left([- (V(s)^\gamma - V(r_k/16)^\gamma) V(s) \mathcal{P}_1(s)]_{r_k/2}^\infty \right. \\ &\quad \left. + \int_{r_k/2}^\infty ((1+\gamma)V(s)^\gamma - V(r_k/16)^\gamma) V'(s) \mathcal{P}_1(s) ds \right) =: c_7 (\text{I} + \text{II}). \end{aligned}$$

By (4.10) we have

$$\lim_{s \rightarrow \infty} (V(s)^\gamma - V(r_k/16)^\gamma) V(s) \mathcal{P}_1(s) \leq c_8 \lim_{s \rightarrow \infty} \frac{V(s)^\gamma - V(r_k/16)^\gamma}{V(s)} = 0,$$

hence

$$\text{I} \leq c_8 \frac{V(r_k/2)^\gamma - V(r_k/16)^\gamma}{V(r_k/2)}.$$

Also, using (4.10) again we have

$$\begin{aligned} \text{II} &\leq c_8 \int_{r_k/2}^\infty ((1+\gamma)V(s)^\gamma - V(r_k/16)^\gamma) \frac{V'(s)}{V(s)^2} ds \\ &= c_8 \left(\frac{1+\gamma}{1-\gamma} V(r_k/2)^\gamma - V(r_k/16)^\gamma \right) \frac{1}{V(r_k/2)}. \end{aligned}$$

Therefore, combining above two inequalities and using (2.9) we get

$$\begin{aligned} |Au_k^-(x)| &\leq c_9 \left(\frac{2}{1-\gamma} V(r_k/2)^\gamma - 2V(r_k/16)^\gamma \right) \frac{1}{V(r_k/2)} \\ &\leq c_9 \left(\frac{2}{1-\gamma} (c_{10} 64^{\alpha_2})^\gamma - 2(c_{10}^{-1} 8^{\alpha_1})^\gamma \right) \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)} \\ &=: c_9 \varepsilon_\gamma \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)} \end{aligned}$$

and hence

$$\|Au_k^+\|_{L^\infty(D_{r_k/4}^{1,1})} \leq c_{11} \left(1 + \varepsilon_\gamma \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)} \right).$$

Note that $\varepsilon_\gamma \rightarrow 0$ as $\gamma \rightarrow 0$.

Now we apply Lemma 4.11 and 4.12 for $u_k^+ \in \mathcal{F}(D, D_{r_k/4}^{1,1})$. Since $u_k = u_k^+$ and $d_D = \psi$ in D_{r_k} , we have

$$\begin{aligned} \sup_{D_{\kappa' r_k/4}^+} \left(\frac{u}{c_2 V(\psi)} - m_k \right) &\leq c_{12} \left(\inf_{D_{\kappa' r_k/4}^+} \left(\frac{u}{c_2 V(\psi)} - m_k \right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \right) \\ &\leq c_{13} \left(\inf_{D_{r_{k+1}}} \left(\frac{u}{c_2 V(\psi)} - m_k \right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \right). \end{aligned}$$

Repeating this procedure with the function $u_k = M_k V(d_D) - c_2^{-1} u$ instead of $u_k = c_2^{-1} u - m_k V(d_D)$, we also have

$$\sup_{D_{\kappa' r_k/4}^+} \left(M_k - \frac{u}{c_2 V(\psi)} \right) \leq c_{14} \left(\inf_{D_{r_{k+1}}} \left(M_k - \frac{u}{c_2 V(\psi)} \right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \right).$$

Adding up these two inequalities, we obtain

$$M_k - m_k \leq c_{15} \left(\inf_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} - \sup_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} + M_k - m_k + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \right).$$

Thus, recalling that $M_k - m_k = V(r_{k+1}/2)^\gamma$, we get

$$\begin{aligned} \sup_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} - \inf_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} &\leq \frac{c_{15} - 1}{c_{15}} V(r_{k+1}/2)^\gamma + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \\ &\leq \left(\frac{c_{15} - 1}{c_{15}} c_{16}^\gamma + c_{17}^\gamma V(\rho_1)^{1-\gamma} + \varepsilon_\gamma \right) V(r_{k+2}/2)^\gamma. \end{aligned}$$

Now we choose γ and ρ_1 small enough so that

$$\frac{c_{15} - 1}{c_{15}} c_{16}^\gamma + c_{17}^\gamma V(\rho_1)^{1-\gamma} + \varepsilon_\gamma \leq 1,$$

and it yields that

$$\sup_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} - \inf_{D_{r_{k+1}}} \frac{u}{c_2 V(\psi)} \leq V(r_{k+2}/2)^\gamma.$$

Therefore, we are able to choose m_{k+1} and M_{k+1} . □

Finally, we prove the Theorem 2.2 using the Lemma 4.10.

Proof of Theorem 2.2 By Remark 3.4, by dividing $\|f\|_{L^\infty(D)}$ on both sides of (2.6) if necessary, we may assume that $\|f\|_{L^\infty(D)} \leq 1$ and $\|u\|_{C(D)} \leq c_1$. We first show that the following holds for any $x \in D$:

$$\left[\frac{u}{V(d_D)} \right]_{C^\beta(B(x,r/2))} \leq \frac{C}{r^\beta V(r)}$$

for each $0 < \beta \leq \alpha_1$, where $r = d_D(x)$. We are going to use the inequality

$$(4.40) \quad \left[\frac{u}{V(d_D)} \right]_{C^\beta} \leq \|u\|_C \left[\frac{1}{V(d_D)} \right]_{C^\beta} + [u]_{C^\beta} \left\| \frac{1}{V(d_D)} \right\|_C.$$

From (3.16) we know that $[u]_{C^{\nu}(B(x,r/2))} \leq c_2$. Thus, we have $[u]_{C^{\beta}(B(x,r/2))} \leq c_3$ for each $0 < \beta \leq \alpha_1$. Since $d_D(y) \geq r/2$ for $y \in B(x, r/2)$, we have

$$\left\| \frac{1}{V(d_D)} \right\|_{C(B(x,r/2))} \leq \frac{c_4}{V(r)}$$

and

$$\begin{aligned} \left[\frac{1}{V(d_D)} \right]_{C^{0,1}(B(x,r/2))} &\leq \sup_{y,z \in B(x,r/2)} \frac{|V(d_D(y))^{-1} - V(d_D(z))^{-1}|}{|y-z|} \\ &\leq \sup_{y,z \in B(x,r/2)} \frac{V'(d^*) |d_D(y) - d_D(z)|}{V(d^*)^2 |y-z|} \\ &\leq c_5 \left(\sup_{y,z \in B(x,r/2)} \frac{1}{d^* V(d^*)} \right) [d]_{C^{0,1}(B(x,r/2))} \\ &\leq \frac{c_6}{rV(r)}, \end{aligned}$$

where d^* is a value in $[d_D(y), d_D(z)]$, so $d^* \geq r/2$. Thus, by interpolation, we obtain

$$\left[\frac{1}{V(d_D)} \right]_{C^{\beta}(B(x,r/2))} \leq c_7 \left\| \frac{1}{V(d_D)} \right\|_{C(B(x,r/2))}^{1-\beta} \left[\frac{1}{V(d_D)} \right]_{C^{0,1}(B(x,r/2))}^{\beta} \leq \frac{c_8}{r^{\beta}V(r)}$$

and it follows from (4.40) that

$$(4.41) \quad \left[\frac{u}{V(d_D)} \right]_{C^{\beta}} \leq \frac{c_1 c_8}{r^{\beta}V(r)} + \frac{c_3 c_4}{V(r)} \leq \frac{c_9}{r^{\beta}V(r)}.$$

Next, let $x, y \in D$ and let us show that

$$\left| \frac{u(x)}{V(d_D(x))} - \frac{u(y)}{V(d_D(y))} \right| \leq C|x-y|^{\alpha}$$

for some $\alpha > 0$. Without loss of generality, we may assume that $r := d_D(x) \geq d_D(y)$. Fix any $0 < \beta \leq \alpha_1$ and let $p > 1 + \alpha_2/\beta$. If $|x-y| \leq r^p/2$, then we have $|x-y| \leq r/2$ and $y \in B(x, r/2)$ since $r \leq 1$. Thus, by (4.41) we obtain

$$\left| \frac{u(x)}{V(d_D(x))} - \frac{u(y)}{V(d_D(y))} \right| \leq c_9 \frac{|x-y|^{\beta}}{r^{\beta}V(r)} \leq c_{10} \frac{|x-y|^{\beta-\beta/p}}{V(|x-y|^{1/p})} \leq c_{11} |x-y|^{\beta-(\beta+\alpha_2)/p}.$$

On the other hand, if $|x-y| \geq r^p/2$, let $x_0, y_0 \in \partial D$ be boundary points satisfying $d_D(x) = |x-x_0|$ and $d_D(y) = |y-y_0|$. Then by the oscillation lemma 4.10 we have

$$(4.42) \quad \left| \frac{u(x)}{V(d_D)(x)} - \frac{u(x_0)}{V(d_D)(x_0)} \right| \leq c_{12} V(d_D(x))^{\gamma}, \quad \left| \frac{u(y)}{V(d_D)(y)} - \frac{u(y_0)}{V(d_D)(y_0)} \right| \leq c_{12} V(d_D(y))^{\gamma}$$

and

$$(4.43) \quad \left| \frac{u(x_0)}{V(d_D)(x_0)} - \frac{u(y_0)}{V(d_D)(y_0)} \right| \leq c_{12} V(d_D(x) + |x-y| + d_D(y))^{\gamma}.$$

Using inequalities (4.42) and (4.43) we obtain

$$\left| \frac{u(x)}{V(d_D)(x)} - \frac{u(y)}{V(d_D)(y)} \right| \leq c_{12} (2V(r)^{\gamma} + V(2r + |x-y|)^{\gamma}) \leq c_{13} |x-y|^{\alpha_1 \gamma/p}.$$

Therefore, taking $\alpha = \min \{ \beta - (\beta + \alpha_2)/p, \alpha_1 \gamma/p \}$ gives the result. \square

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