

MULTIPLY-PERIODIC HYPERSURFACES WITH CONSTANT NONLOCAL MEAN CURVATURE

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ABSTRACT. We study hypersurfaces with fractional mean curvature in N -dimensional Euclidean space. These hypersurfaces are critical points of the fractional perimeter under a volume constraint. We use local inversion arguments to prove existence of smooth branches of multiply-periodic hypersurfaces bifurcating from suitable parallel hyperplanes.

1. INTRODUCTION AND MAIN RESULT

Let $\alpha \in (0, 1)$, $N \geq 2$ and E be an open set in \mathbb{R}^N with C^2 -boundary. The Nonlocal Mean Curvature (NMC) of the set E (or the hypersurface ∂E) is defined as

$$H_E(x) := \text{PV} \int_{\mathbb{R}^N} \frac{\tau_{E^c}(y)}{|x-y|^{N+\alpha}} dy := \lim_{\varepsilon \rightarrow 0} \int_{|y-x| \geq \varepsilon} \frac{\tau_{E^c}(y)}{|x-y|^{N+\alpha}} dy \quad \text{for every } x \in \partial E. \quad (1.1)$$

Here and in the following, we use the notation

$$\tau_{E^c}(y) := 1_{E^c}(y) - 1_E(y),$$

where 1_A denotes the characteristic function of A and $E^c := \mathbb{R}^N \setminus E$. In the first integral PV denotes the principal value sense. The nonlocal mean curvature H_E , renormalized with a positive constant factor $C_{N,\alpha}$, converges locally uniformly to the classical mean curvature, as $\alpha \rightarrow 1$, (see e.g. [1, 3, 14]).

The NMC defined in (1.1) enjoys a geometric expression that can be derived via integration by parts. It is given by

$$H_E(x) = -\frac{2}{\alpha} \int_{\partial E} \frac{(x-y) \cdot \nu_E(y)}{|x-y|^{N+\alpha}} dy, \quad (1.2)$$

where $\nu_E(y)$ denotes the outer unit normal to ∂E at y .

Notice that the integral in (1.2) is absolutely convergent in the Lebesgue sense if, for $\beta > \alpha$, ∂E is of class $C^{1,\beta}$ and $\int_{\partial E} \frac{dy}{(1+|y|)^{N+\alpha-1}} < +\infty$. Moreover with this geometric expression, the NMC of any $C^{1,\beta}$ orientable hypersurface can be defined. We use both expressions (1.1) and (1.2) in the computations.

In this paper, we say that a non empty set E is a Constant Nonlocal Mean Curvature (CNMC) set or E is a set with CNMC, if $H_E : \partial E \rightarrow \mathbb{R}$ is pointwisely equivalent to

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a constant function. In this case ∂E is called a CNMC hypersurface or a hypersurface with CNMC.

It is around 2008 that Caffarelli and Souganidis in [10], and Caffarelli, Roquejoffre, and Savin in [9] introduced the notion of non local mean curvature. See e.g. [9], it is a geometric quantity that appears in the first variation of the fractional perimeter, [22]. In the recent years, several works have been devoted to the study of nonlocal minimal surfaces, while there is a lot to be understood, for instance the classification of stable nonlocal minimal cones. We refer the reader to [7] for a brief description of known results.

In the case of constant nonlocal and *nontrivial* mean curvature there have been few results that appeared in the litterature. In [7], Cabré, Fall, Solà-Morales and Weth established the analogue of the Alexandrov rigidity theorem for bounded CNMC hypersurfaces in \mathbb{R}^N . Namely the sphere is the only smooth and bounded CNMC hypersurface. This result was proved at the same time and independently by Ciruolo, Figalli, Maggi, and Novaga [11].

The question of existence of unbounded CNMC hypersurfaces (as defined above) has been first studied by Cabré, Fall, Solà-Morales and Weth in [7]. Indeed, in [7], the authors constructed a continuous branch of CNMC one-periodic bands in \mathbb{R}^2 bifurcating from a straight band $\{(s, \zeta) \in \mathbb{R} \times \mathbb{R} : |\zeta| < \lambda\}$, for some $\lambda > 0$. This result was generalized and improved very recently in [6] by Cabré, Fall and Weth. Indeed, they established in [6] a nonlocal analogue of the classical result of Delaunay [16] on periodic cylinders with constant mean curvature, the so called onduloids in \mathbb{R}^N , which are one-periodic. They bifurcate smoothly from a cylinder $\{(s, \zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} : |\zeta| < \lambda\}$. We note that the paper [15] by Dávila, del Pino, Dipierro and Valdinoci, uses variational methods to prove the existence of 1-periodic hypersurfaces of Delauney-type in \mathbb{R}^N which minimize a certain renormalized fractional perimeter under a volume constraint. It is an open problem to know if the boundary of these minimizers are CNMC surfaces as defined above. In [8], Cabré, Fall and Weth constructed CNMC sets in \mathbb{R}^N which are the countable union of a perturbed sphere and all its translations through a periodic integer lattice of dimension $M \leq N$. These CNMC sets form a C^2 branch emanating from the unit ball alone, where the parameter in the branch is the distance to the closest lattice point.

The present paper deals with new kind of unbounded multiply-periodic CNMC hypersurfaces with nontrivial NMC. Indeed, we build constant nonlocal mean curvature sets, which are $2\pi\mathbb{Z}^{N-1} \times \frac{1}{\tau}\mathbb{Z}$ -periodic, bifurcating from the translation-invariant union of parallel slabs $E_\lambda^\tau := \{(s, \zeta) \in \mathbb{R}^{N-1} \times \mathbb{R} : |\zeta| < \lambda\} + \frac{1}{\tau}e_N\mathbb{Z}$, for some $\lambda \in (0, \frac{1}{2\tau})$ and τ positive small.

We note that the hypersurfaces ∂E_λ^τ have zero local (or classical) mean curvature. However, this is not the case for the nonlocal mean curvature as can be easily verified from the geometric form of the NMC (1.2). In particular our CNMC hypersurfaces do not have counterparts in the classical theory of constant mean curvature surfaces.

We consider $u : \mathbb{R}^{N-1} \rightarrow (0, \infty)$ a $2\pi\mathbb{Z}^{N-1}$ -periodic function and $e_N = (0, \dots, 0, 1) \in \mathbb{R}^N$. We look for $2\pi\mathbb{Z}^{N-1} \times \frac{1}{\tau}\mathbb{Z}$ -periodic sets with constant nonlocal mean curvature which have the form

$$E_u^\tau = E_u + \frac{1}{\tau}e_N\mathbb{Z} = \bigcup_{q \in \mathbb{Z}} \left(E_u + \frac{q}{\tau}e_N \right), \quad (1.3)$$

where

$$E_u = \{(s, \zeta) \in \mathbb{R}^{N-1} \times \mathbb{R} : |\zeta| < u(s)\}, \quad (1.4)$$

$\tau > 0$, $e_N = (0, \dots, 0, 1)$ and $u \in C^{1,\beta}(\mathbb{R}^{N-1})$ is a $2\pi\mathbb{Z}^{N-1}$ -periodic function.

Provided $\|u\|_{L^\infty(\mathbb{R}^{N-1})} < 1/2\tau$, it is clear that E_u^τ is a set which is $2\pi\mathbb{Z}^{N-1} \times \frac{1}{\tau}\mathbb{Z}$ -periodic. Moreover, for every $\lambda \in (0, \frac{1}{2\tau})$, the set E_λ^τ is a CNMC set.

To state our main results, we introduce first the function spaces where we look for u . The space $C_{p,e}^{k,\gamma}(\mathbb{R}^{N-1})$ is the subspace of the Hölder space $C^{k,\gamma}(\mathbb{R}^{N-1})$ constituted by functions which are 2π -periodic and even in each of their variables. We then define the spaces

$$\begin{aligned} X_{\mathcal{P}} &:= \{u \in C_{p,e}^{1,\beta}(\mathbb{R}^{N-1}) : u \text{ is invariant under coordinate permutations}\}, \\ Y_{\mathcal{P}} &:= \{u \in C_{p,e}^{0,\beta-\alpha}(\mathbb{R}^{N-1}) : u \text{ is invariant under coordinate permutations}\}. \end{aligned}$$

The above spaces are equipped with their respective standard Hölder norms, see (3.1) below.

Our main result is the following.

Theorem 1.1. *Let $N \geq 2$, $\alpha \in (0, 1)$ and $\beta \in (\alpha, 1)$. There exist $\lambda_* > 0$, $\tau_0, b_0 > 0$ and a unique C^1 curve*

$$[0, \tau_0) \times (-b_0, b_0) \rightarrow X_{\mathcal{P}} \times \mathbb{R}_+, \quad (\tau, b) \mapsto (w_{\tau,b}, \lambda_{\tau,b})$$

with the following properties:

- (i) $\lambda_{0,0} = \lambda_*$, $w_{0,0} \equiv \lambda_*$,
- (ii) For all $(\tau, b) \in [0, \tau_0) \times (-b_0, b_0)$, the domains

$$E_{w_{\tau,b}}^\tau = \{(s, z) \in \mathbb{R}^{N-1} \times \mathbb{R} : |z| < w_{\tau,b}(s)\}$$

are constant non local mean curvature sets and

$$H_{E_{w_{\tau,b}}^\tau}(x) = H_{E_{\lambda_{\tau,b}}^\tau} \quad \text{for all } x \in \partial E_{w_{\tau,b}}^\tau.$$

- (iii) For all $(\tau, b) \in [0, \tau_0) \times (-b_0, b_0)$ and $s = (s_1, \dots, s_{N-1}) \in \mathbb{R}^{N-1}$, we have

$$w_{\tau,b}(s) = \lambda_{\tau,b} + b \sum_{i=1}^{N-1} \cos(s_i) + bw_{\tau,b}(s). \quad (1.5)$$

Furthermore, there exists a constant $c > 0$ such that

$$|\lambda_{\tau,b} - \lambda_*| + \|v_{\tau,b}\|_{C^{1,\beta}(\mathbb{R}^{N-1})} \leq c(|\tau|^{1+\alpha} + |b|)$$

and we have

$$\int_{[-\pi, \pi]^{N-1}} v_{\tau, b}(s) \cos(s_i) ds = 0$$

for all $i = 1, \dots, N - 1$ and $(\tau, b) \in [0, \tau_0) \times (-b_0, b_0)$.

The nonlocal mean curvature operator H of the set E_u^τ , is given by

$$H(\tau, u)(s) = H_{E_u^\tau}(s, u(s)) \quad \text{for all } s \in \mathbb{R}^{N-1}.$$

The result in Theorem 1.1 is obtained by solving the equation

$$H(\tau, \lambda + \varphi) - H(\tau, \lambda) = 0 \quad \text{on } Y_{\mathcal{P}}, \quad (1.6)$$

for some $\lambda \in \mathbb{R}$ and $\varphi \in X_{\mathcal{P}}$. This is achieved by means of the implicit function theorem. To proceed, we first provide the expression of the NMC, H_{E_u} , of the set E_u and we compute its linearization at any function u . Next we study the spectral properties of the linearized operator \mathcal{H}_λ with respect to constant functions $u \equiv \lambda > 0$. The operator \mathcal{H}_λ has a non trivial $(N - 1)$ -dimensional kernel. However, its restriction on the subspace of functions that are even in each variable and invariant under coordinates permutations yields an operator with 1-dimension kernel and whose eigenvalues are increasing as functions in the variable λ . Combining these properties with regularity results contained in [20] and [28], we gather the hypotheses that enable us to apply the implicit function theorem.

Let us notice that the equation (1.6) is valid as well for $\tau = 0$ and $H(0, u)$ is nothing, but the non local mean curvature of the set E_u , (see Section 4). As a direct consequence of our construction, we get in Corollary 1.2 below, the existence of a smooth branch of constant nonlocal mean curvature set which are $2\pi\mathbb{Z}^{N-1}$ -periodic and bifurcate from the slab $E_{\lambda_*} := \{(s, \zeta) \in \mathbb{R}^{N-1} \times \mathbb{R} : |\zeta| < \lambda_*\}$, where λ_* is the parameter in Theorem 1.1. In the particular case when $N = 2$, we also recover the branch of CNMC hypersurfaces constructed in [7].

Corollary 1.2. *Let $N \geq 2$, λ_* and b_0 be the positive constants in Theorem 1.1. Then there exists a unique C^1 curve*

$$(-b_0, b_0) \rightarrow X_{\mathcal{P}} \times \mathbb{R}_+, \quad b \mapsto (w_b, \lambda_b)$$

with the following properties:

- (i) $\lambda_0 = \lambda_*$, $w_0 \equiv \lambda_*$,
- (ii) For all $b \in (-b_0, b_0)$, the domains

$$E_{w_b} = \{(s, z) \in \mathbb{R}^{N-1} \times \mathbb{R} : |z| < w_b(s)\}$$

are constant non local mean curvature sets, and

$$H_{E_{w_b}}(x) = H_{E_{\lambda_b}} \quad \text{for all } x \in \partial E_{w_b}.$$

The result in this paper parallels the result of Fall, Minlend and Weth in [21], where the authors built unbounded periodic domains to Serrin's overdetermined boundary value problem. The boundary of these domains bifurcate from generalized slabs. We also quote the related papers [27] and [26], where it is proved the existence of Delauney-type domains in \mathbb{R}^N whose first Dirichlet eigenfunction has constant Neumann data on the boundary.

Further notable related results are contained in [19], where the author considers equilibrium surfaces which are critical points to an energy constituted by the sum of the area functional and a repulsive Coulomb-type potential, arising in diblock copolymer melt. Among other results, the paper [19] establishes the existence of multiply-periodic equilibrium patterns domains bifurcating from parallel slabs. In [14], Davilá, del Pino and Wei proved the existence of an unstable two-sheet nonlocal minimal surface which is symmetric with respect to the horizontal plane, bifurcating from two parallel planes.

The paper is organized as follows. In Section 2, we provide the expression of H_{E_u} which will be used to prove the regularity of the NMC operator. In Section 3, we study the spectral properties of the linearized operator at a constant function $\lambda > 0$ and provide qualitative properties of its eigenvalues, as λ varies. In Section 4, we prove Theorem 1.1 and Corollary 1.2. Finally the proof of the regularity of the NMC operator, respectively the computations of the explicit expression of the first derivative, are done in Section 5.

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2. THE NMC OPERATOR OF PERTURBED SLABS

Let $\alpha \in (0, 1)$ and $\beta \in (\alpha, 1)$. For a positive function $u \in C^{1,\beta}(\mathbb{R}^{N-1})$, we consider the set E_u as defined in (1.4). We first recall the following expression for the NMC of E_u :

$$H_{E_u}(x) = -\frac{2}{\alpha} \int_{\partial E_u} |x - y|^{-(N+\alpha)} (x - y) \cdot \nu_{E_u}(y) dy; \quad (2.1)$$

see e.g. [7]. Here $\nu_{E_u}(y)$ denotes the unit outer normal of ∂E_u and dy is the volume element of ∂E_u . Next, we consider the open set

$$\mathcal{O} := \left\{ u \in C^{1,\beta}(\mathbb{R}^{N-1}) : \inf_{\mathbb{R}^{N-1}} u > 0 \right\}. \quad (2.2)$$

For $u \in \mathcal{O}$, we consider the map $F_u : \mathbb{R}^{N-1} \times \mathbb{R} \rightarrow \mathbb{R}^N$ given by

$$F_u(s, z) = (s, u(s)z).$$

The boundary of E_u ,

$$\partial E_u = \{(s, u(s)\sigma) \in \mathbb{R}^{N-1} \times \mathbb{R} : \sigma \in \{-1, 1\}\},$$

is parameterized by the restriction of F_u to $\mathbb{R}^{N-1} \times \{-1, 1\}$.

2.1. The NMC operator. The following result provides the expression for the NMC of E_u in terms of the above parametrization and the function u .

Lemma 2.1. *Let $u \in \mathcal{O}$. Then the nonlocal mean curvature H_{E_u} —that we will denote by $H(u)(s)$ — at a point $(s, u(s)\theta)$, with $\theta \in \{-1, 1\}$, does not depend on θ and is given by*

$$\begin{aligned} -\frac{\alpha}{2}H(u)(s) &= \int_{\mathbb{R}^{N-1}} \frac{u(s) - u(s-t) - t \cdot \nabla u(s-t)}{\{|t|^2 + (u(s) - u(s-t))^2\}^{(N+\alpha)/2}} dt \\ &\quad - \int_{\mathbb{R}^{N-1}} \frac{u(s) + u(s-t) + t \cdot \nabla u(s-t)}{\{|t|^2 + (u(s) + u(s-t))^2\}^{(N+\alpha)/2}} dt. \end{aligned} \quad (2.3)$$

Moreover, the two integrals above converge absolutely in the Lebesgue sense.

Proof. The proof is similar to the one in [6, Lemma 2.1]. We then skip it. \square

3. PROPERTIES OF THE LINEARIZED NMC OPERATOR

3.1. The linearized NMC operator. In order to compute the linearized nonlocal mean curvature operator, we need to introduce the functional spaces in which we work. Consider the Banach spaces

$$C_p^{k,\gamma}(\mathbb{R}^{N-1}) = \{u \in C^{k,\gamma}(\mathbb{R}^{N-1}) : u \text{ is } 2\pi\mathbb{Z}^{N-1}\text{-periodic}\}$$

and

$$C_{p,e}^{k,\gamma}(\mathbb{R}^{N-1}) = \{u \in C^{k,\gamma}(\mathbb{R}^{N-1}) : u \text{ is } 2\pi\mathbb{Z}^{N-1}\text{-periodic and even in each variable}\},$$

for $k \in \mathbb{N}$ and $\gamma \in (0, 1)$. The norms in two spaces above are the standard $C^{k,\gamma}(\mathbb{R}^{N-1})$ -norms, defined by

$$\|u\|_{C^{k,\gamma}(\mathbb{R}^{N-1})} := \sum_{j=0}^k \|D^j u\|_{L^\infty(\mathbb{R}^{N-1})} + \sup_{\substack{s,r \in \mathbb{R}^{N-1} \\ s \neq r}} \frac{|D^k u(s) - D^k u(r)|}{|s-r|^\gamma}. \quad (3.1)$$

We denote by \mathcal{P} the set of $(N-1) \times (N-1)$ -matrix of permutations. We then define

$$\begin{aligned} X_{\mathcal{P}} &:= \{u \in C_{p,e}^{1,\beta}(\mathbb{R}^{N-1}) : u(s) = u(\mathbf{p}(s)), \quad \text{for all } s \in \mathbb{R}^{N-1} \text{ and } \mathbf{p} \in \mathcal{P}\}, \\ Y_{\mathcal{P}} &:= \{\tilde{u} \in C_{p,e}^{0,\beta-\alpha}(\mathbb{R}^{N-1}) : \tilde{u}(s) = \tilde{u}(\mathbf{p}(s)), \quad \text{for all } s \in \mathbb{R}^{N-1} \text{ and } \mathbf{p} \in \mathcal{P}\}. \end{aligned}$$

The norms of $X_{\mathcal{P}}$ and $Y_{\mathcal{P}}$, are those of $C^{1,\beta}(\mathbb{R}^{N-1})$ and $C^{0,\beta-\alpha}(\mathbb{R}^{N-1})$ respectively.

The map $H : \mathcal{O} \subset C^{1,\beta}(\mathbb{R}^{N-1}) \rightarrow C^{0,\beta-\alpha}(\mathbb{R}^{N-1})$ is smooth in \mathcal{O} by Proposition 3.1, and clearly H sends $X_{\mathcal{P}} \rightarrow Y_{\mathcal{P}}$ by change of variables.

We will use the expression (2.3) to show the smoothness of $H : \mathcal{O} \rightarrow C^{0,\beta-\alpha}(\mathbb{R}^{N-1})$. We now state the following fundamental result which will be proved in Section 5.

Proposition 3.1. *The map $H : \mathcal{O} \subset C^{1,\beta}(\mathbb{R}^{N-1}) \rightarrow C^{0,\beta-\alpha}(\mathbb{R}^{N-1})$ is of class C^∞ . Moreover for every $k \in \mathbb{N}$ and $u \in \mathcal{O}$, there exists a constant $c > 0$, only depending on α, β, N and $\inf_{\mathbb{R}^{N-1}} u$, such that*

$$\|D^k H(u)\| \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c. \quad (3.2)$$

In addition, if $u \in \mathcal{O}$, $\lambda > 0$ and $v, w \in C^{1,\beta}(\mathbb{R}^{N-1})$, then

$$\begin{aligned} DH(u)[v](s) &= 2PV \int_{\mathbb{R}^{N-1}} \frac{v(s) - v(s-t)}{(|t|^2 + (u(s) - u(s-t))^2)^{\frac{N+\alpha}{2}}} dt \\ &\quad - 2 \int_{\mathbb{R}^{N-1}} \frac{v(s) + v(s-t)}{(|t|^2 + (u(s) + u(s-t))^2)^{\frac{N+\alpha}{2}}} dt. \end{aligned} \quad (3.3)$$

Next we need to study properties of the family of linearized operators

$$L_\lambda := \lambda^{1+\alpha} DH(\lambda) \in \mathcal{B}(C_{p,e}^{1,\beta}(\mathbb{R}^{N-1}), C_{p,e}^{0,\beta-\alpha}(\mathbb{R}^{N-1})), \quad \lambda > 0.$$

Here and in the following, $\mathcal{B}(X, Y)$ denotes the space of bounded linear operators between Banach spaces X and Y .

3.2. Study of the linearized NMC operator at constant functions. We now study the behavior of the eigenvalues of the operator L_λ , as $\lambda > 0$ varies. By Proposition 3.1, L_λ is given by

$$L_\lambda v(s) = 2\lambda^{1+\alpha} \left(PV \int_{\mathbb{R}^{N-1}} \frac{v(s) - v(s-t)}{|t|^{N+\alpha}} dt - \int_{\mathbb{R}^{N-1}} \frac{v(s) + v(s-t)}{(|t|^2 + 4\lambda^2)^{\frac{N+\alpha}{2}}} dt \right). \quad (3.4)$$

Lemma 3.2. *Let $\lambda > 0$. The functions $e_k \in C_{p,e}^{1,\beta}(\mathbb{R}^{N-1})$ given by*

$$s = (s_1, \dots, s_{N-1}) \mapsto e_k(s) = \prod_{i=1}^{N-1} \cos(k_i s_i), \quad k = (k_1, \dots, k_{N-1}) \in \mathbb{N}^{N-1} \quad (3.5)$$

are the eigenfunctions of $L_\lambda : C_{p,e}^{1,\beta}(\mathbb{R}^{N-1}) \rightarrow C_{p,e}^{0,\beta-\alpha}(\mathbb{R}^{N-1})$ with

$$L_\lambda e_k = \nu(\lambda|k|)e_k, \quad (3.6)$$

where $|k| = \sqrt{k_1^2 + \dots + k_{N-1}^2}$ and $\nu : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$\nu(R) = 2R^{1+\alpha} \left(\int_{\mathbb{R}^{N-1}} \frac{1 - \cos(t_1)}{|t|^{N+\alpha}} dt - \int_{\mathbb{R}^{N-1}} \frac{1 + \cos(t_1)}{(|t|^2 + 4R^2)^{\frac{N+\alpha}{2}}} dt \right). \quad (3.7)$$

Moreover,

$$\nu'(R) > 0 \quad \text{for every } R > 0, \quad (3.8)$$

$$\lim_{R \rightarrow +\infty} \frac{\nu(R)}{R^{1+\alpha}} = 2 \int_{\mathbb{R}^{N-1}} \frac{1 - \cos(t_1)}{|t|^{N+\alpha}} dt > 0 \quad (3.9)$$

and

$$\nu_0 := \lim_{R \rightarrow 0} \nu(R) = -4 \int_{\mathbb{R}^{N-1}} \frac{1}{(|t|^2 + 4)^{\frac{N+\alpha}{2}}} dt < 0. \quad (3.10)$$

Proof. To get (3.7), we prove by induction on $\ell \in \mathbb{N}$, $\ell \geq 1$ that for any even function E defined on \mathbb{R}^ℓ ,

$$\int_{\mathbb{R}^\ell} E(t) \prod_{i=1}^{\ell} \cos(k_i s_i - k_i t_i) dt = \prod_{i=1}^{\ell} \cos(k_i s_i) \int_{\mathbb{R}^\ell} \prod_{i=1}^{\ell} \cos(k_i t_i) E(t) dt. \quad (3.11)$$

Recall the fact that

$$\cos(k_i s_i - k_i t_i) = \cos(k_i s_i) \cos(k_i t_i) + \sin(k_i s_i) \sin(k_i t_i). \quad (3.12)$$

For $\ell = 1$, (3.11) is obviously true using (3.12), the evenness of E and the oddness of the sine function.

For $\ell = 2$, we have

$$\begin{aligned} & \int_{\mathbb{R}^2} E(t) \prod_{i=1}^2 \cos(k_i s_i - k_i t_i) dt \\ &= \int_{\mathbb{R}} \cos(k_2 s_2 - k_2 t_2) \int_{\mathbb{R}} \left([\cos(k_1 s_1) \cos(k_1 t_1) + \sin(k_1 s_1) \sin(k_1 t_1)] E(t) dt_1 \right) dt_2 \\ &= \int_{\mathbb{R}} \cos(k_2 s_2 - k_2 t_2) \left(\int_{\mathbb{R}} \cos(k_1 s_1) \cos(k_1 t_1) E(t) dt_1 \right) dt_2, \end{aligned} \quad (3.13)$$

where we have used the evenness of E to get the last line.

Arguing as in (3.13), we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} E(t) \prod_{i=1}^2 \cos(k_i s_i - k_i t_i) dt &= \int_{\mathbb{R}} \cos(k_1 s_1) \cos(k_1 t_1) \left(\int_{\mathbb{R}} \cos(k_2 s_2 - k_2 t_2) E(t) dt_2 \right) dt_1 \\ &= \int_{\mathbb{R}} \cos(k_1 s_1) \cos(k_1 t_1) \left(\int_{\mathbb{R}} \cos(k_2 s_2) \cos(k_2 t_2) E(t) dt_2 \right) dt_1 \\ &= \cos(k_1 s_1) \cos(k_2 s_2) \int_{\mathbb{R}^2} \cos(k_1 t_1) \cos(k_2 t_2) E(t) dt \\ &= \prod_{i=1}^2 \cos(k_i s_i) \int_{\mathbb{R}^2} \prod_{i=1}^2 \cos(k_i t_i) E(t) dt. \end{aligned} \quad (3.14)$$

Let us now assume that (3.11) holds true for $\ell - 1$, ($\ell \in \mathbb{N}$), then

$$\begin{aligned}
 & \int_{\mathbb{R}^\ell} E(t) \prod_{i=1}^{\ell} \cos(k_i s_i - k_i t_i) dt \\
 &= \int_{\mathbb{R}} \cos(k_\ell s_\ell - k_\ell t_\ell) \left(\int_{\mathbb{R}^{\ell-1}} E(t) \prod_{i=1}^{\ell-1} \cos(k_i s_i - k_i t_i) dt_1 \dots dt_{\ell-1} \right) dt_\ell \\
 &= \prod_{i=1}^{\ell-1} \cos(k_i s_i) \int_{\mathbb{R}^{\ell-1}} \prod_{i=1}^{\ell-1} \cos(k_i t_i) \left(\int_{\mathbb{R}} E(t) \cos(k_\ell s_\ell - k_\ell t_\ell) dt_\ell \right) dt_1 \dots dt_{\ell-1} \\
 &= \prod_{i=1}^{\ell} \cos(k_i s_i) \int_{\mathbb{R}^\ell} E(t) \prod_{i=1}^{\ell} \cos(k_i t_i) dt. \tag{3.15}
 \end{aligned}$$

With this we conclude that (3.11) holds true for all $\ell \in \mathbb{N}$

Applying (3.11) to the even functions $t \mapsto |t|^{-N-\alpha}$ and $t \mapsto (|t|^2 + 4\lambda^2)^{-\frac{N+\alpha}{2}}$, we get

$$\begin{aligned}
 & \int_{\mathbb{R}^{N-1}} \frac{e_k(s) - e_k(s-t)}{|t|^{N+\alpha}} dt = e_k(s) \int_{\mathbb{R}^{N-1}} \frac{1 - e_k(t)}{|t|^{N+\alpha}} dt, \\
 & \int_{\mathbb{R}^{N-1}} \frac{e_k(s-t)}{(|t|^2 + 4\lambda^2)^{\frac{N+\alpha}{2}}} dt = e_k(s) \int_{\mathbb{R}^{N-1}} \frac{e_k(t)}{(|t|^2 + 4\lambda^2)^{\frac{N+\alpha}{2}}} dt.
 \end{aligned}$$

and it follows that

$$L_\lambda e_k(s) = \lambda^{1+\alpha} \sigma_k(\lambda) e_k(s), \tag{3.16}$$

where we have set

$$\sigma_k(\lambda) := 2 \int_{\mathbb{R}^{N-1}} \frac{1 - e_k(t)}{|t|^{N+\alpha}} dt - 2 \int_{\mathbb{R}^{N-1}} \frac{1 + e_k(t)}{(|t|^2 + 4\lambda^2)^{\frac{N+\alpha}{2}}} dt. \tag{3.17}$$

Now by the change of variables $t = \frac{\bar{t}}{|k|}$, we get

$$\sigma_k(\lambda) = 2|k|^{1+\alpha} \int_{\mathbb{R}^{N-1}} \frac{1 - \prod_{i=1}^{N-1} \cos\left(\frac{\bar{t}_i k_i}{|k|}\right)}{|\bar{t}|^{N+\alpha}} d\bar{t} - 2|k|^{1+\alpha} \int_{\mathbb{R}^{N-1}} \frac{1 + \prod_{i=1}^{N-1} \cos\left(\frac{\bar{t}_i k_i}{|k|}\right)}{(|\bar{t}|^2 + 4|k|^2 \lambda^2)^{\frac{N+\alpha}{2}}} d\bar{t}.$$

We claim that the first integral in the above expression does not depend on k , while the second one is a function of $|k|$. We consider $f : S^{N-2} \rightarrow \mathbb{R}$ given by, for all $\theta \in S^{N-2}$

$$f(\theta) := \int_0^\infty \rho^{-1-\alpha} \int_{S^{N-2}} \left(1 - \prod_{i=1}^{N-1} \cos(\rho(\sigma \cdot e_i)(\theta \cdot e_i)) \right) d\sigma d\rho.$$

Let $\mathcal{R} \in O(N-1)$ be a rotation matrix and denote by \mathcal{R}^t its transpose. By a change of variables $\sigma = \mathcal{R}\tilde{\sigma}$, we get

$$\begin{aligned} f(\mathcal{R}\theta) &= \int_0^\infty \rho^{-1-\alpha} \int_{\mathcal{R}^t S^{N-2}} \left(1 - \prod_{i=1}^{N-1} \cos(\rho(\tilde{\sigma} \cdot \mathcal{R}^t e_i)(\theta \cdot \mathcal{R}^t e_i)) \right) d\tilde{\sigma} d\rho \\ &= \int_0^\infty \rho^{-1-\alpha} \int_{S^{N-2}} \left(1 - \prod_{i=1}^{N-1} \cos(\rho(\sigma \cdot e_i)(\theta \cdot e_i)) \right) d\sigma d\rho \\ &= f(\theta). \end{aligned}$$

Since \mathcal{R} is arbitrary, it then follows that

$$f(\theta) = f(e_1).$$

Using polar coordinates, we get

$$\int_{\mathbb{R}^{N-1}} \frac{1 - \prod_{i=1}^{N-1} \cos\left(\frac{\bar{t}_i k_i}{|k|}\right)}{|\bar{t}|^{N+\alpha}} d\bar{t} = f\left(\frac{k}{|k|}\right) = f(e_1).$$

That is

$$\int_{\mathbb{R}^{N-1}} \frac{1 - \prod_{i=1}^{N-1} \cos\left(\frac{\bar{t}_i k_i}{|k|}\right)}{|\bar{t}|^{N+\alpha}} d\bar{t} = \int_{\mathbb{R}^{N-1}} \frac{1 - \cos(\bar{t}_1)}{|\bar{t}|^{N+\alpha}} d\bar{t}.$$

By a similar reasoning, we also have

$$\int_{\mathbb{R}^{N-1}} \frac{1 + \prod_{i=1}^{N-1} \cos\left(\frac{\bar{t}_i k_i}{|k|}\right)}{(|\bar{t}|^2 + 4|k|^2\lambda^2)^{\frac{N+\alpha}{2}}} d\bar{t} = \int_{\mathbb{R}^{N-1}} \frac{1 + \cos(\bar{t}_1)}{(|\bar{t}|^2 + 4|k|^2\lambda^2)^{\frac{N+\alpha}{2}}} d\bar{t}.$$

The last two equalities above allows to see that $\lambda^{1+\alpha}\sigma_k(\lambda) = \nu(\lambda|k|)$, where ν is the function defined in (3.7). This also yields (3.6).

Now (3.8) and (3.9) follows immediately from the expression of ν . Finally, to see (3.10), we observe that

$$R^{1+\alpha} \int_{\mathbb{R}^{N-1}} \frac{1 + \cos(\bar{t}_1)}{(|\bar{t}|^2 + 4R^2)^{\frac{N+\alpha}{2}}} d\bar{t} = \int_{\mathbb{R}^{N-1}} \frac{1 + \cos(Rt_1)}{(|t|^2 + 4)^{\frac{N+\alpha}{2}}} dt,$$

where we made the change of variable $\bar{t} = Rt$.

□

In the following, we consider the fractional Sobolev spaces

$$H_{p,e}^\sigma := \left\{ v \in H_{loc}^\sigma(\mathbb{R}^{N-1}) : v \text{ is } 2\pi\mathbb{Z}^{N-1}\text{-periodic and even in each variable} \right\} \quad (3.18)$$

for $\sigma \geq 0$, and we put $L_{p,e}^2 := H_{p,e}^0$. Note that $L_{p,e}^2$ is a Hilbert space with scalar product

$$(u, v) \mapsto \langle u, v \rangle_{L_{p,e}^2} := \int_{[-\pi, \pi]^{N-1}} u(t)v(t) dt \quad \text{for } u, v \in L_{p,e}^2.$$

We denote the induced norm by $\|\cdot\|_{L^2_{p,e}}$. Since $\|e_k\|_{L^2_{p,e}} = \pi^{\frac{N-1}{2}}$, it follows that the set $\{\frac{e_k}{\pi^{\frac{N-1}{2}}}, k \in \mathbb{N}^{N-1}\}$ forms a complete orthonormal basis of $L^2_{p,e}$. Moreover, $H_{p,e}^\sigma \subset L^2_{p,e}$ is characterized as the subspace of all functions $v \in L^2_{p,e}$ such that

$$\sum_{k \in \mathbb{N}^{N-1}} (1 + |k|^2)^\sigma \langle v, e_k \rangle_{L^2_{p,e}}^2 < \infty.$$

Therefore, $H_{p,e}^\sigma$ is also a Hilbert space with scalar product

$$(u, v) \mapsto \sum_{k \in \mathbb{N}^{N-1}} (1 + |k|^2)^\sigma \langle u, e_k \rangle_{L^2_{p,e}} \langle v, e_k \rangle_{L^2_{p,e}} \quad \text{for } u, v \in H_{p,e}^\sigma. \quad (3.19)$$

We consider the eigenspaces V_ℓ corresponding to the eigenvalues $\nu(\lambda\ell)$ of the operator L_λ , defined as

$$V_\ell := \text{span} \{e_k : |k| = \ell\} \subset \bigcap_{j \in \mathbb{N}} H_{p,e}^j. \quad (3.20)$$

We also denote their orthogonal L^2 -complements by

$$V_\ell^\perp := \left\{ w \in L^2_{p,e} : \int_{[-\pi, \pi]^{N-1}} v(s)w(s) ds = 0 \quad \text{for every } v \in V_\ell \right\}.$$

Obviously, by Lemma 3.2,

$$L_\lambda v = \nu(\lambda\ell)v \quad \text{for every } v \in V_\ell. \quad (3.21)$$

We note that

$$V_1 \cap X_{\mathcal{P}} = V_1 \cap Y_{\mathcal{P}} = \text{span}\{\bar{v}\} \quad \text{with } \bar{v}(t) = \sum_{i=1}^{N-1} \cos(t_i). \quad (3.22)$$

The following proposition contains important results that we will use for the proof of Theorem 1.1 in the next section. To apply the implicit function theorem, we will need the following result.

Proposition 3.3. *There exists a unique $\lambda_* > 0$, only depending on α, β and N , such that the linear operator $L_* := L_{\lambda_*} : X_{\mathcal{P}} \rightarrow Y_{\mathcal{P}}$ has the following properties.*

(i) *The kernel $N(L_*)$ of L_* is spanned by the function*

$$\bar{v} \in X_{\mathcal{P}}, \quad \bar{v}(t_1, \dots, t_{N-1}) = \cos(t_1) + \dots + \cos(t_{N-1}). \quad (3.23)$$

(ii) *$L_* : X_{\mathcal{P}} \cap V_1^\perp \rightarrow Y_{\mathcal{P}} \cap V_1^\perp$ is an isomorphism.*

Moreover

$$\partial_\lambda \Big|_{\lambda=\lambda_*} L_\lambda \bar{v} = \nu'(\lambda_*) \bar{v} \notin Y_{\mathcal{P}} \cap V_1^\perp. \quad (3.24)$$

Proof. By Lemma 3.2, there exists a unique $\lambda_* > 0$ such that $\nu(\lambda_*) = 0$ and $\nu' > 0$ on $(0, +\infty)$. This with (3.21) and (3.22) imply that $N(L_*) = \text{span}\{\bar{v}\}$. This proves (i) and (3.24).

To prove (ii) we pick $g \in Y_{\mathcal{P}} \cap V_1^\perp \subset L_{p,e}^2$. Then by (3.21), the asymptotics (3.9) and the definition of fractional Sobolev spaces in terms of Fourier coefficients, we deduce that there exists a unique $w \in H_{p,e}^{1+\alpha}$ with

$$\int_{[-\pi,\pi]^{N-1}} \bar{v}(s)w(s) ds = 0 \quad (3.25)$$

such that

$$L_* w = g. \quad (3.26)$$

By the uniqueness, we also have $w(s) = w(\mathbf{p}(s))$, for every $s \in \mathbb{R}^{N-1}$ and $\mathbf{p} \in \mathcal{P}$. We define $P : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ by

$$P(s) := \frac{1}{(|s|^2 + 4\lambda_*^2)^{(N+\alpha)/2}} \quad (3.27)$$

and

$$c_* := \int_{\mathbb{R}^{N-1}} P(s) ds.$$

Then (3.26) can be written as

$$PV \int_{\mathbb{R}^{N-1}} \frac{w(s) - w(s-t)}{|t|^{N+\alpha}} dt = c_* w(s) + P \star w(s) + \frac{1}{2\lambda_*^{1+\alpha}} g(s) \quad \text{for } s \in \mathbb{R}^{N-1},$$

where \star denotes convolution product. This is equivalent to,

$$(-\Delta)^\sigma w(s) - c_* w(s) = f(s) \quad \text{for } s \in \mathbb{R}^{N-1}, \quad (3.28)$$

where $\sigma := \frac{1+\alpha}{2}$ and

$$f := P \star w + \frac{1}{2\lambda_*^{2\sigma}} g. \quad (3.29)$$

We now use the results in [20] and [28] to prove that $w \in C^{1,\beta}(\mathbb{R}^{N-1})$, $\beta \in (\alpha, 1)$.

We start by proving that $P \star w \in L^\infty(\mathbb{R}^{N-1})$. Remark that the function P defined in (3.27) belongs to $L^2(\mathbb{R}^{N-1})$ to $w \in H^{1+\alpha} \subset L_{loc}^2 \mathbb{R}^{N-1}$.

Let $y \in [-\pi, \pi]^{N-1}$. We have

$$\begin{aligned} |P \star w(y)| &\leq \sum_{k \in \mathbb{Z}^{N-1}} \int_{[-\pi,\pi]^{N-1}} P(y-z+2\pi k) |w(z)| dz \\ &\leq \sum_{k \in \mathbb{Z}^{N-1}} \left(\int_{[-\pi,\pi]^{N-1}} P^2(y-z+2\pi k) dz \right)^{1/2} \|w\|_{L^2([- \pi, \pi]^{N-1})}. \end{aligned}$$

Since $y, z \in [-\pi, \pi]^{N-1}$, it follows that

$$|y-z+2\pi k|^2 + 4\lambda_*^2 \geq C(N)|k|^2, \quad \text{for } |k| \geq 2N,$$

where $C(N)$ is a positive constant only depending on N .

It is also plain that

$$|y-z+2\pi k|^2 + 4\lambda_*^2 \geq 4\lambda_*^2 \quad \text{for all } k \in \mathbb{Z}^{N-1}.$$

Thus

$$|P \star w(y)| \leq C(N, \lambda_*, w, \alpha) \left(\sum_{k \in \mathbb{Z}^{N-1}, |k| < 2N} + \sum_{k \in \mathbb{Z}^{N-1}, |k| \geq 2N} \frac{1}{|k|^{N+\alpha}} \right), \quad (3.30)$$

where $C(N, \lambda_*, w, \alpha)$ is a positive constant depending on N , α , w and λ_* . The function $P \star w$ being also $2\pi\mathbb{Z}^{N-1}$ -periodic, we deduce from (3.30) that $P \star w \in L^\infty(\mathbb{R}^{N-1})$ and by [20], it follows that $w \in C^{0, \beta-\alpha}(\mathbb{R}^{N-1})$.

Next, we write (3.31) on the form

$$(-\Delta)^\sigma w(s) = c_* w(s) + f(s) \quad \text{for } s \in \mathbb{R}^{N-1}, \quad (3.31)$$

where f is defined by (3.29). Observe also that $P \star w \in C^{0, \beta-\alpha}(\mathbb{R}^{N-1})$, since $P \in L^1(\mathbb{R}^{N-1})$. We finally apply [28, Proposition 2.1.8] to get $w \in C^{1, \beta}(\mathbb{R}^{N-1})$, and (3.25) yields $w \in X_{\mathcal{P}} \cap V_1^\perp$ which completes the proof. \square

4. PROOF OF THEOREM 1.1

For $\tau \in \mathbb{R} \setminus \{0\}$, $u \in \mathcal{O}$, with $\|u\|_{L^\infty(\mathbb{R}^{N-1})} < \frac{1}{2|\tau|}$, we consider the sets

$$E_u^\tau := E_u + \frac{1}{\tau} e_N \mathbb{Z} = \bigcup_{q \in \mathbb{Z}} \left(E_u + \left(0, \dots, \frac{q}{\tau} \right) \right). \quad (4.1)$$

We recall that

$$E_u = \{(s, \zeta) \in \mathbb{R}^{N-1} \times \mathbb{R} : |\zeta| < u(s)\} = \{(s, zu(s)) \in \mathbb{R}^{N-1} \times \mathbb{R} : |z| < 1\}.$$

It is easy to see that $H_{E_u^\tau}(x + \frac{q}{\tau} e_N) = H_{E_u^\tau}(x)$ for very $x \in \partial E_u$ and $q \in \mathbb{Z}$. Moreover, it is clear that in (4.1), we have a disjoint union.

Proposition 4.1. *Let $\tau \in \mathbb{R} \setminus \{0\}$, $u \in \mathcal{O}$ be such that $\|u\|_{L^\infty(\mathbb{R}^{N-1})} < \frac{1}{2|\tau|}$. Then for every $\theta \in \{1, -1\}$ and $s \in \mathbb{R}^{N-1}$, we have*

$$H_{E_u^\tau}(F_u(s, \theta)) = H(u)(s) - 2 \sum_{q \in \mathbb{Z}_*} \int_{E_1} \frac{u(s-t) dt dz}{\{|t|^2 + (u(s) - u(s-t))z - \frac{q}{\tau}\}^{(N+\alpha)/2}}, \quad (4.2)$$

where $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$. In particular, if $|\lambda| < \frac{1}{2|\tau|}$, then E_λ^τ is a set with constant nonlocal mean curvature.

Proof. Let $s \in \mathbb{R}^{N-1}$ and $\theta \in \{-1, 1\}$. Then putting $x = F_u(s, \theta)$, we then have

$$\begin{aligned} H_{E_u^\tau}(F_u(s, \theta)) &= -\frac{2}{\alpha} \int_{\partial E_u^\tau} \frac{(x-y) \cdot \nu_{E_u^\tau}(y)}{|x-y|^{N+\alpha}} dy \\ &= -\frac{2}{\alpha} \sum_{q \in \mathbb{Z}} \int_{\partial E_u} \frac{(x-y - \frac{q}{\tau} e_N) \cdot \nu_{E_u}(y)}{|x-y - \frac{q}{\tau} e_N|^{N+\alpha}} dy. \end{aligned}$$

This implies that

$$H_{E_u^\tau}(F_u(s, \theta)) = H(u)(s) - \frac{2}{\alpha} \sum_{q \in \mathbb{Z}_*} \int_{\partial E_u} \frac{(x-y - \frac{q}{\tau} e_N) \cdot \nu_{E_u}(y)}{|x-y - \frac{q}{\tau} e_N|^{N+\alpha}} dy.$$

Using the fact that

$$\nabla_y \cdot \left\{ (x - y - \frac{q}{\tau} e_N) |x - y - \frac{q}{\tau} e_N|^{-(N+\alpha)} \right\} = \alpha |x - y - \frac{q}{\tau} e_N|^{-(N+\alpha)}$$

and the divergence theorem, we get

$$H_{E_u^\tau}(F_u(s, \theta)) = H(u)(s) - 2 \sum_{q \in \mathbb{Z}_*} \int_{E_u} \frac{1}{|x - y - \frac{q}{\tau} e_N|^{N+\alpha}} dy,$$

where we have used the fact that $y \mapsto |x - y - \frac{q}{\tau} e_N|^{-(N+\alpha)}$ is smooth in E_u for every $x \in \partial E_u$.

By change of variables $y = F_u(\bar{s}, z) = (\bar{s}, zu(\bar{s}))$, we deduce that

$$H_{E_u^\tau}(F_u(s, \theta)) = H(u)(s) - 2 \sum_{q \in \mathbb{Z}_*} \int_{E_1} \frac{u(\bar{s}) d\bar{s} dz}{|F_u(s, \theta) - F_u(\bar{s}, z) - \frac{q}{\tau} e_N|^{N+\alpha}},$$

recalling that $E_1 = \{(s, z) \in \mathbb{R}^{N-1} \times \mathbb{R} : |z| < 1\} = \mathbb{R}^{N-1} \times (-1, 1)$. Since

$$|F_u(s, \theta) - F_u(\bar{s}, z) - \frac{q}{\tau} e_N|^2 = |s - \bar{s}|^2 + |u(s)\theta - u(\bar{s})z - \frac{q}{\tau}|^2,$$

it follows that

$$H_{E_u^\tau}(F_u(s, \theta)) = H(u)(s) - 2 \sum_{q \in \mathbb{Z}_*} \int_{E_1} \frac{u(s-t) dt dz}{\{|t|^2 + (u(s)\theta - u(s-t)z - \frac{q}{\tau})^2\}^{(N+\alpha)/2}},$$

where we made the change of variable $\bar{s} = s - t$. By the change of variables $z \rightarrow -z$ and using that $\mathbb{Z}_* = -\mathbb{Z}_*$, we get

$$H_{E_u^\tau}(F_u(s, \theta)) = H(u)(s) - 2 \sum_{q \in \mathbb{Z}_*} \int_{E_1} \frac{u(s-t) dt dz}{\{|t|^2 + (u(s)\theta + u(s-t)z + \frac{q}{\tau})^2\}^{(N+\alpha)/2}}.$$

From the two identities above, it is easy to see that

$$H_{E_u^\tau}(F_u(s, 1)) = H_{E_u^\tau}(F_u(s, -1)) \quad \text{for every } s \in \mathbb{R}^{N-1}.$$

This implies, in particular, that if $u \equiv \lambda$, a constant, then E_λ^τ is a CNMC set. \square

We will consider in the following of this section, the parameter $\lambda_* > 0$ given by Proposition 3.3 and we put

$$\tau_1 := \frac{1}{6\lambda_*}.$$

Then, for all $\tau \in (-\tau_1, \tau_1)$ and $u \in B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2) \subset \mathcal{O} \cap X_{\mathcal{P}}$, we have

$$|u(s) + u(s-t)z - \frac{q}{\tau}| \geq \frac{|q|}{2\tau_1} \quad \text{for all } z \in [-1, 1] \text{ and all } s, t \in \mathbb{R}^{N-1}. \quad (4.3)$$

We define the map

$$\mathcal{H} : (-\tau_1, \tau_1) \times B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2) \rightarrow Y_{\mathcal{P}} \in C^{0, \beta-\alpha}(\mathbb{R}^{N-1})$$

given by, for every $u \in B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2)$,

$$\mathcal{H}(\tau, u)(s) = \begin{cases} \sum_{q \in \mathbb{Z}_*} \int_{E_1} \frac{u(s-t) dt dz}{\{|t|^2 + (u(s) - u(s-t)z - q/\tau)^2\}^{\frac{N+\alpha}{2}}} & \text{if } \tau \neq 0, \\ 0 & \text{if } \tau = 0. \end{cases}$$

This implies that, for every $\tau \in (-\tau_1, \tau_1) \setminus \{0\}$, $u \in B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2)$, $\theta \in \{-1, 1\}$ and $s \in \mathbb{R}^{N-1}$,

$$H_{E_t^+}(F_u(s, \theta)) = H(u)(s) - 2\mathcal{H}(\tau, u)(s). \quad (4.4)$$

In the following result, we prove regularity estimates for the map \mathcal{H} .

Proposition 4.2. *The maps \mathcal{H} is of class C^1 on $(-\tau_1, \tau_1) \times B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2)$. Moreover there exists a constant $c = c(N, \alpha, \beta) > 0$ such that for every $u \in B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2)$ and $\tau \in (-\tau_1, \tau_1)$,*

$$\|\mathcal{H}(\tau, u)\|_{C^{0, \beta-\alpha}(\mathbb{R}^{N-1})} \leq c|\tau|^{1+\alpha}. \quad (4.5)$$

In addition, for every $v \in X_{\mathcal{P}}$

$$\begin{aligned} D_u \mathcal{H}(\tau, u)[v](s) &= - \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} \frac{v(s) - v(s-t)}{(|t|^2 + (u(s) - u(s-t) - q/\tau)^2)^{\frac{N+\alpha}{2}}} dt \\ &\quad + \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} \frac{v(s) + v(s-t)}{(|t|^2 + (u(s) + u(s-t) - q/\tau)^2)^{\frac{N+\alpha}{2}}} dt \end{aligned} \quad (4.6)$$

and

$$\|D_u \mathcal{H}(\tau, u)[v]\|_{C^{0, \beta-\alpha}(\mathbb{R}^{N-1})} \leq c\|v\|_{C^{1, \beta}(\mathbb{R}^{N-1})} |\tau|^{1+\alpha}. \quad (4.7)$$

Proof. By (4.3), it is easy to see that $\mathcal{H}(\tau, u)$ is of class C^∞ in $(-\tau_1, \tau_1) \setminus \{0\} \times B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2)$. Also by (4.3), there exists a constant $c > 0$, such that for every $u \in B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2)$, $\tau \in (-\tau_1, \tau_1)$ and $s \in \mathbb{R}^{N-1}$,

$$\begin{aligned} |\mathcal{H}(\tau, u)(s)| &\leq c \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} \frac{dt}{\{|t|^2 + \frac{q^2}{\tau^2}\}^{\frac{N+\alpha}{2}}} \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})} \\ &\leq c|\tau|^{1+\alpha} \sum_{q \in \mathbb{Z}_*} |q|^{-1-\alpha} \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})} \\ &\leq c|\tau|^{1+\alpha} \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})}. \end{aligned}$$

In addition, by the mean value property, we also have

$$\begin{aligned} |\mathcal{H}(\tau, u)(s) - \mathcal{H}(\tau, u)(\bar{s})| &\leq c|\tau|^{1+\alpha} |s - \bar{s}| + c|s - \bar{s}| \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} \frac{\frac{|q|}{|\tau|} dt}{\{|t|^2 + \frac{q^2}{\tau^2}\}^{\frac{N+2+\alpha}{2}}} \\ &\leq c|s - \bar{s}| |\tau|^{1+\alpha}. \end{aligned}$$

We therefore get

$$\|\mathcal{H}(\tau, u)\|_{C^{0, \beta-\alpha}(\mathbb{R}^{N-1})} \leq c|\tau|^{1+\alpha}. \quad (4.8)$$

Now, letting $V_\varepsilon = u + \varepsilon v$, we have

$$\begin{aligned} D_u \mathcal{H}(\tau, u)[v](s) &= \left. \frac{d\mathcal{H}(\tau, u)(V_\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0}(s) = \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} v(s-t) \int_{-1}^1 G(z) dz dt \\ &\quad - \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} \int_{-1}^1 \{v(s) - v(s-t)z\} G'(z) dz dt, \end{aligned} \quad (4.9)$$

where

$$G(z) = \frac{1}{(|t|^2 + (u(s) - u(s-t)z - q/\tau)^2)^{\frac{N+\alpha}{2}}},$$

which satisfies

$$G'(z) = (N + \alpha) \frac{u(s-t)(u(s) - u(s-t)z - q/\tau)}{(|t|^2 + (u(s) - u(s-t)z - q/\tau)^2)^{\frac{N+2+\alpha}{2}}}.$$

Using integration by parts in the last integral of (4.9), we deduce that

$$\begin{aligned} D_u \mathcal{H}(\tau, u)[v](s) &= - \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} \left(\{v(s) - v(s-t)\} G(1) + \{v(s) + v(s-t)\} G(-1) \right) dt \\ &= - \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} \frac{v(s) - v(s-t)}{(|t|^2 + (u(s) - u(s-t) - q/\tau)^2)^{\frac{N+\alpha}{2}}} dt \\ &\quad + \sum_{q \in \mathbb{Z}_*} \int_{\mathbb{R}^{N-1}} \frac{v(s) + v(s-t)}{(|t|^2 + (u(s) + u(s-t) - q/\tau)^2)^{\frac{N+\alpha}{2}}} dt. \end{aligned}$$

This gives (4.6). Moreover, it is easy to derive, from the above expression of $D_u \mathcal{H}(\tau, u)$ and similar arguments as above, the estimate

$$\|D_u \mathcal{H}(\tau, u)[v]\|_{C^{0, \beta-\alpha}(\mathbb{R}^{N-1})} \leq c \|v\|_{C^{1, \beta}(\mathbb{R}^{N-1})} |\tau|^{1+\alpha}.$$

This and (4.8), imply that \mathcal{H} is of class C^1 in $(-\tau_1, \tau_1) \times B_{X_{\mathcal{P}}}(\lambda_*, \lambda_*/2)$. \square

We are now in position to complete the proof of Theorem 1.1. The argument we use is inspired by the proof of the Crandall-Rabinowitz bifurcation theorem, [12, 13].

Proof of Theorem 1.1 completed. Our aim is to solve the equation

$$H(\lambda + \varphi) - 2\mathcal{H}(\tau, \lambda + \varphi) = H(\lambda) - 2\mathcal{H}(\tau, \lambda) \quad \text{on } Y_{\mathcal{P}}, \quad (4.10)$$

For this, we define the map $\Psi : (-\tau_1, \tau_1) \times (3\lambda_*/4, 5\lambda_*/4) \times B_{X_{\mathcal{P}}}(0, \lambda_*/4) \rightarrow Y_{\mathcal{P}}$, given by

$$\Psi(\tau, \lambda, \varphi) := \lambda^{1+\alpha} \{H(\lambda + \varphi) - 2\mathcal{H}(\tau, \lambda + \varphi)\}$$

and

$$\bar{\Psi}(\tau, \lambda, \varphi) := \Psi(\tau, \lambda, \varphi) - \Psi(\tau, \lambda, 0)$$

so that (4.10) becomes equivalent to

$$\bar{\Psi}(\tau, \lambda, \varphi) = 0 \quad \text{on } Y_{\mathcal{P}}. \quad (4.11)$$

It is clear that Ψ and $\bar{\Psi}$ are of class C^1 by Proposition 4.2. We shall use the Implicit Function Theorem to find a solution of the form

$$\varphi(s) = b(\bar{v}(s) + v(s)),$$

where \bar{v} is defined in (3.22). We will apply this theorem to the new C^1 -map

$$f : (-\tau_1, \tau_1) \times (-\beta_*, \beta_*) \times (3\lambda_*/4, 5\lambda_*/4) \times B_{X_{\mathcal{P}}}(0, \gamma_*) \cap (X_{\mathcal{P}} \cap V_1^\perp) \rightarrow Y_{\mathcal{P}},$$

given by

$$f(\tau, b, \lambda, v) := \begin{cases} \frac{\bar{\Psi}(\tau, \lambda, b(\bar{v} + v))}{b} & \text{if } b \neq 0 \\ D_\varphi \bar{\Psi}(\tau, \lambda, 0)(\bar{v} + v) & \text{if } b = 0, \end{cases}$$

where

$$\gamma_* := \|\bar{v}\|_{C^{1,\beta}(\mathbb{R}^{N-1})} \quad \text{and} \quad \beta_* := \frac{\lambda_*}{8\gamma_*}.$$

Observe that

$$\bar{\Psi}(0, \lambda, \varphi) = \lambda^{1+\alpha} [H(\lambda + \varphi) - H(\lambda)].$$

and

$$f(0, 0, \lambda, v) = D_\varphi \bar{\Psi}(0, \lambda, 0)(\bar{v} + v) = \lambda^{1+\alpha} DH(\lambda)(\bar{v} + v) = L_\lambda(\bar{v} + v).$$

In fact, we have the following properties of the map f :

1. $f(0, 0, \lambda_*, 0) = L_{\lambda_*}(\bar{v}) = 0$.
2. The differential of $(\lambda, v) \mapsto f(0, 0, \lambda, v)$ at $(\lambda_*, 0)$ is the mapping

$$G : \mathbb{R} \times (X_{\mathcal{P}} \cap V_1^\perp) \rightarrow Y_{\mathcal{P}}, \quad (\gamma, z) \mapsto L_{\lambda_*}(z) + \gamma \partial_\lambda \Big|_{\lambda=\lambda_*} L_\lambda \bar{v} = L_{\lambda_*}(z) + \gamma \nu'(\lambda_*) \bar{v},$$

where we have used (3.24). The mapping G is an isomorphism by Proposition 3.3 and its inverse is given by

$$y \mapsto \left(\frac{(y, \bar{v})}{\nu'(\lambda_*) \|\bar{v}\|^2}, \quad L_{\lambda_*}^{-1} \left[y - \frac{(y, \bar{v})}{\|\bar{v}\|^2} \bar{v} \right] \right).$$

We apply the implicit function theorem to get the existence of positives constants $\tau_0, b_0 > 0$, only depending on α, β, λ_* , and unique C^1 curves

$$(-\tau_0, \tau_0) \times (-b_0, b_0) \rightarrow (0, \infty) \times (X_{\mathcal{P}} \cap V_1^\perp), \quad (\tau, b) \mapsto (\lambda_{\tau,b}, v_{\tau,b})$$

such that for every $(\tau, b) \in (-\tau_0, \tau_0) \times (-b_0, b_0)$ and $b \neq 0$,

$$\bar{\Psi}(\tau, \lambda_{\tau,b}, b(\bar{v} + v_{\tau,b})) = 0. \tag{4.12}$$

Now, for $b \neq 0$ and $s \in \mathbb{R}^{N-1}$

$$\begin{aligned} f(\tau, b, \lambda_*, 0)(s) &= \frac{\overline{\Psi}(\tau, \lambda, b\overline{v})}{b}(s) \\ &= \int_0^1 D_\varphi \overline{\Psi}(\tau, \lambda_*, b\rho\overline{v})[\overline{v}](s) d\rho \\ &= \lambda_*^{1+\alpha} \int_0^1 \{DH(\lambda_* + \rho b\overline{v})[\overline{v}](s) - 2D_u \mathcal{H}(\tau, \lambda_* + \rho b\overline{v})[\overline{v}](s)\} d\rho. \end{aligned}$$

Therefore by Proposition 3.1 and Proposition 4.2, we get

$$\|f(\tau, b, \lambda_*, 0)\|_{C^{0,\beta-\alpha}(\mathbb{R}^{N-1})} \leq c(|\tau|^{1+\alpha} + |b|).$$

It then follows that

$$|\lambda_{\tau,b} - \lambda_*| + \|v_{\tau,b}\|_{C^{1,\beta}(\mathbb{R}^{N-1})} \leq c(|\tau|^{1+\alpha} + |b|).$$

Finally, letting $w_{\tau,b} := \lambda_{\tau,b} + b(\overline{v} + v_{\tau,b})$, we thus get properties (i)-(iii) of Theorem 1.1.

Finally, notice that (4.12) yields in particular,

$$\lambda_{0,0} = \lambda_* \quad \text{and} \quad v_{0,0} = 0.$$

Moreover, putting $\lambda_b := \lambda_{0,b}$, $v_b := v_{0,b}$ and $u_b := \lambda_b + b(\overline{v} + v_b)$, we get with (4.10) that

$$H[\lambda_b + b(\overline{v} + v_b)] = H(\lambda_b) \quad \text{on } Y_{\mathcal{P}}, \quad (4.13)$$

which shows that for all $b \in (-b_0, b_0)$, the domains

$$E_{u_b} = \{(s, z) \in \mathbb{R}^{N-1} \times \mathbb{R} : |z| < u_b(s)\}$$

are constant nonlocal mean curvature sets, with

$$H_{E_{u_b}}(x) = H_{E_{\lambda(b)}} \quad \text{for every } x \in \partial E_{u_b}.$$

The proof of Corollary 1.2 is also completed.

5. REGULARITY OF THE NMC OPERATOR

The purpose of this section is to prove the regularity of the NMC operator and to derive the result of Proposition 3.1.

The proof of the differentiability of the NMC operator is inspired by [6]. In [6] and [8], the linearized NMC operators were computed only at constant functions. Here, our arguments allow for the computation of the linearized operator at any function. We believe that this might be of interest for the study of certain qualitative of our CNMC hypersurfaces, e.g. the stability or global bifurcation branch.

We recall that

$$-\frac{\alpha}{2}H(u)(s) = \int_{\mathbb{R}^{N-1}} \frac{u(s) - u(s-t) - t \cdot \nabla u(s-t)}{\{|t|^2 + (u(s) - u(s-t))^2\}^{(N+\alpha)/2}} dt \quad (5.1)$$

$$- \int_{\mathbb{R}^{N-1}} \frac{u(s) + u(s-t) + t \cdot \nabla u(s-t)}{\{|t|^2 + (u(s) + u(s-t))^2\}^{(N+\alpha)/2}} dt.$$

We will need to consider the family of maps $H_\varepsilon : \mathcal{O} \rightarrow C^{0,\beta-\alpha}(\mathbb{R}^{N-1})$, for $\varepsilon \geq 0$, given by

$$-\frac{\alpha}{2}H_\varepsilon(u)(s) = \int_{\mathbb{R}^{N-1}} \frac{u(s) - u(s-t) - t \cdot \nabla u(s-t)}{\{|t|^2 + (u(s) - u(s-t))^2 + \varepsilon^2\}^{(N+\alpha)/2}} dt \quad (5.2)$$

$$- \int_{\mathbb{R}^{N-1}} \frac{u(s) + u(s-t) + t \cdot \nabla u(s-t)}{\{|t|^2 + (u(s) + u(s-t))^2 + \varepsilon^2\}^{(N+\alpha)/2}} dt.$$

Here, we notice that by a change of variables formula (see e.g. the proof of Lemma 2.1), we have

$$-\frac{\alpha}{2}H_\varepsilon(u)(s) = \int_{\partial E_u} \frac{(x-y) \cdot \nu_{E_u}(y)}{(|x-y|^2 + \varepsilon^2)^{(N+\alpha)/2}} dy \quad \text{for } x = (s, u(s)) = F_u(s, 1). \quad (5.3)$$

We make the obvious observation that to prove the regularity of H , it suffices to consider $\delta > 0$ and to prove the regularity of

$$H_\varepsilon : \mathcal{O}_\delta \rightarrow C^{0,\beta-\alpha}(\mathbb{R}^{N-1}), \quad \text{where } \mathcal{O}_\delta := \left\{ u \in C^{1,\beta}(\mathbb{R}^{N-1}) : \inf_{\mathbb{R}^{N-1}} u > \delta \right\},$$

for every $\varepsilon \geq 0$. We will prove uniform estimates (with respect to ε) for the higher derivatives of H_ε . This will allow us to get the general expression for first derivative of H in Proposition 3.1. This will be completed in Section 5.1 below.

We fix some notations that will be used throughout this section. For $i = 1, 2, 3, 4$, we define the maps $\Lambda_i : C^{1,\beta}(\mathbb{R}^{N-1}) \times \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \rightarrow \mathbb{R}$, by

$$\Lambda_1(\varphi, s, t) = \frac{\varphi(s) - \varphi(s-t)}{|t|} = \int_0^1 \nabla \varphi(s - \rho t) \cdot \frac{t}{|t|} d\rho,$$

$$\Lambda_2(\varphi, s, t) = \Lambda_1(\varphi, s, t) - \nabla \varphi(s-t) \cdot \frac{t}{|t|} = \int_0^1 (\nabla \varphi(s - \rho t) - \nabla \varphi(s-t)) \cdot \frac{t}{|t|} d\rho, \quad (5.4)$$

$$\Lambda_3(\varphi, s, t) = \varphi(s) + \varphi(s-t) \quad (5.5)$$

and

$$\Lambda_4(\varphi, s, t) = \varphi(s) + \varphi(s-t) + t \cdot \nabla \varphi(s-t). \quad (5.6)$$

Therefore (5.2), becomes

$$-\frac{\alpha}{2}H_\varepsilon(u)(s) = \int_{\mathbb{R}^{N-1}} \frac{\Lambda_2(u, s, t)}{|t|^{N-1+\alpha} \{1 + \Lambda_1(u, s, t)^2 + |t|^{-2}\varepsilon^2\}^{(N+\alpha)/2}} dt \quad (5.7)$$

$$- \int_{\mathbb{R}^{N-1}} \frac{\Lambda_4(u, s, t)}{\{|t|^2 + \Lambda_3(u, s, t)^2 + \varepsilon^2\}^{(N+\alpha)/2}} dt.$$

We observe that for every $s, \bar{s}, t \in \mathbb{R}^{N-1}$, we have

$$|\Lambda_2(\varphi, s, t)| \leq 2\|\varphi\|_{C^{1,\beta}(\mathbb{R}^{N-1})} \min(|t|^\beta, 1) \quad (5.8)$$

and also

$$|\Lambda_2(\varphi, s, t) - \Lambda_2(\varphi, \bar{s}, t)| \leq 2\|\varphi\|_{C^{1,\beta}(\mathbb{R}^{N-1})} \min(|t|^\beta, |s - \bar{s}|^\beta). \quad (5.9)$$

Note also that for every $s, \bar{s}, t \in \mathbb{R}^{N-1}$ and $i = 1, 3, 4$, we have

$$|\Lambda_i(\varphi, s, t)^2 - \Lambda_i(\varphi, \bar{s}, t)^2| \leq 2\|\varphi\|_{C^{1,\beta}(\mathbb{R}^{N-1})}^2 |s - \bar{s}|^\beta \quad (5.10)$$

and

$$|\Lambda_i(\varphi, s, t)^2 - \Lambda_i(\varphi, s, -t)^2| \leq 2\|\varphi\|_{C^{1,\beta}(\mathbb{R}^{N-1})}^2 \min(|t|^\beta, 1). \quad (5.11)$$

For $\varrho > -N$, we let $\mathcal{K}_{\varrho,\varepsilon}, \bar{\mathcal{K}}_{\varrho,\varepsilon} : C^{1,\beta}(\mathbb{R}^{N-1}) \times \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ be defined by

$$\mathcal{K}_{\varrho,\varepsilon}(u, s, t) = \frac{1}{(1 + \Lambda_1(u, s, t)^2 + |t|^{-2\varepsilon^2})^{(N+\varrho)/2}} \quad (5.12)$$

and

$$\bar{\mathcal{K}}_{\varrho,\varepsilon}(u, s, t) = \frac{1}{(|t|^2 + \Lambda_3(u, s, t)^2 + \varepsilon^2)^{(N+\varrho)/2}}, \quad (5.13)$$

in such a way that

$$-\frac{\alpha}{2}H_\varepsilon(u)(s) = \int_{\mathbb{R}^{N-1}} \frac{\Lambda_2(u, s, t)}{|t|^{N-1+\alpha}} \mathcal{K}_{\alpha,\varepsilon}(u, s, t) dt - \int_{\mathbb{R}^{N-1}} \Lambda_4(u, s, t) \bar{\mathcal{K}}_{\alpha,\varepsilon}(u, s, t) dt. \quad (5.14)$$

Using this expression (5.14), we shall show that $H_\varepsilon : \mathcal{O}_\delta \rightarrow C^{0,\beta-\alpha}(\mathbb{R}^{N-1})$ is of class C^∞ for every $\delta > 0$ and $\varepsilon \geq 0$.

For a finite set \mathcal{N} , we let $|\mathcal{N}|$ denote the length (cardinal) of \mathcal{N} . It will be understood that $|\emptyset| = 0$. Let Z be a Banach space and U a nonempty open subset of Z . If $T \in C^k(U, \mathbb{R})$ and $u \in U$, then $D^k T(u)$ is a continuous symmetric k -linear form on Z whose norm is given by

$$\|D^k T(u)\| = \sup_{u_1, \dots, u_k \in Z} \frac{|D^k T(u)[u_1, \dots, u_k]|}{\prod_{j=1}^k \|u_j\|_Z}.$$

If $L : Z \rightarrow \mathbb{R}$ is a linear map, we have

$$D^{|\mathcal{M}|}(LT)(u)[u_i]_{i \in \mathcal{N}} = L(u) D^{|\mathcal{M}|} T(u)[u_i]_{i \in \mathcal{N}} + \sum_{j \in \mathcal{N}} L(u_j) D^{|\mathcal{M}|-1} T(u)[u_i]_{\substack{i \in \mathcal{N} \\ i \neq j}}. \quad (5.15)$$

We let T be as above, $V \subset \mathbb{R}$ open with $T(U) \subset V$ and $g : V \rightarrow \mathbb{R}$ be a k -times differentiable map. The Faà de Bruno formula states that

$$D^k(g \circ T)(u)[u_1, \dots, u_k] = \sum_{\Pi \in \mathcal{P}_k} g^{(|\Pi|)}(T(u)) \prod_{P \in \Pi} D^{|P|} T(u)[u_j]_{j \in P}, \quad (5.16)$$

for $u, u_1, \dots, u_k \in U$, where \mathcal{P}_k denotes the set of all partitions of $\{1, \dots, k\}$, see e.g. [24].

For a function $u : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$, we use the notation

$$[u; s, r] := u(s) - u(r) \quad \text{for } s, r \in \mathbb{R}^{N-1},$$

and we note the obvious equality

$$[uv; s, r] = [u; s, r]v(s) + u(r)[v; s, r] \quad \text{for } u, v : \mathbb{R}^{N-1} \rightarrow \mathbb{R}, s, r \in \mathbb{R}^{N-1}. \quad (5.17)$$

We first give some estimates related to the kernel $\mathcal{K}_{\varrho, \varepsilon}$ and $\overline{\mathcal{K}}_{\varrho, \varepsilon}$ above.

Lemma 5.1. *Let $k \in \mathbb{N}$, $\delta > 0$, $\varrho > -N$ and $\beta \in (0, 1)$.*

(i) *There exists a constant $c = c(\varrho, \beta, k, N) > 1$ such that for all $\varepsilon \geq 0$, $s, r, t \in \mathbb{R}^{N-1}$ and $u \in C^{1, \beta}(\mathbb{R}^{N-1})$, we have*

$$\|D_u^k \mathcal{K}_{\varrho, \varepsilon}(u, s, t)\| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c, \quad (5.18)$$

$$\|[D_u^k \mathcal{K}_{\varrho, \varepsilon}(u, \cdot, t); s, r]\| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c |s - r|^\beta \quad (5.19)$$

and

$$|\mathcal{K}_{\varrho, \varepsilon}(u, s, t) - \mathcal{K}_{\varrho, \varepsilon}(u, s, -t)| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \min(|t|^\beta, 1). \quad (5.20)$$

(ii) *There exists $c = c(\varrho, \beta, k, \delta, N) > 1$ such that for all $\varepsilon \geq 0$, $s, r, t \in \mathbb{R}^{N-1}$ and $u \in \mathcal{O}_\delta$, we have*

$$\|D_u^k \overline{\mathcal{K}}_{\varrho, \varepsilon}(u, s, t)\| \leq \frac{c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c}{(1 + |t|^2)^{(N+\varrho)/2}}, \quad (5.21)$$

$$\|[D_u^k \overline{\mathcal{K}}_{\varrho, \varepsilon}(u, \cdot, t); s, r]\| \leq \frac{c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c |s - r|^\beta}{(1 + |t|^2)^{(N+\varrho)/2}}. \quad (5.22)$$

Proof. Throughout this proof, the letter c stands for different constants greater than one and depending only on ϱ, β, k and δ . Since all the estimates trivially holds true for $k = 0$, we will assume in the following that $k \geq 1$. We define

$$Q : C^{1, \beta}(\mathbb{R}^{N-1}) \times \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \rightarrow \mathbb{R}, \quad Q(u, s, t) = \Lambda_1(u, s, t)^2 + |t|^{-2} \varepsilon^2$$

and

$$g_\varrho \in C^\infty(\mathbb{R}_+, \mathbb{R}), \quad g_\varrho(x) = (1 + x)^{-(N+\varrho)/2},$$

so that

$$\mathcal{K}_{\varrho, \varepsilon}(u, s, t) = g_\varrho(Q(u, s, t)).$$

By (5.11), we then have

$$\begin{aligned} & \left| \mathcal{K}_{\varrho, \varepsilon}(u, s, t) - \mathcal{K}_{\varrho, \varepsilon}(u, s, -t) \right| = |g_\varrho(Q(u, s, t)) - g_\varrho(Q(u, s, -t))| \\ &= \frac{N + \varrho}{2} \left| \int_0^1 \frac{Q(u, s, t) - Q(u, s, -t)}{(1 + \tau Q(u, s, t) + (1 - \tau)Q(u, s, -t))^{(N+\varrho+2)/2}} d\tau \right| \\ &\leq c \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})}^2 \min(|t|^\beta, 1). \end{aligned}$$

This gives (5.20).

By (5.16) and recalling that Q is quadratic in u , we have

$$\begin{aligned} & D_u^k \mathcal{K}_{\varrho, \varepsilon}(u, s, t)[u_1, \dots, u_k] \\ &= \sum_{\Pi \in \mathcal{P}_k^2} g_\varrho^{(|\Pi|)}(Q(u, s, t)) \prod_{P \in \Pi} D_u^{|P|} Q(u, s, t)[u_j]_{j \in P}, \end{aligned} \quad (5.23)$$

where \mathcal{P}_k^2 denotes the set of partitions Π of $\{1, \dots, k\}$ such that $1 \leq |P| \leq 2$ for every $P \in \Pi$. Hence by (5.17) we have

$$\begin{aligned} & [D_u^k \mathcal{K}_{\varrho, \varepsilon}(u, \cdot, t)[u_1, \dots, u_k]; s, r] \\ &= \sum_{\Pi \in \mathcal{P}_k^2} [g_\varrho^{(|\Pi|)}(Q(u, \cdot, t)); s, r] \prod_{P \in \Pi} D_u^{|P|} Q(u, s, t)[u_j]_{j \in P} \\ &+ \sum_{\Pi \in \mathcal{P}_k^2} g_\alpha^{(|\Pi|)}(Q(u, r, t)) \left[\prod_{P \in \Pi} D_u^{|P|} Q(u, \cdot, t)[u_j]_{j \in P}; s, r \right]. \end{aligned} \quad (5.24)$$

For $P \in \Pi$ with $1 \leq |P| \leq 2$, by using inductively (5.17) and (5.10), we find that

$$\begin{aligned} & |[D_u^{|P|} Q(u, \cdot, t)[u_j]_{j \in P}; s, r]| \\ & \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})}^2) |s - r|^\beta \prod_{j \in P} \|u_j\|_{C^{1, \beta}(\mathbb{R}^{N-1})} \end{aligned} \quad (5.25)$$

and

$$|D_u^{|P|} Q(u, s, t)[u_j]_{j \in P}| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})}^2) \prod_{j \in P} \|u_j\|_{C^{1, \beta}(\mathbb{R}^{N-1})}. \quad (5.26)$$

For $\ell \in \mathbb{N}$ and $x > 0$, we have

$$g_\varrho^{(\ell)}(x) = (-1)^\ell 2^{-\ell} \prod_{i=0}^{\ell-1} (N + \varrho + 2i) (1+x)^{-\frac{N+\varrho+2\ell}{2}}.$$

Consequently, for every $u \in \mathcal{O}_\delta$, using (5.25), we have the estimates

$$\begin{aligned} & |[g_{\varrho, \varepsilon}^{(\ell)}(Q(u, \cdot, t)); s, r]| \\ & \leq \left| [Q(u, \cdot, t); s, r] \int_0^1 g_\varrho^{(\ell+1)}(\tau Q(u, s, t) + (1-\tau)Q(u, r, t)) d\tau \right| \\ & \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})}^c) |s - r|^\beta \end{aligned} \quad (5.27)$$

and

$$|g_\varrho^{(\ell)}(Q(u, \cdot, t, p))| \leq 1 \quad (5.28)$$

for $\ell = 1, \dots, k$. Therefore by (5.24), (5.25), (5.26), (5.27) and (5.28), we obtain

$$|[D_u^k \mathcal{K}_{\varrho, \varepsilon}(u, \cdot, t)[u_1, \dots, u_k]; s, r]| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})}^c) |s - r|^\beta \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})}.$$

This yields (5.19). Furthermore we easily deduce from (5.23), (5.26) and (5.28) that

$$|D_u^k \mathcal{K}_{\varrho, \varepsilon}(u, s, t)[u_1, \dots, u_k]| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})},$$

yielding (5.18). The proof of (i) is completed.

We now prove (ii). We define

$$\bar{Q} : C^{1, \beta}(\mathbb{R}^{N-1}) \times \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \rightarrow \mathbb{R}, \quad \bar{Q}(u, s, t) = |t|^2 + \Lambda_3(u, s, t)^2 + \varepsilon^2$$

and

$$\bar{g}_\varrho \in C^\infty(\mathbb{R}_+, \mathbb{R}), \quad \bar{g}_\varrho(x) = x^{-(N+\varrho)/2},$$

so that

$$\bar{\mathcal{K}}_{\varrho, \varepsilon}(u, s, t) = \bar{g}_\varrho(\bar{Q}(u, s, t)).$$

By (5.16) and recalling that \bar{Q} is quadratic in u , we have

$$\begin{aligned} D_u^k \bar{\mathcal{K}}_{\varrho, \varepsilon}(u, s, t)[u_1, \dots, u_k] &= \sum_{\Pi \in \mathcal{P}_k^2} \bar{g}_\varrho^{(|\Pi|)}(\bar{Q}(u, s, t)) \prod_{P \in \Pi} D_u^{|P|} \bar{Q}(u, s, t)[u_j]_{j \in P}, \end{aligned} \quad (5.29)$$

Hence by (5.17) we have

$$\begin{aligned} &[D_u^k \bar{\mathcal{K}}_\alpha(u, \cdot, t)[u_1, \dots, u_k]; s, r] \\ &= \sum_{\Pi \in \mathcal{P}_k^2} [\bar{g}_\varrho^{(|\Pi|)}(\bar{Q}(u, \cdot, t)); s, r] \prod_{P \in \Pi} D_u^{|P|} \bar{Q}(u, s, t)[u_j]_{j \in P} \\ &+ \sum_{\Pi \in \mathcal{P}_k^2} \bar{g}_\varrho^{(|\Pi|)}(\bar{Q}(u, r, t)) \left[\prod_{P \in \Pi} D_u^{|P|} \bar{Q}(u, \cdot, t)[u_j]_{j \in P}; s, r \right]. \end{aligned} \quad (5.30)$$

For $P \in \Pi$ with $1 \leq |P| \leq 2$, by using inductively (5.17) and (5.10), we find that

$$\begin{aligned} &|[D_u^{|P|} \bar{Q}(u, \cdot, t, p)[u_j]_{j \in P}; s, r]| \\ &\leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})}^2)(1 + |t|^2)|s - r|^\beta \prod_{j \in P} \|u_j\|_{C^{1, \beta}(\mathbb{R}^{N-1})} \end{aligned} \quad (5.31)$$

and

$$|D_u^{|P|} \bar{Q}(u, s, t)[u_j]_{j \in P}| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})}^2)(1 + |t|^2) \prod_{j \in P} \|u_j\|_{C^{1, \beta}(\mathbb{R}^{N-1})}. \quad (5.32)$$

For $\ell \in \mathbb{N}$ and $x > 0$, we have

$$\bar{g}_\varrho^{(\ell)}(x) = (-1)^\ell 2^{-\ell} \prod_{i=0}^{\ell-1} (N + \varrho + 2i) x^{-\frac{N+\varrho+2\ell}{2}}.$$

Consequently, for every $u \in \mathcal{O}_\delta$, using (5.31), we have the estimates

$$\begin{aligned}
& | [\bar{g}_\varrho^{(\ell)}(\bar{Q}(u, \cdot, t)); s, r] | \\
& \leq \left| [\bar{Q}(u, \cdot, t); s, r] \int_0^1 \bar{g}_\alpha^{(\ell+1)}(\tau \bar{Q}(u, s, t) + (1-\tau)\bar{Q}(u, r, t)) d\tau \right| \\
& \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})}^2) |s-r|^\beta (1+|t|^2)(|t|^2 + \delta^2)^{-\frac{N+\varrho+2\ell+2}{2}} \\
& \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c \frac{|s-r|^\beta}{(1+|t|^2)^{\frac{N+\varrho+2\ell}{2}}} \tag{5.33}
\end{aligned}$$

and

$$|\bar{g}_\varrho^{(\ell)}(\bar{Q}(u, \cdot, t))| \leq \frac{c}{(1+|t|^2)^{(N+\varrho+2\ell)/2}} \tag{5.34}$$

for $\ell = 0, \dots, k$. Therefore by (5.30), (5.31), (5.32), (5.33) and (5.34), we obtain

$$\begin{aligned}
& | [D_u^k \bar{\mathcal{K}}_{\varrho,\varepsilon}(u, \cdot, t)[u_1, \dots, u_k]; s, r] | \\
& \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c |s-r|^\beta \sum_{\Pi \in \mathcal{P}_k^2} \frac{1}{(1+|t|^2)^{\frac{N+\varrho+2|\Pi|}{2}}} \prod_{P \in \Pi} (1+|t|^2) \prod_{j \in P} \|u_j\|_{C^{1,\beta}(\mathbb{R}^{N-1})} \\
& = \frac{c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c |s-r|^\beta}{(1+|t|^2)^{(N+\varrho)/2}} \sum_{\Pi \in \mathcal{P}_k^2} \prod_{P \in \Pi} \prod_{j \in P} \|u_j\|_{C^{1,\beta}(\mathbb{R}^{N-1})}.
\end{aligned}$$

We then conclude that

$$| [D_u^k \bar{\mathcal{K}}_{\varrho,\varepsilon}(u, \cdot, t)[u_1, \dots, u_k]; s, r] | \leq \frac{c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c |s-r|^\beta}{(1+|t|^2)^{(N+\varrho)/2}} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R}^{N-1})}.$$

This yields (5.22). Furthermore we easily deduce from (5.29), (5.32) and (5.34) that

$$|D_u^k \bar{\mathcal{K}}_{\varrho,\varepsilon}(u, s, t)[u_1, \dots, u_k]| \leq \frac{c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c}{(1+|t|^2)^{(N+\varrho)/2}} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R}^{N-1})},$$

completing the proof of (ii). \square

As in [6], we provide estimates for the candidates to be the derivatives of H . Indeed, letting

$$M_\varepsilon(u, s, t) = \frac{1}{|t|^{N-1+\alpha}} \Lambda_2(u, s, t) \mathcal{K}_{\alpha,\varepsilon}(u, s, t), \quad \bar{M}_\varepsilon(u, s, t) = \Lambda_4(u, s, t) \bar{\mathcal{K}}_{\alpha,\varepsilon}(u, s, t), \tag{5.35}$$

then by (5.15), if $k \geq 1$, we get

$$\begin{aligned} D_u^k M_\varepsilon(u, s, t)[u_i]_{i \in \{1, \dots, k\}} &= \frac{1}{|t|^{N-1+\alpha}} \Lambda_2(u, s, t) D_u^k \mathcal{K}_{\alpha, \varepsilon}(u, s, t)[u_i]_{i \in \{1, \dots, k\}} \\ &\quad + \sum_{j=1}^k \frac{1}{|t|^{N-1+\alpha}} \Lambda_2(u_j, s, t) D_u^{k-1} \mathcal{K}_{\alpha, \varepsilon}(u, s, t)[u_i]_{\substack{i \in \{1, \dots, k\} \\ i \neq j}} \end{aligned} \quad (5.36)$$

and

$$\begin{aligned} D_u^k \overline{M}_\varepsilon(u, s, t)[u_i]_{i \in \{1, \dots, k\}} &= \Lambda_4(u, s, t) D_u^k \overline{\mathcal{K}}_{\alpha, \varepsilon}(u, s, t)[u_i]_{i \in \{1, \dots, k\}} \\ &\quad + \sum_{j=1}^k \Lambda_4(u_j, s, t) D_u^{k-1} \overline{\mathcal{K}}_{\alpha, \varepsilon}(u, s, t)[u_i]_{\substack{i \in \{1, \dots, k\} \\ i \neq j}}. \end{aligned} \quad (5.37)$$

Here, we used that $u \mapsto \Lambda_2(u, \cdot, \cdot)$ and $u \mapsto \Lambda_4(u, \cdot, \cdot)$ are linear, see (5.4) and (5.6). Now our aim is to provide estimates for $s \mapsto \int_{\mathbb{R}^{N-1}} D_u^k M_\varepsilon(u, s, t)[u_i]_{i \in \{1, \dots, k\}}(s) dt$ and $s \mapsto \int_{\mathbb{R}^{N-1}} D_u^k \overline{M}_\varepsilon(u, s, t)[u_i]_{i \in \{1, \dots, k\}}(s) dt$, from which the regularity of H_ε will follow, and

$$-\frac{\alpha}{2} D^k H_\varepsilon(u)(s) = \int_{\mathbb{R}^{N-1}} D^k M_\varepsilon(u, s, t) dt + \int_{\mathbb{R}^{N-1}} D^k \overline{M}_\varepsilon(u, s, t) dt.$$

These estimates will be uniform with respect to ε , since it will be need to get the expression of the derivatives of H stated in Proposition 5.6.

We will need the following technical result.

Lemma 5.2. *Let $N \geq 2$, $k \in \mathbb{N}$, $\delta > 0$, and $\alpha, \beta \in (0, 1)$. Then there exists a constant $c = c(\alpha, \beta, k, \delta, N) > 1$ such that for all $\varepsilon \geq 0$, $s, r, t \in \mathbb{R}^{N-1}$ and $u \in C^{1, \beta}(\mathbb{R}^{N-1})$, we have*

$$|D_u^k M_\varepsilon(u, s, t)[u_i]_{i \in \{1, \dots, k\}}| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \frac{\min(|t|^\beta, 1)}{|t|^{N-1+\alpha}} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})}, \quad (5.38)$$

$$\begin{aligned} |[D_u^k M_\varepsilon(u, \cdot, t)[u_i]_{i \in \{1, \dots, k\}}; s, r]| &\leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \times \\ &\quad \left(\frac{\min(|t|^\beta, |s-r|^\beta)}{|t|^{N-1+\alpha}} + \frac{\min(|t|^\beta, 1)|s-r|^\beta}{|t|^{N-1+\alpha}} \right) \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})}, \end{aligned} \quad (5.39)$$

$$|D_u^k \overline{M}_\varepsilon(u, s, t)[u_i]_{i \in \{1, \dots, k\}}| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \frac{\min(|t|^\beta, 1)}{(1 + |t|^2)^{(N+\alpha)/2}} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})} \quad (5.40)$$

and

$$\begin{aligned} |[D_u^k \overline{M}_\varepsilon(u, \cdot, t)[u_i]_{i \in \{1, \dots, k\}}; s, r]| &\leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \times \\ &\frac{\min(|t|^\beta, 1)|s - r|^\beta}{(1 + |t|^2)^{(N+\alpha)/2}} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})}. \end{aligned} \quad (5.41)$$

Proof. By (5.8) and Lemma 5.1(i), the estimate (5.38) follows. Using inductively (5.17), the estimates (5.8), (5.9) and Lemma 5.1(i), we get (5.39). By similar arguments, the proof of (5.40) and (5.41) follow from Lemma 5.1(ii), (5.10) and (5.17). \square

We prove estimates for all possible candidates for the derivatives of H_ε .

Lemma 5.3. *Let $N \geq 2$, $\delta > 0$, $\alpha \in (0, 1)$, $\beta \in (\alpha, 1)$, $\varepsilon \geq 0$, $u \in \mathcal{O}_\delta$ and $k \in \mathbb{N}$. For $u_1, \dots, u_k \in C^{1, \beta}(\mathbb{R}^{N-1})$, we define the functions $\mathcal{F}_\varepsilon, \overline{\mathcal{F}}_\varepsilon : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ by*

$$\mathcal{F}_\varepsilon(s) = \int_{\mathbb{R}^{N-1}} D^k M_\varepsilon(u, s, t) dt$$

and

$$\overline{\mathcal{F}}_\varepsilon(s) = \int_{\mathbb{R}^{N-1}} D^k \overline{M}_\varepsilon(u, s, t) dt.$$

Then $\mathcal{F}_\varepsilon \in C^{0, \beta - \alpha}(\mathbb{R}^{N-1})$ and $\overline{\mathcal{F}}_\varepsilon \in C^{0, \beta}(\mathbb{R}^{N-1})$. Moreover, there exists a constant $c = c(\alpha, \beta, k, \delta, N) > 1$ such that for every $\varepsilon \geq 0$,

$$\|\mathcal{F}_\varepsilon\|_{C^{0, \beta - \alpha}(\mathbb{R}^{N-1})} \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})} \quad (5.42)$$

and

$$\|\overline{\mathcal{F}}_\varepsilon\|_{C^{0, \beta}(\mathbb{R}^{N-1})} \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})}. \quad (5.43)$$

Proof. Throughout this proof, the letter c stands for different constants greater than one and depending only on α, β, k, δ and N . By (5.38), for every $s \in \mathbb{R}^{N-1}$,

$$|\mathcal{F}_\varepsilon(s)| \leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})} \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})}. \quad (5.44)$$

By (5.39), we have

$$\begin{aligned} |[\mathcal{F}_\varepsilon; s, r]| &\leq c(1 + \|u\|_{C^{1, \beta}(\mathbb{R}^{N-1})})^c \prod_{i=1}^k \|u_i\|_{C^{1, \beta}(\mathbb{R}^{N-1})} \times \\ &\int_{\mathbb{R}^{N-1}} \{ \min(|t|^\beta, |s - r|^\beta) + \min(|t|^\beta, 1)|s - r|^\beta \} |t|^{-(N-1) - \alpha} dt. \end{aligned} \quad (5.45)$$

Assuming $|s - r| \leq 1$, then using polar coordinates $t = \rho\theta$, with $\theta \in S^{N-2}$, we get

$$\begin{aligned} & \left\{ \int_{|t| \leq |s-r|} + \int_{|t| \geq |s-r|} \right\} \left\{ \min(|t|^\beta, |s-r|^\beta) + \min(|t|^\beta, 1)|s-r|^\beta \right\} |t|^{-N+1-\alpha} dt \\ & \leq |S^{N-2}| \int_0^{|s-r|} \rho^{\beta-\alpha-1} d\rho + |S^{N-2}| |s-r|^\beta \int_{|s-r|}^{+\infty} \rho^{-1-\alpha} d\rho \\ & \leq c|s-r|^{\beta-\alpha}. \end{aligned}$$

Using this in (5.45), we then conclude that

$$\|[\mathcal{F}_\varepsilon; s, r]\| \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c |s-r|^{\beta-\alpha} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R}^{N-1})}. \quad (5.46)$$

Letting $|s - r| \geq 1$, we have, by (5.44)

$$\begin{aligned} \|[\mathcal{F}_\varepsilon; s, r]\| & \leq 2 \|\mathcal{F}_\varepsilon\|_{L^\infty(\mathbb{R}^{N-1})} \\ & \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c |s-r|^{\beta-\alpha} \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(\mathbb{R}^{N-1})}. \end{aligned} \quad (5.47)$$

(5.46) and (5.47) together with (5.44) give (5.42).

To prove (5.43), we just integrate the inequality in (5.41) and (5.40) on \mathbb{R}^{N-1} . \square

We are now in position to prove that $H_\varepsilon : \mathcal{O}_\delta \rightarrow C^{0,\beta-\alpha}(\mathbb{R}^{N-1})$ given by (5.2) is smooth.

Proposition 5.4. *For every $\varepsilon \geq 0$ and $\delta > 0$, the map $H_\varepsilon : \mathcal{O}_\delta \subset C^{1,\beta}(\mathbb{R}^{N-1}) \rightarrow C^{0,\beta-\alpha}(\mathbb{R}^{N-1})$ defined by (5.2) is of class C^∞ , and for every $k \in \mathbb{N}$, we have*

$$-\frac{\alpha}{2} D^k H_\varepsilon(u)(s) = \int_{\mathbb{R}^{N-1}} D^k M_\varepsilon(u, s, t) dt - \int_{\mathbb{R}^{N-1}} D^k \overline{M}_\varepsilon(u, s, t) dt, \quad (5.48)$$

where M_ε and \overline{M}_ε are defined in (5.35). Moreover, there exists a positive constant $c = c(\alpha, \beta, \delta, N) > 1$ such that for every $\varepsilon \geq 0$,

$$\|D^k H_\varepsilon(u)\|_{C^{0,\beta-\alpha}(\mathbb{R}^{N-1})} \leq c(1 + \|u\|_{C^{1,\beta}(\mathbb{R}^{N-1})})^c. \quad (5.49)$$

Proof. With Lemma 5.3 at hand, it suffices to follow precisely the arguments in [6] to get the desired result, we skip the details. \square

As a consequence of this result, we have

Corollary 5.5. *Let $k \in \mathbb{N}$, $u \in \mathcal{O}$, $u_1, \dots, u_k \in C^{1,\beta}(\mathbb{R}^{N-1})$. Then, for every $s \in \mathbb{R}^{N-1}$, the map $\varepsilon \mapsto D^k H_\varepsilon(u)[u_1, \dots, u_k](s)$ is continuous on $[0, 1]$, and $D^k H_0(u)[u_1, \dots, u_k](s) = D^k H(u)[u_1, \dots, u_k](s)$.*

Proof. The continuity of the map $\varepsilon \mapsto D^k H_\varepsilon(u)[u_1, \dots, u_k](\cdot)$ follows from Lemma 5.2 and the dominated convergence theorem. The last statement is an immediate consequence of (5.48), Lemma 5.2 and the dominated convergence theorem. \square

5.1. The derivative of the NMC operator. The aim of this section is to complete the proof of Proposition 3.1 by computing the derivatives of the NMC operator H . To do so, we need to recall the expression of the NMC of a set E , in terms of principal integral,

$$H_E(x) = PV \int_{\mathbb{R}^N} \frac{\tau_{E_u^c}(y)}{|x-y|^{N+\alpha}} dy = \lim_{\varepsilon \rightarrow 0} \int_{|x-y| \geq \varepsilon} \frac{\tau_{E_u^c}(y)}{|x-y|^{N+\alpha}} dy,$$

for every $x \in \partial E$, where as before

$$\tau_{E^c}(y) := 1_{E^c}(y) - 1_E(y),$$

with 1_A denotes the characteristic function of A and $E^c := \mathbb{R}^N \setminus E$. While we used the geometric expression to derive the regularity of the NMC operator (1.2), we find more convenient to use PV integral form to compute the full expressions of the linearized operator about nonconstant functions.

Lemma 5.6. *For every $\lambda > 0$, $u \in \mathcal{O}$ and $v, w \in C^{1,\beta}(\mathbb{R}^{N-1})$, we have*

$$\begin{aligned} DH(u)[v](s) &= 2PV \int_{\mathbb{R}^{N-1}} \frac{v(s) - v(s-t)}{(|t|^2 + (u(s) - u(s-t))^2)^{\frac{N+\alpha}{2}}} dt \\ &\quad - 2 \int_{\mathbb{R}^{N-1}} \frac{v(s) + v(s-t)}{(|t|^2 + (u(s) + u(s-t))^2)^{\frac{N+\alpha}{2}}} dt. \end{aligned} \quad (5.50)$$

Proof. We consider H_ε defined by (5.2). Then recalling (5.3), we have

$$H_\varepsilon(u)(s) = -\frac{2}{\alpha} \int_{\partial E_u} \frac{(x-y) \cdot \nu_{E_u}(y)}{(|x-y|^2 + \varepsilon^2)^{(N+\alpha)/2}} dy \quad \text{for } x = (s, u(s)) = F_u(s, 1).$$

For $\varrho, \varepsilon > 0$, we consider the map $h_{\varepsilon, \varrho} : \mathcal{O} \rightarrow L^\infty(\mathbb{R}^{N-1})$ given by

$$h_{\varepsilon, \varrho}(u)(s) = \int_{\mathbb{R}^N} \frac{\tau_{E_u^c}(y)}{(|x-y|^2 + \varepsilon^2)^{\frac{N+\varrho}{2}}} dy = \int_{\mathbb{R}^N} \frac{\tau_{E_u^c}(y)}{(|(s, u(s)) - y|^2 + \varepsilon^2)^{\frac{N+\varrho}{2}}} dy,$$

with $x = (s, u(s)) = F_u(s, 1)$. For $\varepsilon > 0$, we have that

$$\nabla_y \cdot \frac{x-y}{(|x-y|^2 + \varepsilon^2)^{(N+\alpha)/2}} = \frac{\alpha}{(|x-y|^2 + \varepsilon^2)^{(N+\alpha)/2}} - \frac{(N+\alpha)\varepsilon^2}{(|x-y|^2 + \varepsilon^2)^{(N+2+\alpha)/2}}.$$

Multiplying this equality by $\tau_{E_1^c}$, integrating on \mathbb{R}^N and using the divergence theorem, we get

$$H_\varepsilon(u)(s) = h_{\varepsilon, \alpha}(u)(s) - \frac{(N+\alpha)\varepsilon^2}{\alpha} h_{\varepsilon, \alpha+2}(u)(s). \quad (5.51)$$

By the change of variables $y = u(\bar{s})z$ and $\bar{s} = s - t$, we obtain

$$\begin{aligned} h_{\varepsilon, \varrho}(u)(s) &= \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \frac{\tau_{E_1^c}(\bar{s}, z)}{(|s - \bar{s}|^2 + (u(s) - u(\bar{s})z)^2 + \varepsilon^2)^{\frac{N+\varrho}{2}}} u(\bar{s}) dz d\bar{s} \\ &= \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \frac{\tau_{(-1,1)^c}(z)}{(|t|^2 + (u(s) - u(s-t)z)^2 + \varepsilon^2)^{\frac{N+\varrho}{2}}} u(s-t) dz dt. \end{aligned}$$

It is clear that the map $h_{\varepsilon, \varrho} : \mathcal{O} \rightarrow C^{0, \beta - \alpha}(\mathbb{R}^{N-1})$ is differentiable for every $\varepsilon, \varrho > 0$. Now letting $V_\zeta = u + \zeta v$, we have

$$\begin{aligned} Dh_{\varepsilon, \varrho}(u)[v](s) &= \left. \frac{dh_{\varepsilon, \varrho}(V_\zeta)}{d\zeta} \right|_{\zeta=0} (s) = \int_{\mathbb{R}^{N-1}} v(\bar{s}) \int_{\mathbb{R}} \tau_{(-1,1)^c}(z) \Upsilon_\varepsilon(z) dz dt \\ &\quad - \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \tau_{(-1,1)^c}(z) \{v(s) - v(s-t)z\} \Upsilon'_\varepsilon(z) dz dt, \end{aligned} \quad (5.52)$$

where

$$\Upsilon_\varepsilon(z) = \frac{1}{(|t|^2 + (u(s) - u(s-t)z)^2 + \varepsilon^2)^{\frac{N+\varrho}{2}}}.$$

Using integration by parts in (5.52), we deduce that

$$Dh_{\varepsilon, \varrho}(u)[v](s) = 2 \int_{\mathbb{R}^{N-1}} \left(\{-v(s-t) - v(s)\} \Upsilon_\varepsilon(-1) - \{v(s-t) - v(s)\} \Upsilon_\varepsilon(1) \right) dt.$$

We then conclude that

$$\begin{aligned} Dh_{\varepsilon, \varrho}(u)[v](s) &= 2 \int_{\mathbb{R}^{N-1}} \frac{v(s) - v(s-t)}{(|t|^2 + (u(s) - u(s-t))^2 + \varepsilon^2)^{\frac{N+\varrho}{2}}} dt \\ &\quad - 2 \int_{\mathbb{R}^{N-1}} \frac{v(s) + v(s-t)}{(|t|^2 + (u(s) + u(s-t))^2 + \varepsilon^2)^{\frac{N+\varrho}{2}}} dt. \end{aligned} \quad (5.53)$$

We define

$$\Gamma_v(s, t) = v(s) - v(s-t), \quad \bar{\Gamma}_v(s, t) = v(s) + v(s-t) \quad \text{for } v \in C^{1, \beta}(\mathbb{R}^{N-1}), \quad s, t \in \mathbb{R}^{N-1}.$$

Recalling (5.12), (5.13) and (5.5), we have

$$Dh_{\varepsilon, \varrho}(u)[v](s) = 2 \int_{\mathbb{R}^{N-1}} |t|^{-N-\varrho} \Gamma_v(s, t) \mathcal{K}_{\varrho, \varepsilon}(u, s, t) dt - 2 \int_{\mathbb{R}^{N-1}} \bar{\Gamma}_v(s, t) \bar{\mathcal{K}}_{\varrho, \varepsilon}(u, s, t) dt. \quad (5.54)$$

We then deduce from (5.51) and (5.54) that, for every $\varepsilon > 0$,

$$\begin{aligned} DH_\varepsilon(u)[v](s) &= Dh_{\varepsilon, \alpha}(u)[v](s) - \frac{(N+\alpha)\varepsilon^2}{\alpha} Dh_{\varepsilon, \alpha+2}(u)[v](s) \\ &= 2 \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} \Gamma_v(s, t) \mathcal{K}_{\alpha, \varepsilon}(u, s, t) dt - 2 \int_{\mathbb{R}^{N-1}} \bar{\Gamma}_v(s, t) \bar{\mathcal{K}}_{\alpha, \varepsilon}(u, s, t) dt \\ &\quad - \frac{(N+\alpha)\varepsilon^2}{\alpha} \left(2 \int_{\mathbb{R}^{N-1}} |t|^{-N+2-\alpha} \Gamma_v(s, t) \mathcal{K}_{\alpha+2, \varepsilon}(u, s, t) dt - 2 \int_{\mathbb{R}^{N-1}} \bar{\Gamma}_v(s, t) \bar{\mathcal{K}}_{\alpha+2, \varepsilon}(u, s, t) dt \right). \end{aligned} \quad (5.55)$$

For $\varepsilon \in (0, 1]$ and s fixed, we let

$$B_{s, \alpha}(\varepsilon) := 2 \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} \Gamma_v(s, t) \mathcal{K}_{\alpha, \varepsilon}(u, s, t) dt - 2 \int_{\mathbb{R}^{N-1}} \bar{\Gamma}_v(s, t) \bar{\mathcal{K}}_{\alpha, \varepsilon}(u, s, t) dt. \quad (5.56)$$

We claim that $B_{s,\alpha}$ is smooth on $(0, 1]$ and extends continuously at $\varepsilon = 0$, for every fixed s . Indeed, it is clear that $\varepsilon \mapsto \int_{\mathbb{R}^{N-1}} \bar{\Gamma}_v(s, t) \bar{\mathcal{K}}_{\alpha,\varepsilon}(u, s, t) dt$ is smooth on $[0, 1]$ by the dominated convergence theorem, since $u \in \mathcal{O}$. Next, we write

$$\begin{aligned} 2 \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} \Gamma_v(s, t) \mathcal{K}_{\alpha,\varepsilon}(u, s, t) dt &= \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} (\Gamma_v(s, t) + \Gamma_v(s, -t)) \mathcal{K}_{\alpha,\varepsilon}(u, s, t) dt \\ &\quad + \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} (\Gamma_v(s, t) - \Gamma_v(s, -t)) \mathcal{K}_{\alpha,\varepsilon}(u, s, t) dt. \end{aligned} \quad (5.57)$$

We then observe that

$$\begin{aligned} |\Gamma_v(s, t) + \Gamma_v(s, -t)| &= |v(s+t) - 2v(s) + v(s-t)| \\ &= \left| \int_0^1 (\nabla v(s + \rho t) \cdot t - \nabla v(s - \rho t) \cdot t) d\rho \right| \\ &\leq 2 \|v\|_{C^{1,\beta}(\mathbb{R}^{N-1})} \min(|t|, |t|^{1+\beta}). \end{aligned} \quad (5.58)$$

Recalling that $\beta > \alpha$, and since $\mathcal{K}_{\alpha,\varepsilon}(u, s, t) \leq 1$, it then follows from the dominated convergence theorem that, for every fixed $s \in \mathbb{R}^{N-1}$, the function

$$\varepsilon \mapsto \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} (\Gamma_v(s, t) + \Gamma_v(s, -t)) \mathcal{K}_{\alpha,\varepsilon}(u, s, t) dt$$

is smooth on $(0, 1]$ and has a finite limit at $\varepsilon \rightarrow 0$. Now, we have

$$\begin{aligned} &\int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} (\Gamma_v(s, t) - \Gamma_v(s, -t)) \mathcal{K}_{\alpha,\varepsilon}(u, s, t) dt \\ &= \frac{1}{2} \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} (\Gamma_v(s, t) - \Gamma_v(s, -t)) (\mathcal{K}_{\alpha,\varepsilon}(u, s, t) - \mathcal{K}_{\alpha,\varepsilon}(u, s, -t)) dt. \end{aligned}$$

Since $|\Gamma_v(s, t) - \Gamma_v(s, -t)| \leq \min(|t|, 1)$, by (5.20), we have

$$|(\Gamma_v(s, t) - \Gamma_v(s, -t)) (\mathcal{K}_{\alpha,\varepsilon}(u, s, t) - \mathcal{K}_{\alpha,\varepsilon}(u, s, -t))| \leq c \min(|t|^{\beta+1}, |t|).$$

Therefore the function $\varepsilon \mapsto \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} (\Gamma_v(s, t) - \Gamma_v(s, -t)) \mathcal{K}_{\alpha,\varepsilon}(u, s, t) dt$ is smooth on $(0, 1]$ and also has a finite limit as $\varepsilon \rightarrow 0$, by the dominated convergence theorem. The claim is thus proved.

For $s \in \mathbb{R}^{N-1}$ fixed and $\varepsilon \in (0, 1]$, we now have, see (5.55) and (5.56),

$$\begin{aligned} DH_\varepsilon(u)[v](s) &= B_{s,\alpha}(\varepsilon) + \frac{\varepsilon}{\alpha} \frac{d}{d\varepsilon} B_{s,\alpha}(\varepsilon) \\ &= \frac{\alpha-1}{\alpha} B_{s,\alpha}(\varepsilon) + \frac{1}{\alpha} \frac{d}{d\varepsilon} (\varepsilon B_{s,\alpha}(\varepsilon)). \end{aligned}$$

Integrating this from 0 to $\bar{\varepsilon} \in (0, 1]$, we find that

$$\int_0^{\bar{\varepsilon}} DH_\varepsilon(u)[v](s) d\varepsilon = \frac{\alpha-1}{\alpha} \int_0^{\bar{\varepsilon}} B_{s,\alpha}(\varepsilon) d\varepsilon + \frac{1}{\alpha} \left(\bar{\varepsilon} B_{s,\alpha}(\bar{\varepsilon}) - \lim_{\varepsilon \rightarrow 0} \varepsilon B_{s,\alpha}(\varepsilon) \right). \quad (5.59)$$

From the continuity of $B_{s,\alpha}$ on $[0, 1]$, we deduce that

$$\int_0^{\bar{\varepsilon}} DH_{\varepsilon}(u)[v](s)d\varepsilon = \frac{\alpha - 1}{\alpha} \int_0^{\bar{\varepsilon}} B_{s,\alpha}(\varepsilon)d\varepsilon + \frac{1}{\alpha}\bar{\varepsilon}B_{s,\alpha}(\bar{\varepsilon}).$$

Dividing this equality by $\bar{\varepsilon}$ and letting $\bar{\varepsilon} \rightarrow 0$, we then have

$$\lim_{\bar{\varepsilon} \rightarrow 0} DH_{\bar{\varepsilon}}(u)[v](s) = \frac{\alpha - 1}{\alpha} \lim_{\bar{\varepsilon} \rightarrow 0} B_{s,\alpha}(\bar{\varepsilon}) + \frac{1}{\alpha} \lim_{\bar{\varepsilon} \rightarrow 0} B_{s,\alpha}(\bar{\varepsilon}) = \lim_{\bar{\varepsilon} \rightarrow 0} B_{s,\alpha}(\bar{\varepsilon}),$$

where we have used the continuity of $\bar{\varepsilon} \mapsto DH_{\bar{\varepsilon}}(u)[v](s)$ on $[0, 1]$, by Corollary 5.5. Since $DH_{\bar{\varepsilon}}(u)[v](s) \rightarrow DH(u)[v](s)$ as $\bar{\varepsilon} \rightarrow 0$ by Corollary 5.5, we thus have

$$\begin{aligned} DH(u)[v](s) &= \lim_{\bar{\varepsilon} \rightarrow 0} B_{s,\alpha}(\bar{\varepsilon}) \\ &= 2 \lim_{\bar{\varepsilon} \rightarrow 0} \int_{\mathbb{R}^{N-1}} |t|^{-N-\alpha} \Gamma_v(s, t) \mathcal{K}_{\alpha, \bar{\varepsilon}}(u, s, t) dt - 2 \int_{\mathbb{R}^{N-1}} \bar{\Gamma}_v(s, t) \bar{\mathcal{K}}_{\alpha, 0}(u, s, t) dt \\ &= 2PV \int_{\mathbb{R}^{N-1}} \frac{v(s) - v(s-t)}{(|t|^2 + (u(s) - u(s-t))^2)^{\frac{N+\alpha}{2}}} dt - 2 \int_{\mathbb{R}^{N-1}} \frac{v(s) + v(s-t)}{(|t|^2 + (u(s) + u(s-t))^2)^{\frac{N+\alpha}{2}}} dt. \end{aligned}$$

This completes the proof. □

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