On the summability of formal solutions of some linear q-difference-differential equations

Hidetoshi TAHARA

Abstract

The paper considers the summability of formal solutions $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n$ of some analytic linear q-difference-differential equations in the complex domain: the equation is a q-difference equation with respect to the time variable t and is a partial differential equation with respect to the space variables z. The discussion is done by using a new framework of q-Laplace and q-Borel transforms developed by the author.

Key words and phrases: q-difference-differential equations, summability, formal solutions, q-Gevrey asymptotics, q-Laplace transforms.

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1 Introduction

In Tahara [13], the author has introduced a new framework of q-Laplace and q-Borel transforms. In this paper, we will apply its theory to the problem of the summability of formal solutions of q-difference-differential equations of the form (1.1) given below. The strategy of the argument was already explained in [§8, [13]]: this paper gives a systematic study of the problem of summability.

Let q > 1 be fixed, and let $(t, z) = (t, z_1, \ldots, z_d) \in \mathbb{C} \times \mathbb{C}^d$ be the variables. We define the q-difference operator D_q in t by

$$D_q(f(t,z)) = \frac{f(qt,z) - f(t,z)}{qt - t}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ (with $\mathbb{N} = \{0, 1, 2, \ldots\}$) we write $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $\partial_z^{\alpha} = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_d}^{\alpha_d}$ with $\partial_{z_i} = \partial/\partial z_i$ $(i = 1, \ldots, d)$.

Let $m \in \mathbb{N}^* (= \{1, 2, \ldots\})$ and $\sigma > 0$. In this paper, we consider the linear q-difference-differential equation

(1.1)
$$\sum_{j+\sigma|\alpha| \le m} a_{j,\alpha}(t,z) (tD_q)^j \partial_z^{\alpha} X = F(t,z)$$

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under the following assumptions:

(1) $a_{j,\alpha}(t,z)$ $(j + \sigma |\alpha| \leq m)$ and F(t,z) are holomorphic functions in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$;

(2) (1.1) has a formal power series solution

(1.2)
$$\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$$

where \mathcal{O}_R (with R > 0) denotes the set of all holomorphic functions on $D_R = \{z \in \mathbb{C}^d ; |z_i| < R \ (i = 1, ..., d)\}.$

As to the existence of such a formal solution of (1.1), see Remark 2.4. Our basic problem is:

Problem 1.1. Under what condition can we get a true solution W(t, z) of (1.1) which admits $\hat{X}(t, z)$ as a q-Gevrey asymptotic expansion (in the sense of Definition 1.2 given below) ?

For $n \in \mathbb{N}$ we write:

$$[n]_q = \frac{q^n - 1}{q - 1}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q.$$

Of course, $[0]_q = 0$ and $[0]_q! = 1$. For $\lambda \in \mathbb{C} \setminus \{0\}$ and $\epsilon > 0$ we set

$$\begin{aligned} \mathscr{Z}_{\lambda} &= \{ -\lambda(q-1)q^m \, ; \, m \in \mathbb{Z} \}, \\ \mathscr{Z}_{\lambda,\epsilon} &= \bigcup_{m \in \mathbb{Z}} \{ t \in \mathbb{C} \, ; \, |t+\lambda(q-1)q^m| < \epsilon |t| \}. \end{aligned}$$

We note that if $\epsilon > 0$ is sufficiently small the set $\mathscr{Z}_{\lambda,\epsilon}$ is a disjoint union of closed disks. For r > 0 we write $D_r^* = \{t \in \mathbb{C} ; 0 < |t| < r\}$. Following Ramis-Zhang [12] we define:

Definition 1.2. (1) Let $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ and let W(t,z) be a holomorphic function on $(D_r^* \setminus \mathscr{Z}_{\lambda}) \times \overline{D}_R$ for some r > 0. We say that W(t,z)admits $\hat{X}(t,z)$ as a q-Gevrey asymptitoc expansion on $(D_r^* \setminus \mathscr{Z}_{\lambda}) \times D_R$, if there are M > 0 and H > 0 such that

$$\left| W(t,z) - \sum_{n=0}^{N-1} X_n(z) t^n \right| \le \frac{MH^N}{\epsilon} [N]_q! |t|^N$$

holds on $(D_r^* \setminus \mathscr{Z}_{\lambda,\epsilon}) \times D_R$ for any $N = 0, 1, 2, \ldots$ and any sufficiently small $\epsilon > 0$.

(2) If there is a W(t, z) as above, we say that the formal solution $\hat{X}(t, z)$ is G_q -summable in the direction λ .

This problem was already solved in Tahara-Yamazawa [14, 15] by using the framework of q-Laplace and q-Borel transforms developed by Ramis-Zhang [12]

and Zhang [17]. In this paper, we will give a new proof by using q-Laplace and q-Borel transforms introduced in [13].

Similar problems are discussed by Zhang [16], Marotte-Zhang [8], Ramis-Sauloy-Zhang [11] and Dreyfus [1] in the q-difference equations, and by Malek [6, 7], Lastra-Malek [3, 4] and Lastra-Malek-Sanz [5] in the case of q-difference-differential equations. But, their equations are different from ours.

In this paper, we use the notations: $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{N}^* = \{1, 2, ...\}$. For an open set $W \subset \mathbb{C}^d$ we denote by $\mathcal{O}(W)$ the set of all holomorphic functions on W. For an interval $I = (\theta_1, \theta_2) \subset \mathbb{R}$ we write $S_I = \{\xi \in \mathcal{R}(\mathbb{C} \setminus \{0\}); \theta_1 < \arg \xi < \theta_2\}$, where $\mathcal{R}(\mathbb{C} \setminus \{0\})$ denotes the universal covering space of $\mathbb{C} \setminus \{0\}$.

2 Main result

For a holomorphic function $f(t, z) (\not\equiv 0)$ in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d$, we define the order of the zeros of the function f(t, z) at t = 0 (we denote this by $\operatorname{ord}_t(f)$) by

$$\operatorname{ord}_t(f) = \min\{p \in \mathbb{N}; (\partial_t^p f)(0, z) \neq 0 \text{ near } z = 0\}.$$

If $f(t, z) \equiv 0$ we set $\operatorname{ord}_t(f) = \infty$. For $(a, b) \in \mathbb{R}^2$ we write $C(a, b) = \{(x, y) \in \mathbb{R}^2 ; x \leq a, y \geq b\}$. We set $C(a, \infty) = \emptyset$. Then, the *t*-Newton polygon $N_t(1.1)$ of equation (1.1) is defined by

$$N_t(1.1) =$$
the convex hull of $\bigcup_{j+\sigma|\alpha| \le m} C(j, \operatorname{ord}_t(a_{j,\alpha}))$

in \mathbb{R}^2 . Since we are considering (1.1) under the assumption that (1.1) has a formal solution $\hat{X}(t, z)$ in (1.2), without loss of generality we may assume that

$$\min\{\operatorname{ord}_t(a_{j,\alpha}); j+\sigma|\alpha| \le m\} = 0.$$

In this paper, we will consider the equation (1.1) under the following conditions (A_1) , (A_2) and (A_3) :

(A₁) There is an integer m_0 such that $0 \le m_0 < m$ and

$$N_t(1.1) = \{(x, y) \in \mathbb{R}^2 : x \le m, y \ge \max\{0, x - m_0\}\}.$$

 (A_2) The following condition is satisfied:

$$|\alpha| > 0 \Longrightarrow (j, \operatorname{ord}_t(a_{j,\alpha})) \in int(N_t(1.1)),$$

where $int(N_t(1.1))$ denotes the interior of the set $N_t(1.1)$ in \mathbb{R}^2 .

 (A_3) In addition, we have

$$a_{m_0,0}(0,0) \neq 0, \quad \frac{a_{m,0}(t,z)}{t^{m-m_0}}\Big|_{t=0,z=0} \neq 0.$$

By (A₁), we have $a_{m,0}(t,z) = O(t^{m-m_0})$ (as $t \to 0$), and so the second condition in (A₃) makes sense.

The figure of $N_t(1.1)$ is as in Figure 1. In Figure 1, the boundary of $N_t(1.1)$ consists of a horizontal half-line Γ_0 , a segment Γ_1 and a vertical half-line Γ_2 , and k_i is the slope of Γ_i (i = 0, 1, 2). By (A_1) we have $k_1 = 1$.

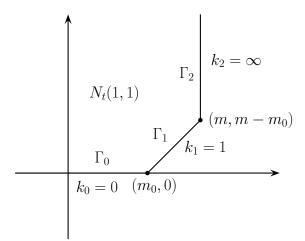


Figure 1: The *t*-Newton polygon of (1.1)

We note:

Lemma 2.1. By (A_1) and (A_2) we have

(2.1)
$$\operatorname{ord}_{t}(a_{j,\alpha}) \geq \begin{cases} \max\{0, j - m_{0}\}, & \text{if } |\alpha| = 0, \\ \max\{1, j - m_{0} + 1\}, & \text{if } |\alpha| > 0. \end{cases}$$

By the assumption, $a_{j,0}(t,z)$ $(m_0 \le j \le m)$ can be expressed in the form

$$a_{j,0}(t,z) = t^{j-m_0}b_{j,0}(t,z)$$

for some holomorphic functions $b_{j,0}(t,z)$ $(m_0 \leq j \leq m)$ satisfying $b_{m_0,0}(0,0) \neq 0$ and $b_{m,0}(0,0) \neq 0$. We set

$$P_0(\lambda, z) = \sum_{m_0 \le j \le m} \frac{b_{j,0}(0, z)}{q^{j(j-1)/2}} \lambda^{j-m_0}$$

and denote by $\lambda_1, \ldots, \lambda_{m-m_0}$ the roots of $P_0(\lambda, 0) = 0$. Since $b_{m_0,0}(0,0) \neq 0$, we have $\lambda_i \neq 0$ for all $i = 1, 2, \ldots, m - m_0$. We set

$$S = \bigcup_{i=1}^{m-m_0} \{\xi = \lambda_i \eta; \eta > 0\} \subset \mathbb{C}_{\xi}$$

which is a candidate of the set of singular directions at z = 0. The role of the set S lies in

Lemma 2.2. For any $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$ we can find a $\delta > 0$, an interval $I = (\theta_1, \theta_2)$ with $\theta_1 < \arg \lambda < \theta_2$ and an R > 0 such that $|P_0(\xi, z)| \ge \delta(1 + |\xi|)^{m-m_0}$ holds on $S_I \times D_R$.

The purpose of this paper is to prove the following result.

Theorem 2.3 (Main theorem). Suppose (A_1) , (A_2) , (A_3) and the additional condition

(2.2)
$$\operatorname{ord}_t(a_{j,\alpha}) \ge j - m_0 + 2, \quad \text{if } m_0 \le j < m \text{ and } |\alpha| > 0.$$

Let $\hat{X}(t,x) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ be a formal solution of (1.1). Then, for any $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$ there are r > 0, $R_1 > 0$, a holomorphic solution W(t,z)of (1.1) on $(D_r^* \setminus \mathscr{Z}_{\lambda}) \times D_{R_1}$ such that W(t,z) admits $\hat{X}(t,z)$ as a q-Gevrey asymptitoc expansion on $(D_r^* \setminus \mathscr{Z}_{\lambda}) \times D_{R_1}$.

Remark 2.4. (1) Set

$$P_1(\lambda; z) = \sum_{0 \le j \le m_0} a_{j,0}(0, z) \lambda^j.$$

If $P_1([n]_q; 0) \neq 0$ holds for any $n \in \mathbb{N}$, the equation (1.1) has a unique formal solution $\hat{X}(t, x) \in \mathcal{O}_R[[t]]$ for some R > 0.

(2) The additional condition (2.2) seems to be a little bit strange, but as is seen in the proof of Theorem 2.3 in §5 we need to use this condition. At present, the author does not know how to remove it.

(3) In the case where the condition (2.2) is not satisfied, as is seen in §6, by setting $q_1 = q^{1/4}$ and $t = \tau^2$ we can transform (1.1) to

(2.3)
$$\sum_{j+\sigma|\alpha|\leq m} A_{j,\alpha}(\tau,z) \left(\frac{1}{[4]_{q_1}} \left((q_1-1)(\tau D_{q_1})^2 + 2(\tau D_{q_1}) \right) \right)^j \partial_z^{\alpha} Y = G(\tau,z)$$

where

$$A_{j,\alpha}(\tau, z) = a_{j,\alpha}(\tau^2, z) \quad (j + \sigma |\alpha| \le m),$$
$$Y(\tau, z) = X(\tau^2, z) = \sum_{n \ge 0} X_n(z)\tau^{2n},$$
$$G(\tau, z) = F(\tau^2, z).$$

Since this equation (2.3) satisifies (2.2), we can apply Theorem 2.3 to (2.3), and obtain the G_q -summability of $Y(\tau, z)$. Thus, the constraint by (2.2) is not a big problem. For details, see §6.

Example 2.5. Let us consider

$$(2.4) \qquad at(t^2 D_q)^2 X + b(t D_q) X + c X + t^{n_1} \partial_z^{\alpha_1} (t D_q) X + t^{n_0} \partial_z^{\alpha_0} X = F(t, z)$$

where $a \neq 0$, $b \neq 0$, $c \in \mathbb{C}$, $n_i \in \mathbb{N}^*$ (i = 0, 1) and $\alpha_i \in \mathbb{N}^*$ (i = 0, 1). Then, this equation satisfies (A₁), (A₂) and (A₃) with $m_0 = 1$, m = 2 and $k_1 = 1$. We note that (A₂) corresponds to the condition " $n_1 \geq 1$ and $n_0 \geq 1$ ", and that (A₃) corresponds to the condition " $a \neq 0$ and $b \neq 0$ ". In this case, (2.2) corresponds to the condition $n_1 \geq 2$. Thus, if $n_1 \geq 2$ we can apply Theorem 2.3 to (2.4).

We note that Theorem 2.3 is already proved in [14]. The purpose of this paper is to give a new proof in the framework of q-Laplace and q-Borel transforms given in [13]. This new proof produces various new tools and techniques which will be very useful in treating other problems, and by this reason, the author believes that it is worthy to write this paper.

The rest part of this paper is organized as follows. In the next §3, we summarize basic results of q-Laplace and q-Borel transforms in [13]. In §4, we do some preparatory discussins which are needed in the proof of Theorem 2.3. In §5, we prove Theorem 2.3 by using a result in §4.. In the last §6, we discuss the case without the condition (2.2).

3 *q*-Laplace and *q*-Borel transforms

In this section, we summarize basic results on q-Laplace and q-Borel transforms developed in [13] with small modifications. We always suppose: q > 1.

3.1 *q*-Laplace transforms

Let $\lambda \in \mathbb{C} \setminus \{0\}$, and set $\lambda q^{\mathbb{Z}} = \{\xi = \lambda q^m ; m \in \mathbb{Z}\}$. For a function $f(\xi, z)$ on $\lambda q^{\mathbb{Z}} \times D_R$, we define the q-Laplace transform $F(t, z) = \mathscr{L}_q^{\lambda}[f](t, z)$ of $f(\xi, z)$ in the direction λ by

(3.1)
$$\mathscr{L}_{q}^{\lambda}[f](t,z) = \int_{\lambda q^{\mathbb{Z}}} \operatorname{Exp}_{q}(-q\xi/t) f(\xi,z) d_{q}\xi,$$

where

$$\operatorname{Exp}_{q}(x) = \sum_{n \ge 0} \frac{q^{n(n-1)/2}}{[n]_{q}!} x^{n} = \frac{1}{\prod_{m \ge 0} (1 - q^{-m-1}(q-1)x)}$$

(the first equality is the definition of $\text{Exp}_q(x)$ and the second equality is from Euler's indentity), and the integral in (3.1) is taken in the following sense:

$$\int_{\lambda q^{\mathbb{Z}}} g(\xi) d_q \xi = \sum_{m \in \mathbb{Z}} g(\lambda q^m) (\lambda q^{m+1} - \lambda q^m),$$

which is a discretization of the classical integral.

In the case $\lambda = 1$ we write \mathscr{L}_q instead of \mathscr{L}_q^1 . If we define the operator M_{λ} by $M_{\lambda}[f](\xi) = f(\lambda\xi)$ we have $\mathscr{L}_q^{\lambda} = \lambda(M_{1/\lambda} \circ \mathscr{L}_q \circ M_{\lambda})$. Since \mathscr{L}_q is investigated quite well in [13], we have the following properties (see Example 3.1, Proposition 3.2 and Proposition 7.1 in [13]).

(3-1-1) If $f = \xi^n \ (n \in \mathbb{N})$, we have $\mathscr{L}_q^{\lambda}[f] = [n]_q! t^{n+1}$.

(3-1-2) If $f(\xi, z)$ is a function on $\lambda q^{\mathbb{Z}} \times D_R$ which is holomorphic in $z \in D_R$ and if

$$|f(\lambda q^n, z)| \le Ch^n [n]_q!$$
 on D_R , $n = 0, 1, 2, ...,$
 $|f(\lambda q^{-m}, z)| \le AB^m$ on D_R , $m = 1, 2, ...$

for some C > 0, h > 0, A > 0 and 0 < B < q, the function $F(t, z) = \mathscr{L}_q^{\lambda}[f](t, z)$ is well-defined as a holomorphic function on $(D_r^* \setminus \mathscr{Z}_{\lambda}) \times D_R$ for a sufficiently small r > 0.

(3-1-3) In addition, F(t, z) has at most simple poles on \mathscr{Z}_{λ} with respect to t, and there is an H > 0 such that

$$|F(t,z)| \leq \frac{H}{\epsilon} |t|^{\alpha}$$
 on $(D_r^* \setminus \mathscr{Z}_{\lambda,\epsilon}) \times D_R$

for any $\epsilon > 0$, where $\alpha = (\log q - \log B) / \log q \ (> 0)$.

(3-1-4) The following result gives a Watson type lemma.

Proposition 3.1 (Watson type lemma). Let $f(\xi, z)$ be a function on $\lambda q^{\mathbb{Z}} \times D_R$, and let $c_k(z)$ (k = 0, 1, 2, ...) be functions on D_R . Suppose that there are C > 0, h > 0, A > 0 and $h_1 > 0$ such that

(3.2)
$$|f(\lambda q^n, z)| \leq Ch^n [n]_q! \text{ on } D_R, \ n = 0, 1, 2, \cdots,$$

(3.3)
$$\left| f(\xi, z) - \sum_{k=0}^{N-1} c_k(z) \xi^k \right| \le A h_1^N |\xi|^N \text{ on } D_R$$

for $\xi = \lambda q^{-m} \ (m = 1, 2, \cdots) \text{ and } N = 0, 1, 2, \ldots$

Then, there are M > 0 and H > 0 such that

$$\left|\mathscr{L}_{q}^{\lambda}[f](t,z) - \sum_{k=0}^{N-1} c_{k}(z)[k]_{q}!t^{k+1}\right| \leq \frac{MH^{N}}{\epsilon}[N]_{q}!|t|^{N+1}$$

on $(D_r^* \setminus \mathscr{Z}_{\lambda,\epsilon}) \times D_R$ for any $\epsilon > 0$ and $N = 0, 1, 2, \dots$

3.2 *q***-Borel transforms**

For a holomorphic function F(t, z) on $(D_r^* \setminus \mathscr{Z}_{\lambda}) \times D_R$ having at most simple poles on \mathscr{Z}_{λ} with respect to t, we define the q-Borel transform $f(\xi, z) = \mathscr{B}_q[F](\xi, z)$ of F(t, z) in the direction λ by

(3.4)
$$\mathscr{B}_{q}^{\lambda}[F](\xi,z) = \frac{1}{2\pi i} \int_{|t|=\rho_{\xi}} F(t,z) \exp_{q}(\xi/t) \frac{dt}{t^{2}}, \quad \xi = \lambda q^{k} \ (k \in \mathbb{Z})$$

where $\rho_{\xi} > 0$ is sufficiently small depending on ξ ,

$$\exp_q(x) = \sum_{n \ge 0} \frac{1}{[n]_q!} x^n = \prod_{m \ge 0} (1 + q^{-m-1}(q-1)x)$$

(the first equality is the definition of $\exp_q(x)$ and the second equality is from Euler's indentity), and the integral in (3.4) is taken as a contour integral along the circle $\{t \in \mathbb{C}; |t| = \rho_{\xi}\}$ in the complex plane.

In the case $\lambda = 1$ we write \mathscr{B}_q instead of \mathscr{B}_q^1 . Since we have $\mathscr{B}_q^{\lambda} = (M_{1/\lambda} \circ \mathscr{B}_q \circ M_{\lambda})/\lambda$ holds, and since \mathscr{B}_q is investigated quite well in [13], we have the following properties (see Example 4.1, Proposition 4.2, Theorem 5.1 and Theorem 5.4 in [13]).

(3-2-1) If $F(t) = t^{n+1}$ $(n \in \mathbb{N})$ we have $\mathscr{B}_q^{\lambda}[F] = \xi^n/[n]_q!$. (3-2-2) If F(t,z) satisfies

$$|F(t,z)| \leq \frac{H}{\epsilon} |t|^{\alpha}$$
 on $(D_r^* \setminus \mathscr{Z}_{\lambda,\epsilon}) \times D_R$, $\forall \epsilon > 0$

for some H > 0 and $\alpha > 0$, the function $f(\xi, z) = \mathscr{B}_q^{\lambda}[F](\xi, z)$ is well-defined as a function on $\lambda q^{\mathbb{Z}} \times D_R$ and it is holomorphic in $z \in D_R$.

(3-2-3) The following gives inversion formulas.

Theorem 3.2 (Inversion formulas). (1) If $f(\xi, z)$ satisfies the assumption in (3-1-2), we have

$$f(\xi, z) = (\mathscr{B}_q^{\lambda} \circ \mathscr{L}_q^{\lambda})[f](\xi, z) \quad on \ \lambda q^{\mathbb{Z}} \times D_R.$$

(2) If F(t, z) satisfies the assumption in (3-2-2), we have

$$F(t,z) = (\mathscr{L}_q^{\lambda} \circ \mathscr{B}_q^{\lambda})[F](t,z) \quad on \ (D_{r_1}^* \setminus \mathscr{Z}_{\lambda}) \times D_R.$$

for some $r_1 > 0$.

By (3-2-1), it will be reasonable to define the formal q-Borel transform $\hat{\mathscr{B}}_q$ in the following way:

$$\hat{\mathscr{B}}_{q}[\hat{F}](\xi,z) = \sum_{n\geq 0} \frac{a_{n}(z)}{[n]_{q}!} \xi^{n}$$
 for $\hat{F}(t,z) = \sum_{n\geq 0} a_{n}(z)t^{n+1}$.

Since $D_q[t^n] = [n]_q t^{n-1}$ holds, this formal q-Borel transform is just fitting to our equation (1.1).

3.3 q-Convolutions

Let $a(\xi, z) = \sum_{k \ge 0} a_k(z)\xi^k$ be a holomorphic function in a neighborhood of the origin of $\mathbb{C}_{\xi} \times \mathbb{C}_z^d$. For a function $f(\xi, z)$ we define the *q*-convolution $(a *_q f)(\xi, z)$ of $a(\xi, z)$ and $f(\xi, z)$ with respect to ξ by

(3.5)
$$(a *_q f)(\xi, z) = \sum_{k \ge 0} \frac{a_k(z)}{q^k} \int_0^{\xi} (\xi - py)_p^k f(q^{-k-1}\xi, z) d_p \xi,$$

where p = 1/q, the integral in (3.5) is taken as *p*-Jackson integral, and $(\xi - py)_p^k$ is defined by the following: $(\xi - py)_p^0 = 1$ and for $k \ge 1$

$$(\xi - py)_p^k = (\xi - py)(\xi - p^2 y) \cdots (\xi - p^k y).$$

We have the following properties:

(3-3-1) By Example 6.1 in [13] we have

$$\xi^m *_q \xi^n = \frac{[m]_q! [n]_q!}{[m+n+1]_q!} \xi^{m+n+1} \text{ for any } m, n \in \mathbb{N}.$$

(3-3-2) For $I = (\theta_1, \theta_2) \subset \mathbb{R}$ and $0 < r \leq \infty$, we write

$$S_I = \{\xi \in \mathcal{R}(\mathbb{C} \setminus \{0\}); 0 < |\xi| < \infty, \theta_1 < \arg \xi < \theta_2\}, S_I(r) = \{\xi \in \mathcal{R}(\mathbb{C} \setminus \{0\}); 0 < |\xi| < r, \theta_1 < \arg \xi < \theta_2\}.$$

Then, by the definition of q-convolution we have

Lemma 3.3. (1) Let $a(\xi, z)$ be a holomorphic function on $D_r \times D_R$, and let $f(\xi, z)$ be a holomorphic function on $S_I(r) \times D_R$ satisfying the following: there is an $0 < \alpha < 1$ such that $\xi^{\alpha}f(\xi, z)$ is bounded on $S_I \times D_R$. Then, $(a *_q f)(\xi, z)$ is well-defined as a holomorphic function on $S_I(qr) \times D_R$.

(2) In addition, if a holomorphic function $A(\xi) = \sum_{k\geq 0} A_k \xi^k$ on D_r and a continuous function F(x) on $x \geq 0$ satisfy

1)
$$a(\xi, z) \ll A(\xi)$$
 in $\mathbb{C}[[\xi]]$ for any $z \in D_R$,
2) $|f(\xi, z)| \le F(|\xi|)$ on $S_I(r) \times D_R$

(where $\sum_{k\geq 0} a_k \xi^k \ll \sum_{k\geq 0} b_k \xi^k$ in $\mathbb{C}[[\xi]]$ means that $|a_k| \leq b_k$ holds for all $k\geq 0$), we have

$$|(a *_q f)(\xi, z)| \le (A *_q F)(|\xi|) \quad on \ S_I(qr) \times D_R.$$

(3-3-3) By Theorems 6.3 and 6.7 in [13] we have

Theorem 3.4 (Convolution theorem). (1) Let $f(\xi, z)$ be a function on $\lambda q^{\mathbb{Z}} \times D_R$ satisfying the condition in (3-1-2), and let $a(\xi, z)$ be a holomorphic function on $\mathbb{C} \times D_R$ with the estimate

(3.6)
$$|a(\xi,z)| \le M|\xi|^{\alpha} \exp\left(\frac{(\log|\xi|)^2}{2\log q}\right) \quad on \ (\mathbb{C}_{\xi} \setminus \{0\}) \times D_R$$

for some M > 0 and $\alpha \in \mathbb{R}$. Then, we have

$$\mathscr{L}_q^{\lambda}[a*_q f](t,z) = \mathscr{L}_q^{\lambda}[a](t,z) \times \mathscr{L}_q^{\lambda}[f](t,z) \quad on \ (D_r^* \setminus \mathscr{Z}_{\lambda}) \times D_R$$

for some r > 0.

(2) Let A(t,z) be a holomorphic function on $D_r \times D_R$ satisfying A(t,z) = O(|t|) (as $|t| \longrightarrow 0$ uniformly on D_R), and let F(t,z) be a holomorphic function on $(D_r^* \setminus \mathscr{Z}_{\lambda}) \times D_R$ having at most simple poles on the set \mathscr{Z}_{λ} with respect to t. Suppose the condition in (3-2-2). Then, we have

$$\mathscr{B}_q^{\lambda}[A \times F](\xi, z) = (\mathscr{B}_q^{\lambda}[A] *_q \mathscr{B}_q^{\lambda}[F])(\xi, z) \quad on \ \lambda q^{\mathbb{Z}} \times D_R.$$

As to the estimate of type (3.6), we have the following result (see Proposition 2.1 in [10]):

Proposition 3.5. Let $\hat{f}(\xi) = \sum_{n\geq 0} a_n \xi^n \in \mathbb{C}[[\xi]]$. The following two conditions are equivalent:

(1) There are A > 0 and H > 0 such that

$$|a_n| \le \frac{AH^n}{[n]_q!}, \quad n = 0, 1, 2, \dots$$

(2) $\hat{f}(\xi)$ is the Taylor expansion at $\xi = 0$ of an entire function $f(\xi)$ satisfying the estimate

$$|f(\xi)| \le M |\xi|^{\alpha} \exp\left(\frac{(\log |\xi|)^2}{2\log q}\right) \quad on \ \mathbb{C}_{\xi} \setminus \{0\}$$

for some M > 0 and $\alpha \in \mathbb{R}$.

3.4 Some other results

In the application to q-difference equations, we need some more results. We summarize such results here.

(3-4-1) If $a(\xi, z) = \sum_{k\geq 0} a_k(z)\xi^k$ and $f(\xi, z) = \sum_{i\geq 0} f_i(z)\xi^i$ are holomorphic functions on $D_r \times D_R$, the q-convolution $(a *_q f)(\xi, z)$ is well-defined as a holomorphic function on $D_{rq} \times D_R$, and its Taylor expansion is given by

(3.7)
$$(a *_q f)(\xi, z) = \sum_{n \ge 0} \left(\sum_{k+i=n} a_k(z) f_i(z) \frac{[k]_q! [i]_q!}{[k+i+1]_q!} \right) \xi^{n+1}$$

(by Proposition 6.2 in [13]).

(3-4-2) By (3.7), it will be reasonable to define the formal q-convolution $(a\hat{*}_q f)(\xi, z)$ of two series $a(\xi, z) = \sum_{k\geq 0} a_k(z)\xi^k$ and $f(\xi, z) = \sum_{i\geq 0} f_i(z)\xi^i$ in $\mathcal{O}_R[[\xi]]$ by

$$(a\hat{*}_q f)(\xi, z) = \sum_{n \ge 0} \left(\sum_{k+i=n} a_k(z) f_i(z) \frac{[k]_q! [i]_q!}{[k+i+1]_q!} \right) \xi^{n+1}.$$

We have:

Lemma 3.6. For two formal series A(t, z) and W(t, z) in $\mathcal{O}_R[[t]] \times t$ we have

(3.8)
$$\hat{\mathscr{B}}_q[A \times W](\xi, z) = \hat{\mathscr{B}}_q[A](\xi, z) \hat{*}_q \hat{\mathscr{B}}_q[W](\xi, z)$$

Proof. Since the summations in (3.8) are formal, to prove (3.8) it is enough to show (3.8) in the case $A(t, z) = a_k(z)t^{k+1}$ and $W(t, z) = w_i(z)t^{i+1}$. In this case, we have $(A \times W)(t, z) = a_k(z)w_i(z)t^{(k+i+1)+1}$ and so

$$\hat{\mathscr{B}}_{q}[A \times W](\xi, z) = \frac{a_{k}(z)w_{i}(z)}{[k+i+1]_{q}!}\xi^{k+i+1}.$$

On the other hand, we have

$$\begin{aligned} \hat{\mathscr{B}}_{q}[A](\xi,z)\hat{*}_{q}\hat{\mathscr{B}}_{q}[W](\xi,z) &= \left(\frac{a_{k}(z)}{[k]_{q}!}\xi^{k}\right)\hat{*}_{q}\left(\frac{w_{i}(z)}{[i]_{q}!}\xi^{i}\right) \\ &= \frac{a_{k}(z)}{[k]_{q}!}\frac{w_{i}(z)}{[i]_{q}!} \times \frac{[k]_{q}![i]_{q}!}{[k+i+1]_{q}!}\xi^{k+i+1} \\ &= \frac{a_{k}(z)w_{i}(z)}{[k+i+1]_{q}!}\xi^{k+i+1}. \end{aligned}$$

Hence we have (3.8).

(3-4-3) If $A(t,z) \ (\in \mathcal{O}_R[[t]] \times t)$ is convergent in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$, we have $\hat{\mathscr{B}}_q[A](\xi,z) = \mathscr{B}_q^{\lambda}[A](\xi,z)$ for any $\lambda \in \mathbb{C} \setminus \{0\}$. If $w(\xi,z) = \hat{\mathscr{B}}_q[W](\xi,z)$ is convergent in a neighborhood of $(0,0) \in \mathbb{C}_{\xi} \times \mathbb{C}_z^d$, the right-hand side of the formula (3.8) is expressed in the form

(3.9)
$$\hat{\mathscr{B}}_{q}[A](\xi,z)\hat{*}_{q}\hat{\mathscr{B}}_{q}[W](\xi,z) = \mathscr{B}_{q}[A](\xi,z) *_{q} w(\xi,z)$$
$$= \mathscr{B}_{q}^{\lambda}[A](\xi,z) *_{q} w(\xi,z)$$

for any $\lambda \in \mathbb{C} \setminus \{0\}$.

(3-4-4) Let $X(t,z) \in \mathcal{O}_R[[t]]$ with $X(0,x) \equiv 0$ and set $u(\xi,z) = \hat{\mathscr{B}}_q[X]$ (ξ,z) : then we have

$$\hat{\mathscr{B}}_{q}[(t^{2}D_{q})^{i}X](\xi,z) = \xi^{i}u(\xi,z), \quad i = 1, 2, \dots$$

4 On a *q*-convolution equation

The main part of the proof of Theorem 2.3 consists of the analysis of a q-convolution equation which is obtained by applying the formal q-Borel transform to (1.1). Hence, in this section we discuss only q-convolution equations first.

Let q > 1, $I = (\theta_1, \theta_2)$ be a non-empty open interval, $0 < r \le \infty$ and R > 0. We set

$$\phi_m(x;h) = \sum_{i=0}^{\infty} \frac{h^i x^{m+i}}{[m+i]_q!}, \quad m = 0, 1, 2, \dots \text{ and } h > 0.$$

Let us consider the q-convolution partial differential equation

(4.1)
$$P(\xi, z)u + \sum_{0 \le i \le m} c_{i,0}(\xi, z) *_q (\xi^i u) + \sum_{i+\sigma |\alpha| \le m, |\alpha| > 0} c_{i,\alpha}(\xi, z) *_q (1 *_q (\xi^i \partial_z^{\alpha} u)) = f(\xi, z)$$

under the following assumptions:

h₁) $\sigma > 0, m_0 \in \mathbb{N}, m \in \mathbb{N}^*, 0 \le m_0 < m \text{ and } 0 < R \le 1;$

h₂) $P(\xi, z)$ is a polynomial in ξ with coefficients in $\mathcal{O}(D_R)$, and there is a $\delta > 0$ such that $|P(\xi, x)| \ge \delta |\xi|^{m_0} (1 + |\xi|)^{m-m_0}$ holds on $S_I(r) \times D_R$.

h₃) $c_{i,\alpha}(\xi, z) \in \mathcal{O}(\mathbb{C} \times D_R) \ (i + \sigma |\alpha| \le m)$ and there are $C_{i,\alpha} > 0 \ (i + \sigma |\alpha| \le m)$ and $h_0 > 0$ such that

$$c_{i,\alpha}(\xi, z) \ll C_{i,\alpha}\phi_{m_0 - i - 1}(\xi; h_0), \quad \text{if } 0 \le i < m_0, \\ c_{i,\alpha}(\xi, z) \ll C_{i,\alpha}\phi_0(\xi; h_0), \quad \text{if } m_0 \le i \le m$$

hold (as formal power series in ξ) for any $z \in D_R$.

Then, we have

Proposition 4.1. Suppose the conditions h_1), h_2) and h_3). Let $f(\xi, z) \in \mathcal{O}(S_I(r) \times D_R)$ and suppose the estimate

$$|f(\xi, z)| \le B\phi_N(|\xi|; h) \quad on \ S_I(r) \times D_R$$

for some B > 0, $h > h_0$ and some $N \in \mathbb{N}^*$ satisfying $N \ge m_0$ and

$$\beta = \frac{1}{[N]_q} \sum_{0 \le i < m_0} \frac{C_{i,0}}{\delta(1 - h_0/h)} < 1.$$

Then, the equation (4.1) has a unique solution $u(\xi, x) \in \mathcal{O}(S_I(r) \times D_R)$ which satisfies the following estimate: for any $0 < R_1 < R$ there are M > 0 and $h_1 > 0$ such that

$$|u(\xi,z)| \le \frac{M}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}} \phi_N(|\xi|;h_1) \quad on \ S_I(r) \times D_{R_1}.$$

The rest part of this section is used to prove this result. In subsections 4.1 and 4.2, we present some preparatory discussions which are needed in the proof of Proposition 4.1, and in subsection 4.3 we give a proof of Proposition 4.1.

4.1 On the functions $\phi_m(x;h)$

In this subsection, let us show some properties of the functions $\phi_m(x;h)$. We note that $\phi_0(x;h) = \exp_a(hx)$ and for $m \ge 1$

$$\phi_m(x;h) = \frac{x^{m-1}}{[m-1]_q!} *_q \exp_q(hx) = \frac{x^{m-1}}{[m-1]_q!} *_q \sum_{i=0}^{\infty} \frac{h^i x^i}{[i]_q!}$$

Lemma 4.2. Let $1 \le k \le n$, 0 < B < h and $0 < h_0 < h$. We have the following results for x > 0.

(4.2)
$$\frac{1}{x^k}\phi_n(x;h) \le \frac{[n-k]_q!}{[n]_q!}\phi_{n-k}(x;h);$$

(4.3)
$$\sum_{m=N}^{\infty} B^m \phi_m(x;h) \le \frac{B^N}{1 - B/h} \phi_N(x;h);$$

(4.4)
$$\phi_m(x;h_0) *_q \phi_n(x;h) \le \frac{1}{1 - h_0/h} \phi_{m+n+1}(x;h).$$

Proof. (4.2) is verified as follows:

$$\frac{1}{x^k}\phi_n(x;h) = \sum_{i\geq 0} \frac{h^i x^{n-k+i}}{[n-k+i]_q!} \times \frac{1}{[n-k+1+i]_q\cdots[n+i]_q}$$
$$\leq \frac{1}{[n-k+1]_q\cdots[n]_q} \sum_{i\geq 0} \frac{h^i x^{n-k+i}}{[n-k+i]_q!} = \frac{[n-k]_q!}{[n]_q!}\phi_{n-k}(x;h).$$

(4.3) is verified as follows:

$$\sum_{m \ge N} B^m \phi_m(x;h) = \sum_{m \ge N} B^m \sum_{i \ge 0} \frac{h^i x^{m+i}}{[m+i]_q!} = \sum_{k \ge N} \frac{h^k x^k}{[k]_q!} \sum_{m+i=k,m \ge N} (B/h)^m$$
$$\leq \sum_{k \ge N} \frac{h^k x^k}{[k]_q!} \times \frac{(B/h)^N}{1 - B/h} = \frac{(B/h)^N}{1 - B/h} \sum_{i \ge 0} \frac{h^{N+i} x^{N+i}}{[N+i]_q!}$$
$$= \frac{B^N}{1 - B/h} \phi_N(x;h).$$

(4.4) is verified as follows:

$$\phi_m(x;h_0) *_q \phi_n(x;h) = \sum_{i \ge 0} \frac{h_0^i x^{m+i}}{[m+i]_q!} *_q \sum_{j \ge 0} \frac{h^j x^{n+j}}{[n+j]_q!}$$
$$= \sum_{i \ge 0, j \ge 0} \frac{h_0^i h^j x^{m+n+i+j+1}}{[m+n+i+j+1]_q!} = \sum_{l \ge 0} \frac{h^l x^{m+n+1+l}}{[m+n+1+l]_q!} \sum_{i+j=l} (h_0/h)^i$$
$$\leq \frac{1}{1-h_0/h} \phi_{m+n+1}(x;h).$$

To see the estimate of $\phi_m(x; h)$, it is enough to use

$$\phi_m(x;h) \le \frac{x^m}{[m]_q!} \exp_q(hx), \quad x > 0$$

and the following result (see Proposition 5.5 in [10]).

Lemma 4.3. Let q > 1. We have

$$\log(\exp_q(x)) = \frac{(\log x)^2}{2\log q} + \left(-\frac{1}{2} + \frac{\log(q-1)}{\log q}\right)\log x + O(1)$$

 $(as \ x \longrightarrow +\infty \ in \ \mathbb{R}).$

As to the estimate of $(a\ast_q f)(\xi,z),$ by Lemma 3.3 and (4.4) of Lemma 4.2 we have

Lemma 4.4. Let $a(\xi, z)$ be a holomorphic function on $\mathbb{C} \times D_R$, and let $f(\xi, z)$ be a holomorphic function on $S_I(r) \times D_R$. If

$$a(\xi, z) \ll A\phi_m(\xi; h_0) \quad in \mathbb{C}[[\xi]] \text{ for any } z \in D_R,$$

$$|f(\xi, z)| \le B\phi_n(|\xi|; h) \quad on \ S_I(r) \times D_R$$

hold for some A > 0, $h > h_0 > 0$ and B > 0, we have

$$|(a *_q f)(\xi, z)| \le \frac{AB}{1 - h_0/h} \phi_{m+n+1}(|\xi|; h) \quad on \ S_I(qr) \times D_R.$$

4.2 On a basic equation

We set

$$\mathscr{H}[w] = P(\xi, z)w + \sum_{0 \le i < m_0} c_{i,0}(\xi, z) *_q (\xi^i w)$$

and consider the equation

(4.5)
$$\mathscr{H}[w] = g(\xi, z).$$

Let N, h and β be as in Proposition 4.1. We have

Lemma 4.5. Let $0 < R_1 \leq R$, and let $g(\xi, z) \in \mathcal{O}(S_I(r) \times D_{R_1})$ satisfy $|g(\xi, z)| \leq C\phi_n(|\xi|; h)$ on $S_I(r) \times D_{R_1}$ for some C > 0 and $n \geq N$. Then, the equation (4.5) has a unique solution $w(\xi, z) \in \mathcal{O}(S_I(r) \times D_{R_1})$ which satisfies the estimate

$$|w(\xi, z)| \le \frac{C}{\delta(1-\beta)|\xi|^{m_0}(1+|\xi|)^{m-m_0}}\phi_n(|\xi|; h) \quad on \ S_I(r) \times D_{R_1}.$$

Proof. We solve the equation (4.5) by the method of successive approximations. We set a formal solution

$$w(\xi, z) = \sum_{k \ge 0} w_k(\xi, z), \quad w_k(\xi, z) \in \mathcal{O}(S_I(r) \times D_{R_1}) \ (k \ge 0)$$

and determine $w_k(\xi, z)$ $(k \ge 0)$ by a solution of the following system of recursive formulas:

(4.6)
$$P(\xi, z)w_0 = f(\xi, z)$$

and for $k \geq 1$

(4.7)
$$P(\xi, z)w_k = -\sum_{0 \le i < m_0} c_{i,0}(\xi, z) *_q (\xi^i w_{k-1}).$$

Since $P(\xi, z) \neq 0$ on $S_I(r) \times D_{R_1}$, we can determine $w_k(\xi, z) \in \mathcal{O}(S_I(r) \times D_{R_1})$ (k = 0, 1, 2, ...) inductively on k. Let us show the convergence of this formal solution. By (4.6) we have $w_0(\xi, z) = f(\xi, z)/P(\xi, z)$ and so by the assumption we have

$$|w_0(\xi, z)| \le \frac{C}{\delta |\xi|^{m_0} (1+|\xi|)^{m-m_0}} \phi_n(|\xi|; h) \text{ on } S_I(r) \times D_{R_1}.$$

Since $0 \le i < m_0$ and $n \ge N$ hold, by (4.2) we have

$$\begin{aligned} |\xi^{i}w_{0}(\xi,z)| &\leq \frac{C}{\delta|\xi|^{m_{0}-i}(1+|\xi|)^{m-m_{0}}}\phi_{n}(|\xi|;h) \\ &\leq \frac{C}{\delta(1+|\xi|)^{m-m_{0}}}\frac{[n-(m_{0}-i)]_{q}!}{[n]_{q}!}\phi_{n-(m_{0}-i)}(|\xi|;h) \\ &\leq \frac{C}{\delta}\frac{1}{[n]_{q}}\phi_{n-(m_{0}-i)}(|\xi|;h) \leq \frac{C}{\delta}\frac{1}{[N]_{q}}\phi_{n-(m_{0}-i)}(|\xi|;h) \end{aligned}$$

and so by Lemma 4.4 we have

$$|(c_{i,0}(\xi,z)*_q(\xi^i w_0)|(\xi,z) \le \frac{C_{i,0}}{(1-h_0/h)} \frac{C}{\delta} \frac{1}{[N]_q} \phi_n(|\xi|;h) \quad \text{on } S_I(r) \times D_{R_1}.$$

Therefore, we have

$$\sum_{0 \le i < m_0} c_{i,0}(\xi, z) *_q (\xi^i w_0) \bigg| \le \sum_{0 \le i < m_0} \frac{C_{i,0}}{(1 - h_0/h)} \frac{C}{\delta} \frac{1}{[N]_q} \phi_n(|\xi|; h).$$

Thus, by (4.7) (with k = 1) and the assumption h_2) we have

$$|w_1(\xi, z)| \le \frac{1}{\delta |\xi|^{m_0} (1+|\xi|)^{m-m_0}} \sum_{0 \le i < m_0} \frac{C_{i,0}}{(1-h_0/h)} \frac{C}{\delta} \frac{1}{[N]_q} \phi_n(|\xi|; h)$$
$$= \frac{C\beta}{\delta |\xi|^{m_0} (1+|\xi|)^{m-m_0}} \phi_n(|\xi|; h) \quad \text{on } S_I(r) \times D_{R_1}.$$

Repeating the same argument as above, we have the estimates

$$|w_k(\xi, z)| \le \frac{C\beta^k}{\delta |\xi|^{m_0} (1+|\xi|)^{m-m_0}} \phi_n(|\xi|; h) \quad \text{on } S_I(r) \times D_{R_1}$$

for k = 0, 1, 2, ... Since $0 \le \beta < 1$ is supposed, this shows that the formal solution is convergent to a true solution $w(\xi, z)$ on $S_I(r) \times D_{R_1}$ and it satisfies

$$|w(\xi, z)| \le \frac{C}{\delta(1-\beta)|\xi|^{m_0}(1+|\xi|)^{m-m_0}}\phi_n(|\xi|; h) \text{ on } S_I(r) \times D_{R_1}.$$

This proves the existence part of Lemma 4.5. The uniqueness of the solution can be proved in the same way. $\hfill \Box$

In subsection 4.3 we will use the norm $||w(\xi)||_{\rho} = \sup_{z \in D_{\rho}} |w(\xi, z)|$ and the following Nagumo's lemma (see Nagumo [9] or Lemma 5.1.3 in Hörmander [2]).

Lemma 4.6. If a holomorphic function $\varphi(z)$ on D_R satisfies

$$\|\varphi\|_{\rho} \leq \frac{A}{(R-\rho)^{a}} \quad for \ any \ 0 < \rho < R$$

for some A > 0 and $a \ge 0$, we have the estimates

$$\left\|\partial_{z_i}\varphi\right\|_{\rho} \leq \frac{(a+1)eA}{(R-\rho)^{a+1}} \quad for any \ 0 < \rho < R \ and \ i = 1, \dots, d.$$

4.3 **Proof of Proposition 4.1**

Let $f(\xi, z)$, B, N and h be as in Proposition 4.1. First, let us construct a formal solution $u(\xi, z)$ of (4.1) in the form

$$u(\xi, z) = \sum_{n \ge N} u_n(\xi, z), \quad u_n(\xi, z) \in \mathcal{O}(S_I(r) \times D_R) \ (n \ge N)$$

so that $u_n(\xi, z)$ $(n \ge N)$ are solutions of the following recursive formulas:

(4.8)
$$\mathscr{H}[u_N] = f(\xi, z)$$

and for $n \ge N+1$

(4.9)
$$\mathscr{H}[u_n] = -\sum_{\substack{m_0 \le i \le m}} c_{i,0}(\xi, z) *_q (\xi^i u_{n-1}) \\ -\sum_{\substack{i+\sigma \mid \alpha \mid \le m, |\alpha| > 0}} c_{i,\alpha}(\xi, z) *_q (1 *_q (\xi^i \partial_z^{\alpha} u_{n-1})).$$

We set $L = [m/\sigma]$ (the integer part of m/σ). Let us show

Lemma 4.7. $u_n(\xi, z) \in \mathcal{O}(S_I(r) \times D_R)$ $(n \ge N)$ are uniquely determined inductively on n so that (4.8) and (4.9) are satisfied. In addition, there are M > 0 and H > 0 such that

(4.10)
$$||u_n(\xi)||_{\rho} \leq \frac{MH^n}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}(R-\rho)^{L(n-1)}}\phi_n(|\xi|;h) \quad on \ S_I(r)$$

holds for any $0 < \rho < R$ and any $n \ge N$.

Proof. By applying Lemma 4.5 to the equation (4.8) we have a unique solution $u_N(\xi, z) \in \mathcal{O}(S_I(r) \times D_R)$ such that

$$|u_N(\xi, z)| \le \frac{B}{\delta(1-\beta)|\xi|^{m_0}(1+|\xi|)^{m-m_0}}\phi_N(|\xi|; h) \quad \text{on } S_I(r) \times D_R.$$

Therefore, if M and H satisfy $MH^N \ge B/(\delta(1-\beta))$ we have (4.10) for n = N and any $0 < \rho < R$.

Let us show the general case by induction on n. Suppose that (4.10) is already proved for any $0 < \rho < R$. Then, by Lemma 4.6 we have

$$(4.11) \quad \|\partial_{z}^{\alpha}u_{n}(\xi)\|_{\rho} \leq \frac{MH^{n}e^{|\alpha|}(L(n-1)+1)\cdots(L(n-1)+|\alpha|)}{|\xi|^{m_{0}}(1+|\xi|)^{m-m_{0}}(R-\rho)^{L(n-1)+|\alpha|}}\phi_{n}(|\xi|;h)$$

$$\leq \frac{MH^{n}(eL)^{|\alpha|}n^{|\alpha|}}{|\xi|^{m_{0}}(1+|\xi|)^{m-m_{0}}(R-\rho)^{Ln}}\phi_{n}(|\xi|;h) \quad \text{on } S_{I}(r)$$

for any $|\alpha| \leq L$ and $0 < \rho < R$.

If $0 \leq i < m_0$ and $|\alpha| > 0$, by (4.11) and (4.2) we have

$$\begin{split} \|\xi^{i}\partial_{z}^{\alpha}u_{n}(\xi)\|_{\rho} &\leq \frac{MH^{n}(eL)^{|\alpha|}n^{|\alpha|}}{(R-\rho)^{Ln}} \frac{1}{|\xi|^{m_{0}-i}}\phi_{n}(|\xi|;h) \\ &\leq \frac{MH^{n}(eL)^{|\alpha|}}{(R-\rho)^{Ln}} \frac{n^{|\alpha|}}{[n]_{q}}\phi_{n-(m_{0}-i)}(|\xi|;h) \\ &\leq \frac{MH^{n}(eL)^{|\alpha|}c_{0}}{(R-\rho)^{Ln}}\phi_{n-(m_{0}-i)}(|\xi|;h) \quad \text{on } S_{I}(r) \end{split}$$

where

(4.12)
$$c_0 = \sup_{|\alpha| \le L, n \ge 1} \frac{n^{|\alpha|}}{[n]_q} = \sup_{|\alpha| \le L, n \ge 1} \frac{n^{|\alpha|}(q-1)}{q^n - 1} < \infty.$$

Hence, we have

$$\|1*_q (\xi^i \partial_z^{\alpha} u_n)\|_{\rho} \le \frac{M H^n (eL)^{|\alpha|} c_0}{(R-\rho)^{Ln}} \phi_{n-(m_0-i)+1}(|\xi|;h) \quad \text{on } S_I(r)$$

and so in the case $0 \leq i < m_0$ and $|\alpha| > 0$ we obtain

(4.13)
$$||c_{i,\alpha} *_q (1 *_q (\xi^i \partial_z^{\alpha} u_n))||_{\rho} \leq \frac{C_{i,\alpha}}{1 - h_0/h} \frac{M H^n (eL)^{|\alpha|} c_0}{(R - \rho)^{Ln}} \phi_{n+1}(|\xi|;h)$$

on $S_I(r)$ for any $0 < \rho < R$.

If $m_0 \leq i \leq m$ and $|\alpha| = 0$, by (4.11) we have

$$\|\xi^i u_n(\xi)\|_{\rho} \le \frac{MH^n}{(R-\rho)^{Ln}} \phi_n(|\xi|;h) \quad \text{on } S_I(r)$$

and so we have

(4.14)
$$||c_{i,0} *_q (\xi^i u_n)||_{\rho} \le \frac{C_{i,0}}{1 - h_0/h} \frac{MH^n}{(R - \rho)^{Ln}} \phi_{n+1}(|\xi|;h) \text{ on } S_I(r)$$

for any $0 < \rho < R$

If $m_0 \leq i \leq m$ and $|\alpha| > 0$, by the condition $i + \sigma |\alpha| \leq m$ we have i < m. In this case, by (4.11) and (4.2) we have

$$\begin{split} \|\xi^{i}\partial_{z}^{\alpha}u_{n}(\xi)\|_{\rho} &\leq \frac{MH^{n}(eL)^{|\alpha|}n^{|\alpha|}}{(R-\rho)^{Ln}} \frac{|\xi|^{i-m_{0}+1}}{(1+|\xi|)^{m-m_{0}}} \frac{1}{|\xi|}\phi_{n}(|\xi|;h) \\ &\leq \frac{MH^{n}(eL)^{|\alpha|}}{(R-\rho)^{Ln}} \frac{n^{|\alpha|}}{[n]_{q}}\phi_{n-1}(|\xi|;h) \\ &\leq \frac{MH^{n}(eL)^{|\alpha|}c_{0}}{(R-\rho)^{Ln}}\phi_{n-1}(|\xi|;h) \quad \text{on } S_{I}(r) \end{split}$$

and so

$$\|1*_q (\xi^i \partial_z^{\alpha} u_n)\|_{\rho} \le \frac{M H^n(eL)^{|\alpha|} c_0}{(R-\rho)^{Ln}} \phi_n(|\xi|;h) \quad \text{on } S_I(r).$$

By applying $c_{i,\alpha}*_q$ to this estimate we obtain

$$(4.15) \|c_{i,\alpha} *_q (1 *_q (\xi^i \partial_z^{\alpha} u_n))\|_{\rho} \le \frac{C_{i,\alpha}}{1 - h_0/h} \frac{M H^n(eL)^{|\alpha|} c_0}{(R - \rho)^{Ln}} \phi_{n+1}(|\xi|;h)$$

on $S_I(r)$ for any $0 < \rho < R$

Thus, by (4.13), (4.14), (4.15) and by setting $\Lambda = \{(i, \alpha); i + \sigma | \alpha | \le m\} \setminus \{(i, 0); 0 \le i < m_0\}$ we have

 $\|$ RHS of (4.9) (with *n* replaced by n+1) $\|_{\rho}$

$$\leq \sum_{(i,\alpha)\in\Lambda} \frac{C_{i,\alpha}}{1 - h_0/h} \frac{MH^n(eL)^{|\alpha|}c_0}{(R - \rho)^{Ln}} \phi_{n+1}(|\xi|;h) \quad \text{on } S_I(r)$$

and by applying Lemma 4.5 to the equation (4.9) (with n replaced by n+1) we have a unique solution $u_{n+1}(\xi, z) \in \mathcal{O}(S_I(r) \times D_R)$ such that

$$\begin{aligned} \|u_{n+1}\|_{\rho} &\leq \frac{1}{\delta(1-\beta)|\xi|^{m_0}(1+|\xi|)^{m-m_0}} \\ & \times \sum_{(i,\alpha)\in\Lambda} \frac{C_{i,\alpha}}{1-h_0/h} \frac{MH^n(eL)^{|\alpha|}c_0}{(R-\rho)^{L_n}} \phi_{n+1}(|\xi|;h) \quad \text{on } S_I(r) \end{aligned}$$

for any $0 < \rho < R$. Thus, if we take H > 0 sufficiently large so that

$$H \geq \frac{1}{\delta(1-\beta)} \sum_{(i,\alpha) \in \Lambda} \frac{C_{i,\alpha}(eL)^{|\alpha|} c_0}{1 - h_0/h}$$

we have (4.10) (with *n* replaced by n + 1). This proves Lemma 4.7.

In Lemma 4.7, by taking H > 0 large enough we may suppose that $2H \ge h$ holds: then we have $2H/(R - \rho)^L > h$ for any $0 < \rho < R$. By Lemma 4.7 and

(4.3) we have

$$\begin{split} &\sum_{n\geq N} \|u_n(\xi)\|_{\rho} \\ &\leq \sum_{n\geq N} \frac{MH^n}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}(R-\rho)^{L(n-1)}} \phi_n(|\xi|;h) \\ &\leq \frac{M(R-\rho)^L}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}} \sum_{n\geq N} \left(\frac{H}{(R-\rho)^L}\right)^n \phi_n(|\xi|;2H/(R-\rho)^L) \\ &\leq \frac{M(R-\rho)^L}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}} \times 2 \times \left(\frac{H}{(R-\rho)^L}\right)^N \phi_N(|\xi|;2H/(R-\rho)^L). \end{split}$$

This shows that the sum $\sum_{n\geq N} u_n(\xi, z)$ is convergent to a true solution $u(\xi, z)$ of (4.1) on $S_I(r) \times D_R$ and it satisfies

$$\|u(\xi)\|_{\rho} \le \frac{2M}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}} \frac{H^N}{(R-\rho)^{L(N-1)}} \phi_N\Big(|\xi|; \frac{2H}{(R-\rho)^L}\Big)$$

on $S_I(r)$ for any $0 < \rho < R$. This proves the existence part of Proposition 4.1.

Lastly, let us show the uniqueness of the solution. To do so, it is enough to prove the following result:

Proposition 4.8. Let $0 < R_1 < R$. Suppose that $u(\xi, z) \in \mathcal{O}(S_I(r) \times D_{R_1})$ satisfies

$$P(\xi, z)u + \sum_{0 \le i \le m} c_{i,0}(\xi, z) *_{q} (\xi^{i}u) + \sum_{i+\sigma|\alpha| \le m, |\alpha| > 0} c_{i,\alpha}(\xi, z) *_{q} (1 *_{q} (\xi^{i}\partial_{z}^{\alpha}u)) = 0 \quad on \ S_{I}(r) \times D_{R_{I}}$$

and

$$|u(\xi, z)| \le \frac{M}{|\xi|^{m_0} (1+|\xi|)^{m-m_0}} \phi_N(|\xi|; h_1) \quad on \ S_I(r) \times D_{R_1}$$

for some M > 0 and $h_1(>h_0)$. Then, we have $u(\xi, z) = 0$ on $S_I(r) \times D_{R_1}$.

Proof. By the same argument as in the proof of Lemma 4.7 we can show that there are M > 0 and H > 0 such that

(4.16)
$$\|u(\xi)\|_{\rho} \leq \frac{MH^n}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}(R_1-\rho)^{L(n-1)}}\phi_n(|\xi|;h_1)$$

on $S_I(r)$ for any $0 < \rho < R$ and any $n \ge N$. Since $[n+i]_q! \ge [n]_q! [i]_q!$ holds, we have

$$\phi_n(|\xi|;h_1) \le \frac{|\xi|^n}{[n]_q!} \exp_q(h_1|\xi|).$$

Therefore, by (4.16) we have

$$\|u(\xi)\|_{\rho} \leq \frac{M(R-\rho)^{L}}{|\xi|^{m_{0}}(1+|\xi|)^{m-m_{0}}} \Big(\frac{H}{(R_{1}-\rho)^{L}}\Big)^{n} \frac{|\xi|^{n}}{[n]_{q}!} \exp_{q}(h_{1}|\xi|)$$

on $S_I(r)$ for any $0 < \rho < R$. Since $[n]_q! \ge q^{n(n-1)/2}$ (where p = 1/q) holds, by letting $n \longrightarrow \infty$ we obtain $||u(\xi)||_{\rho} = 0$ on $S_I(r)$ for any $0 < \rho < R_1$. This proves that $u(\xi, z) = 0$ holds on $S_I(r) \times D_{R_1}$.

5 Proof of Theorem 2.3

In this section, we will prove Theorem 2.3. In the next subsection 5.1, we give estimates of the coefficients $X_n(z)$ $(n \ge 0)$ of the formal solution, and in subsection 5.2 we prove Theorem 2.3 by using Proposition 4.1.

5.1 Estimates of the formal solution

Let us show

Proposition 5.1. Suppose the conditions (2.1) and $a_{m_0,0}(0,0) \neq 0$. Then, if equation (1.1) has a formal solution $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_{R_0}[[t]]$ (with $R_0 > 0$), we can find $0 < R < R_0$, C > 0 and $\overline{h} > 0$ such that $|X_n(z)| \leq Ch^n[n]_q!$ holds on D_R for any $n = 0, 1, 2, \ldots$

Proof. We set $a_{j,0}^0(t,z) = a_{j,0}(t,z) - a_{j,0}(0,z)$ for $0 \le j \le m_0$, and $a_{j,\alpha}^0(t,z) = a_{j,\alpha}(t,z)$ for (j,α) with $j > m_0$ or $|\alpha| > 0$. We set

$$P_1(\lambda; z) = \sum_{0 \le j \le m_0} a_{j,0}(0, z) \lambda^j.$$

Then, we have $\operatorname{ord}_t(a_{j,\alpha}^0) \ge p_{j,\alpha}$ with

$$p_{j,\alpha} = \begin{cases} 1, & \text{if } 0 \le j \le m_0, \\ j - m_0, & \text{if } m_0 < j \le m \text{ and } |\alpha| = 0, \\ j - m_0 + 1, & \text{if } m_0 < j \le m \text{ and } |\alpha| > 0 \end{cases}$$

and the equation (1.1) is expressed in the form

(5.1)
$$P_1(tD_q; z)X + \sum_{j+\sigma|\alpha| \le m} a_{j,\alpha}^0(t, z)(tD_q)^j \partial_z^{\alpha} X = F(t, z).$$

Since $a_{m_0,0}(0,0) \neq 0$ holds, by taking R > 0 sufficiently small and by taking $N \in \mathbb{N}^*$ sufficiently large, we can take $\delta > 0$ such that

(5.2)
$$|P_1([n]_q; z)| \ge \delta (1 + [n]_q)^{m_0}$$
 on D_R for any $n \ge N$.

We set

$$F(t,z) = \sum_{n \ge 0} F_n(z)t^n,$$

$$a_{j,\alpha}^0(t,z) = \sum_{n \ge p_{j,\alpha}} a_{j,\alpha,n}(z)t^n \quad (j+\sigma|\alpha| \le m).$$

Then, by (5.1) we have the relation:

(5.3)
$$P_1([n]_q; z) X_n = F_n(z) - \sum_{j+\sigma |\alpha| \le m} \sum_{p_{i,\alpha} \le l \le n} a_{j,\alpha,l}(z) ([n-l]_q)^j \partial_z^{\alpha} X_{n-l}$$

for any $n = 0, 1, 2, \ldots$ By taking R > 0 sufficiently small we may assume that $X_n(z)$ $(n \ge 0)$, $F_n(z)$ $(n \ge 0)$ and $a_{j,\alpha,n}(z)$ $(n \ge p_{j,\alpha})$ are all bounded holomorphic functions on D_R . In addition, we may assume that $|F_n(z)| \le Ah^n$ $(n \ge 0)$ and $|a_{j,\alpha,n}(z)| \le Ah^n$ $(n \ge p_{j,\alpha})$ hold on D_R for some A > 0 and h > 0. To prove Proposition 5.1 it is sufficient to show the following lemma.

Lemma 5.2. Let $L = [m/\sigma]$. There are M > 0 and H > 0 such that

(5.4)
$$||X_n||_{\rho} \leq \frac{MH^n[n]_q!}{(R-\rho)^{Ln}} \text{ on } D_R \text{ for any } 0 < \rho < R$$

holds for any n = 0, 1, 2, ...

Proof. Let N be as in (5.2). Since $X_n(z)$ $(0 \le n \le N)$ are bounded holomorphic functions on D_R , by taking M > 0 and H > 0 sufficiently large we may suppose that (5.4) is satisfied for all $0 \le n \le N$.

Let us show the general case by induction on n. Let n > N, and suppose that (5.4) (with n replaced by p) is already proved for all p < n. Then, by applying Lemma 4.6 to the estimate (5.4) (with n replaced by n - l) we have

$$\begin{aligned} \|\partial_{z}^{\alpha}X_{n-l}\|_{\rho} &\leq \frac{MH^{n-l}[n-l]_{q}! \times e^{|\alpha|}(L(n-l)+1)\cdots(L(n-l)+|\alpha|)}{(R-\rho)^{L(n-l)+|\alpha|}} \\ &\leq \frac{MH^{n-l}[n-l]_{q}! \times (eL)^{|\alpha|}(n-l+1)^{|\alpha|}}{(R-\rho)^{Ln}} \end{aligned}$$

for any $0 < \rho < R.$ Therefore, by (5.2), (5.3) and the condition $0 < R - \rho < 1$ we have

$$(5.5) \quad ||X_{n}||_{\rho} \\ \leq \frac{1}{\delta(1+[n]_{q})^{m_{0}}} \Big[Ah^{n} + \sum_{j+\sigma|\alpha| \leq m} \sum_{p_{j,\alpha} \leq l \leq n} Ah^{l} ([n-l]_{q})^{j} ||\partial_{z}^{\alpha} X_{n-l}||_{\rho} \Big] \\ \leq \frac{1}{\delta(1+[n]_{q})^{m_{0}}} \Big[Ah^{n} + \sum_{j+\sigma|\alpha| \leq m} \sum_{p_{j,\alpha} \leq l \leq n} Ah^{l} ([n-l]_{q})^{j} \times \\ \times \frac{MH^{n-l}[n-l]_{q}! \times (eL)^{|\alpha|}(n-l+1)^{|\alpha|}}{(R-\rho)^{Ln}} \Big] \\ \leq \frac{MH^{n}}{\delta(R-\rho)^{Ln}} \Big[\frac{A}{M} \Big(\frac{h}{H} \Big)^{n} + A \sum_{j+\sigma|\alpha| \leq m} K_{j,\alpha}(n) (eL)^{|\alpha|} \sum_{p_{j,\alpha} \leq l \leq n} \Big(\frac{h}{H} \Big)^{l} \Big]$$

for any $0 < \rho < R$, where

$$K_{j,\alpha}(n) = \frac{([n-p_{j,\alpha}]_q)^j [n-p_{j,\alpha}]_q ! (n-p_{j,\alpha}+1)^{|\alpha|}}{(1+[n]_q)^{m_0}}.$$

In the case $0 \leq j \leq m_0$ we have $p_{j,\alpha} = 1$ and so

$$K_{j,\alpha}(n) = \frac{([n-1]_q)^j}{(1+[n]_q)^{m_0}} \frac{(n-1+1)^{|\alpha|}}{[n]_q} \times [n]_q! \le c_0[n]_q!$$

where c_0 is the one in (4.12). In the case $m_0 < j \le m$ and $|\alpha| = 0$ we have $p_{j,\alpha} = j - m_0$ and so

$$K_{j,\alpha}(n) \le ([n-p_{j,\alpha}]_q)^{j-m_0} [n-p_{j,\alpha}]_q!$$

= $([n-(j-m_0)]_q)^{j-m_0} \times [n-(j-m_0)]_q! \le [n]_q! \le c_0[n]_q!.$

In the case $m_0 < j \le m$ and $|\alpha| > 0$ we have $p_{j,\alpha} = j - m_0 + 1$ and so

$$K_{j,\alpha}(n) \leq ([n - p_{j,\alpha}]_q)^{j - m_0} [n - p_{j,\alpha}]_q! (n - p_{j,\alpha} + 1)^{|\alpha|}$$

= $([n - (j - m_0 + 1)]_q)^{j - m_0} \times [n - (j - m_0 + 1)]_q!$
 $\times (n - (j - m_0 + 1) + 1)^{|\alpha|}$
 $\leq [n - 1]_q! \times (n - (j - m_0 + 1) + 1)^{|\alpha|}$
= $[n]_q! \times \frac{(n - (j - m_0 + 1) + 1)^{|\alpha|}}{[n]_q} \leq c_0[n]_q!.$

Therefore, by applying these estimates to (5.5) we have

$$\begin{aligned} \|X_n\|_{\rho} &\leq \frac{MH^n c_0[n]_q!}{\delta(R-\rho)^{Ln}} \Big[\frac{A}{M} \Big(\frac{h}{H} \Big)^n + A \sum_{j+\sigma|\alpha| \leq m} (eL)^{|\alpha|} \sum_{p_{j,\alpha} \leq l \leq n} \Big(\frac{h}{H} \Big)^l \Big] \\ &\leq \frac{MH^n c_0[n]_q!}{\delta(R-\rho)^{Ln}} \Big[\frac{A}{M} \Big(\frac{h}{H} \Big)^n + A \sum_{j+\sigma|\alpha| \leq m} (eL)^{|\alpha|} \frac{(h/H)^{p_{j,\alpha}}}{1-h/H} \Big] \end{aligned}$$

for any $0 < \rho < R$. Thus, if the condition H > h and

(5.6)
$$\frac{c_0}{\delta} \left[\frac{A}{M} + A \sum_{j+\sigma|\alpha| \le m} (eL)^{|\alpha|} \frac{(h/H)^{p_{j,\alpha}}}{1 - h/H} \right] \le 1$$

hold, we have the result (5.4). By taking M > 0 and H > 0 sufficiently large, we can get the condition (5.6). This proves Lemma 5.2.

This completes the proof of Proposition 5.1.

Example 5.3. Let us consider

(5.7)
$$(tD_q + 1)X = \frac{a}{1-z}t + t(tD_q)^2 X + bt\partial_z^{\alpha} X,$$

where a > 0, b > 0 and $\alpha \in \mathbb{N}^*$. This equation is a particular case of equations of type (1.1) with $m_0 = 1$ and m = 2. The unique formal solution is given by $\hat{X}(t,z) = \sum_{n \ge 1} X_n(z)t^n$ with $X_1(z) = a/(([1]_q + 1)(1 - z))$ and

$$X_{n+1}(z) = \frac{([1]_q^2 + b\partial_z^{\alpha}) \cdots ([n]_q^2 + b\partial_z^{\alpha})}{([1]_q + 1)([2]_q + 1) \cdots ([n+1]_q + 1)} \Big(\frac{a}{(1-z)}\Big), \quad n \ge 1.$$

It is easy to see that

$$X_{n+1}(z) \gg \frac{[n]_q!}{2^{n+1}[n+1]_q} \frac{a}{(1-z)} \gg \frac{[n]_q!}{2^{n+1}(1+q)^n} \frac{a}{(1-z)}.$$

Thus, in the case (5.7), we can see that the estimates in Proposition 5.1 is best possible.

5.2 Proof of Theorem 2.3

Suppose (A₁), (A₂), (A₃) and (2.2). Let $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n$ be a formal solution of (1.1). Let $\mu \in \mathbb{N}^*$ be sufficiently large and set

$$\begin{aligned} X^{0}(t,z) &= \sum_{n \ge \mu} X^{0}_{n}(z) t^{n+1} \quad \text{with } X^{0}_{n}(z) = X_{n+1}(z) \quad (n \ge \mu), \\ F^{0}(t,z) &= F(t,z) - \sum_{j+\sigma |\alpha| \le m} a_{j,\alpha}(t,z) (tD_{q})^{j} \partial_{z}^{\alpha} \sum_{0 \le n \le \mu} X_{n}(z) t^{n}. \end{aligned}$$

Then, $X^0(t, z)$ is a formal solution of the equation

(5.8)
$$\sum_{j+\sigma|\alpha| \le m} a_{j,\alpha}(t,z) (tD_q)^j \partial_z^{\alpha} X^0 = F^0(t,z)$$

5.2.1 Some formulas

First, let us show

Lemma 5.4. (1) For $n \in \mathbb{N}^*$ we have

$$t^n(tD_q) = \frac{1}{q^n}(tD_q - [n]_q)t^n.$$

(2) For $n \in \mathbb{N}^*$ and $1 \leq i < n$ we have

$$(t^2 D_q)t^{n-i} = q^{n-i}t^{n-i}(t^2 D_q) + [n-i]_q t^{n-i+1}.$$

(3) For $n \in \mathbb{N}^*$ we have

$$t^{n}(tD_{q})^{n} = \frac{1}{q^{n(n-1)/2}} \sum_{i=1}^{n} H_{n,i} t^{n-i} (t^{2}D_{q})^{i},$$

where $H_{n,n} = 1$ $(n \ge 1)$ and $H_{n,i}$ $(1 \le i < n)$ are constants determined by the recurrence formula:

$$H_{n,i} = q^{n-i} H_{n-1,i-1} + ([n-1-i]_q - [n-1]_q) H_{n-1,i-1}$$

Proof. We know that $D_q(f(t)g(t)) = D_q(f(t))g(t) + f(qt)D_q(g(t))$ holds. Hence, we have

$$\begin{aligned} (tD_q)(t^n f(t)) &= t([n]_q t^{n-1} f(t) + (qt)^n D_q(f(t))) \\ &= [n]_q t^n f(t) + q^n t^n (tD_q)(f(t)), \end{aligned}$$

that is, $(tD_q)t^n = [n]_q t^n + q^n t^n (tD_q)$. This leads us to (1). The result (2) is verified in the same way.

Let us show (3). The case n = 1 is clear. Let us show the general case by induction on n. Suppose that (3) is already proved. Then, by (1) and (2) we have

$$t^{n+1}(tD_q)^{n+1} = t(t^n(tD_q))(tD_q)^n = t\left(\frac{1}{q^n}(tD_q - [n]_q)t^n\right)(tD_q)^n$$

$$= \frac{1}{q^n}(t^2D_q - [n]_qt) \times \frac{1}{q^{n(n-1)/2}}\sum_{i=1}^n H_{n,i}t^{n-i}(t^2D_q)^i$$

$$= \frac{1}{q^{n(n+1)/2}} \left[\sum_{i=1}^n H_{n,i}\left(q^{n-i}t^{n-i}(t^2D_q) + [n-i]_qt^{n-i+1}\right)(t^2D_q)^i - \sum_{i=1}^n [n]_qH_{n,i}t^{n-i+1}(t^2D_q)^i\right].$$

This shows (3) with n replaced by n + 1.

5.2.2 A reduction

We set $b_{j,0}(t,z) = a_{j,0}(t,z)$ (for $0 \le j < m_0$), $b_{j,0}(t,z) = t^{-(j-m_0)}a_{j,0}(t,z)$ (for $m_0 \le j \le m$), $b_{j,\alpha}(t,z) = t^{-1}a_{j,\alpha}(t,z)$ (for $0 \le j < m_0$ and $|\alpha| > 0$), and $b_{j,\alpha}(t,z) = t^{-(j-m_0+2)}a_{j,\alpha}(t,z)$ (for $m_0 \le j < m$ and $|\alpha| > 0$). Then, by (2.1) and (2.2) we see that $b_{j,\alpha}(t,z)$ $(j + \sigma |\alpha| \le m)$ are holomorphic functions in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$.

By multiplying (5.8) by t^{m_0} we have

$$\begin{split} \sum_{0 \leq j < m_0} t^{m_0 - j} b_{j,0}(t,z) t^j (tD_q)^j X^0 &+ \sum_{m_0 \leq j \leq m} b_{j,0}(t,z) t^j (tD_q)^j X^0 \\ &+ \sum_{0 \leq j < m_0, |\alpha| > 0} t^{m_0 - j + 1} b_{j,\alpha}(t,z) t^j (tD_q)^j \partial_z^{\alpha} X^0 \\ &+ \sum_{m_0 \leq j < m, |\alpha| > 0} t^2 b_{j,\alpha}(t,z) t^j (tD_q)^j \partial_z^{\alpha} X^0 \\ &= t^{m_0} F^0(t,z). \end{split}$$

Therefore, by setting $b_{j,0}^*(t,z) = t^{m_0-j}b_{j,0}(t,z)$ (for $0 \le j < m_0$), $b_{j,0}^*(t,z) = b_{j,0}(t,z)$ (for $m_0 \le j \le m$), $b_{j,\alpha}^*(t,z) = t^{m_0-j+1}b_{j,\alpha}(t,z)$ (for $0 \le j < m_0$ and $|\alpha| > 0$), and $b_{j,\alpha}^*(t,z) = t^2 b_{j,\alpha}(t,z)$ (for $m_0 \le j < m$ and $|\alpha| > 0$) we have

(5.9)
$$\sum_{j+\sigma|\alpha| \le m} b_{j,\alpha}^*(t,z) t^j (tD_q)^j \partial_z^{\alpha} X^0 = t^{m_0} F^0(t,z).$$

Hence, by (3) of Lemma 5.4 we have

$$\sum_{j+\sigma|\alpha| \le m} b_{j,\alpha}^*(t,z) \frac{1}{q^{j(j-1)/2}} \sum_{i=1}^j H_{j,i} t^{j-i} (t^2 D_q)^i \partial_z^\alpha X^0 = t^{m_0} F^0(t,z).$$

This shows

Lemma 5.5. The equation (5.9) can be expressed in the form

(5.10)
$$\sum_{i+\sigma|\alpha| \le m} A_{i,\alpha}(t,z) (t^2 D_q)^i \partial_z^{\alpha} X^0 = t^{m_0} F^0(t,z)$$

for some holomorphic functions $A_{i,\alpha}(t,z)$ $(i + \sigma |\alpha| \le m)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$. In addition, we have $\operatorname{ord}_t(A_{i,0}) \ge m_0 - i$ (for $0 \le i < m_0$), $\operatorname{ord}_t(A_{i,0}) \ge 0$ (for $m_0 \le i \le m$), $\operatorname{ord}_t(A_{i,\alpha}) \ge m_0 - i + 1$ (for $0 \le i < m_0$ and $|\alpha| > 0$), $\operatorname{ord}_t(A_{i,\alpha}) \ge 2$ (for $m_0 \le i < m$ and $|\alpha| > 0$), and $A_{i,0}(0,z) = b_{i,0}(0,z)/q^{i(i-1)/2}$ (for $m_0 \le i \le m$).

5.2.3 q-Convolution equation

By Lemma 5.5 we see that the equation (5.10) is written in the form

$$\begin{split} \sum_{0 \leq i < m_0} t^{m_0 - i} A^0_{i,0}(t,z) (t^2 D_q)^i X^0 \\ &+ \sum_{m_0 \leq i \leq m} A_{i,0}(0,z) (t^2 D_q)^i X^0 + \sum_{m_0 \leq i \leq m} t A^0_{i,0}(t,z) (t^2 D_q)^i X^0 \\ &+ \sum_{0 \leq i < m_0, |\alpha| > 0} t^{m_0 - i + 1} A^0_{i,\alpha}(t,z) (t^2 D_q)^i \partial^{\alpha}_z X^0 \\ &+ \sum_{m_0 \leq i < m, |\alpha| > 0} t^2 A^0_{i,\alpha}(t,z) (t^2 D_q)^i \partial^{\alpha}_z X^0 = t^{m_0} F^0(t,z) \end{split}$$

for some holomorphic functions $A^0_{i,\alpha}(t,z)$ $(i + \sigma |\alpha| \le m)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$. We set

$$u(\xi, z) = \hat{\mathscr{B}}_q[X^0](\xi, z) = \sum_{n \ge \mu} \frac{X_n^0(z)}{[n]_q!} \xi^n.$$

By Proposition 5.1 we know that $u(\xi, z)$ is a holomorphic function in a neighborhood of $(0,0) \in \mathbb{C}_{\xi} \times \mathbb{C}_{z}^{d}$. By applying *q*-formal Borel transform $\hat{\mathscr{B}}_{q}$ to the

above equation and by using (3.9) and (3-4-4) we have

$$\begin{split} \sum_{0 \leq i < m_0} & \mathscr{B}_q[t^{m_0 - i} A^0_{i,0}(t, z)] *_q (\xi^i u) \\ &+ \sum_{m_0 \leq i \leq m} A_{i,0}(0, z)(\xi^i u) + \sum_{m_0 \leq i \leq m} \mathscr{B}_q[tA^0_{i,0}(t, z)] *_q (\xi^i u) \\ &+ \sum_{0 \leq i < m_0, |\alpha| > 0} \mathscr{B}_q[t^{m_0 - i} A^0_{i,\alpha}(t, z)] *_q (1 *_q (\xi^i \partial_z^{\alpha} u)) \\ &+ \sum_{m_0 \leq i < m, |\alpha| > 0} \mathscr{B}_q[tA^0_{i,\alpha}(t, z)] *_q (1 *_q (\xi^i \partial_z^{\alpha} u)) \\ &= \mathscr{B}_q[t^{m_0} F^0(t, z)]. \end{split}$$

Thus, by setting

$$c_{i,\alpha}(\xi, z) = \mathscr{B}_q[t^{m_0-i}A^0_{i,\alpha}(t, z)] \quad (\text{for } 0 \le i < m_0),$$

$$c_{i,\alpha}(\xi, z) = \mathscr{B}_q[tA^0_{i,\alpha}(t, z)] \quad (\text{for } m_0 \le i \le m),$$

$$P(\xi, z) = \sum_{m_0 \le i \le m} A_{i,0}(0, z)\xi^i,$$

$$f(\xi, z) = \mathscr{B}_q[t^{m_0}F^0(t, z)]$$

we have a q-convolution partial differential equation

(5.11)
$$P(\xi, z)u + \sum_{0 \le i \le m} c_{i,0}(\xi, z) *_q (\xi^i u) + \sum_{i+\sigma |\alpha| \le m, |\alpha| > 0} c_{i,\alpha}(\xi, z) *_q (1 *_q (\xi^i \partial_z^{\alpha} u)) = f(t, z).$$

By the definition of $c_{i,\alpha}(\xi, z)$ and $f(\xi, z)$ we see that they are holomorphic functions on $\mathbb{C}_{\xi} \times D_R$ for some R > 0 and we have

$$\begin{aligned} |f(\xi, z)| &\leq C\phi_N(|\xi|; h) \quad \text{on } \mathbb{C} \times D_R \quad (\text{with } N = m_0 + \mu), \\ c_{i,\alpha}(\xi, z) &\ll C_{i,\alpha}\phi_{m_0 - i - 1}(\xi; h_0) \quad \text{for any } z \in D_R \quad (0 \leq i < m_0), \\ c_{i,\alpha}(\xi, z) &\ll C_{i,\alpha}\phi_0(\xi; h_0) \quad \text{for any } z \in D_R \quad (m_0 \leq i \leq m) \end{aligned}$$

for some C > 0, $h > h_0 > 0$ and $C_{i,\alpha} > 0$ $(i + \sigma |\alpha| \le m)$. We note:

$$P(\xi, z) = \sum_{m_0 \le i \le m} A_{i,0}(0, z)\xi^i = \sum_{m_0 \le i \le m} \frac{b_{i,0}(0, z)}{q^{i(i-1)/2}}\xi^i = \xi^{m_0} P_0(\xi, z)$$

where $P_0(\xi, z)$ is the one appearing in Lemma 2.2.

5.2.4 Holomorphic extension of $u(\xi, z)$

Take any $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$. By Lemm 2.2 we have a $\delta > 0$, an interval $I = (\theta_1, \theta_2)$ with $\theta_1 < \arg \lambda < \theta_2$ and an R > 0 such that

$$|P(\xi, z)| \ge \delta |\xi|^{m_0} (1+|\xi|)^{m-m_0}$$
 on $S_I \times D_R$.

Since μ is taken sufficiently large, we may suppose that $N = m_0 + \mu$ satisfies

$$\frac{1}{[N]_q} \sum_{0 \le i < m_0} \frac{C_{i,0}}{\delta(1 - h_0/h)} < 1$$

Thus, we can apply Proposition 4.1 to the equation (5.11). This shows that $u(\xi, z)$ has an analytic extension $u^*(\xi, z)$ to the domain $S_I \times D_{R_1}$ (for some $R_1 > 0$) as a solution of (5.11), and we have the estimate

(5.12)
$$|u^*(\xi, z)| \le \frac{M_1}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}} \phi_N(|\xi|; h_1) \quad \text{on } S_I(r) \times D_{R_1}$$

for some $M_1 > 0$ and $h_1 > 0$.

5.2.5 Completion of the proof of Theorem 2.3

To complete the proof of Theorem 2.3 it is enough to show Lemma 5.6 given below. If this is true, by setting

$$W(t,z) = \sum_{0 \le n \le \mu} X_n(z) t^n + \mathscr{L}_q^{\lambda}[u^*](t,z)$$

we have a true solution of (1.1) desired in Theorem 2.3.

Lemma 5.6. There are C > 0, h > 0, A > 0 and 0 < B < q such that

(5.13)
$$|u^*(\lambda q^n, z)| \le Ch^n [n]_q!$$
 on D_{R_1} for $n = 0, 1, 2, \dots$

(5.14)
$$|u^*(\lambda q^{-m}, z)| \le AB^m$$
 on D_{R_1} for $m = 1, 2, \dots$

Proof. Since $u^*(\xi, z)$ is a holomorphic function in a neighborhood of $(0, 0) \in \mathbb{C}_{\xi} \times \mathbb{C}_{z}^{d}$, by setting B = 1 we have the condition (5.14) for a sufficiently large A > 0. By (5.12) and Lemma 4.3 we have

$$\begin{aligned} |u^*(\xi, z)| &\leq \frac{M_1}{|\xi|^{m_0}(1+|\xi|)^{m-m_0}} \frac{|\xi|^N}{[N]_q!} \exp_q(h_1|\xi|) \\ &\leq \frac{M_1|\xi|^{N-m_0}}{[N]_q!} \times \\ &\times K_1 \exp\left(\frac{(\log(h_1|\xi|))^2}{2\log q} + \left(-\frac{1}{2} + \frac{\log(q-1)}{\log q}\right) \log(h_1|\xi|)\right) \\ &\leq M_2 \exp\left(\frac{(\log|\xi|)^2}{2\log q} + \alpha \log|\xi|\right) \quad \text{on } S_I \times D_{R_1} \end{aligned}$$

for some $K_1 > 0$, $M_2 > 0$ and $\alpha \in \mathbb{R}$. Hence, we obtain

$$|u^*(\lambda q^n, z)| \le M_2(|\lambda|q^\alpha)^n q^{n^2/2} \exp\left(\frac{(\log|\lambda|)^2}{2\log q} + \alpha \log|\lambda|\right) \quad \text{on } D_{R_1}$$

for $n = 0, 1, 2, \dots$ Since $q^{n(n-1)/2} \leq [n]_q!$ holds, we have the result (5.13).

6 The case without (2.2)

In Theorem 2.3, we have shown the G_q -summability of the formal solution (1.2) under the additional assumption (2.2). Let us consider here the case without the assumption (2.2). We note:

Lemma 6.1. Let f(t) be a function in t, and let $n \in \mathbb{N}^*$. We set $F(\tau) = f(t)$ with $t = \tau^n$; then we have

(6.1)
$$tD_q(f)(t) = \frac{1}{[n]_{q^{1/n}}} \tau D_{q^{1/n}}(F)(\tau)$$

Proof. By the definition we have

$$tD_q(f)(t) = t \times \frac{f(qt) - f(t)}{(q-1)t} = \tau^n \times \frac{f(q\tau^n) - f(\tau^n)}{(q-1)\tau^n}$$
$$= \tau^n \times \frac{F(q^{1/n}\tau) - F(\tau)}{(q-1)\tau^n}$$
$$= \frac{q^{1/n} - 1}{(q^{1/n})^n - 1} \times \tau \times \frac{F(q^{1/n}\tau) - F(\tau)}{(q^{1/n} - 1)\tau} = \frac{1}{[n]_{q^{1/n}}} \tau D_{q^{1/n}}(F)(\tau).$$

Lemma 6.2. Let $n \in \mathbb{N}^*$: we have

(6.2)
$$tD_{q^n} = \frac{q-1}{q^n-1} \sum_{i=0}^{n-1} ((q-1)tD_q+1)^i (tD_q).$$

Note that D_{q^n} in the left-hand side is q^n -derivative and D_q in the right-hand side is q-derivative.

Proof. By the definition we have

$$tD_{q^{n}}(f)(t) = t \times \frac{f(q^{n}t) - f(t)}{(q^{n} - 1)t}$$

= $t \times \frac{q - 1}{q^{n} - 1} \times \frac{(f(q^{n}t) - f(q^{n-1}t)) + \dots + (f(qt) - f(t))}{(q - 1)t}$
= $\frac{q - 1}{q^{n} - 1} (q^{n-1}tD_{q}(f)(q^{n-1}t) + \dots + qtD_{q}(f)(qt) + tD_{q}(f)(t)).$

Therefore, by using the operator σ_q defined by $\sigma_q(f(t)) = f(qt)$ we have

$$tD_{q^n}(f)(t) = \frac{q-1}{q^n-1} (\sigma_q^{n-1} + \dots + \sigma_q + 1)(tD_q)(f)(t).$$

Since $\sigma_q = (q-1)tD_q + 1$ holds, we have (6.2).

Corollary 6.3. For any $m \in \mathbb{N}^*$ and $n \in \mathbb{N}$ we have

(6.3)
$$[m]_{q^n} = \frac{1}{[n]_q} \sum_{i=0}^{n-1} ((q-1)[m]_q + 1)^i [m]_q.$$

Proof. By applying (6.2) to t^m we have this result.

Discussion in the case without (2.2). We set $t = \tau^2$ and $q_1 = q^{1/4}$. Then, by applying Lemmas 6.1 and 6.2 we can see that our equation (1.1) is written in the form

(6.4)
$$\sum_{j+\sigma|\alpha|\leq m} A_{j,\alpha}(\tau,z) \left(\frac{1}{[4]_{q_1}} \left((q_1-1)(\tau D_{q_1})^2 + 2(\tau D_{q_1}) \right) \right)^j \partial_z^{\alpha} Y = G(\tau,z)$$

where

$$A_{j,\alpha}(\tau, z) = a_{j,\alpha}(\tau^2, z) \quad (j + \sigma |\alpha| \le m),$$

$$Y(\tau, z) = X(\tau^2, z) = \sum_{n \ge 0} X_n(z)\tau^{2n},$$

$$G(\tau, z) = F(\tau^2, z).$$

In this case, the *t*-Newton polygon $N_t(6.4)$ of (6.4) is

$$N_t(6.4) = \{(x, y) \in \mathbb{R}^2 ; x \le 2m, y \ge \max\{0, x - 2m_0\}\},\$$

and we have

$$\operatorname{ord}_t(A_{j,\alpha}) \ge \begin{cases} \max\{0, 2j - 2m_0\}, & \text{if } |\alpha| = 0, \\ \max\{2, 2j - 2m_0 + 2\}, & \text{if } |\alpha| > 0. \end{cases}$$

Therefore, the condition corresponding to (2.2) is satisfied. Thus, we can apply Theorem 2.3 to the q_1 -difference equation (6.4) and we have G_{q_1} -summability of the formal solution $Y(\tau, z)$.

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Hidetoshi Tahara Department of Information and Communication Sciences, Sophia University, Kioicho, Chiyoda-ku, Tokyo 102-8554, Japan. Email: h-tahara@sophia.ac.jp