Strong instability of standing waves for nonlinear Schrödinger equations with attractive inverse power potential

Noriyoshi Fukaya and Masahito Ohta

Abstract

We study the strong instability of standing waves $e^{i\omega t}\phi_\omega(x)$ for nonlinear Schrödinger equations with an L^2 -supercritical nonlinearity and an attractive inverse power potential, where $\omega \in \mathbb{R}$ is a frequency, and $\phi_{\omega} \in H^1(\mathbb{R}^N)$ is a ground state of the corresponding stationary equation. Recently, for nonlinear Schrödinger equations with a harmonic potential, Ohta (2018) proved that if $\partial_{\lambda}^{2} S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} \leq 0$, then the standing wave is strongly unstable, where S_ω is the action, and $\phi_\omega^\lambda(x) :=$ $\lambda^{N/2} \phi_\omega(\lambda x)$ is the scaling, which does not change the L^2 -norm. In this paper, we prove the strong instability under the same assumption as the above-mentioned in inverse power potential case. Our proof is applicable to nonlinear Schrödinger equations with other potentials such as an attractive Dirac delta potential.

1 Introduction

In this paper, we consider the nonlinear Schrödinger equation with an attractive inverse power potential

(NLS)
$$
i\partial_t u = -\Delta u - \frac{\gamma}{|x|^{\alpha}} u - |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,
$$

where

(1.1)
$$
N \in \mathbb{N}
$$
, $\gamma > 0$, $0 < \alpha < \min\{2, N\}$, $1 + \frac{4}{N} < p < 1 + \frac{4}{N - 2}$,

²⁰¹⁰ *Mathematics Subject Classification*. 35Q55, 35B35

and $u: \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is an unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. Here, $1 + 4/(N - 2)$ stands for ∞ if $N = 1$ or 2.

Let us consider the Cauchy problem for [\(NLS\)](#page-0-0). Since the potential $V(x) := -\gamma |x|^{-\alpha}$ belongs to $(L^r + L^{\infty})(\mathbb{R}^N)$ for some $r > \min\{1, N/2\}$ under the assumption [\(1.1\)](#page-0-1), the multiplication operator $v \mapsto V(x)v$ is continuous from $H^1(\mathbb{R}^N)$ to $(L^{\rho'}+L^2)(\mathbb{R}^N)$ for some $\rho \in [2, 2N/(N-2))$, and thus, the potential energy $\int_{\mathbb{R}^N} V(x)|v(x)|^2 dx$ is well-defined on $H^1(\mathbb{R}^N)$. Therefore, the local well-posedness of [\(NLS\)](#page-0-0) in the energy space $H^1(\mathbb{R}^N)$ follows from the standard theory, e.g. [\[3,](#page-14-0) Theorems 3.3.5, 3.3.9, Proposition 4.2.3]. More precisely, for each $u_0 \in H^1(\mathbb{R}^N)$, there exist a maximal interval $I_{\text{max}} = [0, T^+) \subset \mathbb{R}$ with $T^+ = T^+(u_0) \in (0,\infty]$ and a unique solution $u \in C(I_{\max}, H^1(\mathbb{R}^N))$ of [\(NLS\)](#page-0-0) with $u(0) = u_0$ such that if $T^+ < \infty$, then $\lim_{t \nearrow T^+} ||u(t)||_{H^1} = \infty$. Here, if $T^+ < \infty$, we say that the solution $u(t)$ *blows up in finite time*. Moreover, [\(NLS\)](#page-0-0) satisfies the two conservation laws

$$
E(u(t)) = E(u_0), \quad ||u(t)||_{L^2} = ||u_0||_{L^2}
$$

for all $t \in I_{\text{max}}$, where

$$
E(v) := \frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|v(x)|^2}{|x|^{\alpha}} dx - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1}
$$

is the energy.

By a *standing wave*, we mean a solution of [\(NLS\)](#page-0-0) with the form $e^{i\omega t}\phi(x)$, where $\omega \in \mathbb{R}$ is a frequency, and $\phi \in H^1(\mathbb{R}^N)$ is a nontrivial solution of the stationary equation

(1.2)
$$
-\Delta \phi + \omega \phi - \frac{\gamma}{|x|^{\alpha}} \phi - |\phi|^{p-1} \phi = 0, \quad x \in \mathbb{R}^{N}.
$$

Eq. [\(1.2\)](#page-1-0) can be written as $S'_{\omega}(\phi) = 0$, where

$$
S_{\omega}(v) := E(v) + \frac{\omega}{2} ||v||_{L^2}^2
$$

is the action. The following existence and variational characterization of ground states by using the Nehari functional

$$
K_{\omega}(v) := \partial_{\lambda} S_{\omega}(\lambda v)|_{\lambda=1} = \langle S'_{\omega}(v), v \rangle
$$

= $\|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - \gamma \int_{\mathbb{R}^N} \frac{|v(x)|^2}{|x|^{\alpha}} dx - \|v\|_{L^{p+1}}^{p+1}$

are known (see [\[6,](#page-14-1) Remarks 1.2 and 1.3]), where a *ground state* is a nontrivial solution of [\(1.2\)](#page-1-0) with the least action.

Proposition 1.1. *Assume* [\(1.1\)](#page-0-1) *and*

$$
(1.3) \qquad \omega > \omega_0 := -\inf \left\{ \|\nabla v\|_{L^2}^2 - \gamma \int_{\mathbb{R}^N} \frac{|v(x)|^2}{|x|^\alpha} dx \; \middle| \; v \in H^1(\mathbb{R}^N), \ \|v\|_{L^2} = 1 \right\}.
$$

Then the set of ground states

$$
\mathcal{G}_{\omega} := \{ \phi \in \mathcal{F}_{\omega} \mid S_{\omega}(\phi) \le S_{\omega}(v) \text{ for all } v \in \mathcal{F}_{\omega} \}
$$

is not empty, where

$$
\mathcal{F}_{\omega} := \{ \phi \in H^1(\mathbb{R}^N) \setminus \{0\} \mid S_{\omega}'(\phi) = 0 \}
$$

is the set of all nontrivial solutions of [\(1.2\)](#page-1-0)*. Moreover, if* $\phi \in \mathcal{G}_{\omega}$ *, then*

(1.4)
$$
S_{\omega}(\phi) = \inf \{ S_{\omega}(v) \mid v \in H^{1}(\mathbb{R}^{N}) \setminus \{0\}, K_{\omega}(v) = 0 \}.
$$

For the sake of completeness, we give a proof of Proposition [1.1](#page-2-0) in Section [2](#page-5-0) by using the argument in [\[8,](#page-14-2) Section 3].

In the present paper, we study the strong instability of the standing wave solution $e^{i\omega t}\phi_\omega$ of [\(NLS\)](#page-0-0), where $\omega > \omega_0$ and $\phi_\omega \in \mathcal{G}_\omega$. We recall the definitions of stability and instability of standing waves.

Definition 1.2. Let $e^{i\omega t}\phi$ be a standing wave solution of [\(NLS\)](#page-0-0).

• We say that $e^{i\omega t}\phi$ is *stable* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N)$ satisfies $||u_0 - \phi||_{H^1} < \delta$, then the solution $u(t)$ of [\(NLS\)](#page-0-0) with $u(0) = u_0$ exists globally in time, and satisfies

$$
\sup_{t\geq 0}\inf_{\theta\in\mathbb{R}}\|u(t)-e^{i\theta}\phi\|_{H^1}<\varepsilon.
$$

- We say that $e^{i\omega t}\phi$ is *unstable* if $e^{i\omega t}\phi$ is not stable.
- We say that $e^{i\omega t}\phi$ is *strongly unstable* if for each $\varepsilon > 0$, there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $||u_0 - \phi||_{H^1} < \varepsilon$ and the solution $u(t)$ of [\(NLS\)](#page-0-0) with $u(0) = u_0$ blows up in finite time.

Here, we state some known results related to our works. The stability and instability of standing waves with a ground state profile for nonlinear Schrödinger equations have been studied by many researchers. For [\(NLS\)](#page-0-0) in the nonpotential case $\gamma = 0$, Berestycki and Cazenave [\[1\]](#page-14-3) proved the strong instability for any $\omega > 0$ when $1+4/N \leq p \leq 1+4/(N-2)$ (for the case $p = 1 + 4/N$, see also [\[22\]](#page-15-0)). Cazenave and Lions [\[4\]](#page-14-4) proved the stability for any $\omega > 0$ if $1 < p < 1 + 4/N$. For abstract Hamiltonian systems including nonlinear Schrödinger equations, Grillakis, Shatah, and Strauss [\[10,](#page-14-5) [11\]](#page-14-6) gave sufficient conditions for the stability and instability, that is, if $\partial_{\omega} ||\phi_{\omega}||_{L^2}^2 > 0$, the standing wave is stable, and if $\partial_{\omega} ||\phi_{\omega}||_{L^2}^2 < 0$, the standing wave is unstable (see also [\[20,](#page-15-1) [21,](#page-15-2) [23\]](#page-15-3)). For the nonlinear Schrödinger equation with a general potential

(1.5)
$$
i\partial_t u = -\Delta u + \tilde{V}(x)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,
$$

Rose and Weinstein [\[19\]](#page-15-4) proved the stability for $\omega > \tilde{\omega}_0$ sufficiently closed to $\tilde{\omega}_0$ even when $1 + 4/N \leq p < 1 + 4/(N - 2)$ by using the criteria of Grillakis, Shatah, and Strauss [\[10\]](#page-14-5), where $-\tilde{\omega}_0$ is the smallest eigenvalue of the Schrödinger operator $-\Delta + V$. In [\[6\]](#page-14-1), Ohta and Fukuizumi improved the stability results of Rose and Weinstein, and in [\[7\]](#page-14-7), they proved the instability for sufficiently large ω when $1 + 4/N < p < 1 + 4/(N-2)$ by using the sufficient condition of Ohta [\[15\]](#page-15-5), that is, if $\partial_{\lambda}^{2} \tilde{S}_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} < 0$, the standing wave is unstable, where \tilde{S}_{ω} is the action corresponding to [\(1.5\)](#page-3-0), and $v^{\lambda}(x) := \lambda^{N/2} v(\lambda x)$ is the scaling, which does not change the L^2 -norm (see also $[8, 9]$ $[8, 9]$ in the Dirac delta potential case and [\[5\]](#page-14-9) in the harmonic potential case). For the nonlinear Schrödinger equation with an attractive Dirac delta potential

(1.6)
$$
i\partial_t u = -\partial_x^2 u - \tilde{\gamma}\delta(x)u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R},
$$

Ohta and Yamaguchi [\[18\]](#page-15-6) proved the strong instability of the standing wave with positive energy $E(\phi_\omega) > 0$ when $\tilde{\gamma} > 0$ and $p > 5$, and as a corollary, they proved the strong instability for sufficiently large ω (see also [\[17\]](#page-15-7) for related works). Recently, for the nonlinear Schrödinger equation with a harmonic potential

(1.7)
$$
i\partial_t u = -\Delta u + |x|^2 u - |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,
$$

Ohta [\[16\]](#page-15-8) proved the strong instability under the same assumption $\partial^2_{\lambda} \tilde{S}_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} \leq 0$ as in [\[15\]](#page-15-5) when $1 + 4/N < p < 1 + 4/(N - 2)$.

In view of the graph of $\lambda \mapsto \tilde{S}_{\omega}(\phi_{\omega}^{\lambda})$, we see that $\tilde{E}(\phi_{\omega}) > 0$ implies $\partial_{\lambda}^{2} \tilde{S}_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} < 0$. Therefore, the question naturally arises whether the standing wave is strongly unstable or not in the case $\tilde{E}(\phi_\omega) \leq 0$ and $\partial_\lambda^2 \tilde{S}_\omega(\phi_\omega^\lambda)|_{\lambda=1} \leq 0$ for [\(1.6\)](#page-3-1). However, the proof for (1.7) in [\[16\]](#page-15-8) is not applicable to (1.6) .

In this paper, we consider the strong instability of standing waves under the same assumption as in [\[16\]](#page-15-8). In order to treat more general potentials with suitable properties related to the scaling $\lambda \mapsto v^{\lambda}$, we study the nonlinear Schrödinger equation [\(NLS\)](#page-0-0) with an inverse power potential. Now, we state our main result.

Theorem 1.3. *Assume* [\(1.1\)](#page-0-1)*,* $\omega > \omega_0$ *, and that* $\phi_{\omega} \in \mathcal{G}_{\omega}$ *satisfies* $\partial_{\lambda}^2 S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} \leq 0$ *, where* $\phi_{\omega}^{\lambda}(x) = \lambda^{N/2} \phi_{\omega}(\lambda x)$. Then the standing wave solution $e^{i\omega t} \phi_{\omega}$ of [\(NLS\)](#page-0-0) is strongly *unstable.*

It is proven in [\[7,](#page-14-7) Section 2] that the assumption $\partial^2_{\lambda} S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} \leq 0$ is satisfied for sufficiently large ω . Therefore, we have the following corollary.

Corollary 1.4. *Assume* [\(1.1\)](#page-0-1)*. Then there exists* $\omega_1 > \omega_0$ *such that if* $\omega \geq \omega_1$ *and* $\phi_{\omega} \in \mathcal{G}_{\omega}$, the standing wave solution $e^{i\omega t} \phi_{\omega}$ of [\(NLS\)](#page-0-0) is strongly unstable.

Remark 1.5*.* Theorem [1.3](#page-3-3) can be extended to more general settings. The important feature used in the proof of Theorem [1.3](#page-3-3) is that the energy satisfies

(1.8)
$$
E(v) = \frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{1}{2}G(v) - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1},
$$

$$
(1.9) \tG(v) \ge 0, \tG(\lambda v) = \lambda^2 G(v), \tG(v^{\lambda}) = \lambda^{\alpha} G(v), \t||v^{\lambda}||_{L^{p+1}}^{p+1} = \lambda^{\beta} ||v||_{L^{p+1}}^{p+1}
$$

with $\beta > 2 > \alpha > 0$. Since the energy of [\(1.6\)](#page-3-1) satisfies [\(1.8\)](#page-4-0) and [\(1.9\)](#page-4-1) with $G(v)$ $\gamma |v(0)|^2$, $\alpha = 1$, and $\beta = (p-1)/2$, the proof is applicable to [\(1.6\)](#page-3-1) for $p > 5$. This gives an improvement of the result of Ohta and Yamaguchi [\[18\]](#page-15-6).

The proof of blowup for nonlinear Schrödinger equations relies on the virial identity

(1.10)
$$
\frac{d^2}{dt^2} ||xu(t)||_{L^2}^2 = 8Q(u(t)),
$$

where Q is the functional on $H^1(\mathbb{R}^N)$ defined by

$$
Q(v) = \|\nabla v\|_{L^2}^2 - \frac{\gamma \alpha}{2} \int_{\mathbb{R}^N} \frac{|v(x)|^2}{|x|^{\alpha}} dx - \frac{N(p-1)}{2(p+1)} \|v\|_{L^{p+1}}^{p+1}.
$$

Note that

$$
S_{\omega}(v^{\lambda}) = \frac{\lambda^{2}}{2} \|\nabla v\|_{L^{2}}^{2} + \frac{\omega}{2} \|v\|_{L^{2}}^{2} - \frac{\gamma \lambda^{\alpha}}{2} \int_{\mathbb{R}^{N}} \frac{|v(x)|^{2}}{|x|^{\alpha}} dx - \frac{\lambda^{N(p-1)/2}}{p+1} \|v\|_{L^{p+1}}^{p+1},
$$

$$
Q(v) = \partial_{\lambda} S_{\omega}(v^{\lambda})|_{\lambda=1}.
$$

Since $x \cdot \nabla V(x) = \gamma \alpha |x|^{-\alpha} \in (L^q + L^{\infty})(\mathbb{R}^N)$ for some $q > \min\{1, N/2\}$ under the assumption [\(1.1\)](#page-0-1), from the standard theory [\[3,](#page-14-0) Proposition 6.5.1], we obtain the local well-posedness of the Cauchy problem for [\(NLS\)](#page-0-0) in the weighted space

$$
\Sigma := \{ v \in H^1(\mathbb{R}^N) \mid ||xv||_{L^2} < \infty \},\
$$

and the virial identity [\(1.10\)](#page-4-2) holds for all $t \in I_{\text{max}}$.

To prove Theorem [1.3,](#page-3-3) we introduce the set

$$
\mathcal{B}_{\omega} = \left\{ v \in H^{1}(\mathbb{R}^{N}) \middle| \begin{array}{c} S_{\omega}(v) < S_{\omega}(\phi_{\omega}), & ||v||_{L^{2}} \leq ||\phi_{\omega}||_{L^{2}}, \\ ||v||_{L^{p+1}} > ||\phi_{\omega}||_{L^{p+1}}, & Q(v) < 0 \end{array} \right\}.
$$

Then we have the following blowup result.

Theorem 1.6. *Assume* [\(1.1\)](#page-0-1)*,* $\omega > \omega_0$ *, and that* $\phi_{\omega} \in \mathcal{G}_{\omega}$ *satisfies* $\partial_{\lambda}^2 S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} \leq 0$ *. If* $u_0 \in \mathcal{B}_{\omega} \cap \Sigma$, then the solution $u(t)$ of [\(NLS\)](#page-0-0) with $u(0) = u_0$ blows up in finite time.

Theorem [1.3](#page-3-3) follows from Theorem [1.6](#page-5-1) and the fact that the ground state ϕ_{ω} belongs to the closure of $\mathcal{B}_{\omega} \cap \Sigma$ in H^1 -topology.

The key to the proof of Theorem [1.6](#page-5-1) is Lemma [3.2](#page-10-0) below. The same assertion of Lemma [3.2](#page-10-0) is proven in $[16, \text{Lemma } 4]$ for (1.7) . In $[16, \text{Lemma } 4]$, the proof is divided into two cases $||x\phi_\omega||^2_{L^2} \le ||xv||^2_{L^2}$ and $||xv||^2_{L^2} \le ||x\phi_\omega||^2_{L^2}$. Although the first case is easy to treat, the second case is more complicated. In the second case, the inequality $||xv||_{L^2}^2 \le ||x\phi_\omega||_{L^2}^2$ is used to obtain upper bounds for the potential energy. However, in our case, this argument does not work well because the sign of the potential is different from that of [\(1.7\)](#page-3-2). In our proof here, to obtain upper bounds for the potential energy, we use the inequality coming out of the variational characterization of the ground state (see Lemma [2.6](#page-9-0) [\(i\)](#page-9-1) below).

We remark that in [\[16,](#page-15-8) [18\]](#page-15-6), they consider

$$
\left\{ v \in H^{1}(\mathbb{R}^{N}) \; \middle| \; \tilde{E}(v) < \tilde{E}(\phi_{\omega}), \; \|v\|_{L^{2}} = \|\phi_{\omega}\|_{L^{2}}, \; \sum_{\omega} \in \mathbb{R}^{N} \; \text{and} \; \tilde{E}(v) < \tilde{E}(\phi_{\omega}) \leq \tilde{E}(\phi_{\omega}) \right\} \cap \Sigma
$$

as the set of initial data of blowup solutions. On the other hand, in our definition of \mathcal{B}_{ω} , we use the action S_ω instead of the energy E in order to treat more general initial data.

We finally remark that the assumption $\partial^2_{\lambda} S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} \leq 0$ is not a necessary condition for the instability because it is known for (1.6) that there exist unstable standing waves satisfying $\partial^2_\lambda \tilde{S}_\omega(\phi^\lambda_\omega)|_{\lambda=1} > 0$ (see [\[18,](#page-15-6) Section 4]). It is an open problem whether the standing wave is strongly unstable or not in this case.

This paper is organized as follows. In Section [2,](#page-5-0) we give a proof of Proposition [1.1](#page-2-0) and prove a useful lemma (Lemma [2.6](#page-9-0) below). In Section [3,](#page-10-1) we prove Theorem [1.6.](#page-5-1) In Section [4,](#page-13-0) we prove Theorem [1.3.](#page-3-3)

2 Existence and Variational Characterization of ground states

The aim of this section is to prove Proposition [1.1](#page-2-0) and Lemma [2.6](#page-9-0) below. Here, we assume [\(1.1\)](#page-0-1) and $\omega > \omega_0$, where ω_0 is defined in [\(1.3\)](#page-2-1). Hereafter, we denote

(2.1)
$$
G(v) = \gamma \int_{\mathbb{R}^N} \frac{|v(x)|^2}{|x|^{\alpha}} dx.
$$

We define

$$
d(\omega) = \inf \{ S_{\omega}(v) \mid v \in H^1(\mathbb{R}^N) \setminus \{0\}, K_{\omega}(v) = 0 \},
$$

$$
\mathcal{M}_{\omega} = \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} \mid K_{\omega}(v) = 0, S_{\omega}(v) = d(\omega) \}.
$$

Note that since $-\omega_0$ is the smallest eigenvalue of the Schrödinger operator $-\Delta - \gamma |x|^{-\alpha}$, under the assumption $\omega > \omega_0$, we have the equivalence of norms

$$
\sqrt{L_{\omega}(v)} \simeq ||v||_{H^1},
$$

where

$$
L_{\omega}(v) = \|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - G(v).
$$

First, we show that ground states of (1.2) are characterized as the minimizers for S_ω under the constraint $K_{\omega} = 0$.

Lemma 2.1. $\mathcal{M}_{\omega} \subset \mathcal{G}_{\omega}$.

Proof. Let $\phi \in \mathcal{M}_{\omega}$. Then by $L_{\omega}(\phi) - ||\phi||_{L^{p+1}}^{p+1} = K_{\omega}(\phi) = 0$, we have

(2.3)
$$
\langle K'_{\omega}(\phi), \phi \rangle = 2L_{\omega}(\phi) - (p+1) \|\phi\|_{L^{p+1}}^{p+1} = -(p-1) \|\phi\|_{L^{p+1}}^{p+1} < 0.
$$

Therefore, there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S'_{\omega}(\phi) = \eta K'_{\omega}(\phi)$. Moreover, since

$$
\eta \langle K'_{\omega}(\phi), \phi \rangle = \langle S'_{\omega}(\phi), \phi \rangle = K_{\omega}(\phi) = 0,
$$

it follows from [\(2.3\)](#page-6-0) that $\eta = 0$, which implies $S'_{\omega}(\phi) = 0$.

Furthermore, if $v \in H^1(\mathbb{R}^N)$ satisfies $v \neq 0$ and $S'_\n\omega(v) = 0$, then by $K_\n\omega(v) =$ $\langle S'_\n\omega(v), v \rangle = 0$ and the definition of \mathcal{M}_ω , we have $S_\omega(\phi) \leq S_\omega(v)$. Thus, we obtain $\phi \in \mathcal{G}_{\omega}$. This completes the proof. \Box

Lemma 2.2. *If* \mathcal{M}_{ω} *is not empty, then* $\mathcal{G}_{\omega} \subset \mathcal{M}_{\omega}$ *.*

Proof. Let $\phi \in \mathcal{G}_{\omega}$. Since \mathcal{M}_{ω} is not empty, we take $\psi \in \mathcal{M}_{\omega}$. Then by Lemma [2.1,](#page-6-1) we have $\psi \in \mathcal{G}_{\omega}$. Therefore, if $v \in H^1(\mathbb{R}^N)$ satisfies $v \neq 0$ and $K_{\omega}(v) = 0$, then $S_\omega(\phi) = S_\omega(\psi) \leq S_\omega(\nu)$. This implies $\phi \in \mathcal{M}_\omega$. This completes the proof. □

Next, we show that \mathcal{M}_{ω} is not empty. By using

(2.4)
$$
S_{\omega}(v) = \frac{1}{2} K_{\omega}(v) + \frac{p-1}{2(p+1)} ||v||_{L^{p+1}}^{p+1}
$$

$$
= \frac{1}{p+1} K_{\omega}(v) + \frac{p-1}{2(p+1)} L_{\omega}(v),
$$

we rewrite

(2.5)
$$
d(\omega) = \inf \left\{ \frac{p-1}{2(p+1)} ||v||_{L^{p+1}}^{p+1} \middle| v \in H^1(\mathbb{R}^N) \setminus \{0\}, K_{\omega}(v) = 0 \right\}
$$

(2.6) =
$$
\inf \left\{ \frac{p-1}{2(p+1)} L_{\omega}(v) \middle| v \in H^1(\mathbb{R}^N) \setminus \{0\}, K_{\omega}(v) = 0 \right\}.
$$

Lemma 2.3. *If* $K_\omega(v) < 0$ *, then*

$$
\frac{p-1}{2(p+1)}||v||_{L^{p+1}}^{p+1} > d(\omega), \quad \frac{p-1}{2(p+1)}L_{\omega}(v) > d(\omega).
$$

In particular,

(2.7)
$$
d(\omega) = \inf \left\{ \frac{p-1}{2(p+1)} ||v||_{L^{p+1}}^{p+1} \middle| v \in H^1(\mathbb{R}^N) \setminus \{0\}, K_{\omega}(v) \le 0 \right\} = \inf \left\{ \frac{p-1}{2(p+1)} L_{\omega}(v) \middle| v \in H^1(\mathbb{R}^N) \setminus \{0\}, K_{\omega}(v) \le 0 \right\}.
$$

Proof. Let

$$
\lambda_1 = \left(\frac{L_{\omega}(v)}{\|v\|_{L^{p+1}}^{p+1}}\right)^{1/(p-1)},
$$

where note that $L_{\omega}(v) > 0$ by [\(2.2\)](#page-6-2). Then since $K_{\omega}(\lambda v) = \lambda^2 L_{\omega}(v) - \lambda^{p+1} ||v||_{L^{p+1}}^{p+1}$ and $K_{\omega}(v) < 0$, we have $K_{\omega}(\lambda_1 v) = 0$ and $0 < \lambda_1 < 1$. Therefore, by [\(2.5\)](#page-7-0),

$$
d(\omega) \le \frac{p-1}{2(p+1)} \|\lambda_1 v\|_{L^{p+1}}^{p+1} = \lambda_1^{p+1} \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} < \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1}.
$$

Similarly, by using [\(2.6\)](#page-7-1), we obtain $d(\omega) < \frac{p-1}{2(p+1)} L_{\omega}(v)$. This completes the proof. \Box

It is well known that in the nonpotential case $\gamma = 0$, the set of all minimizers

$$
\mathcal{M}_{\omega}^{0} := \{ v \in H^{1}(\mathbb{R}^{N}) \setminus \{0\} \mid K_{\omega}^{0}(v) = 0, S_{\omega}^{0}(v) = d^{0}(\omega) \}
$$

is not empty (see e.g. $[12, 14]$ $[12, 14]$), where

$$
S_{\omega}^{0}(v) = \frac{1}{2} ||\nabla v||_{L^{2}}^{2} + \frac{\omega}{2} ||v||_{L^{2}}^{2} - \frac{1}{p+1} ||v||_{L^{p+1}}^{p+1},
$$

\n
$$
K_{\omega}^{0}(v) = ||\nabla v||_{L^{2}}^{2} + \omega ||v||_{L^{2}}^{2} - ||v||_{L^{p+1}}^{p+1},
$$

\n
$$
d^{0}(\omega) = \inf \{ S_{\omega}^{0}(v) \mid v \in H^{1}(\mathbb{R}^{N}) \setminus \{0\}, K_{\omega}^{0}(v) = 0 \}
$$

\n
$$
= \inf \left\{ \frac{p-1}{2(p+1)} ||v||_{L^{p+1}}^{p+1} \middle| v \in H^{1}(\mathbb{R}^{N}) \setminus \{0\}, K_{\omega}^{0}(v) = 0 \right\}.
$$

Lemma 2.4. $d^{0}(\omega) > d(\omega) > 0$.

Proof. First, we show $d^0(\omega) > d(\omega)$. Since \mathcal{M}^0_{ω} is not empty, we take $\psi \in \mathcal{M}^0_{\omega}$. Since

$$
K_{\omega}(\psi) = K_{\omega}^{0}(\psi) - G(\psi) = -G(\psi) < 0,
$$

by Lemma [2.3,](#page-7-2) we have

$$
d(\omega) < \frac{p-1}{2(p+1)} \|\psi\|_{L^{p+1}}^{p+1} = d^0(\omega).
$$

Next, we show that $d(\omega) > 0$. Let $v \in H^1(\mathbb{R}^N)$ satisfy $v \neq 0$ and $K_\omega(v) = 0$. By the Sobolev embedding, [\(2.2\)](#page-6-2), and $L_{\omega}(v) = ||v||_{L^{p+1}}^{p+1}$, we have

$$
||v||_{L^{p+1}}^2 \le C_1 ||v||_{H^1}^2 \le C_2 L_{\omega}(v) = C_2 ||v||_{L^{p+1}}^{p+1}.
$$

for some $C_1, C_2 > 0$. Since $v \neq 0$, we have $||v||_{L^{p+1}} \geq C_2^{-1/(p-1)}$ $2^{(-1/(p-1))}$. Taking the infimum over *v*, we obtain $d(\omega) > 0$. This completes the proof. □

Lemma 2.5. *Let* $(v_n)_n \subset H^1(\mathbb{R}^N)$ *be a minimizing sequence for* $d(\omega)$ *, that is,*

$$
v_n \neq 0, \quad K_{\omega}(v_n) = 0, \quad S_{\omega}(v_n) \to d(\omega).
$$

Then there exist a subsequence $(v_{n_k})_k$ *of* $(v_n)_n$ *and* $v_0 \in H^1(\mathbb{R}^N)$ *such that* $v_{n_k} \to v_0$ *in* $H^1(\mathbb{R}^N)$, $K_\omega(v_0) = 0$, and $S_\omega(v_0) = d(\omega)$. In particular, \mathcal{M}_ω is not empty.

Proof. First, by $K_\omega(v_n) = 0$, $S_\omega(v_n) \to d(\omega)$, and [\(2.4\)](#page-6-3), we have

(2.8)
$$
\frac{p-1}{2(p+1)}L_{\omega}(v_n) = \frac{p-1}{2(p+1)}||v_n||_{L^{p+1}}^{p+1} \to d(\omega).
$$

Therefore, it follows from [\(2.2\)](#page-6-2) that $(v_n)_n$ is bounded in $H^1(\mathbb{R}^N)$. This implies that there exist a subsequence of $(v_n)_n$, which is still denoted by $(v_n)_n$, and $v_0 \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v_0$ weakly in $H^1(\mathbb{R}^N)$.

Next, we show $v_0 \neq 0$. Since $v_n \neq 0$, letting

$$
\lambda_n = \left(\frac{\|\nabla v_n\|_{L^2}^2 + \omega \|v_n\|_{L^2}^2}{\|v_n\|_{L^{p+1}}^{p+1}}\right)^{1/(p-1)} = \left(\frac{L_\omega(v_n) + G(v_n)}{\|v_n\|_{L^{p+1}}^{p+1}}\right)^{1/(p-1)},
$$

then we have $\lambda_n > 0$ and $K^0_\omega(\lambda_n v_n) = 0$. Moreover, by [\(2.8\)](#page-8-0) and the weak continuity of the potential energy (cf. [\[13,](#page-15-10) Theorem 11.4]), we obtain

.

(2.9)
$$
\lim_{n \to \infty} \lambda_n = \left(\frac{d(\omega) + \frac{p-1}{2(p+1)} G(v_0)}{d(\omega)} \right)^{1/(p-1)}
$$

By Lemma [2.4,](#page-8-1) $K^0_\omega(\lambda_n v_n) = 0$, and the definition of $d^0(\omega)$, it follows that

$$
d(\omega) < d^0(\omega) \le \frac{p-1}{2(p+1)} \|\lambda_n v_n\|_{L^{p+1}}^{p+1} = \lambda_n^{p+1} \frac{p-1}{2(p+1)} \|v_n\|_{L^{p+1}}^{p+1}
$$

for all $n \in \mathbb{N}$. Therefore, taking the limit, by [\(2.8\)](#page-8-0), [\(2.9\)](#page-8-2), and $d(\omega) > 0$, we obtain $G(v_0) > 0$. This implies $v_0 \neq 0$.

Finally, we show the strong convergence of $(v_n)_n$ in $H^1(\mathbb{R}^N)$. Taking a subsequence of $(v_n)_n$ if necessary, we may assume that $v_n \to v_0$ a.e. in \mathbb{R}^N . Then by using the Brezis–Lieb Lemma [\[2\]](#page-14-11), we have

(2.10)
$$
L_{\omega}(v_n) - L_{\omega}(v_n - v_0) \rightarrow L_{\omega}(v_0),
$$

$$
(2.11) \t\t -K_{\omega}(v_n - v_0) \to K_{\omega}(v_0),
$$

where we used $K_\omega(v_n) = 0$ in [\(2.11\)](#page-9-2). Since $L_\omega(v_0) > 0$ by $v_0 \neq 0$, it follows from [\(2.10\)](#page-9-3) and [\(2.8\)](#page-8-0) that

$$
\frac{p-1}{2(p+1)}\lim_{n\to\infty}L_{\omega}(v_n-v_0)<\frac{p-1}{2(p+1)}\lim_{n\to\infty}L_{\omega}(v_n)=d(\omega).
$$

From this and [\(2.7\)](#page-7-3), we have $K_\omega(v_n - v_0) > 0$ for large *n*. Therefore, by [\(2.11\)](#page-9-2), we obtain $K_\omega(v_0) \leq 0$, and thus, by [\(2.7\)](#page-7-3) and the weak lower semicontinuity of norms,

$$
d(\omega) \le \frac{p-1}{2(p+1)} L_{\omega}(v_0) \le \frac{p-1}{2(p+1)} \lim_{n \to \infty} L_{\omega}(v_n) = d(\omega).
$$

This and [\(2.10\)](#page-9-3) imply that $L_{\omega}(v_n - v_0) \to 0$, and therefore, $v_n \to v_0$ in $H^1(\mathbb{R}^N)$. This completes the proof. \Box

Finally, we give a useful lemma for the proof of Theorem [1.6.](#page-5-1)

Lemma 2.6. *Let* $\phi \in \mathcal{G}_{\omega}$ *. If* $v \in H^1(\mathbb{R}^N)$ *satisfies* $||v||_{L^{p+1}} = ||\phi||_{L^{p+1}}$ *, then the following hold.*

- (i) $K_{\omega}(v) \geq 0$,
- (ii) $S_{\omega}(v) > S_{\omega}(\phi)$.

Proof. Inequality [\(i\)](#page-9-1) follows from Lemma [2.3](#page-7-2) and $d(\omega) = \frac{p-1}{2(p+1)} ||\phi||_{L^{p+1}}^{p+1}$. Inequality [\(ii\)](#page-9-4) follows from (2.4) and (i) . \Box

3 Blowup solutions

In this section, we prove Theorem [1.6.](#page-5-1) Throughout this section, we impose the same assumption as in Theorem [1.6,](#page-5-1) that is, we assume (1.1) , $\omega > \omega_0$, and

$$
(3.1) \t\t \partial_{\lambda}^{2} S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} = \|\nabla \phi_{\omega}\|_{L^{2}}^{2} - \frac{\alpha(\alpha-1)}{2}G(\phi_{\omega}) - \frac{\beta(\beta-1)}{p+1} \|\phi_{\omega}\|_{L^{p+1}}^{p+1} \leq 0,
$$

where $v^{\lambda}(x) = \lambda^{N/2} v(\lambda x)$, *G* is defined in [\(2.1\)](#page-5-2), and

$$
\beta = \frac{N(p-1)}{2}.
$$

By using this notation, we have

(3.2)
$$
S_{\omega}(v^{\lambda}) = \frac{\lambda^{2}}{2} \|\nabla v\|_{L^{2}}^{2} + \frac{\omega}{2} \|v\|_{L^{2}}^{2} - \frac{\lambda^{\alpha}}{2} G(v) - \frac{\lambda^{\beta}}{p+1} \|v\|_{L^{p+1}}^{p+1},
$$

(3.3)
$$
Q(v^{\lambda}) = \lambda^2 \|\nabla v\|_{L^2}^2 - \frac{\alpha \lambda^{\alpha}}{2} G(v) - \frac{\beta \lambda^{\beta}}{p+1} \|v\|_{L^{p+1}}^{p+1} = \lambda \partial_{\lambda} S_{\omega}(v^{\lambda}),
$$

(3.4)
$$
K_{\omega}(v^{\lambda}) = \lambda^{2} \|\nabla v\|_{L^{2}}^{2} + \omega \|v\|_{L^{2}}^{2} - \lambda^{\alpha} G(v) - \lambda^{\beta} \|v\|_{L^{p+1}}^{p+1}.
$$

Here, we define

$$
\mathcal{A}_{\omega} = \{ v \in H^{1}(\mathbb{R}^{N}) \mid S_{\omega}(v) < S_{\omega}(\phi_{\omega}), \ \|v\|_{L^{2}} \le \|\phi_{\omega}\|_{L^{2}}, \ \|v\|_{L^{p+1}} > \|\phi_{\omega}\|_{L^{p+1}} \}.
$$

Recall that

$$
\mathcal{B}_{\omega} = \{ v \in \mathcal{A}_{\omega} \mid Q(v) < 0 \}.
$$

Lemma 3.1. *If* $u_0 \in A_\omega$, then the solution $u(t)$ of [\(NLS\)](#page-0-0) with $u(0) = u_0$ satisfies $u(t) \in \mathcal{A}_{\omega}$ *for all* $t \in I_{\max}$.

Proof. Since *E* and $\|\cdot\|_{L^2}$ are conserved quantities of [\(NLS\)](#page-0-0), we have $\|u(t)\|_{L^2} \le \|\phi_\omega\|_{L^2}$ and $S_\omega(u(t)) < S_\omega(\phi_\omega)$ for all $t \in I_{\text{max}}$. By Lemma [2.6](#page-9-0) [\(ii\)](#page-9-4), it follows that $||u(t)||_{L^{p+1}} \neq$ $\|\phi_{\omega}\|_{L^{p+1}}$ for all $t \in I_{\max}$. Therefore, by $\|u_0\|_{L^{p+1}} > \|\phi_{\omega}\|_{L^{p+1}}$ and the continuity of the solution $u(t)$, we obtain $||u(t)||_{L^{p+1}} > ||\phi_{\omega}||_{L^{p+1}}$ for all $t \in I_{\max}$. This completes the \Box proof.

The following is the key lemma for our proof.

Lemma 3.2. *Let* $v \in H^1(\mathbb{R}^N)$ *satisfy*

$$
||v||_{L^2} \le ||\phi_\omega||_{L^2}, \quad ||v||_{L^{p+1}} \ge ||\phi_\omega||_{L^{p+1}}, \quad Q(v) \le 0.
$$

Then

(3.5)
$$
\frac{Q(v)}{2} \leq S_{\omega}(v) - S_{\omega}(\phi_{\omega}).
$$

In particular, if $u_0 \in \mathcal{B}_{\omega}$ *, then the solution* $u(t)$ *of* [\(NLS\)](#page-0-0) *with* $u(0) = u_0$ *satisfies* $u(t) \in \mathcal{B}_{\omega}$ *for all* $t \in I_{\max}$.

Proof. Let

$$
\lambda_0 = \left(\frac{\|\phi_\omega\|_{L^{p+1}}^{p+1}}{\|v\|_{L^{p+1}}^{p+1}}\right)^{1/\beta}
$$

.

Then we have

$$
0 < \lambda_0 \le 1, \quad \|v^{\lambda_0}\|_{L^2} = \|v\|_{L^2} \le \|\phi_\omega\|_{L^2}, \quad \|v^{\lambda_0}\|_{L^{p+1}}^{p+1} = \lambda_0^\beta \|v\|_{L^{p+1}}^{p+1} = \|\phi_\omega\|_{L^{p+1}}^{p+1}.
$$

Here, we define

$$
f(\lambda) = S_{\omega}(v^{\lambda}) - \frac{\lambda^2}{2} Q(v)
$$

= $-\frac{1}{2} \left(\lambda^{\alpha} - \frac{\alpha \lambda^2}{2} \right) G(v) + \frac{\omega}{2} ||v||_{L^2}^2 - \frac{1}{p+1} \left(\lambda^{\beta} - \frac{\beta \lambda^2}{2} \right) ||v||_{L^{p+1}}^{p+1}$

for $\lambda \in (0,1]$. If we have $f(\lambda_0) \leq f(1)$, then it follows from Lemma [2.6](#page-9-0) [\(ii\)](#page-9-4), $Q(v) \leq 0$, and $f(\lambda_0) \leq f(1)$ that

(3.6)
$$
S_{\omega}(\phi_{\omega}) \leq S_{\omega}(v^{\lambda_0}) \leq S_{\omega}(v^{\lambda_0}) - \frac{\lambda_0^2}{2}Q(v) \leq S_{\omega}(v) - \frac{Q(v)}{2},
$$

which is the desired inequality [\(3.5\)](#page-10-2).

In what follows, we prove $f(\lambda_0) \leq f(1)$, which is rewritten as

(3.7)
$$
G(v) \leq \frac{2(2\lambda_0^{\beta} - \beta\lambda_0^2 - 2 + \beta)}{(p+1)(\alpha\lambda_0^2 - 2\lambda_0^{\alpha} - \alpha + 2)} ||v||_{L^{p+1}}^{p+1}.
$$

By $\alpha K_{\omega}(\phi_{\omega}) - (\alpha + 1)Q(\phi_{\omega}) = 0$ and [\(3.1\)](#page-10-3), we have

$$
\alpha \omega \|\phi_{\omega}\|_{L^{2}}^{2} = \|\nabla \phi_{\omega}\|_{L^{2}}^{2} - \frac{\alpha(\alpha - 1)}{2}G(\phi_{\omega}) + \left(\alpha - \frac{\beta(\alpha + 1)}{p + 1}\right) \|\phi_{\omega}\|_{L^{p+1}}^{p+1}
$$

$$
\leq \left(\alpha + \frac{\beta(\beta - \alpha - 2)}{p + 1}\right) \|\phi_{\omega}\|_{L^{p+1}}^{p+1}.
$$

Therefore, it follows from $||v||_{L^2} \le ||\phi_\omega||_{L^2}$ and $||\phi_\omega||_{L^{p+1}}^{p+1} = \lambda_0^\beta ||v||_{L^{p+1}}^{p+1}$ that

(3.8)
$$
\omega \|v\|_{L^2}^2 \le \left(1 + \frac{\beta(\beta - \alpha - 2)}{(p+1)\alpha}\right) \lambda_0^{\beta} \|v\|_{L^{p+1}}^{p+1}.
$$

By using Lemma [2.6](#page-9-0) [\(i\)](#page-9-1) for v^{λ_0} , [\(3.4\)](#page-10-4), [\(3.8\)](#page-11-0), and $Q(v) \le 0$, we have

$$
G(v) \leq \lambda_0^{2-\alpha} \|\nabla v\|_{L^2}^2 + \lambda_0^{-\alpha} \omega \|v\|_{L^2}^2 - \lambda_0^{\beta-\alpha} \|v\|_{L^{p+1}}^{p+1}
$$

\n
$$
\leq \lambda_0^{2-\alpha} \|\nabla v\|_{L^2}^2 + \frac{\beta(\beta-\alpha-2)}{(p+1)\alpha} \lambda_0^{\beta-\alpha} \|v\|_{L^{p+1}}^{p+1}
$$

\n
$$
\leq \frac{\alpha}{2} \lambda_0^{2-\alpha} G(v) + \frac{\beta}{p+1} \left(\lambda_0^{2-\alpha} + \frac{\beta-\alpha-2}{\alpha} \lambda^{\beta-\alpha}\right) \|v\|_{L^{p+1}}^{p+1},
$$

and thus,

(3.9)
$$
G(v) \leq \frac{2\beta}{(p+1)(2-\alpha\lambda_0^{2-\alpha})} \left(\lambda_0^{2-\alpha} + \frac{\beta-\alpha-2}{\alpha}\lambda^{\beta-\alpha}\right) ||v||_{L^{p+1}}^{p+1}.
$$

In view of (3.7) and (3.9) , we only have to show that

$$
\frac{\beta}{2-\alpha\lambda^{2-\alpha}}\left(\lambda_0^{2-\alpha}+\frac{\beta-\alpha-2}{\alpha}\lambda^{\beta-\alpha}\right)\leq \frac{2\lambda_0^{\beta}-\beta\lambda_0^2-2+\beta}{\alpha\lambda_0^2-2\lambda_0^{\alpha}-\alpha+2}
$$

for all $\lambda \in (0,1)$, which is equivalent to

$$
g_1(\lambda) := \frac{(2 - \alpha\lambda^{2-\alpha})(2\lambda^{\beta} - \beta\lambda^2 - 2 + \beta)}{\beta\lambda^{\beta-\alpha}(\alpha\lambda^2 - 2\lambda^{\alpha} - \alpha + 2)} - \frac{1}{\lambda^{\beta-2}} - \frac{\beta - \alpha - 2}{\alpha} \ge 0.
$$

Since $\lim_{\lambda \nearrow 1} g_1(\lambda) = 0$, it suffices to show that

$$
g_1'(\lambda) = \frac{2(1 - \lambda^{2-\alpha})}{\beta \lambda^{\beta-\alpha+1}(\alpha \lambda^2 - 2\lambda^{\alpha} - \alpha + 2)^2} \left(2\alpha(2-\alpha)\lambda^{\beta} - \alpha\beta(\beta-\alpha)\lambda^2 + 2\beta(\beta-2)\lambda^{\alpha} - (\beta-\alpha)(\beta-2)(2-\alpha)\right) \le 0
$$

for all $\lambda \in (0, 1)$, which holds if we have

$$
g_2(\lambda) := 2\alpha(2-\alpha)\lambda^{\beta} - \alpha\beta(\beta-\alpha)\lambda^2 + 2\beta(\beta-2)\lambda^{\alpha} - (\beta-\alpha)(\beta-2)(2-\alpha) \le 0.
$$

Since $g_2(1) = 0$, it is enough to show that

$$
g_2'(\lambda) = 2\alpha \beta \lambda^{\alpha-1} \left((2-\alpha)\lambda^{\beta-\alpha} - (\beta-\alpha)\lambda^{2-\alpha} + \beta - 2) \right) \ge 0
$$

for all $\lambda \in (0,1)$. This is equivalent to

$$
g_3(\lambda) := (2 - \alpha)\lambda^{\beta - \alpha} - (\beta - \alpha)\lambda^{2 - \alpha} + \beta - 2 \ge 0.
$$

Since $g_3(1) = 0$ and

$$
g_3'(\lambda) = -(\beta - \alpha)(2 - \alpha)\lambda^{1-\alpha}(1 - \lambda^{\beta-2}) \le 0,
$$

we have $g_3(\lambda) \geq 0$ for all $\lambda \in (0,1)$. Therefore, we obtain $f(\lambda_0) \leq f(1)$, and thus, the inequality [\(3.6\)](#page-11-2) follows.

The last claim of Lemma [3.2](#page-10-0) follows from Lemma [3.1](#page-10-5) and [\(3.5\)](#page-10-2). This completes the proof. \Box

Proof of Theorem [1.6.](#page-5-1) Let $u_0 \in \mathcal{B}_{\omega} \cap \Sigma$ and $u(t)$ be the solution of [\(NLS\)](#page-0-0) with $u(0) = u_0$. Then by the virial identity [\(1.10\)](#page-4-2), Lemma [3.2,](#page-10-0) and the conservation of S_ω , we have

$$
(3.10) \frac{d^2}{dt^2} ||xu(t)||_{L^2}^2 = 8Q(u(t)) \le 16(S_{\omega}(u(t)) - S_{\omega}(\phi_{\omega})) = 16(S_{\omega}(u_0) - S_{\omega}(\phi_{\omega})) < 0
$$

for all $t \in I_{\max}$.

If $T^+ = \infty$, then it follows from [\(3.10\)](#page-12-1) that $||xu(t)||_{L^2}$ becomes negative for large *t*. This is a contradiction. Thus, the solution $u(t)$ blows up in finite time. \Box

4 Strong instability of standing waves

In this section, we prove Theorem [1.3.](#page-3-3) Here, we impose the same assumption as in Theorem [1.3.](#page-3-3)

Lemma 4.1. $\phi_{\omega}^{\lambda} \in \mathcal{B}_{\omega}$ for all $\lambda > 1$.

Proof. By the definition of the scaling $\lambda \mapsto \phi_{\omega}^{\lambda}$, we have $\|\phi_{\omega}^{\lambda}\|_{L^2} = \|\phi_{\omega}\|_{L^2}$ and $\|\phi_{\omega}^{\lambda}\|_{L^{p+1}} =$ $\lambda^{\beta/(p+1)} \|\phi_{\omega}\|_{L^{p+1}} > \|\phi_{\omega}\|_{L^{p+1}}$ for all $\lambda > 1$, where $\beta = N(p-1)/2 > 2$.

Now, we show that $S_\omega(\phi_\omega^\lambda) < S_\omega(\phi_\omega)$ and $Q(\phi_\omega^\lambda) < 0$ for all $\lambda > 1$. In view of [\(3.2\)](#page-10-6), the function $S_\omega(\phi_\omega^\lambda)$ of λ has the form $S_\omega(\phi_\omega^\lambda) = A\lambda^2 + B - C\lambda^\alpha - D\lambda^\beta$ with some $A, B, C, D > 0$. By $\partial_{\lambda} S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} = 0$ and the assumption $\partial_{\lambda}^{2} S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} \leq 0$, we have $-\beta(\beta-2)D \leq -\alpha(2-\alpha)C$. This leads to

$$
\partial_{\lambda}^{3} S_{\omega}(\phi_{\omega}^{\lambda}) = \alpha(\alpha - 1)(2 - \alpha) C \lambda^{\alpha - 3} - \beta(\beta - 1)(\beta - 2) D \lambda^{\beta - 3}
$$

$$
\leq -\alpha(2 - \alpha) \lambda^{\alpha - 3} ((\beta - 1)\lambda^{\beta - \alpha} - (\alpha - 1)) C < 0
$$

for all $\lambda > 1$. Therefore, for $\lambda > 1$, it follows that $\partial_{\lambda}^{2} S_{\omega}(\phi_{\omega}^{\lambda}) < \partial_{\lambda}^{2} S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} \leq 0$, $\partial_{\lambda}S_{\omega}(\phi_{\omega}^{\lambda}) < \partial_{\lambda}S_{\omega}(\phi_{\omega}^{\lambda})|_{\lambda=1} = 0$, and thus $S_{\omega}(\phi_{\omega}^{\lambda}) < S_{\omega}(\phi_{\omega})$. Moreover, by differentiating [\(3.3\)](#page-10-7), we have $\partial_{\lambda}Q(\phi_{\omega}^{\lambda}) = \partial_{\lambda}S_{\omega}(\phi_{\omega}^{\lambda}) + \lambda \partial_{\lambda}^{2}S_{\omega}(\phi_{\omega}^{\lambda}) < 0$. This implies $Q(\phi_{\omega}^{\lambda}) < Q(\phi_{\omega}) = 0$. This completes the proof. \Box

Now, we prove the main theorem.

Proof of Theorem [1.3.](#page-3-3) Let $\varepsilon > 0$. Then since $\phi_{\omega}^{\lambda} \to \phi_{\omega}$ in $H^{1}(\mathbb{R}^{N})$ as $\lambda \searrow 1$, there exists $\lambda_0 > 1$ such that $\|\phi_\omega - \phi_\omega^{\lambda_0}\|_{H^1} < \varepsilon/2$. Let $\chi \in C^\infty[0, \infty)$ be a function satisfying $0 \leq \chi \leq 1$, $\chi(r) = 1$ if $0 \leq r \leq 1$, and $\chi(r) = 0$ if $r \geq 2$. For $M > 0$, we define a cutoff function $\chi_M \in C_c^{\infty}(\mathbb{R}^N)$ by $\chi_M(x) = \chi(|x|/M)$. Then we see that $\chi_M \phi_{\omega}^{\lambda_0} \to \phi_{\omega}^{\lambda_0}$ in $H^1(\mathbb{R}^N)$ as $M \to \infty$. Moreover, we have $\chi_M \phi^{\lambda_0}_{\omega} \in \Sigma$ and $\|\chi_M \phi^{\lambda_0}_{\omega}\|_{L^2} \leq \|\phi^{\lambda_0}_{\omega}\|_{L^2} = \|\phi_{\omega}\|_{L^2}$ for all $M > 0$. Therefore, by Lemma [4.1](#page-13-1) and the continuity of S_ω , $\|\cdot\|_{L^{p+1}}$, and Q , there exists $M_0 > 0$ such that $\|\phi_\omega^{\lambda_0} - \chi_{M_0} \phi_\omega^{\lambda_0}\|_{H^1} < \varepsilon/2$ and $\chi_{M_0} \phi_\omega^{\lambda_0} \in \mathcal{B}_\omega \cap \Sigma$. Thus, we obtain $\|\chi_{M_0}\phi_\omega^{\lambda_0} - \phi_\omega\|_{H^1} < \varepsilon$, and by Theorem [1.6,](#page-5-1) the solution $u(t)$ with $u(0) = \chi_{M_0}\phi_\omega^{\lambda_0}$ blows up in finite time. Hence, the standing wave solution $e^{i\omega t}\phi_\omega$ of [\(NLS\)](#page-0-0) is strongly unstable. \Box

Acknowledgements The authors would like to express their deepest gratitude to Yusuke Shimabukuro for valuable discussions. This work was supported by JSPS KAK-ENHI Grant Numbers 15K04968 and 26247013.

References

- [1] H. Berestycki and T. Cazenave, *Instabilité des états stationaires dans les équations de Schrödinger et de Klein–Gordon non linéaires*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), 489–492.
- [2] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.
- [3] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
- [4] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.
- [5] R. Fukuizumi, *Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential*, Discrete Contin. Dynam. Systems **7** (2001), 525–544.
- [6] R. Fukuizumi and M. Ohta, *Stability of standing waves for nonlinear Schrödinger equations with potentials*, Differential Integral Equations **16** (2003), 111–128.
- [7] R. Fukuizumi and M. Ohta, *Instability of standing waves for nonlinear Schrödinger equations with potentials*, Differential Integral Equations **16** (2003), 691–706.
- [8] R. Fukuizumi, M. Ohta, and T. Ozawa, *Nonlinear Schrödinger equation with a point defect*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 837–845.
- [9] R. H. Goodman, P. J. Holmes, and M. I. Weinstein, *Strong NLS soliton-defect interactions*, Phys. D **192** (2004), 215–248.
- [10] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, J. Funct. Anal. **74** (1987), 160–197.
- [11] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry II*, J. Funct. Anal. **94** (1990), 308–348.
- [12] S. Le Coz, *Standing waves in nonlinear Schrödinger equations*, Analytical and Numerical Aspects of Partial Differential Equations, de Gruyter, Berlin, (2009), 151– 192.
- [13] E. H. Lieb and M. Loss, Analysis, Second edition, Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
- [14] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 223–283.
- [15] M. Ohta, *Instability of standing waves for the generalized Davey–Stewartson system*, Ann. Inst. H. Poincaré Phys. Théor. **62** (1995), 69–80.
- [16] M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with harmonic potential*, Funkcial. Ekvac. **61** (2018), 135–143.
- [17] M. Ohta and T. Yamaguchi, *Strong instability of standing waves for nonlinear Schrödinger equations with double power nonlinearity*, SUT J. Math. **51** (2015), 49–58.
- [18] M. Ohta and T. Yamaguchi, *Strong instability of standing waves for nonlinear Schrödinger equations with a delta potential*, Harmonic analysis and nonlinear partial differential equations, 79–92, RIMS Kôkyûroku Bessatsu, **B56**, Res. Inst. Math. Sci. (RIMS), Kyoto, 2016.
- [19] H. A. Rose and M. I. Weinstein, *On the bound states of the nonlinear Schrödinger equation with a linear potential*, Phys. D **30** (1988), 207–218.
- [20] J. Shatah, *Stable standing waves of nonlinear Klein–Gordon Equations*, Comm. Math. Phys. **91** (1983), 313–327.
- [21] J. Shatah and W. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys. **100** (1985), 173–190.
- [22] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1982/83), 567–576.
- [23] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), 51–67.

Noriyoshi Fukaya

Department of Mathematics, Graduate School of Science, Tokyo University of Science,

1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

E-mail address: 1116702@ed.tus.ac.jp

Masahito Ohta

Department of Mathematics, Tokyo University of Science, 1-3 Kagurazaka, Shinjukuku, Tokyo 162-8601, Japan

E-mail address: mohta@rs.tus.ac.jp