

EXISTENCE OF SOLITARY WAVES SOLUTIONS FOR INTERNAL WAVES IN TWO-LAYERS SYSTEMS

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ABSTRACT. The aim of this paper is to establish the existence of solitary wave solutions for two classes of two-layers systems modeling the propagation of internal waves. More precisely we will consider the Boussinesq-Full dispersion system and the Intermediate Long Wave (ILW) system together with its Benjamin-Ono (BO) limit.

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1. INTRODUCTION

This paper is concerned with the existence and properties of solitary wave solutions to some internal waves models. Such models were systematically and rigorously (in the sense of consistency) derived from the two-layers system with a rigid lid in [11] (see also [22] and the survey article [33]). They depend of course on the various regimes determined by the wave lengths, heights of the layers,.. that determine the suitable small parameters used in the derivation of the asymptotic models.

For instance, it is shown [11] that in the so-called Boussinesq-Full dispersion regime and in the absence of surface tension, the two-layers system for internal waves is consistent with the *three-parameter family* of Boussinesq-Full dispersion systems (henceforth B-FD systems)

$$\begin{cases} (1 - \mu b \Delta) \partial_t \zeta + \frac{1}{\gamma} \nabla \cdot ((1 - \epsilon \zeta) \mathbf{v}_\beta) \\ - \frac{\sqrt{\mu}}{\gamma^2} |D| \coth(\sqrt{\mu_2} |D|) \nabla \cdot \mathbf{v}_\beta + \frac{\mu}{\gamma} \left(a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2} |D|) \right) \Delta \nabla \cdot \mathbf{v}_\beta = 0, \\ (1 - \mu d \Delta) \partial_t \mathbf{v}_\beta + (1 - \gamma) \nabla \zeta - \frac{\epsilon}{2\gamma} \nabla |\mathbf{v}_\beta|^2 + \mu c (1 - \gamma) \Delta \nabla \zeta = 0, \end{cases} \quad (1.1)$$

where $\mathbf{v}_\beta = (1 - \mu\beta\Delta)^{-1}\mathbf{v}$ (\mathbf{v} being the horizontal velocity) and the constants a, b, c and d are defined as

$$a = \frac{1}{3}(1 - \alpha_1 - 3\beta), \quad b = \frac{1}{3}\alpha_1, \quad c = \beta\alpha_2, \quad d = \beta(1 - \alpha_2),$$

with $\alpha_1 \geq 0$, $\beta \geq 0$, and $\alpha_2 \leq 1$. Note that $a + b + c + d = \frac{1}{3}$.

To describe precisely the asymptotic regime we are considering we recall the definition of a few parameters (see [11] for more details). The upper (resp. lower) layer is indexed by 1 (resp. 2.) $\gamma = \frac{\rho_1}{\rho_2}$ is the ratio of densities. $d = \frac{d_1}{d_2}$ is the ratio of the typical depth of the two layers. The typical elevation of the wave is denoted by a , the typical horizontal wavelength is λ . Then, one set

$$\epsilon = \frac{a}{d_1}, \quad \mu = \frac{d_1^2}{\lambda^2}, \quad \epsilon_2 = \frac{a}{d_2} = \epsilon\delta, \quad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.$$

The Boussinesq-Full dispersion regime is obtained by assuming that $\mu \sim \epsilon$ and $\mu_2 \sim 1$ (and thus $\delta \sim \epsilon^{1/2}$) in addition to $\epsilon \ll 1$ and $\epsilon_2 \ll 1$. The three free parameters in (1.1) arise from a double use of the so-called BBM (Benjamin-Bona-Mahony) trick and of a one-parameter choice of the velocity (see [11] for details).

Remark 1.1. If in addition to $\epsilon \ll 1$ and $\epsilon_2 \ll 1$ one assumes that $\mu \sim \epsilon$ and $\mu_2 \sim \epsilon_2$ and thus $\delta \sim 1$, one obtains the *Boussinesq-Boussinesq* regime, leading to Boussinesq systems similar to those introduced in [8, 9, 10] and studied in [35, 16, 34].

Remark 1.2. In oceanographic applications the surface tension effects are very weak and can be ignored when deriving the aforementioned asymptotic models (they appear as a lower order effect). In situations where they are small but of the order of the "small" parameters involved in the asymptotic expansions (the ϵ 's or the μ 's) one has to add a cubic "capillary" term to the equation for \mathbf{v} , proportional to $\Delta\nabla\zeta$, but also add higher order terms in ϵ, μ leading to a somewhat more complicated system (see [24]) that we will not consider here.

It is easily checked that (1.1) is linearly well posed when (see [23])

$$a \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0.$$

Throughout the paper we will assume that this condition holds.

The local well-posedness of the Cauchy problem for (1.1) was considered by Cung in [23] in the following cases

- (1) $b > 0, d > 0, a \leq 0, c < 0$;
- (2) $b > 0, d > 0, a \leq 0, c = 0$;
- (3) $b = 0, d > 0, a \leq 0, c = 0$;
- (4) $b = 0, d > 0, a \leq 0, c < 0$;
- (5) $b > 0, d = 0, a \leq 0, c = 0$.

Note that all the linearly well-posed systems are not covered here, in particular those where both the coefficients b and d vanish (but never happens in the modeling of internal waves without surface tension due to the modeling constraint $a + b + c + d = \frac{1}{3}$). Note also that long time existence (that is on time scales of order $\frac{1}{\epsilon} \sim \frac{1}{\mu}$) is still an open problem and will be considered elsewhere.

On the other hand (1.1) is an Hamiltonian system when $b = d$. The Hamiltonian structure is given by

$$\partial_t \begin{pmatrix} \zeta \\ \mathbf{v}_\beta \end{pmatrix} + J \text{grad } H(\zeta, \mathbf{v}_\beta) = 0, \quad (1.2)$$

where

$$J = (1 - \mu b \Delta)^{-1} \begin{pmatrix} 0 & \nabla \cdot \\ \nabla & 0 \end{pmatrix}$$

and

$$\begin{aligned} H(\zeta, \mathbf{v}_\beta) = \int_{\mathbb{R}} & \left(\frac{1-\gamma}{2} \zeta^2 + \frac{1}{2} |\mathbf{v}_\beta|^2 - \frac{\epsilon}{2\gamma} \zeta |\mathbf{v}_\beta|^2 - \frac{\mu c}{2} (1-\gamma) |\nabla \zeta|^2 \right. \\ & \left. - \frac{a\mu}{2\gamma} |\nabla \mathbf{v}_\beta|^2 + \frac{\sqrt{\mu}}{2\gamma^2} |L_1^{1/2} \mathbf{v}_\beta|^2 + \frac{\mu}{2\gamma^3} |L_2^{1/2} \mathbf{v}_\beta|^2 \right), \end{aligned} \quad (1.3)$$

where $L_1 = |D| \coth(\sqrt{\mu}|D|)$ and $L_2 = \coth(\sqrt{\mu}|D|)$.

We are interested here in the existence of solitary waves solutions for the B-FD systems in one space dimension. Denoting now v the velocity, the system writes :

$$\begin{cases} J_b \partial_t \zeta + \mathcal{L}_{\mu_2} \partial_x v - \frac{\epsilon}{\gamma} \partial_x (\zeta v) = 0 \\ J_d \partial_t v + (1-\gamma) J_c \partial_x \zeta - \frac{\epsilon}{2\gamma} \partial_x (v^2) = 0, \end{cases} \quad (1.4)$$

where \mathcal{L}_{μ_2} is the self-adjoint operator defined for $\mu_2 \in (0, +\infty]$ by

$$\mathcal{L}_{\mu_2} = \frac{1}{\gamma} - \frac{\sqrt{\mu}}{\gamma^2} |D| \coth(\sqrt{\mu_2}|D|) + \frac{\mu}{\gamma} \left(a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2}|D|) \right) \partial_x^2, \quad (1.5)$$

We note that for $\mu_2 = +\infty$ the linear operator $\mathcal{L}_{\mu_2} \equiv \mathcal{L}_{+\infty}$ becomes

$$\mathcal{L}_{+\infty} = \frac{1}{\gamma} - \frac{\sqrt{\mu}}{\gamma^2} |D| + \frac{\mu}{\gamma} \left(a - \frac{1}{\gamma^2} \right) \partial_x^2. \quad (1.6)$$

The Bessel-type operators J_b, J_d, J_c are defined by

$$J_b = 1 - \mu b \partial_x^2, \quad J_d = 1 - \mu d \partial_x^2, \quad J_c = 1 + \mu c \partial_x^2.$$

The solitary waves solutions $\zeta(x, t) = \xi(x - \omega t), v(x, t) = \nu(x - \omega t)$ vanishing at infinity should satisfy in the variable $z = x - \omega t, \xi = \xi(z), \nu = \nu(z)$ the system

$$\begin{cases} -\omega J_b \xi + \mathcal{L}_{\mu_2} \nu = \frac{\epsilon}{\gamma} \xi \nu \\ -\omega J_d \nu + (1-\gamma) J_c \xi = \frac{\epsilon}{2\gamma} \nu^2. \end{cases} \quad (1.7)$$

The following theorem establishes our results on the existence of solitary waves solutions for systems (1.4) in the Hamiltonian case $b = d$. More exactly, we assume

$$a + c + 2b = \frac{1}{3}, \quad a, c \leq 0, \quad b \geq 0, \quad (1.8)$$

implying thus $b \geq \frac{1}{6}$ and $\frac{1}{3} - 2b \leq a \leq 0$ and $\frac{1}{3} - 2b \leq c \leq 0$.

Theorem 1.1. *Let $\gamma \in (0, 1)$ and $\epsilon > 0$. For $b = d > 0$ and $a, c \leq 0$, the systems (2.1) have a non-trivial solitary waves solutions $(\xi, \nu) \in H^\infty(\mathbb{R}) \times H^\infty(\mathbb{R})$ provided:*

- (a) case $\mu_2 = +\infty$: if $|\omega| < (1 - \gamma)\min\{1, \frac{|c|}{b}\}$,
- (b) case μ_2 finite: let ω such that $|\omega| < (1 - \gamma)\min\{1, \frac{|c|}{b}\}$ and define

$$\alpha_0 \equiv \frac{1}{\gamma} - |\omega| - \frac{1}{\beta_0} > 0 \quad (1.9)$$

where $\beta_0 \equiv -4\gamma^4(b|\omega| + \frac{1}{\gamma}(a - \frac{1}{\gamma^2}))$. Then existence of solitary waves is insured by choosing μ_2 satisfying $\frac{\sqrt{\mu}}{\sqrt{\mu_2}} < \gamma^2\alpha_0$.

Remark 1.1. The statement in Theorem 1.1 deserves to be clarified at least in some points.

- (1) The positivity of α_0 in (1.9) is deduced from the condition on the speed velocity ω , $|\omega| < (1 - \gamma)\min\{1, \frac{|c|}{b}\}$.
- (2) For a fixed $\gamma \in (0, 1)$, in (1.4), our approach only gives the existence of solitary waves solutions by choosing the depth parameter μ_2 in an interval of the form $(\mu_0, +\infty)$.
- (3) The regularity of the solitary wave solutions is proved in Theorems 4.1 and 4.2 below.

The proof of Theorem 1.1 is based on the Concentration-Compactness Principle (see Lions [32]). Thus, we consider the family of minimization problems

$$I_\lambda = \inf\{E_{\mu_2}(\xi, \nu) : (\xi, \nu) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \text{ and } F(\xi, \nu) = \lambda\} \quad (1.10)$$

where $\lambda > 0$ and the functionals E_{μ_2} and F are defined as

$$E_{\mu_2}(\xi, \nu) = \int_{\mathbb{R}} \frac{(1 - \gamma)}{2} \xi J_c \xi + \frac{1}{2} \nu \mathcal{L}_{\mu_2} \nu - \omega \xi J_b \nu dx \quad (1.11)$$

and $F(\xi, \nu) = r \int_{\mathbb{R}} \xi \nu^2 dx$, with $r = \frac{\epsilon}{2\gamma}$. For the case $\mu_2 = +\infty$, $E_{+\infty}$ is defined in the same form as E_{μ_2} in (1.11) by changing the operator \mathcal{L}_{μ_2} by $\mathcal{L}_{+\infty}$ defined in (1.6).

The paper is organized as follows. In the next section we prove Theorem (1.1) in the case $\mu_2 = +\infty$ while we treat the case where μ_2 is finite in Section 3. We prove in Section 4 the algebraic decay of the solitary waves.

Finally the last two Sections are devoted to two systems that are the two ways propagation versions of the classical Intermediate Long Wave (ILW) and Benjamin-Ono (BO)

equations. Using the parameters introduced above, the ILW regime corresponds to $\mu \sim \epsilon \ll 1$ and $\mu_2 \sim 1$ (and thus $\delta^2 \sim \mu \sim \epsilon_2$).

If furthermore the lower layer depth is infinite, that is $\delta = 0$ (and thus $\mu_2 = \infty$, $\epsilon_2 = 0$), one obtains the BO regime. Again we refer to [11] for details on the justification of those asymptotic regimes.

The Intermediate Long wave (one parameter family of) systems write in one spatial dimension

$$\begin{cases} \mathcal{W}\partial_t\zeta + \mathcal{Z}\partial_x v - \frac{\epsilon}{\gamma}\partial_x(\zeta v) = 0 \\ \partial_t v + (1 - \gamma)\partial_x\zeta - \frac{\epsilon}{2\gamma}\partial_x(v^2) = 0, \end{cases} \quad (1.12)$$

where \mathcal{W} and \mathcal{Z} are self-adjoint operators defined for $\gamma \in (0, 1)$, $\beta > 1$, $\mu, \mu_2, \epsilon > 0$ by

$$\mathcal{W} = 1 + g(D), \quad \mathcal{Z} = \frac{1}{\gamma} \left(1 + \frac{\beta - 1}{\gamma} \sqrt{\mu} |D| \coth(\sqrt{\mu_2} |D|) \right), \quad (1.13)$$

with $|D| = \mathcal{H}\partial_x$, where \mathcal{H} represents the Hilbert transform, and

$$g(D) = \frac{\beta}{\gamma} \sqrt{\mu} |D| \coth(\sqrt{\mu_2} |D|).$$

On the other hand the Benjamin-Ono systems write, again in spatial dimension one:

$$\begin{cases} \mathcal{D}\partial_t\zeta + \mathcal{B}\partial_x v - \frac{\epsilon}{\gamma}\partial_x(\zeta v) = 0 \\ \partial_t v + (1 - \gamma)\partial_x\zeta - \frac{\epsilon}{2\gamma}\partial_x(v^2) = 0, \end{cases} \quad (1.14)$$

where \mathcal{D} and \mathcal{B} are self-adjoint operators defined for $\gamma \in (0, 1)$, $\beta > 1$ and $\mu, \epsilon > 0$ by

$$\mathcal{D} = 1 + \frac{\beta}{\gamma} \sqrt{\mu} |D|, \quad \mathcal{B} = \frac{1}{\gamma} \left(1 + \frac{\beta - 1}{\gamma} \sqrt{\mu} |D| \right), \quad (1.15)$$

with $|D| = \mathcal{H}\partial_x$, where \mathcal{H} represents the Hilbert transform.

For both systems $\beta > 1$ is a free parameter stemming from the use of the BBM trick.

Long time existence (that is on time scales of order $1/\epsilon$) of the Cauchy problem, in spatial dimension one and two has been obtained for both families of systems by Li Xu ([36]).

When one restricts to wave propagating in one direction, all systems (1.12), resp. (1.14) yield the Intermediate Long Wave Equation (ILW) (see [30]) resp. the Benjamin-Ono equation (BO) (see [7]). Both of those equations are completely integrable and have explicit solitary waves, algebraically decaying at infinity for BO and exponentially for ILW. We refer to [29] and the references therein for details. The BO solitary wave is shown to be unique (modulo obvious symmetries) in [5]. A similar result for the ILW solitary wave is proven in [4].

The asymptotic stability of the BO solitary waves (and of explicit multi-solitons) is established in [28]. The orbital stability of the ILW solitary wave is proven in [1, 2], see also [6, 13, 14]. We do not know of asymptotic stability results for the 1 or N-soliton similar to the corresponding ones for the BO equation. Those results should be in some

sense easier than the corresponding ones for BO since the exponential decay of the solitons should make the spectral analysis of the linearized operators easier.

On the other hand we do not know of previous existence results for ILW or BO *systems* in (1.12) and (1.14), respectively. In Sections 5 and 6 we prove the existence of even solitary waves for both the systems by implicit functions theorem arguments and by perturbation theory of closed linear operators.

In section 8 we establish some open problems associated to the systems B-FD (1.4), (1.12) and (1.14). In Appendix A we prove the global existence of small solutions for (1.4) for the cases $b = d > 0$, $a \leq 0$ and $c < 0$.

Notations. The norm of standard Sobolev spaces $H^s(\mathbb{R})$ is denoted $\|\cdot\|_s$, while the norm of the Lebesgue space $L^2(\mathbb{R})$ is denoted $\|\cdot\|$, $s \geq 0$. We will denote $|\cdot|_p$ the norm in the Lebesgue space $L^p(\mathbb{R})$, $1 \leq p \leq \infty$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R})$. We will denote \hat{f} the Fourier transform of a tempered distribution f . For any $s \in \mathbb{R}$, we define $|D|^s f$ by its Fourier transform $\widehat{|D|^s f}(y) = |y|^s \hat{f}(y)$. Similarly, we define $L_1 = |D| \coth(\sqrt{\mu_2}|D|)$ and $L_2 = \coth(\sqrt{\mu_2}|D|)$ by its Fourier transform $\widehat{L_1 f}(y) = |y| \coth(\sqrt{\mu_2}|y|) \hat{f}(y)$ and $\widehat{L_2 f}(y) = \coth(\sqrt{\mu_2}|y|) \hat{f}(y)$, respectively.

2. EXISTENCE OF SOLITARY WAVE SOLUTIONS FOR THE FULL DISPERSION-BOUSSINESQ SYSTEM WHEN $\mu_2 = +\infty$

In this section we give the proof of the existence of solitary waves solutions of (2.1) in the case $\mu_2 = +\infty$. Thus, we will find a non-trivial solution, (ξ, ν) , for the system

$$\begin{cases} -\omega J_b \xi + \mathcal{L}_{+\infty} \nu = 2r \xi \nu \\ -\omega J_b \nu + (1 - \gamma) J_c \xi = r \nu^2, \end{cases} \quad (2.1)$$

with $\gamma \in (0, 1)$ and $r = \frac{\epsilon}{2\gamma}$.

Our approach for the existence will be via the Concentration-Compactness Principle, thus we call $\{(\xi_n, \nu_n)\}_{n \geq 1}$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ a minimizing sequence for I_λ if it satisfies $F(\xi_n, \nu_n) = \lambda$ for all n and $\lim_{n \rightarrow \infty} E_{+\infty}(\xi_n, \nu_n) = I_\lambda$

For a, b, c satisfying the relations in (1.8), the following Lemma establishes the coercivity property of the functional $E_{+\infty}$.

Proposition 2.1. *Let $\gamma \in (0, 1)$ fixed. Then,*

(a) $E_{+\infty} : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow \mathbb{R}$ is well defined and satisfies

$$E_{+\infty}(\xi, \nu) \leq c_0(\|J_c^{1/2} \xi\|^2 + \|\nu\|_1^2) + c_1(\|J_b^{1/2} \xi\|^2 + \|J_b^{1/2} \nu\|^2).$$

(b) For $|\omega| < (1 - \gamma) \min\{1, \frac{|c|}{b}\}$, we have

$$E_{+\infty}(\xi, \nu) \geq C \|(\xi, \nu)\|_{1 \times 1}^2 \quad (2.2)$$

where $C = C(b, c, \mu, |\omega|, \gamma) > 0$.

(c) $I_\lambda \in (0, +\infty)$.

- (d) All minimizing sequences for I_λ are bounded in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.
(e) For all $\theta \in (0, \lambda)$, we have the sub-additivity property of I_λ ,

$$I_\lambda < I_\theta + I_{\lambda-\theta}. \quad (2.3)$$

The most delicate statement in the Proposition 2.1 is item (b). The next Lemma shows firstly that the functional $E_{+\infty}$ is non-negative.

Lemma 2.1. *Let $\gamma \in (0, 1)$ fixed and a, b, c satisfying the relations in (1.8). Then, for $|\omega| < (1 - \gamma)\min\{1, \frac{|c|}{b}\}$, we have $E_{+\infty}(\xi, \nu) \geq 0$ for all $(\xi, \nu) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$.*

Proof. To start with, since J_b is a positive operator we obtain from the Cauchy-Schwarz inequality for positive quadratic forms that

$$\left| \omega \int_{\mathbb{R}} \xi J_b \nu dx \right| \leq |\omega| \langle J_b \xi, \xi \rangle^{1/2} \langle J_b \nu, \nu \rangle^{1/2} \leq \frac{|\omega|}{2} \langle J_b \xi, \xi \rangle + \frac{|\omega|}{2} \langle J_b \nu, \nu \rangle. \quad (2.4)$$

Next, we show the following two inequalities under the condition $|\omega| \leq (1 - \gamma)\min\{1, \frac{|c|}{b}\}$:

$$\frac{|\omega|}{2} \langle J_b \xi, \xi \rangle \leq \frac{(1 - \gamma)}{2} \langle J_c \xi, \xi \rangle, \quad \text{and,} \quad \frac{|\omega|}{2} \langle J_b \nu, \nu \rangle \leq \frac{1}{2} \langle \mathcal{L}_{\mu_2} \nu, \nu \rangle \quad (2.5)$$

We divide the proof in various steps:

- (1) since the second order polynomial $p(x) = [-(1 - \gamma)\mu c - |\omega|\mu b]x^2 + (1 - \gamma) - |\omega|$ achieves positive values for all $x \in \mathbb{R}$, under the conditions $|\omega| \leq (1 - \gamma)$ and $b|\omega| \leq (1 - \gamma)|c|$, we obtain from Plancherel's theorem

$$\langle [(1 - \gamma)J_c - |\omega|J_b]\xi, \xi \rangle = \int_{\mathbb{R}} |\widehat{\xi}(x)|^2 p(x) dx \geq 0.$$

- (2) Since the symbol of $\mathcal{L}_{\mu_2} - |\omega|J_b$ is given by the even function,

$$f(x) = \frac{1}{\gamma} - \frac{\sqrt{\mu}}{\gamma^2} |x| - \mu \left[b|\omega| + \frac{1}{\gamma} \left(a - \frac{1}{\gamma^2} \right) \right] x^2, \quad (2.6)$$

we will show that under the condition $|\omega| \leq (1 - \gamma)\min\{1, \frac{|c|}{b}\}$, $\min_{x \in \mathbb{R}} f(x) = f(x_0) > 0$, with x_0 being the only positive critical point for f and

$$f(x_0) = \frac{1}{\gamma} - |\omega| - \frac{1}{4\gamma^4 \beta_0} \quad (2.7)$$

where $\beta_0 \equiv -[b|\omega| + \frac{1}{\gamma}(a - \frac{1}{\gamma^2})]$ (see (1.9)). In fact, for $x > 0$ we have that the only critical point of $f(x)$, x_0 , is given by the relation

$$2\sqrt{\mu}\beta_0 x_0 = \frac{1}{\gamma^2}.$$

We note that β_0 needs to be strictly positive. Moreover, the profile of f is a strictly convex function from the relation $f''(x) = 2\sqrt{\mu}\beta_0$. The idea now is to show that

$f(x_0) > 0$, which is equivalent to check that the mapping $M : I(0) \rightarrow \mathbb{R}$ defined by (with γ fixed)

$$M(\omega) = 4(1 - \gamma|\omega|)[1 - b\gamma^3|\omega| - a\gamma^2] - 1 \quad (2.8)$$

is strictly positive for $\omega \in I(0)$ satisfying $|\omega| \leq (1 - \gamma) \min\{1, \frac{|c|}{b}\}$. Before, we show that the unique positive critical point ω_0 of M given by

$$\omega_0 = \frac{1 + (b - a)\gamma^2}{2b\gamma^3} \equiv q(\gamma)$$

does not belong to the admissible interval for the speed velocity ω . In fact, note that q is a strictly decreasing positive function with $\inf_{\gamma \in (0,1)} q(\gamma) = \frac{1-a+b}{2b}$. Now, for $|c| \leq b$ we have that $\omega_0 > \frac{|c|}{b}$, indeed, since

$$\frac{1 - a + b}{2b} > \frac{|c|}{b} \Leftrightarrow \frac{2}{3} + 3b > -3c$$

our claim follows immediately. Next, for $|c| \geq b$ we obtain $\omega_0 > 1$. Thus, since $M(0) > 0$ and M is a convex function we observe that in order to show that $M > 0$ on $I(0)$ is sufficient to check that $M(1 - \gamma) > 0$ for $|c| \geq b$ and $M((1 - \gamma)\frac{|c|}{b}) > 0$ for $|c| \leq b$. Thus, we consider the case $|c| \leq b$, then since $1 - \gamma(1 - \gamma)\frac{|c|}{b} \geq 1 - \gamma(1 - \gamma)$, $1 - \frac{1}{3}(1 - \gamma)\gamma^3 > 0$ and the polynomial $\tau(\gamma) = 4(1 - \gamma + \gamma^2)\left[1 - \frac{1}{3}(1 - \gamma)\gamma^3\right] - 1 > 0$ for all $\gamma \in (0, 1)$, we obtain

$$M\left((1 - \gamma)\frac{|c|}{b}\right) \geq -4a\gamma^2(1 - \gamma + \gamma^2)^2 + \tau(\gamma) > 0.$$

This concludes the proof of the Lemma. \square

Proof. [Proposition 2.1] Item (a) is an immediate consequence of the Cauchy-Schwarz inequality. For the coercivity property (2.2), we deduce from (2.4)

$$E_{+\infty}((\xi, \nu)) \geq \frac{1}{2} \langle ((1 - \gamma)J_c - |\omega|J_b)\xi, \xi \rangle + \frac{1}{2} \langle (\mathcal{L}_{+\infty} - |\omega|J_b)\nu, \nu \rangle \equiv I + II. \quad (2.9)$$

Next, we estimate the term I . Suppose $|c| \geq b$, then $|\omega| < 1 - \gamma$. Thus

$$\begin{aligned} 2I &= \int_{\mathbb{R}} [(1 - \gamma) - |\omega|]\xi^2(x) - c\mu(\xi'(x))^2((1 - \gamma) + \frac{b}{c}|\omega|)dx \\ &\geq [(1 - \gamma) - |\omega|] \int_{\mathbb{R}} \xi^2 - c\mu(\xi')^2 dx = [(1 - \gamma) - |\omega|] \|J_c^{1/2}\xi\|^2. \end{aligned} \quad (2.10)$$

For the term II , we have first from Cauchy-Schwarz inequality and from Young's inequality that for all $\eta > 0$,

$$\|D|^{1/2}\nu\|^2 \leq \|\nu\| \|\nu'\| \leq \eta \|\nu\|^2 + \frac{1}{4\eta} \|\nu'\|^2.$$

Then, for $-\alpha \equiv \frac{\mu}{\gamma}(a - \frac{1}{\gamma^2})$ we have

$$2II \geq \left(\frac{1}{\gamma} - |\omega| - \eta \frac{\sqrt{\mu}}{\gamma^2}\right) \|\nu\|^2 + \left(\alpha - \mu b |\omega| - \frac{1}{4\eta} \frac{\sqrt{\mu}}{\gamma^2}\right) \|\nu'\|^2. \quad (2.11)$$

Next, for $\beta_0 \equiv -[b|\omega| + \frac{1}{\gamma}(a - \frac{1}{\gamma^2})] > 0$ we choose

$$\eta = \frac{\gamma^2}{\sqrt{\mu}} \frac{1}{4\gamma^4\beta_0} + \epsilon_0 \frac{\gamma^2}{\sqrt{\mu}} \quad (2.12)$$

for ϵ_0 chosen such that $f(x_0) > \epsilon_0 > 0$ (see (2.7)). Thus

$$\left(\frac{1}{\gamma} - |\omega| - \eta \frac{\sqrt{\mu}}{\gamma^2}\right) \|\nu\|^2 = (f(x_0) - \epsilon_0) \|\nu\|^2, \quad (2.13)$$

and

$$\left(\alpha - \mu b |\omega| - \frac{1}{4\eta} \frac{\sqrt{\mu}}{\gamma^2}\right) \|\nu'\|^2 = \frac{4\mu\epsilon\gamma^4\beta_0^2}{1 + 4\epsilon\gamma^4\beta_0} \|\nu'\|^2 \equiv \theta \|\nu'\|^2. \quad (2.14)$$

Therefore, for $|c| \geq b$ we get that

$$E_{+\infty}((\xi, \nu)) \geq \frac{1}{2} [(1 - \gamma) - |\omega|] \|J_c^{1/2}\xi\|^2 + \frac{1}{2} \min\{f(x_0) - \epsilon_0, \theta\} \|\nu\|_1^2. \quad (2.15)$$

We note that the $\|J_c^{1/2}\xi\|$ is equivalent to $\|\xi\|_1$.

Next, we consider the case $|c| \leq b$. Thus $|\omega| < (1 - \gamma)\frac{|c|}{b}$ and the expression I in (2.9) is estimated as

$$\begin{aligned} 2I &= \int_{\mathbb{R}} [(1 - \gamma) - |\omega|] \xi^2(x) + \mu b (\xi'(x))^2 \left(\frac{|c|}{b}(1 - \gamma) - |\omega|\right) dx \\ &\geq \left[\frac{|c|}{b}(1 - \gamma) - |\omega|\right] \int_{\mathbb{R}} \xi^2 + \mu b (\xi')^2 dx = \left[\frac{|c|}{b}(1 - \gamma) - |\omega|\right] \|J_b^{1/2}\xi\|^2. \end{aligned} \quad (2.16)$$

The estimative of the term II in (2.9) is the same as in the case $|c| \geq b$. Therefore, for $|c| \leq b$ we get that

$$E_{+\infty}((\xi, \nu)) \geq \frac{1}{2} \left[\frac{|c|}{b}(1 - \gamma) - |\omega|\right] \|J_b^{1/2}\xi\|^2 + \frac{1}{2} \min\{f(x_0) - \epsilon_0, \theta\} \|\nu\|_1^2. \quad (2.17)$$

We note that $\|J_b^{1/2}\xi\|$ is equivalent to $\|\xi\|_1$. This concludes item (b). Now, item (c) follows immediately from the coercivity property (2.2) and the inequality

$$0 < \lambda = \int_{\mathbb{R}} \xi \nu^2 dx \leq |\nu|_{\infty} \|\nu\| \|\xi\| \leq \|(\xi, \nu)\|_{1 \times 1}^3.$$

Indeed $I_{\lambda} \geq C\lambda^{2/3} > 0$. Next, item (d) follows immediately from (2.2). Finally, from the property $I_{\tau\lambda} = \tau^{2/3}I_{\lambda}$ for all $\tau > 0$ and from $I_{\lambda} > 0$ we obtain the sub-additivity property (2.3). This concludes the proof of the Proposition. \square

In order to show the existence of solutions of equation (2.1), we will prove that a minimizing sequence for the problem (1.10) converges (modulo translations) to a function in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ satisfying the constraint $F(\xi, \nu) = \lambda$. To do so we will use the following result, it which is the key tool in our analysis (see Lemma 1.1 in Lions [32]).

Lemma 2.2. (*The Concentration-Compactness Principle*)

Let $\{\rho_n\}_{n \geq 1}$ be a sequence of non-negative functions in $L^1(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} \rho_n(x) dx = \sigma$ for all n and some $\sigma > 0$. Then there exists a subsequence $\{\rho_{n_k}\}_{k \geq 1}$ satisfying one of the following three conditions:

(1) **(Compactness)** there are $y_k \in \mathbb{R}$ for $k = 1, 2, \dots$, such that $\rho_{n_k}(\cdot + y_k)$ is tight, i.e. for any $\epsilon > 0$, there is $R > 0$ large enough such that

$$\int_{|x-y_k| \leq R} \rho_{n_k}(x) dx \geq \sigma - \epsilon;$$

(2) **(Vanishing)** for any $R > 0$,

$$\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq R} \rho_{n_k}(x) dx = 0;$$

(3) **(Dichotomy)** there exists $\theta_0 \in (0, \sigma)$ such that for any $\epsilon > 0$, there exists $k_0 \geq 1$ and $\rho_{k,1}, \rho_{k,2} \in L^1(\mathbb{R})$, with $\rho_{k,1}, \rho_{k,2} \geq 0$, such that for $k \geq k_0$,

$$|\rho_{n_k} - (\rho_{k,1} + \rho_{k,2})|_{L^1} \leq \epsilon,$$

$$|\int_{\mathbb{R}} \rho_{k,1} dx - \theta_0| \leq \epsilon, \quad |\int_{\mathbb{R}} \rho_{k,2} dx - (\sigma - \theta_0)| \leq \epsilon,$$

$$\text{supp } \rho_{k,1} \cap \text{supp } \rho_{k,2} = \emptyset, \quad \text{dist}(\text{supp } \rho_{k,1}, \text{supp } \rho_{k,2}) \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Remark 2.1. In Lemma 2.2 above, the condition $\int_{\mathbb{R}} \rho_n(x) dx = \sigma$ can be replaced by $\int_{\mathbb{R}} \rho_n(x) dx = \lambda_n$ where $\lambda_n \rightarrow \sigma > 0$ as $n \rightarrow \infty$. It is enough to replace ρ_n by $\sigma \rho_n / \lambda_n$ and apply the lemma.

The next step in our analysis is to rule out the possibilities of vanishing and dichotomy in Lemma 2.2. The following classical Lemma (see Lemma I.1 in [32]) is the key tool to rule out vanishing.

Lemma 2.3. Let $\{(\xi_n, \nu_n)\}_{n \geq 1}$, be a bounded sequence in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Assume that for some $R > 0$,

$$Q_n(R) = \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} |\nu_n(x)|^2 dx \rightarrow 0$$

as $n \rightarrow \infty$. Then, $\int_{\mathbb{R}} \xi_n(x) \nu_n^2(x) dx \rightarrow 0$ as $n \rightarrow \infty$.

Proof. From Hölder inequality and from the embedding $H^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$, $p \geq 2$, we have for $C > 0$ that

$$\left| \int_{\mathbb{R}} \xi_n(x) \nu_n^2(x) dx \right| \leq C \|\xi_n\|_1 \|\nu_n\|_{L^3}^2 \leq C_1 \delta_n^{1/3} \|\xi_n\|_1 \|\nu_n\|_1^{4/3}$$

with $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. This achieves the proof. \square

Let $\{(\xi_n, \nu_n)\}_{n \geq 1}$ be a minimizing sequence for problem (1.10) and consider the sequence of non-negative functions in $L^1(\mathbb{R})$

$$\rho_n(x) = |\xi_n(x)|^2 + |\xi_n'(x)|^2 + |\nu_n(x)|^2 + |\nu_n'(x)|^2.$$

Let $\lambda_n = \int_{\mathbb{R}} \rho_n(x) dx$. Since $\lambda_n = \|(\xi_n, \nu_n)\|_{1 \times 1}^2$, λ_n is bounded and $\lambda_n \geq \lambda^{2/3}$. Assume that $\lambda_n \rightarrow \sigma$ as $n \rightarrow \infty$. Then, from Lemma 2.2 there exists a subsequence $\{\rho_{n_k}\}_{k \geq 1}$ of $\{\rho_n\}_{n \geq 1}$ which satisfies either vanishing or dichotomy. If vanishing occurs, then

$$\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} |\nu_{n_k}(x)|^2 dx = 0, \quad \text{for any } R > 0,$$

hence, by Lemma 2.3 we obtain $\lim_{k \rightarrow \infty} F(\xi_{n_k}, \nu_{n_k}) = 0$, which is a contradiction.

If dichotomy occurs, there exist $\theta \in (0, \sigma)$ and $L^1(\mathbb{R})$ -functions $\rho_{k,1}, \rho_{k,2}$ satisfying item (3) of Lemma 2.2. Moreover, we may assume that the supports of $\rho_{k,1}$ and $\rho_{k,2}$ are disjoint as follows:

$$\text{supp } \rho_{k,1} \subset (y_k - R_0, y_k + R_0), \quad \text{supp } \rho_{k,2} \subset (-\infty, y_k - 2R_k) \cup (y_k + 2R_k, +\infty) \quad (2.18)$$

for some fixed R_0 , a sequence $\{y_k\}_{k \geq 1}$ and $R_k \rightarrow +\infty$, as $k \rightarrow \infty$.

Now, denoting $\mathbf{h}_{n_k} = (\xi_{n_k}, \nu_{n_k})$ we obtain the following splitting functions $\mathbf{h}_{k,1}$ and $\mathbf{h}_{k,2}$ of \mathbf{h}_{n_k} . Consider $\varphi, \zeta \in C^\infty(\mathbb{R})$ with $0 \leq \varphi, \zeta \leq 1$ such that $\zeta(x) = 1$, for $|x| \leq 1$ and $\zeta(x) = 0$ for $|x| \geq 2$, $\varphi(x) = 1$, for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$. Denote,

$$\zeta_k(x) = \zeta\left(\frac{x - y_k}{R_1}\right), \quad \text{and} \quad \varphi_k(x) = \varphi\left(\frac{x - y_k}{R_k}\right),$$

for $x \in \mathbb{R}$ and $R_1 > R_0$, so that $\text{supp } \zeta_k \subset \{x : |x - y_k| \leq 2R_1\}$ and $\text{supp } \varphi_k \subset \{x : |x - y_k| \leq 2R_k\}$. Now, we define $\mathbf{h}_{k,1} = \zeta_k \mathbf{h}_{n_k}$ and $\mathbf{h}_{k,2} = \varphi_k \mathbf{h}_{n_k}$. Therefore, from the relations

$$\begin{aligned} \int_{y_k - R_0}^{y_k + R_0} |\rho_{n_k}(x) - \rho_{k,1}(x)| dx &\leq \epsilon, \\ \int_{|x - y_k| \geq 2R_k} |\rho_{n_k}(x) - \rho_{k,2}(x)| dx &\leq \epsilon, \end{aligned} \quad (2.19)$$

$$\int_{R_0 \leq |x - y_k| \leq 2R_k} \rho_{n_k}(x) dx \leq \epsilon,$$

$|\zeta_k'|_\infty \leq \frac{1}{R_1} |\zeta|_\infty$, $|\varphi_k'|_\infty \leq \frac{1}{R_k} |\varphi|_\infty$, and by using a classical argument we can see that for R_1 large enough

$$N_0 = \left| \|\zeta_k \xi_{n_k}\|_1^2 + \|\zeta_k \nu_{n_k}\|_1^2 - \int_{\mathbb{R}} \rho_{k,1}(x) dx \right| \leq \epsilon, \quad (2.20)$$

and

$$N_1 = \left| \|\varphi_k \xi_{n_k}\|_1^2 + \|\varphi_k \nu_{n_k}\|_1^2 - \int_{\mathbb{R}} \rho_{k,2}(x) dx \right| \leq \epsilon. \quad (2.21)$$

Next, we define $\mathbf{w}_k = \mathbf{h}_{n_k} - (\mathbf{h}_{k,1} + \mathbf{h}_{k,2})$. Then, for $\chi_k = 1 - \zeta_k - \varphi_k$ we obtain

$$\|\mathbf{w}_k\|_{1 \times 1} = O(\epsilon). \quad (2.22)$$

Indeed, since $\text{supp } \chi_k \subset \{x : R_1 \leq |x - y_k| \leq 2R_k\} \leq 2R_k$ we obtain from (2.19) that

$$\|\chi_k \xi_k\|_1^2 \leq 2(|\chi_k|_\infty^2 + |\chi_k'|_\infty^2) \int_{R_1 \leq |x - y_k| \leq 2R_k} \rho_{n_k}(x) dx = O(\epsilon).$$

Similarly, we obtain $\|\chi_k \nu_k\|_1^2 = O(\epsilon)$. Next, since $\int_{\mathbb{R}} \zeta_k^3 \xi_{n_k} \nu_{n_k}^2 dx$ is bounded, there is a subsequence of $\{\mathbf{h}_{k,1}\}$, still denoted by $\{\mathbf{h}_{k,1}\}$, such that for $k \geq k_0 > 0$ and $\theta \in \mathbb{R}$

$$\left| \int_{\mathbb{R}} \zeta_k^3 \xi_{n_k} \nu_{n_k}^2 dx - \theta \right| \leq \epsilon. \quad (2.23)$$

Moreover, since $F(\xi_{n_k}, \nu_{n_k}) = \lambda$, $\zeta_k \varphi_k = 0$, $\chi_k^3 = 1 - \zeta_k^3 - \varphi_k^3$ we obtain for $k \geq k_0$,

$$\left| \int_{\mathbb{R}} \varphi_k^3 \xi_{n_k} \nu_{n_k}^2 dx - (\lambda - \theta) \right| \leq \left| \int_{\mathbb{R}} \chi_k^3 \xi_{n_k} \nu_{n_k}^2 dx \right| + \epsilon \leq \|\mathbf{w}_k\|_{1 \times 1}^3 + \epsilon = O(\epsilon). \quad (2.24)$$

Next, we will show that for R_1, R_k , large enough, we have

$$E_{+\infty}(\mathbf{h}_{n_k}) = E_{+\infty}(\mathbf{h}_{k,1}) + E_{+\infty}(\mathbf{h}_{k,2}) + O(\epsilon). \quad (2.25)$$

Indeed, denoting $(a_k, b_k) \equiv \mathbf{w}_k = (\chi_k \xi_{n_k}, \chi_k \nu_k)$ one easily checks that

$$E_{+\infty}(\mathbf{h}_{n_k}) = E_{+\infty}(\mathbf{w}_k) + E_{+\infty}(\mathbf{h}_{k,1}) + E_{+\infty}(\mathbf{h}_{k,2}) + N_3 + N_4, \quad (2.26)$$

where for positive constants c_1, c_2, c_3, c_5, c_6 and $c_4 < 0$

$$\begin{aligned} N_3 = & \int_{\mathbb{R}} (\zeta_k + \varphi_k) [(2c_1 a_k - \omega b_k) \xi_{n_k} + (c_2 b_k - \omega a_k) \nu_{n_k}] \\ & + ((\zeta_k \xi_{n_k})' + (\varphi_k \xi_{n_k})') (2c_3 a_k' - \omega c_4 b_k') + ((\zeta_k \nu_{n_k})' + (\varphi_k \nu_{n_k})') (2c_5 b_k' - \omega c_4 a_k') \end{aligned} \quad (2.27)$$

and

$$N_4 = -2c_6 \int_{\mathbb{R}} [\zeta_k \nu_{n_k} + \varphi_k \nu_{n_k}] D(b_k) + \varphi_k \nu_{n_k} D(\zeta_k \nu_{n_k}) dx. \quad (2.28)$$

Moreover $N_3 = O(\epsilon)$ and $N_4 = O(\epsilon)$. Indeed, from Cauchy-Schwarz inequality, Proposition 2.1 (item (d)) and (2.22) we obtain for R_k large enough that

$$|N_3| \leq C \|\mathbf{w}_k\|_{1 \times 1} \|\mathbf{h}_{n_k}\|_{1 \times 1} = O(\epsilon).$$

Now, concerning N_4 we observe first from Cauchy-Schwarz inequality, Proposition 2.1 and (2.22), that

$$\int_{\mathbb{R}} [\zeta_k \nu_{n_k} + \varphi_k \nu_{n_k}] D(b_k) dx \leq 2 \|D(b_k)\| \|\nu_{n_k}\| \leq C \|\mathbf{w}_k\|_{1 \times 1} \|\mathbf{h}_{n_k}\|_{1 \times 1} = O(\epsilon). \quad (2.29)$$

Next, since $D = \mathcal{H}\partial_x$, where \mathcal{H} is the Hilbert-transform, we have by using commutators, $\zeta_k \varphi_k = 0$ and Calderon commutator Theorem (see [17]-[21]) that for R_1 large

$$\begin{aligned} \left| \int_{\mathbb{R}} \varphi_k \nu_{n_k} D(\zeta_k \nu_{n_k}) dx \right| &= \left| \int_{\mathbb{R}} \varphi_k \nu_{n_k} (\mathcal{H}(\zeta'_k \nu_{n_k}) + [\mathcal{H}, \zeta_k] \nu'_{n_k} + \zeta_k \mathcal{H} \nu'_{n_k}) dx \right| \\ &\leq C \|\nu_{n_k}\| (\|[\mathcal{H}, \zeta_k] \nu'_{n_k}\| + \|\zeta'_k \nu_{n_k}\|) \\ &\leq C \|\nu_{n_k}\|^2 |\zeta'_k|_{\infty} = C \frac{|\zeta'_k|_{\infty}}{R_1} \|\nu_{n_k}\|^2 = O(\epsilon). \end{aligned} \quad (2.30)$$

Thus, from (2.29)-(2.30) we obtain $N_4 = O(\epsilon)$.

Now, from Proposition 2.1, item (a), and (2.22) it follows that for R_1, R_k large enough, $E_{+\infty}(\mathbf{w}_k) \leq C \|\mathbf{w}_k\|_{1 \times 1} = O(\epsilon)$, and so we obtain the key relation (2.25). Moreover,

$$I_{\lambda} \geq \liminf_{k \rightarrow \infty} E_{+\infty}(\mathbf{h}_{n_k}) \geq \liminf_{k \rightarrow \infty} E_{+\infty}(\mathbf{h}_{k,1}) + \liminf_{k \rightarrow \infty} E_{+\infty}(\mathbf{h}_{k,2}) + O(\epsilon). \quad (2.31)$$

The next step in our analysis is to rule out dichotomy. Indeed, recalling that $\theta = \lim_{k \rightarrow \infty} \int_{\mathbb{R}} \zeta_k^3 \xi_{n_k} \nu_{n_k}^2 dx$, we consider the following cases:

- 1) Suppose $\theta = 0$. Then, from (2.24) we have for k large that $F(\mathbf{h}_{k,2}) > \lambda/2 > 0$. Therefore, for k fixed we consider $d_k > 0$ such that $F(d_k \mathbf{h}_{k,2}) = \lambda$. Moreover,

$$|d_k - 1| \leq \left(\frac{2}{\lambda}\right)^{1/3} |\lambda^{1/3} - F(\mathbf{h}_{k,2})^{1/3}| \leq C_1 \epsilon$$

with C_1 independent of $\mathbf{h}_{k,2}$ and ϵ . Hence, $d_k \rightarrow 1$, as $k \rightarrow \infty$, and

$$I_{\lambda} \leq E_{+\infty}(d_k \mathbf{h}_{k,2}) = d_k^2 E_{+\infty}(\mathbf{h}_{k,2}) = E_{+\infty}(\mathbf{h}_{k,2}) + O(\epsilon). \quad (2.32)$$

Thus, from the coercivity property of $E_{+\infty}$ (see item (b)-Proposition 2.1) and from (2.20) and item (3) in Lemma 2.2 we obtain for $\theta_0 > 0$,

$$\liminf_{k \rightarrow \infty} E_{+\infty}(\mathbf{h}_{k,1}) \geq C \liminf_{k \rightarrow \infty} \|\mathbf{h}_{k,1}\|_{1 \times 1}^2 \geq C \liminf_{k \rightarrow \infty} |\rho_{k,1}|_{L^1} + O(\epsilon) \geq C \theta_0 + O(\epsilon)$$

and therefore from (2.31)-(2.32)

$$I_{\lambda} \geq C \theta_0 + I_{\lambda} + O(\epsilon).$$

Thus, letting $\epsilon \rightarrow 0$ in the last relation leads to the contradiction $I_{\lambda} \geq C \theta_0 + I_{\lambda}$.

- 2) Suppose $\lambda > \theta > 0$. Using the previous argument combined with item (3) in Lemma 2.2 and (2.21), we can prove that $I_{\lambda} \geq I_{\theta} + I_{\lambda-\theta} + O(\epsilon)$ and for $\epsilon \rightarrow 0$ we obtain

$$I_{\lambda} \geq I_{\theta} + I_{\lambda-\theta}.$$

But, from (2.3) we obtain a contradiction.

- 3) Case $\theta < 0$. From (2.24) follows that $\lim_{k \rightarrow \infty} F(\mathbf{h}_{k,2}) = \lambda - \theta > \lambda/2$. Thus, by a similar analysis as in item 1) we get a contradiction.
- 4) Case $\theta = \lambda$. Then $F(\mathbf{h}_{k,1}) > \lambda/2$ for k large, and so by using (2.21) and again, an analysis as in item 1) leads to a contradiction.

5) Case $\theta > \lambda$. Let consider $e_k > 0$ such that $F(e_k \mathbf{h}_{k,1}) = \lambda$. Hence, $e_k \rightarrow (\lambda/\theta)^{1/3}$, as $k \rightarrow \infty$. Then, for k large and from the positivity of $E_{+\infty}$

$$I_\lambda \leq E_{+\infty}(e_k \mathbf{h}_{k,1}) = e_k^2 E_{+\infty}(\mathbf{h}_{k,1}) < E_{+\infty}(\mathbf{h}_{k,1}). \quad (2.33)$$

Thus, from the coercivity property of $E_{+\infty}$, from (2.21) and item (3) in Lemma 2.2 we obtain for $\theta_0 > 0$

$$\liminf_{k \rightarrow \infty} E_{+\infty}(\mathbf{h}_{k,2}) \geq C \liminf_{k \rightarrow \infty} \|\mathbf{h}_{k,2}\|_{1 \times 1}^2 \geq C \liminf_{k \rightarrow \infty} |\rho_{k,2}|_{L^1} + O(\epsilon) \geq C\theta_0 + O(\epsilon)$$

and therefore from (2.31)-(2.33)

$$I_\lambda \geq C\theta_0 + I_\lambda + O(\epsilon).$$

Thus, letting $\epsilon \rightarrow 0$ in the last relation leads to the contradiction $I_\lambda \geq C\theta_0 + I_\lambda$.

Since vanishing and dichotomy have been ruled out, it follows from Lemma 2.2 that there exists a sequence $\{y_k\}_{k \geq 1} \subset \mathbb{R}$ such that for any $\epsilon > 0$, there are $R > 0$, large enough, and $k_0 > 0$ such that for $k \geq k_0$

$$\int_{|x-y_k| \leq R} \rho_{n_k}(x) dx \geq \sigma - \epsilon, \quad \int_{|x-y_k| \geq R} \rho_{n_k}(x) dx = O(\epsilon),$$

and for $P_k = (-\infty, y_k - R] \cup [y_k + R, \infty)$

$$\left| \int_{P_k} \zeta_{n_k} \nu_{n_k}^2 dx \right| \leq |\zeta_{n_k}|_{L^3} \left(\int_{P_k} |\nu_{n_k}|^3 dx \right)^{2/3} \leq C \|\zeta_{n_k}\|_1 |\nu_{n_k}|_\infty^{2/3} \left(\int_{P_k} \rho_{n_k}(x) dx \right)^{2/3} = O(\epsilon),$$

where we have used that every minimizing sequence is bounded in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

Then, it follows that

$$\left| \int_{|x-y_k| \leq R} \zeta_{n_k}(x) \nu_{n_k}^2(x) dx - \lambda \right| \leq \epsilon \quad (2.34)$$

Let $\tilde{\mathbf{h}}_{n_k}(x) = (\tilde{\zeta}_{n_k}(x), \tilde{\nu}_{n_k}(x)) = (\zeta_{n_k}(x - y_k), \nu_{n_k}(x - y_k))$. Then $\{\tilde{\mathbf{h}}_{n_k}\}_{k \geq 1}$ is bounded in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and therefore $\{\tilde{\mathbf{h}}_{n_k}\}_{k \geq 1}$ (up to a subsequence) converges weakly in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ to a vector-valued function $\tilde{\mathbf{h}} = (\zeta_0, \nu_0)$. Then it follows from (2.34) that for $k \geq k_0$,

$$\lambda \geq \int_{-R}^R \tilde{\zeta}_{n_k}(x) \tilde{\nu}_{n_k}^2(x) dx \geq \lambda - \epsilon, \quad (2.35)$$

and so by the compact embedding of $H^1(-R, R)$ in $L^2(-R, R)$ we have the relation $\lambda \geq \int_{-R}^R \zeta_0(x) \nu_0^2(x) dx \geq \lambda - \epsilon$. Thus for $\epsilon = \frac{1}{j}$, $j \in \mathbb{N}$, there exists $R_j > j$ such that

$$\lambda \geq \int_{-R_j}^{R_j} \zeta_0(x) \nu_0^2(x) dx \geq \lambda - \frac{1}{j}, \quad (2.36)$$

and consequently as $j \rightarrow \infty$, we finally have that $F(\zeta_0, \nu_0) = \lambda$. Furthermore, from the weak lower semicontinuity of $E_{+\infty}$ and the invariance of $E_{+\infty}$ by translations, we have

$$I_\lambda = \liminf_{k \rightarrow \infty} E_{+\infty}(\tilde{\zeta}_{n_k}, \tilde{\nu}_{n_k}) \geq E_{+\infty}(\zeta_0, \nu_0) \geq I_\lambda.$$

Thus, the vector-valued function $\tilde{\mathbf{h}} = (\zeta_0, \nu_0)$ solves the variational problem (1.10) and therefore there exists $K > 0$ (Lagrange multiplier) such that

$$\begin{cases} (1 - \gamma)J_c \xi_0 - \omega J_b \nu_0 = Kr\nu_0^2, \\ \mathcal{L}_{+\infty} \nu_0 - \omega J_b \xi_0 = 2Kr\xi\nu \end{cases} \quad (2.37)$$

with $r = \frac{\varepsilon}{2\gamma}$. We note that $K > 0$ since $3K\lambda = 2E_{+\infty}(\zeta_0, \nu_0) = I_\lambda > 0$. Thus, if we consider $(\zeta, \nu) = K(\zeta_0, \nu_0)$ then from (2.37) we have that (ζ, ν) solves problem (2.1). This concludes part (a) of Theorem 1.1.

3. EXISTENCE OF SOLITARY WAVE SOLUTIONS WHEN μ_2 IS FINITE

In this section we give the proof of the existence of solitary waves solutions for system (1.7) when μ_2 is finite and satisfies appropriate conditions. We will find a non-trivial solution (ξ, ν) by solving the minimization problem (1.10).

We start the analysis by establishing the coercivity property of E_{μ_2} . For a, b, c satisfying the relations in (1.8) with $b = d$ we obtain:

Proposition 3.1. *Let $\gamma \in (0, 1)$ fixed and $|\omega| < (1 - \gamma)\min\{1, \frac{|c|}{b}\}$. Then, for $(\xi, \nu) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, we have*

(a) $E_{\mu_2}(\xi, \nu)$ is well defined and satisfies

$$E_{\mu_2}(\xi, \nu) \leq c_0(\|J_c^{1/2}\xi\|^2 + \|\nu\|_1^2) + c_1(\|J_b^{1/2}\xi\|^2 + \|J_b^{1/2}\nu\|^2).$$

(b) $\langle ((1 - \gamma)J_c - |\omega|J_b)\xi, \xi \rangle \geq 0$.

(c) Let $\alpha_0 > 0$ defined in (b)-Theorem 1.1. Then for every fixed γ , we can choose μ_2 satisfying $\frac{\sqrt{\mu}}{\sqrt{\mu_2}} < \gamma^2 \alpha_0$ and such that

$$\langle (\mathcal{L}_{\mu_2} - |\omega|J_b)\nu, \nu \rangle \geq 0. \quad (3.1)$$

(d) For $C = C(b, c, \mu, \mu_2, |\omega|, \gamma) > 0$, we have

$$E_{\mu_2}(\xi, \nu) \geq C\|(\xi, \nu)\|_{1 \times 1}^2. \quad (3.2)$$

(e) $I_\lambda \in (0, +\infty)$.

(f) All minimizing sequences for I_λ are bounded in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

(g) For all $\theta \in (0, \lambda)$, we have the sub-additivity property of I_λ ,

$$I_\lambda < I_\theta + I_{\lambda - \theta}. \quad (3.3)$$

Proof. The proof of items (a)-(b)-(e)-(f)-(g) are similar to those given in Proposition 2.1. For inequality (3.1), we consider the symbol associated to the linear operator $\mathcal{L}_{\mu_2} - |\omega|J_b$ given by the following even function

$$f_{\mu_2}(x) = \frac{1}{\gamma} - \frac{\sqrt{\mu}}{\gamma^2}|x| \coth(\sqrt{\mu_2}|x|) - \frac{\mu}{\gamma} \left(a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2}|x|) \right) x^2 - |\omega| - \mu b |\omega| x^2.$$

Next, since for $x > 0$ we observe that

$$x \coth(\sqrt{\mu_2}x) \leq \frac{1}{\sqrt{\mu_2}} + x, \quad (3.4)$$

and that for all x , $\coth^2(\sqrt{\mu_2}|x|) - 1 \geq 0$.

Then it follows that

$$f_{\mu_2}(x) \geq f_{+\infty}(x) - \frac{\sqrt{\mu}}{\gamma^2} \frac{1}{\sqrt{\mu_2}} \geq f_{+\infty}(x_0) - \frac{\sqrt{\mu}}{\gamma^2} \frac{1}{\sqrt{\mu_2}} > 0, \quad (3.5)$$

where the positivity property is deduced from (2.7) and from the condition on μ_2 .

For the coercivity property (3.2), we only need to show that

$$\langle (\mathcal{L}_{\mu_2} - |\omega|J_b)\nu, \nu \rangle \geq C \|\nu\|_1^2. \quad (3.6)$$

Indeed,

$$\begin{aligned} \langle (\mathcal{L}_{\mu_2} - |\omega|J_b)\nu, \nu \rangle &= \int_{\mathbb{R}} \left(\left(\frac{1}{\gamma} - |\omega| \right) \nu^2(x) - \frac{\sqrt{\mu}}{\gamma^2} \nu(x) |D| \coth(\sqrt{\mu_2}|D|) \nu(x) \right. \\ &\quad \left. + \frac{\mu}{\gamma} \nu(x) \left(a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2}|D|) \right) \partial_x^2 \nu(x) + \mu b |\omega| \nu(x) \partial_x^2 \nu(x) \right) dx. \end{aligned} \quad (3.7)$$

Next, we estimative the integral terms in (3.7). From (3.4) and Plancherel Theorem we obtain

$$- \int_{\mathbb{R}} \frac{\sqrt{\mu}}{\gamma^2} \nu(x) |D| \coth(\sqrt{\mu_2}|D|) \nu(x) \geq - \frac{\sqrt{\mu}}{\gamma^2} \frac{1}{\sqrt{\mu_2}} \|\nu\|^2 - \frac{\sqrt{\mu}}{\gamma^2} \| |D|^{1/2} \nu \|^2. \quad (3.8)$$

Now,

$$\begin{aligned} &\int_{\mathbb{R}} \nu(x) \left(a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2}|D|) \right) \partial_x^2 \nu(x) dx \\ &= - \left(a - \frac{1}{\gamma^2} \right) \|\nu'\|^2 + \frac{1}{\gamma^2} \int_{\mathbb{R}} \nu(x) [1 - \coth^2(\sqrt{\mu_2}|D|)] \partial_x^2 \nu(x) dx \\ &\geq - \left(a - \frac{1}{\gamma^2} \right) \|\nu'\|^2, \end{aligned} \quad (3.9)$$

where we have used that $\int_{\mathbb{R}} \nu(x) [1 - \coth^2(\sqrt{\mu_2}|D|)] \partial_x^2 \nu(x) dx \geq 0$.

Thus, by following a similar analysis as in the proof of (2.2) we obtain for η defined in (2.12) that

$$\begin{aligned} \langle (\mathcal{L}_{\mu_2} - |\omega|J_b)\nu, \nu \rangle &= \left(\frac{1}{\gamma} - |\omega| - \frac{\sqrt{\mu}}{\gamma^2} \frac{1}{\sqrt{\mu_2}} - \eta \frac{\sqrt{\mu}}{\gamma^2} \right) \|\nu\|^2 + \left(\alpha - \mu b |\omega| - \frac{1}{4\eta} \frac{\sqrt{\mu}}{\gamma^2} \right) \|\nu'\|^2 \\ &= r \|\nu\|^2 + \theta \|\nu'\|^2, \end{aligned} \quad (3.10)$$

where $\alpha \equiv -\frac{\mu}{\gamma}(a - \frac{1}{\gamma^2})$, $\theta > 0$ is defined in (2.14) and ϵ_0 is chosen such that

$$r = f(x_0) - \frac{\sqrt{\mu}}{\gamma^2} \frac{1}{\sqrt{\mu_2}} - \epsilon_0 > 0.$$

Thus, from the analysis above, we obtain the coercivity property of E_{μ_2} ,

$$2E_{\mu_2}(\xi, \nu) \geq \min \left\{ (1 - \gamma) - |\omega|, (1 - \gamma) \frac{|c|}{b} - |\omega| \right\} \|J_b^{1/2} \xi\|^2 + \min\{r, \theta\} \|\nu\|_1^2. \quad (3.11)$$

This ends the proof of the Proposition. \square

The proof of existence of non-trivial solutions of equation (1.7) follows the same strategy as in the case $\mu_2 = +\infty$. We use the Concentration-Compactness's Lemma 2.2 with the $H^1(\mathbb{R}) \times H^1(\mathbb{R}) - \rho_n$ type sequence defined in Section 2 to show the existence of a minimum of (1.10). The analysis for ruling-out vanishing and dichotomy follows *mutatis mutandis* the case of an infinite depth μ_2 and the more delicate part to be obtained is the key relation (2.25) for the functional E_{μ_2} .

Thus, by using the same notations as in Section 2, we have that for $(a_k, b_k) \equiv \mathbf{w}_k = (\chi_k \xi_{n_k}, \chi_k \nu_{n_k})$

$$E_{\mu_2}(\mathbf{h}_{n_k}) = E_{\mu_2}(\mathbf{w}_k) + E_{\mu_2}(\mathbf{h}_{k,1}) + E_{\mu_2}(\mathbf{h}_{k,2}) + N_5 + N_6 + N_7, \quad (3.12)$$

where for positive constants c_1, c_2, c_3, c_6 , and $c_4 < 0$,

$$\begin{aligned} N_5 &= \int_{\mathbb{R}} (\zeta_k + \varphi_k) [(2c_1 a_k - \omega b_k) \xi_{n_k} + (c_2 b_k - \omega a_k) \nu_{n_k}] \\ &\quad + ((\zeta_k \xi_{n_k})' + (\varphi_k \xi_{n_k})') (2c_3 a_k' - \omega c_4 b_k') - \omega c_4 a_k' ((\zeta_k \nu_{n_k})' + (\varphi_k \nu_{n_k})'), \end{aligned} \quad (3.13)$$

$$N_6 = -2c_6 \int_{\mathbb{R}} b_k |D| \coth(\sqrt{\mu_2} |D|) (\zeta_k \nu_{n_k} + \varphi_k \nu_{n_k}) + \zeta_k \nu_{n_k} |D| \coth(\sqrt{\mu_2} |D|) (\varphi_k \nu_{n_k}) dx, \quad (3.14)$$

and for $J(|D|) = a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2} |D|)$ and $c_7 > 0$ we have

$$N_7 = c_7 \int_{\mathbb{R}} b_k J(|D|) \partial_x^2 (\zeta_k \nu_{n_k} + \varphi_k \nu_{n_k}) + \zeta_k \nu_{n_k} J(|D|) \partial_x^2 (\varphi_k \nu_{n_k}) dx \quad (3.15)$$

and such that $N_5 = O(\epsilon)$, $N_6 = O(\epsilon)$ and $N_7 = O(\epsilon)$.

Indeed, from Cauchy-Schwarz inequality, Proposition 2.1 (item (d)) and (2.22) we obtain for R_k large that

$$|N_5| \leq C \|\mathbf{w}_k\|_{1 \times 1} \|\mathbf{h}_{n_k}\|_{1 \times 1} = O(\epsilon).$$

Now, with regard to N_6 we observe first from relation (3.4), Cauchy-Schwarz inequality, Proposition 2.1 and (2.22) that the first integral term in (3.14) can be estimated as

$$\begin{aligned} \int_{\mathbb{R}} b_k |D| \coth(\sqrt{\mu_2}|D|)(\zeta_k \nu_{n_k} + \varphi_k \nu_{n_k}) dx &= \int_{\mathbb{R}} b_k (|D| \coth(\sqrt{\mu_2}|D|) - |D|)[(\zeta_k + \varphi_k) \nu_{n_k}] \\ &+ \int_{\mathbb{R}} \nu_{n_k} (\zeta_k + \varphi_k) |D| b_k dx \\ &\leq \frac{2}{\sqrt{\mu_2}} \int_{\mathbb{R}} |b_k| |\nu_{n_k}| dx + 2 \int_{\mathbb{R}} |D| |b_k| |\nu_{n_k}| dx \leq C \|\mathbf{h}_{n_k}\|_{1 \times 1} \|\mathbf{w}_k\|_{1 \times 1} = O(\epsilon). \end{aligned} \quad (3.16)$$

Next, we define the bounded linear operator $\mathcal{T} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$\mathcal{T} = \coth(\sqrt{\mu_2}|D|) - \frac{1}{\sqrt{\mu_2}|D|}, \quad (3.17)$$

and so since $\zeta_k \varphi_k \equiv 0$ we have that the second integral term in (3.14) can be written as

$$\begin{aligned} \int_{\mathbb{R}} \zeta_k \nu_{n_k} |D| \coth(\sqrt{\mu_2}|D|)(\varphi_k \nu_{n_k}) dx &= \int_{\mathbb{R}} \zeta_k \nu_{n_k} \left[|D| \coth(\sqrt{\mu_2}|D|) - \frac{1}{\sqrt{\mu_2}} \right] (\varphi_k \nu_{n_k}) dx \\ &= \int_{\mathbb{R}} \zeta_k \nu_{n_k} \mathcal{T} |D| (\varphi_k \nu_{n_k}) dx \equiv L. \end{aligned} \quad (3.18)$$

Now, we estimate L . Since $D = \mathcal{H}\partial_x$, we have by using commutators,

$$L = \int_{\mathbb{R}} \zeta_k \nu_{n_k} \mathcal{T}(\varphi_k \mathcal{H}\nu'_{n_k}) + \zeta_k \nu_{n_k} \mathcal{T}[\mathcal{H}, \varphi_k] \nu'_{n_k} + \zeta_k \nu_{n_k} \mathcal{T}\mathcal{H}(\varphi'_k \nu_{n_k}) dx \equiv L_1 + L_2 + L_3. \quad (3.19)$$

Next, since the symbol, m , associated with the nonlocal operator \mathcal{T} ,

$$m(x) = \coth(\sqrt{\mu_2}|x|) - \frac{1}{\sqrt{\mu_2}|x|},$$

satisfies that $m \in C^\infty(\mathbb{R} - \{0\})$ and for all $x \neq 0$

$$\left| \frac{d^k}{dx^k} m(x) \right| \leq C_k \frac{1}{|x|^k}$$

with $k \geq 0$, it follows from Coifman-Meyer Theorem (see [21]) and from $\zeta_k \varphi_k \equiv 0$ that for R_k large,

$$L_1 = \int_{\mathbb{R}} \zeta_k \nu_{n_k} [\mathcal{T}, \varphi_k] \mathcal{H}\nu'_{n_k} \leq \|\nu_{n_k}\| \|[\mathcal{T}, \varphi_k] \mathcal{H}\nu'_{n_k}\| \leq C |\varphi'_k|_\infty \|\nu_{n_k}\|^2 \leq C \frac{|\varphi'|_\infty}{R_k} \|\mathbf{h}_{n_k}\|_{1 \times 1}^2 = O(\epsilon). \quad (3.20)$$

Next, from the bounded property of \mathcal{T} and Calderon commutator Theorem ([17]) we obtain for R_k large

$$L_2 \leq \|\nu_{n_k}\| \|[\mathcal{H}, \varphi_k] \nu'_{n_k}\| \leq C |\varphi'_k|_\infty \|\nu_{n_k}\|^2 = O(\epsilon). \quad (3.21)$$

Similarly, we obtain

$$L_3 \leq \|\nu_{n_k}\| \|\varphi_k \nu_{n_k}\| \leq C |\varphi'_k|_\infty \|\nu_{n_k}\|^2 = O(\epsilon). \quad (3.22)$$

Thus, from (3.20)-(3.21)-(3.22) we obtain $L = O(\epsilon)$.

Now, we estimate the term N_7 . By denoting $J_0(|D|) = 1 - \coth^2(\sqrt{\mu_2}|D|)$, we deduce that the linear operator $J_0(|D|)\partial_x^2$ is $L^2(\mathbb{R})$ -bounded since the associated symbol $(\coth^2(\sqrt{\mu_2}|y|) - 1)y^2$ is bounded on the real line, and so from Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \int_{\mathbb{R}} b_k J(|D|) \partial_x^2 (\zeta_k \nu_{n_k} + \varphi_k \nu_{n_k}) dx &= \int_{\mathbb{R}} \left(a - \frac{1}{\gamma^2}\right) b_k \partial_x^2 (\zeta_k \nu_{n_k} + \varphi_k \nu_{n_k}) dx \\ &+ \frac{1}{\gamma^2} \int_{\mathbb{R}} b_k J_0(|D|) \partial_x^2 (\zeta_k \nu_{n_k} + \varphi_k \nu_{n_k}) dx \\ &\leq C(\|b'_k\| \|\nu'_{n_k}\| + \|b_k\| \|\nu_{n_k}\|) \leq C \|\mathbf{w}_k\|_{1 \times 1} \|\mathbf{h}_{n_k}\|_{1 \times 1} = O(\epsilon). \end{aligned} \quad (3.23)$$

Next, we define the bounded linear operator $\mathcal{T}_1 : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$\mathcal{T}_1 = \coth^2(\sqrt{\mu_2}|D|) - \frac{1}{\mu_2|D|^2}, \quad (3.24)$$

with symbol m_1 , defined by $m_1(x) = \coth^2(\sqrt{\mu_2}|x|) - \frac{1}{\mu_2|x|^2}$.

Then, since $m_1 \in C^\infty(\mathbb{R} - \{0\})$ and for all $x \neq 0$

$$\left| \frac{d^k}{dx^k} m_1(x) \right| \leq C_k \frac{1}{|x|^k}$$

with $k \geq 0$, it follows from Coifman-Meyer [21] and from the fact that ζ_k and φ_k and its derivatives have disjoint support, that for R_k large enough,

$$\begin{aligned} \int_{\mathbb{R}} \zeta_k \nu_{n_k} J(|D|) \partial_x^2 (\varphi_k \nu_{n_k}) dx &= \frac{1}{\gamma^2} \int_{\mathbb{R}} \zeta_k \nu_{n_k} \left[|D|^2 \coth^2(\sqrt{\mu_2}|D|) - \frac{1}{\mu_2} \right] (\varphi_k \nu_{n_k}) dx \\ &= -\frac{1}{\gamma^2} \int_{\mathbb{R}} \zeta_k \nu_{n_k} \mathcal{T}_1 \partial_x^2 (\varphi_k \nu_{n_k}) dx = \frac{1}{\gamma^2} \int_{\mathbb{R}} (\zeta_k \nu_{n_k})' \mathcal{T}_1 (\varphi_k \nu_{n_k}) + (\zeta_k \nu_{n_k}) \mathcal{T}_1 (\varphi_k \nu'_{n_k}) dx \\ &\leq C \frac{|\varphi'|_\infty}{R_k} \|\nu_{n_k}\|_1^2 + \frac{1}{\gamma^2} \int_{\mathbb{R}} (\zeta_k \nu_{n_k})' [\mathcal{T}_1, \varphi_k] \nu'_{n_k} dx \\ &\leq C \frac{|\varphi'|_\infty}{R_k} \|\nu_{n_k}\|_1^2 + C \|\nu_{n_k}\|_1 \|\varphi'_k\| \|\nu_{n_k}\| \\ &\leq C \frac{1}{R_k} \|\mathbf{h}_{n_k}\|_{1 \times 1}^2 = O(\epsilon). \end{aligned} \quad (3.25)$$

Thus, for R_1, R_k , large enough, we have from (3.12)

$$E_{\mu_2}(\mathbf{h}_{n_k}) = E_{\mu_2}(\mathbf{h}_{k,1}) + E_{\mu_2}(\mathbf{h}_{k,2}) + O(\epsilon). \quad (3.26)$$

Therefore, we obtain that $I_\lambda \geq I_\theta + I_{\lambda-\theta}$ for some $\theta \in (0, \lambda)$ and so we obtain a contradiction with (3.3). Thus, the compactness alternative of Lions's Lemma 2.2 will imply the

existence of a minimum for the problem (1.10) and then we obtain nontrivial solutions for equation (1.7).

4. REGULARITY AND ASYMPTOTICS OF THE SOLITARY WAVES

The spatial decay of solitary waves at infinity is essentially governed by the behavior of the linear part in the solitary wave equation. Paley-Wiener type arguments show that no exponential decay is possible when the symbol of the linear operator is not smooth. One expects then an algebraic decay rate, which is the case of the Benjamin-Ono solitary wave (decaying as $1/x^2$) or of the KP I equation (see [15]). The situation should be thus quite different for the solitary waves of (1.4) depending on the value on the finiteness of μ_2 . When μ_2 is finite the nonlocal operator has a smooth symbol and one expects the exponential decay of solitary waves as it is the case for the ILW equation.

On the other hand when μ_2 is infinite the nonlocal operator involves the Hilbert transform whose symbol has a singularity at the origin and one expects an algebraic decay, as for the BO equation.

Technically following an idea in [12] one writes the equation for solitary waves as a convolution with the inverse of a linear operator whose symbol determines the decay rate of the solitary waves.

4.1. Regularity and decay of solitary wave solutions for μ_2 infinite. We will prove here that the solutions for system (2.1) have a polynomial decay. More exactly, we have the following theorem

Theorem 4.1. *Let (ξ, ν) be a solution of system (2.1) given by Theorem 1.1. Then $\xi, \nu \in H^\infty(\mathbb{R})$. Moreover, there exists a constant $D \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,*

$$x^2|\xi(x)| \leq D, \quad \text{and} \quad x^2|\nu(x)| \leq D.$$

Remark 4.1. The decay rate of the solitary wave is exactly the same as that of the (explicit) solitary wave of the Benjamin-Ono equation.

Proof. For (ξ, ν) solution of system (2.1) we have the following relation for ξ

$$(1 - \gamma)\xi = \omega J_c^{-1} J_b \nu + r J_c^{-1}(\nu^2) \tag{4.1}$$

and thus from the first equation in (2.1), we have the relation for ν

$$(1 - \gamma)\mathcal{L}_{+\infty}\nu = G(\nu) \tag{4.2}$$

with $G(\nu) = \omega^2 J_c^{-1} J_b^2 \nu + \omega r J_c^{-1} J_b(\nu^2) + 2\omega r \nu J_c^{-1} J_b \nu + 2r^2 \nu J_c^{-1}(\nu^2)$. We note that by the regularity properties of ν , $G(\nu) \in L^2(\mathbb{R})$. Therefore, by using a bootstrap argument we obtain from (4.2) that $\nu \in H^\infty(\mathbb{R})$ and so from the second equation in (2.1) follows $\xi \in H^\infty(\mathbb{R})$.

Next, equation (4.2) can be written in the convolution form

$$\nu = \frac{1}{1 - \gamma} \mathcal{L}_{+\infty}^{-1}[G(\nu)] = \frac{1}{\beta_1(1 - \gamma)} K * G(\nu) \tag{4.3}$$

where $\beta_1 = -\frac{\mu}{\gamma}(a - \frac{1}{\gamma^2})$ and the kernel K is the inverse Fourier transform of

$$\widehat{K}(y) = \frac{1}{y^2 - \ell|y| + c} \quad (4.4)$$

for $\ell = \frac{\sqrt{\mu}}{\beta_1\gamma^2}$ and $c = \frac{1}{\beta_1\gamma}$. So, via the Residue Theorem (see [6]), it follows that K is determined explicitly by

$$K(x) = -\frac{2\ell}{\sqrt{2\pi}} \int_0^{+\infty} \frac{ye^{-|x|y}}{(c-y^2)^2 + \ell^2y^2} dy + \frac{2\sqrt{2\pi}}{\sqrt{4c-\ell^2}} e^{-\frac{\sqrt{4c-\ell^2}}{2}|x|} \cos(\ell x/2). \quad (4.5)$$

We note that $4c - \ell^2 > 0$, because $a \leq 0$ and $1 - a\gamma^2 > \frac{1}{4}$. We also have that $K \in C^\infty(\mathbb{R} - \{0\})$. Now, according to the theory in Bona&Li [12] the asymptotic properties of ν satisfying (4.3) are essentially based on those of the kernel K . In the present context, since $\widehat{K} \in H^s(\mathbb{R})$ for $s > 3/2$, and from (4.5)

$$\lim_{|x| \rightarrow +\infty} x^2 K(x) = -\frac{2\ell}{c^2 \sqrt{2\pi}}.$$

Then, it follows that ν satisfies the relation

$$\lim_{|x| \rightarrow \infty} x^2 \nu(x) = D,$$

where D is a constant. Now, with regard to the polynomial decay of ξ we will see that if $f \in H^n(\mathbb{R})$, satisfies $|x^2 f(x)| \leq C$ for $x \in \mathbb{R}$, then $|x^2 J_c^{-1} f(x)| \leq C_1$ for $x \in \mathbb{R}$. Indeed, since the Fourier transform of $h(x) = \frac{1}{1-\mu c x^2}$ is given by

$$\widehat{h}(y) = \frac{\pi}{\sqrt{-\mu c}} e^{-\sqrt{-\mu c}|y|}$$

we have for $\theta = \sqrt{-\mu c}$

$$\begin{aligned} \frac{\theta}{\pi} |x^2 J_c^{-1} f(x)| &= \left| \int_{\mathbb{R}} e^{-\theta|x-y|} x^2 f(y) dy \right| \leq \int_{\mathbb{R}} e^{-\theta|x-y|} |x-y|^2 |f(y)| dy + \int_{\mathbb{R}} e^{-\theta|x-y|} y^2 |f(y)| dy \\ &\leq |f|_\infty \int_{\mathbb{R}} e^{-\theta|p|} p^2 dp + |p^2 f|_\infty \int_{\mathbb{R}} e^{-\theta|p|} dp < \infty. \end{aligned} \quad (4.6)$$

Therefore, we obtain that $|x^2 J_c^{-1}(\nu^2)(x)| \leq D_1$ for all $x \in \mathbb{R}$. Similarly, by using integration by parts we can obtain $|x^2 J_c^{-1} J_b \nu(x)| \leq D_2$ for all $x \in \mathbb{R}$. Thus, from (4.1) we have $|x^2 \xi(x)| \leq D_3$ for all $x \in \mathbb{R}$.

This achieves the proof of the Theorem. \square

4.2. Regularity and decay of solitary wave solutions for μ_2 finite. As aforementioned we expect smoothness and exponential decay of the solitary waves solutions here since the symbols of the linear dispersive parts are smooth. We proceed as in the case μ_2 infinite. More exactly, we have the following theorem.

Theorem 4.2. *Let (ξ, ν) be a solution of system (1.7) given by Theorem 1.1-(b). Then $\xi, \nu \in H^\infty(\mathbb{R})$. Moreover, there exists a constant $M \in \mathbb{R}^+$ such that for all $x \in \mathbb{R}$,*

$$e^{\sigma|x|}|\nu(x)| \leq M, \quad \text{and} \quad e^{\sigma_0|x|}|\xi(x)| \leq M,$$

with $\sigma > 0$ such that $\sigma^2 = -\frac{1}{a\mu}(1 + \frac{\mu}{\mu_2\gamma^2} - \frac{\sqrt{\mu}}{\gamma\sqrt{\mu_2}})$ and with $\sigma_0 \in (0, \sigma]$ and $\sigma_0 < \sqrt{-c\mu}$.

Proof. For (ξ, ν) solution of system (1.7) we have the following relation for ξ

$$(1 - \gamma)\xi = \omega J_c^{-1} J_b \nu + \frac{\epsilon}{2\gamma} J_c^{-1}(\nu^2) \equiv G_0(\nu). \quad (4.7)$$

Thus, we deduce from (1.7) that ν should satisfy the equation

$$(1 - \gamma)\mathcal{L}_{\mu_2}\nu = \frac{\epsilon}{2\gamma}\nu G_0(\nu) + \omega J_b G_0(\nu) \equiv G_1(\nu). \quad (4.8)$$

Now, since $\nu \in H^1(\mathbb{R})$ it follows immediately that $G_1(\nu) \in L^2(\mathbb{R})$. Next we show that $\nu \in H^2(\mathbb{R})$. Actually, from the definition for \mathcal{L}_{μ_2} in (1.5) we can split it in two operators \mathcal{M}, \mathcal{N} , such that $\gamma\mathcal{L}_{\mu_2} = \mathcal{M} + \mathcal{N}$ with

$$\mathcal{M} = 1 - \frac{\sqrt{\mu}}{\gamma\sqrt{\mu_2}} - \frac{\sqrt{\mu}}{\gamma}|D| \left[\coth(\sqrt{\mu_2}|D|) - \frac{1}{\sqrt{\mu_2}|D|} \right], \quad (4.9)$$

and

$$\mathcal{N} = \frac{\mu}{\gamma^2} \left[\coth^2(\sqrt{\mu_2}|D|) - \frac{1}{\mu_2|D|^2} - a\gamma^2 \right] |D|^2 + \frac{\mu}{\mu_2\gamma^2}. \quad (4.10)$$

Now, from the boundedness of $\mathcal{T} = \coth(\sqrt{\mu_2}|D|) - \frac{1}{\sqrt{\mu_2}|D|}$ on $L^2(\mathbb{R})$ it follows from (4.9) that $\mathcal{M}\nu \in L^2(\mathbb{R})$. By defining $\mathcal{T}_2 = \frac{\mu}{\gamma^2} \left[\coth^2(\sqrt{\mu_2}|D|) - \frac{1}{\mu_2|D|^2} - a\gamma^2 \right] |D|^2$ we obtain from (4.8) that

$$\mathcal{T}_2\nu = \frac{\gamma}{1 - \gamma} G_1(\nu) - \mathcal{M}\nu - \frac{\mu}{\mu_2\gamma^2}\nu \in L^2(\mathbb{R}). \quad (4.11)$$

Thus, since $\coth^2(\sqrt{\mu_2}|x|) - \frac{1}{\mu_2|x|^2} - a\gamma^2 \geq -a\gamma^2$, we obtain from $a < 0$ the relation $\|\mathcal{T}_2\nu\|^2 \geq \mu^2 a^2 \int |y|^4 |\hat{\nu}(y)|^2 dy$ and therefore $\nu \in H^2(\mathbb{R})$. Thus, a bootstrap argument applied to equation (4.8) implies $\nu \in H^\infty(\mathbb{R})$ and from (4.1) follows immediately that $\xi \in H^\infty(\mathbb{R})$.

Next, we prove the exponential decay of the profile ν . Indeed, from (4.8) follows the relation

$$-a\mu(|D|^2 + \sigma^2)\nu = \frac{\gamma}{1 - \gamma} G_1(\nu) + \frac{\sqrt{\mu}}{\gamma}|D|\mathcal{T}\nu - \frac{\mu}{\gamma^2}|D|^2\mathcal{T}_1\nu \equiv G_2(\nu) \in L^2(\mathbb{R}), \quad (4.12)$$

where \mathcal{T}_1 was defined in (3.24) and $\sigma > 0$ is such that $\sigma^2 = -\frac{1}{a\mu}(1 + \frac{\mu}{\mu_2\gamma^2} - \frac{\sqrt{\mu}}{\gamma\sqrt{\mu_2}})$ (we note from Theorem 1.1-(b) that $1 > \frac{\sqrt{\mu}}{\gamma\sqrt{\mu_2}}$). Thus, from (4.12) we obtain

$$\nu = -\frac{1}{a\sigma\mu}K_1 \star G_2(\nu), \quad (4.13)$$

where K_1 is the inverse Fourier transform of

$$\widehat{K_1}(y) = \frac{\sigma}{\sigma^2 + y^2}.$$

Namely, $K_1(x) = \pi e^{-\sigma|x|}$. Thus, from Bona&Li in [12] follows the relation

$$\lim_{|x| \rightarrow \infty} e^{\sigma|x|}\nu(x) = C, \quad (4.14)$$

where C is a constant.

Now, with regard to the exponential decay of ξ in (4.7) we have that if $g \in H^n(\mathbb{R})$, satisfies $|e^{\sigma|x|}g(x)| \leq C_1$ for $x \in \mathbb{R}$, then $|e^{\sigma_0|x|}J_c^{-1}g(x)| \leq C_2$ for $x \in \mathbb{R}$ with $\sigma_0 \in (0, \sigma]$ and $\sigma_0 < \sqrt{-c\mu}$. Indeed, for $\theta = \sqrt{-c\mu}$ we have

$$\begin{aligned} \frac{\theta}{\pi}|e^{\sigma_0|x|}J_c^{-1}g(x)| &\leq \int_{\mathbb{R}} e^{(\sigma_0-\theta)|x-y|}e^{\sigma_0|y|}|g(y)|dy \\ &\leq \sup_{y \in \mathbb{R}}|e^{\sigma_0|y|}g(y)| \int_{\mathbb{R}} e^{(\sigma_0-\theta)|p|}dp < \infty. \end{aligned} \quad (4.15)$$

Therefore, we obtain that $|e^{\sigma_0|x|}J_c^{-1}(\nu^2)(x)| \leq D_1$ and $|e^{\sigma_0|x|}J_c^{-1}(J_b\nu)(x)| \leq D_2$ for all $x \in \mathbb{R}$ (we note from (4.13) that $-a\sigma\mu J_b\nu = K_1 \star (J_b G_2(\nu))$ and so $|e^{\sigma|x|}J_b\nu(x)| \leq C_4$ for $x \in \mathbb{R}$). Thus, from (4.7) we have $|e^{\sigma_0|x|}\xi(x)| \leq D_3$ for all $x \in \mathbb{R}$.

This achieves the proof of the Theorem. \square

5. EXISTENCE OF SOLITARY WAVES SOLUTIONS FOR BO SYSTEMS

In this section we show the existence of even solitary waves solutions for the Benjamin-Ono system (henceforth BO-systems)

Solitary waves solutions for (1.14), that is of the form

$$\zeta(x, t) = \xi(x - ct), \quad v(x, t) = \nu(x - ct), \quad c \in \mathbb{R}$$

and (ξ, ν) vanish at infinity, it will satisfy the systems

$$\begin{cases} -c\mathcal{D}\xi + \mathcal{B}\nu = \frac{\epsilon}{\gamma}\xi\nu, \\ \xi = \frac{1}{1-\gamma}\left(c\nu + \frac{\epsilon}{2\gamma}\nu^2\right). \end{cases} \quad (5.1)$$

Our approach of the existence of a smooth curve $c \in (-\delta, \delta) \rightarrow (\xi_c, \nu_c)$ of solutions for (5.1) will be based on the Implicit Function Theorem.

Indeed, for $s \geq 0$, let $H_e^s(\mathbb{R})$ denote the closed subspace of all even functions in $H^s(\mathbb{R})$. Our existence theorem is the following,

Theorem 5.1. *Let $\gamma \in (0, 1)$. Then there exists $\delta > 0$ such that for $c \in (-\delta, \delta)$, equation (5.1) has a solution $(\xi_c, \nu_c) \in H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$, and the map $c \in (-\delta, \delta) \rightarrow (\xi_c, \nu_c)$ is smooth. In particular, $(\xi_c(x), \nu_c(x))$ converges to $(\xi_0(x), \nu_0(x))$ as $c \rightarrow 0$, uniformly for $x \in \mathbb{R}$, where ν_0 is defined by the unique (modulo translations) even and positive solution of*

$$\alpha D\nu_0 + \frac{1}{\gamma}\nu_0 - \eta\nu_0^3 = 0, \quad (5.2)$$

and $\xi_0 = \frac{1}{1-\gamma}\frac{\epsilon}{2\gamma}\nu_0^2$. Here, $\alpha = \frac{\beta-1}{\gamma^2}\sqrt{\mu}$ and $\eta = \frac{\epsilon^2}{2\gamma^2(1-\gamma)}$.

Proof. Let $X_e = H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$ and define a map $G : \mathbb{R} \times X_e \rightarrow L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R})$ by

$$G(c, \xi, \nu) = (-c\mathcal{D}\xi + \mathcal{B}\nu - \frac{\epsilon}{\gamma}\xi\nu, -c\nu + (1-\gamma)\xi - \frac{\epsilon}{2\gamma}\nu^2).$$

Then, from [26], we obtain that there is a unique even solution (ξ_0, ν_0) of equation $G(0, \xi_0, \nu_0) = 0$, with ν_0 satisfying (5.2) and ξ_0, ν_0 positive. Next, a calculation shows that the Fréchet derivative $G_{(\xi, \nu)} = \partial G(c, \xi, \nu) / \partial(\xi, \nu)$ exists on $\mathbb{R} \times X_e$ and is defined as a map from $\mathbb{R} \times X_e$ to $B(X_e; L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R}))$ by

$$G_{(\xi, \nu)}(c, \xi, \nu) = \begin{pmatrix} -c\mathcal{D} - \frac{\epsilon}{\gamma}\nu & \mathcal{B} - \frac{\epsilon}{\gamma}\xi \\ 1 - \gamma & -\frac{\epsilon}{\gamma}\nu - c \end{pmatrix}$$

Moreover the linear operator $\mathcal{T}_0 = G_{(\xi, \nu)}(0, \xi_0, \nu_0)$ with domain $D(\mathcal{T}_0) = H^1(\mathbb{R}) \times H^1(\mathbb{R})$ has a one-dimensional kernel, $Ker(\mathcal{T}_0)$, generated by $(\xi'_0, \nu'_0)^t$. Indeed, by considering $c = 0$ in (5.1) we have that (ξ'_0, ν'_0) satisfies

$$\mathcal{B}\nu'_0 = \frac{\epsilon}{\gamma}(\xi'_0\nu_0 + \xi_0\nu'_0), \quad \text{and,} \quad \xi'_0 = \frac{1}{1-\gamma}\frac{\epsilon}{\gamma}\nu_0\nu'_0,$$

and hence $(\xi'_0, \nu'_0)^t \in Ker(\mathcal{T}_0)$.

Next, suppose $(\phi, \psi)^t \in Ker(\mathcal{T}_0)$. Then, since ϕ satisfies

$$\phi = \frac{\epsilon}{\gamma(1-\gamma)}\nu_0\psi, \quad (5.3)$$

we deduce that ψ belongs to the kernel of the linear operator

$$\mathcal{M}_0 = \alpha D + \frac{1}{\gamma} - 3\eta\nu_0^2.$$

Thus, from [26] we have that $Ker(\mathcal{M}_0) = [\nu'_0]$ and the number of negative eigenvalues of \mathcal{M}_0 is exactly one. Then, $\psi = \theta\nu'_0$ and from (5.3) $\phi = \theta\xi_0$.

Now, since $(\xi'_0, \nu'_0) \notin X_e$ we obtain that $\mathcal{T}_0 : X_e \rightarrow L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R})$ is invertible. Moreover, since G and $G_{(\xi, \nu)}$ are smooth maps on their domains, we have from the Implicit Function Theorem that there exist a number $\delta > 0$ and a smooth map $c \in (-\delta, \delta) \rightarrow (\xi_c, \nu_c) \in X_e$ such that $G(c, \xi_c, \nu_c) = 0$ for all $c \in (-\delta, \delta)$. This proves the theorem. \square

Theorem 5.1 shows the existence of a smooth curve of solutions for (5.1) bifurcating from the “positive” profile (ξ_0, ν_0) with ν_0 being the positive solution for (5.2). Since $-\nu_0$ is the negative solution associated to (5.2), a similar analysis to that used in the proof of Theorem 5.1 shows the following existence result of solitary waves solutions for (5.1).

Theorem 5.2. *Let $\gamma \in (0, 1)$. Then there exists $\delta_1 > 0$ such that for $c \in (-\delta_1, \delta_1)$, equation (5.1) has a solution $(\xi_{c,1}, \nu_{c,1}) \in H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$, and the correspondence $c \in (-\delta_1, \delta_1) \rightarrow (\xi_{c,1}, \nu_{c,1})$ define a smooth map. In particular, for $c \rightarrow 0$, $(\xi_{c,1}(x), \nu_{c,1}(x))$ converges to $(\xi_0(x), -\nu_0(x))$, uniformly for $x \in \mathbb{R}$, where ν_0 is the unique (modulo translations) even and positive solution of (5.2) and*

$$\xi_0 = \frac{1}{1-\gamma} \frac{\epsilon}{2\gamma} \nu_0^2.$$

Moreover, from Theorem 5.2 we obtain that $\delta_1 = \delta$ and

$$(\xi_{c,1}, \nu_{c,1}) = (\xi_c, -\nu_c), \quad \text{for } c \in (-\delta, \delta).$$

We note that Theorems 5.1 and 5.2 show the bidirectional nature of the system (1.14).

6. EXISTENCE OF SOLITARY WAVES SOLUTIONS FOR ILW SYSTEMS

In this section we show the existence of even solitary waves solutions for the Intermediate Long Wave systems (henceforth ILW-systems)

Solitary waves solutions for (1.12), that is of the form

$$\zeta(x, t) = \xi(x - ct), \quad v(x, t) = \nu(x - ct), \quad c \in \mathbb{R}$$

and (ξ, ν) vanish at infinity, it will satisfy the systems

$$\begin{cases} -c\mathcal{W}\xi + \mathcal{Z}\nu = \frac{\epsilon}{\gamma}\xi\nu, \\ \xi = \frac{1}{1-\gamma} \left(c\nu + \frac{\epsilon}{2\gamma}\nu^2 \right). \end{cases} \quad (6.1)$$

Our approach of the existence of a smooth curve $c \in (-\delta, \delta) \rightarrow (\xi_c, \nu_c)$ of solutions for (5.1) will be based again in the Implicit Function Theorem. First, we prove that for $c = 0$, system (6.1) has a smooth curve $\mu_2 \rightarrow (\xi_{0,\mu_2}, \nu_{0,\mu_2}) \in H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$ with μ_2 sufficiently large.

Theorem 6.1. *Let $\gamma \in (0, 1)$ and $c = 0$ in (6.1). Then there exists $\sigma > 0$ sufficiently large such that for $\mu_2 \in (\sigma, +\infty)$, equation (6.1) has a solution $(\xi_{0,\mu_2}, \nu_{0,\mu_2}) \in H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$, and the correspondence $\mu_2 \in (\sigma, +\infty) \rightarrow (\xi_{0,\mu_2}, \nu_{0,\mu_2})$ define a smooth map. In particular, for $\mu_2 \rightarrow +\infty$, $(\xi_{0,\mu_2}(x), \nu_{0,\mu_2}(x))$ converges to $(\xi_0(x), \nu_0(x))$, uniformly for $x \in \mathbb{R}$, where ξ_0, ν_0 are defined in Theorem 5.1.*

Proof. Let $X_e = H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$ and define a map $H : \mathbb{R} \times X_e \rightarrow L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R})$ by

$$P(a, \xi, \nu) = \left(\frac{1}{\gamma}\nu - \frac{\epsilon}{\gamma}\xi\nu + \alpha|a|\tau_{|a|}[|D|\coth(|D|)\tau_{\frac{1}{|a|}}\nu], (1-\gamma)\xi - \frac{\epsilon}{2\gamma}\nu^2 \right),$$

where τ_d is the linear dilation operator, $\tau_d f(x) \equiv f(dx)$. For $d = 0$, $\tau_{\frac{1}{d}} f \equiv 0$. We note from the relation,

$$0 \leq x \coth\left(\frac{1}{d}x\right) - x \leq d, \quad \text{for all } x \geq 0 \text{ and } d \geq 0 \quad (6.2)$$

that $|D|\coth(|D|)v \in L^2(\mathbb{R})$ for $v \in H^1(\mathbb{R})$ and

$$|a|\tau_{|a|}[|D|\coth(|D|)\tau_{\frac{1}{|a|}}\nu] \rightarrow |D|\nu, \quad \text{as } a \rightarrow 0$$

in $L^2(\mathbb{R})$, because of the relation

$$|a|\tau_{|a|}[|D|\coth(|D|)\tau_{\frac{1}{|a|}}\nu] = |D|\coth\left(\frac{1}{|a|}|D|\right)\nu \quad (6.3)$$

Then from [26], we obtain that there is a unique even solution (ξ_0, ν_0) of equation

$$(0, 0) = P(0, \xi_0, \nu_0) = \left(\frac{1}{\gamma}\nu_0 - \frac{\epsilon}{\gamma}\xi_0\nu_0 + \alpha|D|\nu_0, (1 - \gamma)\xi_0 - \frac{\epsilon}{2\gamma}\nu_0^2\right),$$

with ν_0 satisfying (5.2) and ξ_0, ν_0 positive.

Next, we calculate the Fréchet derivative $\mathcal{J}(a, \xi, \nu) = \partial P(a, \xi, \nu)/\partial(\xi, \nu)$ which is a map from $\mathbb{R} \times X_e$ to $B(X_e; L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R}))$

$$\mathcal{J}(a, \xi, \nu) = \begin{pmatrix} -\frac{\epsilon}{\gamma}\nu & \frac{1}{\gamma} - \frac{\epsilon}{\gamma}\xi + \alpha|a|\tau_{|a|}[|D|\coth(|D|)\tau_{\frac{1}{|a|}}] \\ 1 - \gamma & -\frac{\epsilon}{\gamma}\nu \end{pmatrix}$$

and so the linear operator $\mathcal{J}_0 = \mathcal{J}(0, \xi_0, \nu_0)$ defined with domain $H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$ has a trivial kernel (see proof of Theorem 5.2). Therefore, $\mathcal{J}_0 : X_e \rightarrow L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R})$ is invertible. Moreover, since P and \mathcal{J} are continuous maps on their domains, we have from the Implicit Function Theorem that there exist a number $\sigma_1 > 0$ and a continuous map $a \in (-\sigma_1, \sigma_1) \rightarrow (\xi_{0,a}, \nu_{0,a}) \in X_e$ such that $P(a, \xi_{0,a}, \nu_{0,a}) = 0$ for all $a \in (-\sigma_1, \sigma_1)$. We note that since P and \mathcal{J} are smooth maps on the domain $(0, +\infty)$, then $a \in (0, \sigma_1) \rightarrow (\xi_{0,a}, \nu_{0,a}) \in X_e$ is also a smooth map.

Now, from relation (6.3) we obtain for $a = \frac{1}{\sqrt{\mu_2}}$ and $\mu_2 > \sigma$ that $(\xi_{0,a}, \nu_{0,a}) \equiv (\xi_{0,\mu_2}, \nu_{0,\mu_2})$ satisfies (6.1) with $c = 0$. This shows the theorem. \square

Next, we show the existence of a smooth curve of solution for (6.1) depending of the velocity c .

Theorem 6.2. *Let $\gamma \in (0, 1)$ and μ_2 sufficiently large. Then there exists $\eta > 0$ such that for $c \in (-\eta, \eta)$, equation (6.1) has a solution $(\xi_{c,\mu_2}, \nu_{c,\mu_2}) \in H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$, and the correspondence $c \in (-\eta, \eta) \rightarrow (\xi_{c,\mu_2}, \nu_{c,\mu_2})$ define a smooth map. In particular, for $c \rightarrow 0$, $(\xi_{c,\mu_2}(x), \nu_{c,\mu_2}(x))$ converges to $(\xi_{0,\mu_2}(x), \nu_{0,\mu_2}(x))$, uniformly for $x \in \mathbb{R}$, where $(\xi_{0,\mu_2}, \nu_{0,\mu_2})$ is the solution of (6.1) with $c = 0$ and defined by Theorem 6.1.*

Proof. Let $X_e = H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$ and define a map $H : \mathbb{R} \times X_e \rightarrow L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R})$ by

$$Q(c, \xi, \nu) = (-c(1 + g(D))\xi + \frac{1}{\gamma}\nu - \frac{\epsilon}{\gamma}\xi\nu + \alpha|D|\coth(\sqrt{\mu_2}|D|)\nu, (1 - \gamma)\xi - c\nu - \frac{\epsilon}{2\gamma}\nu^2).$$

From Theorem 6.1 we obtain $Q(0, \xi_{0,\mu_2}, \nu_{0,\mu_2}) = 0$. Next, the Fréchet derivative $\mathcal{Z}(c, \xi, \nu) = \partial Q(c, \xi, \nu)/\partial(\xi, \nu)$ which is a map from $\mathbb{R} \times X_e$ to $B(X_e; L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R}))$ is given by

$$\mathcal{Z}(c, \xi, \nu) = \begin{pmatrix} -c(1 + g(D)) - \frac{\epsilon}{\gamma}\nu & \frac{1}{\gamma} - \frac{\epsilon}{\gamma}\xi + \alpha|D|\coth(\sqrt{\mu_2}|D|) \\ 1 - \gamma & -c - \frac{\epsilon}{\gamma}\nu \end{pmatrix}$$

Next, we will see that $\mathcal{Z}_{\mu_2} \equiv \mathcal{Z}(0, \xi_{0,\mu_2}, \nu_{0,\mu_2})$ has a trivial kernel on X_e for μ_2 sufficiently large. Indeed, initially we have that on $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, $(\xi'_{0,\mu_2}, \nu'_{0,\mu_2}) \in \text{Ker}(\mathcal{Z}_{\mu_2})$ for all μ_2 . Now, we show that \mathcal{Z}_{μ_2} converges to \mathcal{T}_0 , as $\mu_2 \rightarrow +\infty$, in the topology of generalized convergence on the set \mathcal{C} of all closed operators on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, namely, for $\widehat{\delta}(S, T)$ denoting a *metric gap* between the closed operators S and T with $D(S), D(T) \subset L^2(\mathbb{R}) \times L^2(\mathbb{R})$ (see the Appendix), we have

$$\lim_{\mu_2 \rightarrow +\infty} \widehat{\delta}(\mathcal{Z}_{\mu_2}, \mathcal{T}_0) = 0. \quad (6.4)$$

Indeed, for

$$\mathcal{V}_{\mu_2} = \begin{pmatrix} -\frac{\epsilon}{\gamma}\nu_{0,\mu_2} & -\frac{\epsilon}{\gamma}\xi_{0,\mu_2} \\ 1 - \gamma & -\frac{\epsilon}{\gamma}\nu_{0,\mu_2} \end{pmatrix}, \quad \mathcal{V}_{+\infty} = \begin{pmatrix} -\frac{\epsilon}{\gamma}\nu_0 & -\frac{\epsilon}{\gamma}\xi_0 \\ 1 - \gamma & -\frac{\epsilon}{\gamma}\nu_0 \end{pmatrix}$$

and

$$\mathcal{L}_{\mu_2} = \begin{pmatrix} 0 & \alpha|D|\coth(\sqrt{\mu_2}|D|) - \alpha|D| \\ 0 & 0 \end{pmatrix}, \quad \mathcal{S}_{+\infty} = \begin{pmatrix} 0 & \mathcal{B} \\ 0 & 0 \end{pmatrix}$$

with \mathcal{B} defined in (1.15), we have $\mathcal{Z}_{\mu_2} = \mathcal{L}_{\mu_2} + \mathcal{S}_{+\infty} + \mathcal{V}_{\mu_2}$ and $\mathcal{T}_0 = \mathcal{S}_{+\infty} + \mathcal{V}_{+\infty}$. Then, from the relations (a)-(b) in Theorem 10.1 we have

$$\begin{aligned} \widehat{\delta}(\mathcal{Z}_{\mu_2}, \mathcal{T}_0) &= \widehat{\delta}(\mathcal{L}_{\mu_2} + \mathcal{S}_{+\infty} + \mathcal{V}_{\mu_2}, \mathcal{S}_{+\infty} + (\mathcal{V}_{+\infty} - \mathcal{V}_{\mu_2}) + \mathcal{V}_{\mu_2}) \\ &\leq 2(1 + \|\mathcal{V}_{\mu_2}\|_{B(L^2)}^2) \widehat{\delta}(\mathcal{L}_{\mu_2} + \mathcal{S}_{+\infty}, \mathcal{S}_{+\infty} + (\mathcal{V}_{+\infty} - \mathcal{V}_{\mu_2})) \\ &\leq 2(1 + \|\mathcal{V}_{\mu_2}\|_{B(L^2)}^2) [\widehat{\delta}(\mathcal{L}_{\mu_2} + \mathcal{S}_{+\infty}, \mathcal{S}_{+\infty}) + \widehat{\delta}(\mathcal{S}_{+\infty}, \mathcal{S}_{+\infty} + (\mathcal{V}_{+\infty} - \mathcal{V}_{\mu_2}))] \\ &\leq 2(1 + \|\mathcal{V}_{\mu_2}\|_{B(L^2)}^2) [\|\mathcal{L}_{\mu_2}\|_{B(L^2)} + \|\mathcal{V}_{\mu_2} - \mathcal{V}_{+\infty}\|_{B(L^2)}] \rightarrow 0 \end{aligned} \quad (6.5)$$

as $\mu_2 \rightarrow +\infty$, where we are used Theorem 6.1 for obtaining

$$\lim_{\mu_2 \rightarrow +\infty} \|\mathcal{V}_{\mu_2} - \mathcal{V}_{+\infty}\|_{B(L^2)} = 0,$$

and inequality (6.2) for obtaining via Plancherel's Theorem that

$$\begin{aligned} \|\mathcal{L}_{\mu_2}(f, g)^t\|^2 &= \alpha \int_{\mathbb{R}} \|\xi\| \coth(\sqrt{\mu_2}|\xi|) - \|\xi\|^2 |\widehat{g}(\xi)|^2 d\xi \\ &\leq \frac{\alpha}{\mu_2} \|g\|^2 \rightarrow 0 \end{aligned} \tag{6.6}$$

as $\mu_2 \rightarrow +\infty$.

Hence the results of section IV-4 of Kato [27] imply that the eigenvalues of \mathcal{Z}_{μ_2} depend continuously on μ_2 . In particular, since zero is a simple eigenvalue of \mathcal{T}_0 , the eigenvalue zero with eigenfunction $(\xi'_{0,\mu_2}, \nu'_{0,\mu_2})$ is simple for \mathcal{Z}_{μ_2} and all μ_2 sufficiently large. Also, for these values of μ_2 , zero is not a eigenvalue of \mathcal{Z}_{μ_2} in X_e , and therefore the mapping $\mathcal{Z}_{\mu_2} : X_e \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})$ is invertible. Moreover, since Q and $\mathcal{Z}_{(c,\xi,\nu)}$ are smooth maps on their domains, we have from the Implicit Function Theorem that there exist a number $\eta > 0$ and a smooth map $c \in (-\eta, \eta) \rightarrow (\xi_{c,\mu_2}, \nu_{c,\mu_2}) \in X_e$ such that $Q(c, \xi_{c,\mu_2}, \nu_{c,\mu_2}) = 0$ for all $c \in (-\eta, \eta)$ and μ_2 sufficiently large. This proves the theorem. \square

7. PROPERTIES OF THE ILW AND BO SYSTEMS SOLITARY WAVES

Similarly to the case of Boussinesq-Full dispersion systems we establish here qualitative properties of the solitary wave solutions to the ILW and BO systems.

Concerning the smoothness of the solitary wave solutions of the ILW and BO systems, one cannot apparently implement a bootstrap argument from (5.1) or (6.1). Since the solitary wave solutions to both the BO and ILW equation are smooth ($H^\infty(\mathbb{R})$), one has instead to apply the implicit function theorem in the Sobolev space $H^n(\mathbb{R})$ where n is arbitrary large.

On the other hand, the proof of the decay properties of the solitary waves follows the lines of the proof of the corresponding properties for the Boussinesq-Full dispersion solitary waves (see Section 4) yielding that the solitary waves of the ILW system decay exponentially while those of the BO system have a algebraic decay $1/x^2$. We give some details below.

7.1. Decay of solitary wave solutions for the BO system. We will prove that the solutions for BO system (5.1) have a polynomial decay. More exactly, we have the following result.

Theorem 7.1. *Let (ξ, ν) be a solution of system (5.1) given by Theorem 5.1 (or by Theorem 5.2). Then there exists a constant $N \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,*

$$x^2 |\xi(x)| \leq N, \quad \text{and} \quad x^2 |\nu(x)| \leq N.$$

Proof. If (ξ, ν) is a solution of system (5.1) we have the following relation for ν

$$(\alpha + |D|)\nu = \gamma\alpha(P(\nu) + cH(\nu)) \equiv G_3(\nu) \in L^2(\mathbb{R}) \tag{7.1}$$

where $\alpha = \frac{\gamma}{(\beta-1)\sqrt{\mu}} > 0$, $P(\nu) = \frac{\epsilon}{\gamma(1-\gamma)}\left(c\nu + \frac{\epsilon}{2\gamma}\nu^2\right)$ and $H(\nu) = \frac{1}{1-\gamma}\mathcal{D}\left(c\nu + \frac{\epsilon}{2\gamma}\nu^2\right)$. Hence, equation (7.1) can be written in convolution form

$$\nu = K_2 * G_3(\nu) \quad (7.2)$$

where the kernel K_2 is the inverse Fourier transform of

$$\widehat{K}_2(y) = \frac{1}{|y| + \alpha} \quad (7.3)$$

So, via the Residue Theorem, it follows that K_2 is determined explicitly by

$$K_2(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{+\infty} \frac{ye^{-|x|y}}{\alpha^2 + y^2} dy, \quad (7.4)$$

and consequently

$$\lim_{|x| \rightarrow +\infty} x^2 K_2(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{\alpha^2}.$$

Then, it follows that ν satisfies the relation

$$\lim_{|x| \rightarrow \infty} x^2 \nu(x) = D_1,$$

where D_1 is a constant. Moreover, from the second equation in (5.1) follows immediately that $x^2|\xi(x)| \leq N$ for all $x \in \mathbb{R}$. This finishes the theorem. \square

7.2. Decay of solitary wave solutions for the ILW system. We will prove that the solutions for ILW system (6.1) have an exponential decay. More exactly, we have the following result.

Theorem 7.2. *Let (ξ, ν) be a solution of system (6.1) given by Theorem 6.2. Then there are positive constants $\eta_1, N_1 > 0$ such that for all $x \in \mathbb{R}$,*

$$e^{\eta_1|x|}|\xi(x)| \leq N_1, \quad \text{and} \quad e^{\eta_1|x|}|\nu(x)| \leq N.$$

Proof. For $(\xi, \nu) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ solution of system (6.1) we will see that $\mathcal{W}\xi \in L^2(\mathbb{R})$. Indeed, from the relation

$$\mathcal{W}\xi = \left(1 + \frac{\beta\sqrt{\mu}}{\gamma\sqrt{\mu_2}}\right)\xi + \frac{\beta\sqrt{\mu}}{\gamma}\mathcal{T}|D|\xi$$

where $\mathcal{T} = \coth(\sqrt{\mu_2}|D|) - \frac{1}{\sqrt{\mu_2}|D|}$ is a bounded operator on $L^2(\mathbb{R})$, we obtain immediately that $\mathcal{W}\xi \in L^2(\mathbb{R})$. Moreover, from (6.1) follows that $\gamma\mathcal{Z}\nu = G_4(\nu) \in L^2(\mathbb{R})$. Thus, we obtain the relation

$$\nu = K_3 * G_4(\nu) \quad (7.5)$$

where the kernel K_3 is the inverse Fourier transform of

$$\widehat{K}_3(y) = \frac{1}{1 + \frac{\beta-1}{\gamma}\sqrt{\mu}|y|\coth(\sqrt{\mu_2}|y|)} = \theta \frac{\sinh(\sqrt{\mu_2}y)}{\sqrt{\mu_2}y \cosh(\sqrt{\mu_2}y) + \theta \sinh(\sqrt{\mu_2}y)}. \quad (7.6)$$

where $\theta = \frac{\gamma\sqrt{\mu_2}}{(\beta-1)\sqrt{\mu}} > 0$. Now, in order to find K_3 we define the following function h_θ ,

$$\widehat{h}_\theta(y) = \frac{\sinh(y)}{y\cosh(y) + \theta\sinh(y)}. \quad (7.7)$$

Then, the meromorphic function on the right hand-side of (7.7) has countably many poles located at point $y = i\eta$ where $\eta \in \mathbb{R}$ and $\eta \neq 0$ satisfy the relation $-\eta = \theta \tan \eta$. Thus an application of the residue theorem implies that its inverse Fourier transform can be expressed as

$$h_\theta(x) = \sqrt{2\pi} \sum_{m=1}^{\infty} \frac{\tan \eta_m}{\eta_m \tan \eta_m - \theta - 1} e^{-\eta_m |x|} \quad (7.8)$$

where $\{i\eta_m\}_{m \geq 1}$ comprise the poles of \widehat{h}_θ on the positive imaginary axis, numbered so that $\eta_m < \eta_{m+1}$ for $m = 1, 2, 3, \dots$. More exactly, $\eta_m \in (\frac{(2m-1)}{2}\pi, m\pi)$.

Now, from the relation $K_3(x) = \theta \frac{1}{\sqrt{\mu_2}} h_\theta(\frac{x}{\sqrt{\mu_2}})$ we obtain from (7.8) for any σ with $0 < \sigma \leq \eta_1$ that

$$\lim_{|x| \rightarrow +\infty} e^{\sigma|x|} K_3(x) = \frac{\sqrt{2\pi}\gamma}{(\beta-1)\sqrt{\mu}} \frac{\tan \eta_1}{\eta_1 \tan \eta_1 - \theta - 1}$$

Then, from (7.5) follows that ν satisfies the relation

$$\lim_{|x| \rightarrow \infty} e^{\eta_1|x|} \nu(x) = D_2,$$

where D_2 is a constant. Moreover, from the second equation in (6.1) follows immediately $e^{\eta_1|x|} |\xi(x)| \leq N_1$ for all $x \in \mathbb{R}$. This finishes the theorem. \square

8. CONCLUSION AND OPEN PROBLEMS

We have establish the existence of solitary wave solutions for the Boussinesq-Full Dispersion, Intermediate Long Wave and Benjamin-Ono systems for a rather restricted range of parameters. We indicate below some related open questions that will be addressed in subsequent works.

- Complete the Cauchy theory for the B-FD systems, in particular long time existence issues.
- Existence in other ranges of parameters for the existence of solitary waves for the B-FD, BO and ILW systems.
- Non existence of solitary waves for some ranges of parameters.
- Stability (including transverse stability) issues.

9. APPENDIX A

In this Appendix we state and prove the global existence of small solutions for the one-dimensional Hamiltonian ($b = d$) system (1.1) for some specific choices of the coefficients, inspired by a similar result for the classical Boussinesq systems (see [9] section 4.2).

We thus consider the one-dimensional version of (1.1) where we skip the underscore β , that is

$$\begin{cases} (1 - \mu b \partial_x^2) \partial_t \zeta + \frac{1}{\gamma} ((1 - \epsilon \zeta) v)_x \\ - \frac{\sqrt{\mu}}{\gamma^2} |D| \coth(\sqrt{\mu_2} |D|) v_x + \frac{\mu}{\gamma} (a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2} |D|) v_{xxx}) = 0, \\ (1 - \mu d \partial_x^2) \partial_t v + (1 - \gamma) \zeta_x - \frac{\epsilon}{\gamma} v v_x + \mu c (1 - \gamma) \zeta_{xxx} = 0, \end{cases} \quad (9.1)$$

leading to the Hamiltonian

$$\begin{aligned} H(\zeta, v) = \int_{\mathbb{R}} & \left(\frac{1 - \gamma}{2} \zeta^2 + \frac{1}{2} |v|^2 - \frac{\epsilon}{2\gamma} \zeta v^2 - \frac{\mu c}{2} (1 - \gamma) |\zeta_x|^2 \right. \\ & \left. - \frac{a\mu}{2\gamma} |v_x|^2 + \frac{\sqrt{\mu}}{2\gamma^2} |L_1^{1/2} v|^2 + \frac{\mu}{2\gamma^3} |L_2^{1/2} v|^2 \right), \end{aligned} \quad (9.2)$$

Throughout this Appendix we will assume that

$$b = d > 0, \quad a \leq 0, \quad c < 0.$$

As in [9] we start with the non degenerate case

$$b = d > 0, \quad a < 0, \quad c < 0. \quad (9.3)$$

Theorem 9.1. *Assume that (9.3) holds and suppose that $(\zeta_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$ is such that*

$$|H(\zeta_0, v_0)| < \frac{\gamma^2 (1 - \gamma) \sqrt{\mu |c|}}{\epsilon^2}, \quad \inf_{x \in \mathbb{R}} (1 - \frac{\epsilon}{\gamma} \zeta_0(x)) > 0. \quad (9.4)$$

Then the corresponding solution (ζ, v) of the one-dimensional system is global in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ and is furthermore uniformly bounded in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

Proof. We follow closely the argument used in [9] for some Hamiltonian Boussinesq systems. In order to avoid technicalities we will focus on the case $s = 1$.

Using (9.4) and the fact that the local solution is continuous in time with value in $H^1(\mathbb{R})$ we infer that there exists $t_0 > 0$ such that the local solution satisfies $1 - \frac{\epsilon}{2\gamma} \zeta(x, t) > 0$ for all $x \in \mathbb{R}$ and $0 \leq t < t_0$ so that for the same values of (x, t) we have

$$\begin{aligned} \zeta^2(x, t) & \leq \int_{\mathbb{R}} |\zeta \zeta_x| dx = \frac{1}{\sqrt{\mu |c|}} \int_{\mathbb{R}} \sqrt{\mu |c|} |\zeta \zeta_x| dx \leq \frac{1}{2\sqrt{\mu |c|}} \int_{\mathbb{R}} (\zeta^2 + \mu |c| \zeta_x^2) dx \\ & \leq \frac{1}{(1 - \gamma) \sqrt{\mu |c|}} H((\zeta(\cdot, t), v(\cdot, t))) \leq \frac{1}{(1 - \gamma) \sqrt{\mu |c|}} H(\zeta_0, v_0) \equiv \alpha^2 \end{aligned} \quad (9.5)$$

Assuming (9.4), it follows that $\alpha^2 < \frac{\gamma^2}{\epsilon^2}$, implying that

$$\sup_{x \in \mathbb{R}} |\zeta(x, t)| \leq \alpha < \frac{\gamma}{\epsilon}, \quad \text{for } 0 \leq t \leq t_0. \quad (9.6)$$

Therefore, one has $1 - \frac{\epsilon}{\gamma} \zeta(x, t) > 0$ for all $x \in \mathbb{R}$. Moreover, as long as the solution continues to exist in $H^1(\mathbb{R})$, this positive lower bound on $1 - \frac{\epsilon}{\gamma}$ continues to hold, independently of $t \geq 0$, by reapplying the above argument.

The uniform H^1 bound results then for the conservation of the Hamiltonian. Once the $H^1(\mathbb{R})$ norm of ζ and v are known to be uniformly bounded a standard argument (see the proof of Theorem 4.2 in [9]) implies that the $H^s(\mathbb{R}, s > 1)$ of ζ and v remain bounded on bounded time intervals provided $(\zeta_0, v_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R})$. \square

We now turn to the degenerate case

$$b = d > 0, \quad a = 0, \quad c < 0. \quad (9.7)$$

The result is now

Theorem 9.2. *Assume that (9.7) holds and suppose that $(\zeta_0, v_0) \in H^{s+1}(\mathbb{R}) \times H^{s+1/2}(\mathbb{R})$, $s \geq 1$ is such that (9.4) holds.*

Then the corresponding solution (ζ, v) of the one-dimensional system is global in $H^{s+1}(\mathbb{R}) \times H^{s+1/2}(\mathbb{R})$ and is furthermore uniformly bounded in $H^1(\mathbb{R}) \times H^{1/2}(\mathbb{R})$.

Proof. The proof is similar to that of Theorem 9.1 (see also Theorem 4.3 in [9]). \square

Remark 9.1. It has been recently proven ([31]) that for some of the Hamiltonian Boussinesq abcd systems having global small solutions (see [9]), those solutions decay to zero, locally strongly in the energy space, uniformly in proper subsets of the light cone $|x| \leq |t|$. It would be interesting to extend those scattering results to the systems studied in this Appendix.

10. APPENDIX B

In this Appendix, we state some facts from perturbation theory of closed linear operator on Hilbert spaces that we have used along this work (see Kato [27] for details).

We consider $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ the Hilbert space with norm defined by $\|(f, g)\|^2 = \|f\|^2 + \|g\|^2$, and for any closed operator T on $L^2(\mathbb{R})$ with domain $D(T)$, its graph, $\mathcal{G}(T) = \{(f, g) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) : f \in D(T), T(f) = g\}$. Then a metric $\widehat{\delta}$ on $\mathcal{C} \equiv \mathcal{C}(L^2(\mathbb{R}))$, the space of closed operator on $L^2(\mathbb{R})$, may be defined as follows: for any $S, T \in \mathcal{C}$,

$$\widehat{\delta}(S, T) = \|P_S - P_T\|_{B(L^2 \times L^2)},$$

where P_S and P_T are the orthogonal projections on $\mathcal{G}(S)$ and $\mathcal{G}(T)$, respectively, and

$\|\cdot\|_{B(L^2 \times L^2)}$ denotes the operator norm on the space of bounded operators on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. The results of section IV-4 of Kato [27] imply the following.

Theorem 10.1. *Let $S, T \in \mathcal{C}$, and suppose A is a bounded operator on $L^2(\mathbb{R})$ with operator norm $\|A\|_{B(L^2)}$. Then*

- (a) $\widehat{\delta}(T + A, T) \leq \|A\|_{B(L^2)}$.
- (b) $\widehat{\delta}(S + A, T + A) \leq 2(1 + \|A\|_{B(L^2)}^2)\widehat{\delta}(S, T)$.

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