SPECTRAL THEORY OF RANK ONE PERTURBATIONS OF NORMAL COMPACT OPERATORS

ANTON BARANOV

ABSTRACT. We construct a functional model for rank one perturbations of compact normal operators acting in a certain Hilbert spaces of entire functions generalizing de Branges spaces. Using this model we study completeness and spectral synthesis problems for such perturbations. Previously, in [10] the spectral theory of rank one perturbations was developed in the selfadjoint case. In the present paper we extend and significantly simplify most of known results in the area. We also prove an Ordering Theorem for invariant subspaces with common spectral part. This result is essentially new even for rank one perturbations of compact selfadjoint operators.

1. INTRODUCTION

1.1. Spectral synthesis problem. One of the basic question in the abstract operator theory is whether a linear operator \mathcal{L} from a given class has a complete set of eigenvectors or generalized eigenvectors (that is, elements of Ker $(\mathcal{L} - \lambda I)^n$ for some $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$). If the answer is positive, then the next question arises, whether it is possible to reconstruct all \mathcal{L} -invariant subspaces from its generalized eigenvectors. Namely, given an \mathcal{L} -invariant subspace \mathcal{M} , the question is whether

(1.1)
$$\mathcal{M} = \overline{\operatorname{Span}} \left\{ x \in \mathcal{M} : x \in \bigcup_{\lambda, n} \operatorname{Ker} \left(\mathcal{L} - \lambda I \right)^n \right\},$$

i.e., whether \mathcal{M} coincides with the closed linear span of the generalized eigenvectors it contains. All subspaces are assumed to be closed.

A continuous linear operator \mathcal{L} in a separable Hilbert (or Banach, or Frechét) space H is said to admit *spectral synthesis* if for any invariant subspace \mathcal{M} of \mathcal{L} we have (1.1). The notion of the spectral synthesis for a general operator goes back to J. Wermer [39], who showed, in particular, that any compact normal operator in a Hilbert space admits spectral

Theorems 2.1–2.6 and the results of Sections 3–6 were obtained with the support of Russian Science Foundation project 14-21-00035. Theorems 2.7–2.8 and the results of Sections 7–8 were obtained as a part of joint grant of Russian Foundation for Basic Research (project 17-51-150005-NCNI-a) and CNRS, France (project PRC CNRS/RFBR 2017-2019 "Noyaux reproduisants dans des espaces de Hilbert de fonctions analytiques").

synthesis. Moreover, Wermer proved that a normal operator \mathcal{A} with simple eigenvalues λ_n does not admit spectral synthesis if and only if the set $\{\lambda_n\}$ carries a complex measure orthogonal to polynomials, i.e., there exists a nontrivial sequence $\{\mu_n\} \in \ell^1$ such that $\sum_n \mu_n \lambda_n^k = 0, \ k \in \mathbb{N}_0$. Existence of such measures follows from classical Wolff's example [40] of a Cauchy transform vanishing outside of the disc: there exist $\lambda_n \in \mathbb{D} = \{z \in \mathbb{C} :$ $|z| < 1\}$ and $\{\mu_n\} \in \ell^1$ such that $\sum_n \frac{\mu_n}{z - \lambda_n} \equiv 0, \ |z| > 1$.

The first example of a compact operator which does not admit spectral synthesis was implicitly given by H. Hamburger [20] (even before Wermer's paper). Further results were obtained in the 1970s by N. Nikolski [33] and A. Markus [32]. E.g., Nikolski [33] proved that any Volterra operator can be a part of a complete compact operator (recall that *Volterra operator* is a compact operator whose spectrum is $\{0\}$).

1.2. Rank one perturbations. One of the major subareas of spectral theory deals with "small" perturbations of "good" operators (e.g., trace class or Schatten class perturbations of selfadjoint operators). However, even in this case there are few general results about completeness and synthesis. The best studied are the cases of dissipative operators and of *weak perturbations* in the sense of Keldysh and Matsaev (for a survey of these results see [18] or [10] and references therein).

We study spectral properties of rank one perturbations of compact normal operators. While compact normal operators are among the simplest infinite-dimensional operators (being unitarily equivalent to a diagonal operator in ℓ^2), we will see that the spectral theory of their rank one perturbations is highly nontrivial.

Let \mathcal{A} be a bounded cyclic normal operator in a Hilbert space H. Then, by the Spectral Theorem, \mathcal{A} is unitarily equivalent to the operator of multiplication by z in $L^2(\nu)$ for some finite compactly supported positive Borel measure ν . In what follows we will always identify H with $L^2(\nu)$ and \mathcal{A} with multiplication by the variable z.

For $a, b \in H$ consider the rank one perturbation $\mathcal{L} = \mathcal{A} + a \otimes b$ of \mathcal{A} ,

$$\mathcal{L}x = \mathcal{A}x + (x, b)a, \qquad x \in H.$$

The goal of the present paper is to study the spectral properties of rank one perturbations in the case when \mathcal{A} is compact. In particular, we are interested in completeness of (generalized) eigenvectors of \mathcal{L} (in which case we say that \mathcal{L} is complete), relations between completeness of \mathcal{L} and its adjoint \mathcal{L}^* , and the spectral synthesis for \mathcal{L} . Earlier, in [10] we considered rank one perturbations of selfadjoint operators. Here a unified treatment for the normal operators case will be presented. Not only will we extend the results of [5, 10] to the case of normal operators, but also give substantially simplified proofs of several results from [5, 10].

One of the main novel features of this paper is the proof of the fact that, for a rank one perturbation, the lattice of all its invariant subspaces with fixed spectral part is totally ordered. This result is essentially new even for perturbations of selfadjoint operators. For the case of normal operators we will have to impose some conditions on the spectrum of the unperturbed operator.

1.3. Notations. In what follows we write $U(x) \leq V(x)$ if there is a constant C such that $U(x) \leq CV(x)$ holds for all x in the set in question. We write $U(x) \approx V(x)$ if both $U(x) \leq V(x)$ and $V(x) \leq U(x)$. The standard Landau notations O and o also will be used.

For an entire function f we denote by f^* the conjugate entire function $f^*(z) = \overline{f(\overline{z})}$. The zero set of an entire function f (ignoring multiplicities) will be denoted by \mathcal{Z}_f . We denote by D(z, R) the disc with center z of radius R. By \mathcal{P}_n we denote the set of all polynomials of degree at most n. By \mathbb{N}_0 we denote the set of $n \in \mathbb{Z}$ such that $n \geq 0$.

1.4. Acknowledgements. This paper is a continuation of our work with Dmitry Yakubovich on spectral theory of rank one perturbations of selfadjoint operators which we started 10 years ago. I am deeply grateful to Dmitry for introducing me to the field of functional models and for our continued fruitfull colaboration. His influence on the subject and the results of the present paper is very important. I also want to thank my friends and coauthors Evgeny Abakumov, Yurii Belov and Alexander Borichev for their numerous ideas and insights which are widely used in this paper. And I want to express my deep gratitude to my Teachers – the late Professor Victor Havin and Professor Nikolai Nikolski – who guided me through the exciting world of Spectral Function Theory.

2. Main results.

In what follows we use the following notation. Given a bounded linear operator \mathcal{U} we denote by $\mathcal{E}(\mathcal{U})$ the subspace generated by all generalized eigenvectors (also known as *root vectors*) of \mathcal{U} . Similarly, given a \mathcal{U} -invariant subspace \mathcal{M} , we denote by $\mathcal{E}(\mathcal{M}, \mathcal{U})$ its *spectral part*

$$\mathcal{E}(\mathcal{M},\mathcal{U}) = \overline{\operatorname{Span}} \left\{ x \in \mathcal{M} : x \in \bigcup_{\lambda,n} \operatorname{Ker} (\mathcal{U} - \lambda I)^n \right\}.$$

With this notation, the spectral synthesis problem for \mathcal{U} is whether $\mathcal{M} = \mathcal{E}(\mathcal{M}, \mathcal{U})$ for any \mathcal{U} -invariant subspace \mathcal{M} .

Now we are able to state the main results of the paper. As we will see, there are two (apparently different) reasons for completeness or spectral synthesis for rank one perturbations (sometimes understood up to finite-dimensional complement): *nonvanishing moments* and *domination*.

Throughout this section we always assume that \mathcal{A} is a cyclic compact normal operator, i.e., the operator of multiplication by z in $H = L^2(\nu)$, where $\nu = \sum_n \nu_n \delta_{s_n}$, $s_n \in \mathbb{C}$, $s_n \neq 0$.

We will impose one more restriction on \mathcal{A} :

there exists
$$p > 0$$
 such that $\sum_{n} |s_n|^p < \infty$,

that is, \mathcal{A} belongs to some Schatten ideal \mathfrak{S}_p . Equivalently, this means that the sequence $t_n = s_n^{-1}$ has a finite convergence exponent and, thus, is the zero set of some entire function of finite order. All main theorems except Theorems 2.2, 2.6 and 2.7 apply only to this case.

We identify the elements of H with sequences. For $a = (a_n), b = (b_n) \in H$ we consider the associated rank one perturbation of \mathcal{A} ,

$$\mathcal{L} = \mathcal{A} + a \otimes b.$$

We write $a \in zL^2(\nu)$, if $a \in \mathcal{A}H$, that is, $\sum_n |a_n|^2 |s_n|^{-2}\nu_n = \sum_n |a_n|^2 |t_n|^2\nu_n < \infty$. However, as we will see many results will depend on the products $a_n \bar{b}_n$ and so it will be convenient to introduce the following notation: we write $a \in zL^2(\nu)$ if $\sum_{n:b_n\neq 0} |a_n|^2 |t_n|^2\nu_n < \infty$, and $b \in zL^2(\nu)$ if $\sum_{n:a_n\neq 0} |b_n|^2 |t_n|^2\nu_n < \infty$.

2.1. Completeness of the operator and its adjoint. The first result shows that \mathcal{L} and \mathcal{L}^* are (nearly) complete when certain moments of the sequence $a_n \bar{b}_n \nu_n$ do not vanish.

Theorem 2.1. Let \mathcal{A} be a normal operator in the class \mathfrak{S}_p and let $\mathcal{L} = \mathcal{A} + a \otimes b$ be its rank one perturbation.

1. Assume that $\sum_n |a_n b_n t_n| \nu_n < \infty$ and

(2.1)
$$\sum_{n} a_n \bar{b}_n t_n \nu_n \neq -1.$$

Then both \mathcal{L} and \mathcal{L}^* are complete.

2. Assume that there exists $N \in \mathbb{N}$ such that $\sum_n |a_n b_n| |t_n|^{N+1} \nu_n < \infty$,

(2.2)
$$\sum_{n}^{n} a_{n} \overline{b}_{n} t_{n} \nu_{n} = -1,$$
$$\sum_{n}^{n} a_{n} \overline{b}_{n} t_{n}^{k} \nu_{n} = 0, \quad k = 2, \dots N,$$
$$\sum_{n}^{n} a_{n} \overline{b}_{n} t_{n}^{N+1} \nu_{n} \neq 0.$$

Then dim $(\mathcal{E}(\mathcal{L}))^{\perp} \leq N$ and dim $(\mathcal{E}(\mathcal{L}^*))^{\perp} \leq N$.

3. If under conditions (2.2) we have $b \notin zL^2(\nu)$, then \mathcal{L} is complete. If $a \notin zL^2(\nu)$, then \mathcal{L}^* is complete.

Statement 1 of Theorem 2.1 is a variation on the Keldysh–Matsaev theorems which deal with the so-called weak perturbations of the form $\mathcal{A}(I + S)$ or $(I + S)\mathcal{A}$ where S is a compact operator of some class (note, however, that perturbations satisfying (2.1) need not be weak). For related results see [10, Theorem 1.1, Proposition 3.1]. A result similar to Statement 2 was proved in [11, Theorem 1.1]. However, in all these results a restriction on geometry of the spectrum $\{s_n\}$ was imposed – it was either real or contained in a finite union of rays. Here we get rid of any geometrical restrictions by using new estimates of Cauchy transforms of planar measures from [7] (see Subsection 3.2).

It was shown in [11, Theorem 1.3] that for Hadamard-lacunary spectra one can prove a result converse to Statement 2 in Theorem 2.1: if the operator is incomplete with infinite defect, then all moments are zero. This statement is no longer true when the lacunarity condition is relaxed (see [11, Theorem 1.4]).

The second result gives conditions sufficient for completeness of \mathcal{L}^* when completeness of \mathcal{L} is known. Here the crucial assumption is the domination of the vector b by a.

Theorem 2.2. Let \mathcal{A} be a compact normal operator (not necessarily in \mathfrak{S}_p for some p > 0). Assume that $\mathcal{L} = \mathcal{A} + a \otimes b$ is complete and there exists $N \in \mathbb{N}_0$ such that

(2.3)
$$\sum_{n} \frac{|b_n|^2}{|a_n|^2 |t_n|^{2N}} < \infty.$$

Then dim $(\mathcal{E}(\mathcal{L}^*))^{\perp} \leq N$.

If, moreover, $a \notin zL^2(\nu)$, then \mathcal{L}^* is complete.

2.2. Spectral synthesis. Here we state our positive results about the possibility of the spectral synthesis. As in the previous section, nonvanishing moments and a domination condition will play the role. The first result is similar to some results of Markus [32, §2], but again we do not need any restrictions on the spectrum location.

Theorem 2.3. Let \mathcal{A} be a normal operator in the class \mathfrak{S}_p and let $\mathcal{L} = \mathcal{A} + a \otimes b$ be its rank one perturbation. Assume that either $a_n \neq 0$ for any n and $a \in zL^2(\nu)$, or $b_n \neq 0$ for any n and $b \in zL^2(\nu)$. If (2.1) is satisfied, then \mathcal{L} admits spectral synthesis.

Theorem 2.4. Let \mathcal{A} be a normal operator in the class \mathfrak{S}_p and let $\mathcal{L} = \mathcal{A} + a \otimes b$ be its rank one perturbation. Let either $a_n \neq 0$ for any n or $b_n \neq 0$ for any n. Assume that there exists $N \in \mathbb{N}_0$ such that $\sum_n |a_n b_n| |t_n|^{N+1} \nu_n < \infty$ and either N = 0 and condition (2.1) is satisfied or $N \geq 1$ and conditions (2.2) are satisfied. Then for any \mathcal{L} -invariant subspace \mathcal{M} we have

$$\dim \left(\mathcal{M} \ominus \mathcal{E}(\mathcal{M}, \mathcal{L}) \right) \le (N+1)^2.$$

To state a "domination" result we need the following definition. We say that the sequence $T = \{t_n\} \subset \mathbb{C}$ is *power separated* if there exists M > 0 such that

(2.4) $\operatorname{dist}(t_n, T \setminus \{t_n\}) \gtrsim |t_n|^{-M}.$

Condition (2.4) implies that T has a finite convergence exponent.

Theorem 2.5. Let $\mathcal{L} = \mathcal{A} + a \otimes b$ be complete and let $T = \{t_n\}$ be power separated with exponent M. Assume that a and b satisfy (2.3) and (for the same N)

(2.5)
$$|a_n|^2 \nu_n \gtrsim |t_n|^{-2N-2}$$

Then for any \mathcal{L} -invariant subspace \mathcal{M} we have

$$\dim \left(\mathcal{M} \ominus \mathcal{E}(\mathcal{M}, \mathcal{L}) \right) \le (M + N + 1)^2.$$

Remark. Conditions (2.3) and (2.5) in Theorem 2.5 can be replaced by a slightly stronger assumption $|a_n|^2 \nu_n \gtrsim |t_n|^{-2N}$, which obviously implies (2.3).

2.3. Counterexamples. We now turn to negative results. One of the main points of the present paper (as well as of [10]) is that already for such small class as rank one perturbations of normal operators one has a very rich and complicated spectral structure. We construct counterexamples showing that completeness of the adjoint operator or spectral synthesis may fail with any finite or infinite defect.

Theorem 2.6. Let \mathcal{A} be a compact normal operator with simple point spectrum and trivial kernel and let $N \in \mathbb{N} \cup \{\infty\}$. Then the following statements hold true.

1. There exists a rank one perturbation \mathcal{L} of \mathcal{A} such that ker $\mathcal{L} = \ker \mathcal{L}^* = 0$ and \mathcal{L} is complete, but \mathcal{L}^* is not complete and

$$\dim \left(\mathcal{E}(\mathcal{L}^*) \right)^\perp = N.$$

- 2. There exists a rank one perturbation \mathcal{L} of \mathcal{A} such that
 - (i) ker $\mathcal{L} = \ker \mathcal{L}^* = 0;$
 - (ii) both \mathcal{L} and \mathcal{L}^* are complete;
 - (iii) L does not admit spectral synthesis and, moreover, there exists L-invariant subspace M such that

$$\dim \mathcal{M} \ominus \mathcal{E}(\mathcal{M}, \mathcal{L}) = N.$$

Note that, for a bounded operator \mathcal{B} , if ker $\mathcal{B} \neq 0$, but ker $\mathcal{B}^* = 0$, then \mathcal{B}^* is not complete. An explicit example of a complete compact operator \mathcal{B} such that ker $\mathcal{B} =$ ker $\mathcal{B}^* = 0$, while \mathcal{B}^* is not complete, was given by Deckard, Foiaş and Pearcy [15]. However, in their examples one cannot conclude that the corresponding operator is a finite rank perturbation of a normal operator. Surprisingly, one can find such examples among rank one perturbations of a compact normal operator with an arbitrary spectrum.

A concrete example of a rank one perturbation \mathcal{L} of a compact normal operator such that ker $\mathcal{L} = \ker \mathcal{L}^* = 0$, \mathcal{L} is complete, but \mathcal{L}^* is not, can be extracted from the results by A.A. Lunyov and M.M. Malamud [29, Section 4]. The operator in this example is realized as the inverse to a two-dimensional first order differential operator with specially chosen boundary conditions. A version of this example was presented in [10, Appendix 1].

At the same time, Lunyov and Malamud [30] showed that for a class of dissipative realizations of Dirac-type differential operators, completeness property is equivalent to the spectral synthesis property.

2.4. Ordered structure of invariant subspaces. Assume that both a compact operator \mathcal{L} and its adjoint \mathcal{L}^* are complete, but the spectral synthesis fails. If we denote by $\{x_n\}_{n \in \mathbb{N}}$ the (generalized) eigenvectors of \mathcal{L} and by $\{y_n\}$ the eigenvectors of \mathcal{L}^* this means that there exist an \mathcal{L} -invariant subspace \mathcal{M} and $N_1 \subset N$ such that $x_n \in \mathcal{M}$ if and only if $n \in N_1$, but $\overline{\text{Span}}\{x_n : n \in N_1\} \neq \mathcal{M}$. It is not difficult to show that $\mathcal{M}^{\perp} \supset \{y_n : n \in \mathbb{N} \setminus N_1\}$ (see, e.g., the proof of Lemma 4.2 in [32]). Hence,

(2.6)
$$\overline{\operatorname{Span}}\{x_n : n \in N_1\} \subset \mathcal{M} \subset \{y_n : n \in N_2\}^{\perp},$$

where $N_2 = N \setminus N_1$. We say that all invariant subspaces \mathcal{M} satisfying (2.6) for a fixed set N_1 have common spectral part $\overline{\text{Span}}\{x_n : n \in N_1\}$.

A natural problem is to describe all non-spectral invariant subspaces or, at least, to find some structural properties of them. Apparently, in general, one cannot expect any structure of the lattice. However, for the case of rank one perturbations of normal operators, there are good reasons to believe that the set of invariant subspaces \mathcal{M} satisfying (2.6) (i.e., the subspaces with the common spectral part) is totally ordered by inclusion, that is, for any $\mathcal{M}_1, \mathcal{M}_2$ satisfying (2.6) one has $\mathcal{M}_1 \subset \mathcal{M}_2$ or $\mathcal{M}_2 \subset \mathcal{M}_1$. We state this as a conjecture.

Conjecture. Let \mathcal{L} be a rank one perturbation of a compact normal operator. Then the set of all invariant subspaces with fixed common spectral part is totally ordered by inclusion.

We prove this conjecture in two cases: for rank one perturbations of compact *selfadjoint* operators (without any additional restrictions on the spectrum such as membership in a Schatten class) and for the case of Schatten-class normal operators under certain conditions on the location of the spectrum.

Theorem 2.7. Let \mathcal{A} be a compact selfadjoint operator with simple point spectrum and trivial kernel and let $\mathcal{L} = \mathcal{A} + a \otimes b$ be its rank one perturbation such that $b_n \neq 0$ for any n. Assume that \mathcal{L} and \mathcal{L}^* are complete. Then the set of all invariant subspaces with fixed common spectral part is totally ordered by inclusion.

In the case of normal operators we establish the ordered structure of invariant subspaces with fixed common spectral part when one of the following conditions holds:

- (i) $\mathbf{Z}: T$ is the zero set of some entire function of zero exponential type;
- (ii) Π : T lies in some strip and has finite convergence exponent;
- (iii) \mathbf{A}_{γ} : T lies in some angle of size $\pi\gamma$, $0 < \gamma < 1$, and the convergence exponent of T is less than γ^{-1} .

Theorem 2.8. Let \mathcal{A} be a compact normal operator with simple point spectrum $\{s_n\}$ such that its inverse spectrum $T = \{t_n\}$ satisfies one of the conditions \mathbb{Z} , Π or \mathbb{A}_{γ} . Let $\mathcal{L} = \mathcal{A} + a \otimes b$ be its rank one perturbation such that $b_n \neq 0$ for any n and assume that \mathcal{L} and \mathcal{L}^* are complete. Then the set of all invariant subspaces with fixed common spectral part is totally ordered by inclusion.

In [1] the theory of Cauchy–de Branges spaces $\mathcal{H}(T, A, \mu)$ (see the definition in Subsection 2.5) was developed which generalizes the theory of classical de Branges spaces. In particular, in the cases \mathbf{Z} , $\mathbf{\Pi}$ and \mathbf{A}_{γ} an ordering theorem for *nearly invariant subspaces*

of the backward shift (see the definition in Section 7) in $\mathcal{H}(T, A, \mu)$ was obtained similar to the classical de Branges' Ordering Theorem [14, Theorem 35]. On the other hand, it is shown in [1] that in general there is no ordered structure for nearly invariant subspaces in Cauchy–de Branges spaces.

As we will see, the functional model translates the ordering problem for invariant subspaces of rank one perturbations into the ordering problem for nearly invariant subspaces in the spaces $\mathcal{H}(T, A, \mu)$. The proofs of Theorems 2.7 and 2.8 are variations on the beautiful idea used by L. de Branges in the proof of [14, Theorem 35].

2.5. Functional model. The model for rank one perturbations constructed in [10] acted in some de Branges space. De Branges spaces' theory is a deep and important field which has numerous applications in operator theory and in spectral theory of differential operators. For the basics of de Branges theory we refer to L. de Branges' classical monograph [14] and to [36]; some further results and applications can be found in [31, 35]. In the normal case their role is played by a more general class of *spaces of Cauchy transforms* that we will call *Cauchy-de Branges spaces*.

Let $T = \{t_n\}_{n=1}^{\infty} \subset \mathbb{C}$, where t_n are distinct, let $|t_n| \to \infty$ as $n \to \infty$, and let $\mu = \sum_n \mu_n \delta_{t_n}$ be a positive measure such that $\sum_n \frac{\mu_n}{|t_n|^2 + 1} < \infty$. Also let A be an entire function which has only simple zeros and whose zero set \mathcal{Z}_A coincides with T. With any such T, A and μ we associate the Cauchy–de Branges space $\mathcal{H}(T, A, \mu)$ of entire functions,

$$\mathcal{H}(T, A, \mu) := \left\{ f : f(z) = A(z) \sum_{n} \frac{a_n \mu_n^{1/2}}{z - t_n}, \quad a = \{a_n\} \in \ell^2 \right\}$$

equipped with the norm $||f||_{\mathcal{H}(T,A,\mu)} := ||a||_{\ell^2}$. Note that the series in the definition of $\mathcal{H}(T,A,\mu)$ converge absolutely and uniformly on compact sets.

The spaces $\mathcal{H}(T, A, \mu)$ were introduced in full generality by Yu. Belov, T. Mengestie, and K. Seip [13]. Essentially, they are spaces of Cauchy transforms. We need the function A to get rid of poles and make the elements entire, but the spaces with the same T, μ and different A's are isomorphic. In what follows we will usually (but not always) assume that T has a finite convergence exponent and A in this case will be chosen to be some canonical product of the corresponding order. We call the pair (T, μ) the spectral data for $\mathcal{H}(T, A, \mu)$.

Each space $\mathcal{H}(T, A, \mu)$ is a reproducing kernel Hilbert space. It is noted in [13] that if \mathcal{H} is a reproducing kernel Hilbert space of entire functions such that \mathcal{H} has the *division* property (that is, $\frac{f(z)}{z-w} \in \mathcal{H}$ whenever $f \in \mathcal{H}$ and f(w) = 0) and there exists a Riesz basis of reproducing kernels in \mathcal{H} , then $\mathcal{H} = \mathcal{H}(T, A, \mu)$ (as sets with equivalence of norms) for some choice of the parameters. Note that the functions $\overline{A'(t_n)}\mu_n \cdot \frac{A(z)}{z-t_n}$ form an orthogonal

basis in $\mathcal{H}(T, A, \mu)$ and are the reproducing kernels at the points t_n . Reproducing kernels at other points can be written in a standard way using this orthogonal basis, but we do not have a good explicit formula for them. The reproducing kernel of the space $\mathcal{H}(T, A, \mu)$ at the point λ will be denoted by k_{λ} .

In the case when $T \subset \mathbb{R}$ and A is real on \mathbb{R} , the space $\mathcal{H}(T, A, \mu)$ is a de Branges space. This follows, e.g., from the axiomatic description of de Branges spaces [14, Theorem 23]. In a recent preprint [1] certain properties of de Branges spaces (e.g., ordered structure of subspaces) are extended to a class of Cauchy–de Branges spaces.

Following de Branges, we say that an entire function G is associated to the space $\mathcal{H}(T, A, \mu)$ and write $G \in \operatorname{Assoc}(T, A, \mu)$ if, for any $F \in \mathcal{H}(T, A, \mu)$ and $w \in \mathbb{C}$, we have

$$\frac{F(w)G(z) - G(w)F(z)}{z - w} \in \mathcal{H}(T, A, \mu).$$

Equivalently, this means that $G \in \mathcal{P}_1\mathcal{H}(T, A, \mu)$. We write $F \in \mathcal{P}_n\mathcal{H}(T, A, \mu)$ if $F(z) = \sum_{j=0}^n z^j F_j(z), F_j \in \mathcal{H}(T, A, \mu)$. Finally, if G has zeros, then the inclusion $G \in \operatorname{Assoc}(T, A, \mu)$ is equivalent to $\frac{G(z)}{z-\lambda} \in \mathcal{H}(T, A, \mu)$ for some (any) $\lambda \in \mathcal{Z}_G$. Note that, in particular, $A \in \operatorname{Assoc}(T, A, \mu) \setminus \mathcal{H}(T, A, \mu)$.

Now we are able to formulate the functional model of rank one perturbations of normal operators. Here and in what follows (except a part of Section 3) we assume that \mathcal{A} is a compact normal operator in a Hilbert space H with simple point spectrum $\{s_n\}, s_n \neq 0$. We identify H with $L^2(\nu)$, where $\nu = \sum_n \nu_n \delta_{s_n}$, and \mathcal{A} with multiplication by z in $L^2(\nu)$. The elements of $L^2(\nu)$ are identified with sequences., i.e., for $a \in L^2(\nu)$ we write $a = (a_n)$, where $a_n = a(s_n)$.

Theorem 2.9. Let \mathcal{A} be multiplication by z in $L^2(\nu)$, $\nu = \sum_n \nu_n \delta_{s_n}$. Put $t_n = s_n^{-1}$. Let $\mathcal{L} = \mathcal{A} + a \otimes b$ be a rank one perturbation of \mathcal{A} such that $b = \{b_n\} \in L^2(\nu)$ is a cyclic vector for \mathcal{A} , i.e., $b_n \neq 0$ for any n. Then there exist

- a positive measure $\mu = \sum_{n} \mu_n \delta_{t_n}$ such that $\sum_{n} \frac{\mu_n}{|t_n|^2 + 1} < \infty$;
- a space $\mathcal{H}(T, A, \mu)$;
- an entire function $G \in Assoc (T, A, \mu)$ with G(0) = 1

such that \mathcal{L} is unitarily equivalent to the model operator $\mathcal{T}_G : \mathcal{H}(T, A, \mu) \to \mathcal{H}(T, A, \mu)$,

$$(\mathcal{T}_G f)(z) = \frac{f(z) - f(0)G(z)}{z}, \qquad f \in \mathcal{H}(T, A, \mu).$$

Conversely, for any space $\mathcal{H}(T, A, \mu)$ and the function $G \in \operatorname{Assoc}(T, A, \mu)$ with G(0) = 1the corresponding operator \mathcal{T}_G is a model of a rank one perturbation for some compact normal operator \mathcal{A} with spectrum $\{s_n\}$, $s_n = t_n^{-1}$. As we will see in Lemma 4.1, the assumption $b_n \neq 0$ does not lead to the loss of generality when we study completeness of eigenvectors of \mathcal{L} and \mathcal{L}^* .

The model itself is by no means original. In the case of selfadjoint operators it was constructed in [10], but many similar models for rank one perturbations (in slightly different situations or under some additional restrictions) were known previously. We will mention the model of V. Kapustin [23] for rank one perturbations of unitary operators and the model of G. Gubreev and A. Tarasenko for selfadjoint operators [19]. The latter model is especially close to ours with the same operator \mathcal{T}_G as the model operator (in de Branges space setting).

The main novelty of the present work is not in the model but in its applications: combined with recent developments in the theory of reproducing kernel Hilbert spaces of entire functions from [4, 5, 6, 7, 10] it leads to a more or less complete understanding of completeness and spectral synthesis problems for rank one perturbations of normal operators.

2.6. Organization of the paper. The paper is organized as follows. In Section 3 we construct the functional model for rank one perturbations. Completeness of \mathcal{L} and \mathcal{L}^* is studied in Section 4, while in Section 5 positive results on the spectral synthesis (Theorems 2.3–2.5) are proved. In Section 6 we prove Theorem 2.6. The Ordering Theorem for invariant subspaces of rank one perturbations with common spectral part is established in Section 7. Finally, in Section 8 we discuss the description of compact normal operators which have a Volterra rank one perturbation.

3. Functional model

In this section we prove Theorem 2.9. In fact, we construct a similar model for general normal (not necessarily compact) operators.

3.1. General normal operators. Let \mathcal{A} be the operator of multiplication by z in $L^2(\nu)$ where ν is a finite measure with compact closed support K and assume that K has zero planar Lebesgue measure. Here we do not assume that ν is an atomic measure. By the classical Hartogs–Rosenthal theorem, we have the following uniqueness property:

$$u \in L^2(\nu)$$
 and $\int \frac{u(\zeta)d\nu(\zeta)}{\zeta - z} = 0$ for all $z \in \mathbb{C} \setminus K \implies u = 0$.

Note that Wolff's example shows that there exist atomic measures with closed support $K = \{|z| \leq 1\}$ such that the Cauchy transform of some nonzero function from $L^2(\nu)$ is identically zero in $\mathbb{C} \setminus K$.

Consider the space of all Cauchy transforms

$$\mathcal{C}(K,\nu) = \left\{ f(z) = \int \frac{u(\zeta)d\nu(\zeta)}{\zeta - z}, \ u \in L^2(\nu) \right\},\$$

considered as the space of analytic functions in $\mathbb{C} \setminus K$ and equipped with the norm $||f||_{\mathcal{C}(K,\nu)} = ||u||_{L^2(\nu)}$. Then $\mathcal{C}(K,\nu)$ is a Hilbert space.

Note that for any $f \in \mathcal{C}(K,\nu)$ the mapping $f \mapsto (zf)_{\infty}$, where $(zf)_{\infty} = \lim_{|z|\to\infty} zf(z) = -\int ud\nu$ is a bounded linear functional on $\mathcal{C}(K,\nu)$.

Theorem 3.1. Let \mathcal{A} be multiplication by z in $L^2(\nu)$, and $\mathcal{L} = \mathcal{A} + a \otimes b$ be a rank one perturbation of \mathcal{A} such that $a, b \in L^2(\nu)$ and $b \neq 0$ ν -a.e. Put $\sigma = |b|^2 \nu$. Then there exists a function β analytic in $\mathbb{C} \setminus K$ with the properties

- (i) $\beta \notin \mathcal{C}(K, \sigma);$
- (ii) $\frac{\beta(z)-\beta(\lambda)}{z-\lambda} \in \mathcal{C}(K,\sigma)$ for any $\lambda \in \mathbb{C} \setminus K$;

(iii)
$$\beta(\infty) = 1$$
,

such that \mathcal{L} is unitarily equivalent to the model operator $\mathcal{M}_{\beta} : \mathcal{C}(K, \sigma) \to \mathcal{C}(K, \sigma)$,

$$(\mathcal{M}_{\beta}f)(z) = zf(z) - (zf)_{\infty}\beta(z), \qquad f \in \mathcal{C}(K,\sigma)$$

Conversely, for any space $C(K, \sigma)$, where σ is a finite Borel measure, and for any function β having the properties (i)–(iii), the corresponding operator \mathcal{M}_{β} is a model of a rank one perturbation of some normal cyclic operator \mathcal{A} .

Proof. By the resolvent identity, for $u \in L^2(\nu)$,

$$(\mathcal{A} - zI)^{-1}u - (\mathcal{L} - zI)^{-1}u = ((\mathcal{L} - zI)^{-1}u, b)(\mathcal{A} - zI)^{-1}a$$

and so

$$\left((\mathcal{A}-zI)^{-1}u,b\right)-\left((\mathcal{L}-zI)^{-1}u,b\right)=\left((\mathcal{L}-zI)^{-1}u,b\right)\left(\mathcal{A}-zI\right)^{-1}a,b\right).$$

Thus,

$$\left((\mathcal{L}-zI)^{-1}u,b\right)=(\beta(z))^{-1}\left((\mathcal{A}-zI)^{-1}u,b\right),$$

where

(3.1)
$$\beta(z) = 1 + \left((\mathcal{A} - zI)^{-1}a, b \right) = 1 + \int \frac{a(\zeta)\bar{b}(\zeta)}{\zeta - z} d\nu(\zeta).$$

The mapping

$$V: u \mapsto \left((\mathcal{A} - zI)^{-1}u, b \right) = \int \frac{u(\zeta)b^{-1}(\zeta)}{\zeta - z} d\sigma(\zeta)$$

is a unitary map from $L^2(\nu)$ to $\mathcal{C}(K, \sigma)$. Let $z, w \in \rho(\mathcal{L}) \cap \rho(\mathcal{A})$ (where $\rho(\mathcal{U})$ denotes the resolvent set for an operator \mathcal{U}) and let $\beta(w) \neq 0$. Then

$$V(\mathcal{L} - wI)^{-1}u(z) = \beta(z)\left((\mathcal{L} - zI)^{-1}(\mathcal{L} - wI)^{-1}u, b\right)$$
$$= \beta(z)\frac{\left((\mathcal{L} - zI)^{-1}u, b\right) - \left((\mathcal{L} - wI)^{-1}u, b\right)}{z - w}$$
$$= \frac{(Vu)(z) - \frac{\beta(z)}{\beta(w)}(Vu)(w)}{z - w}$$

Now let \mathcal{M}_{β} be the model operator on $\mathcal{C}(K, \sigma)$. Let us compute $(\mathcal{M}_{\beta} - wI)^{-1}$ assuming $w \in \rho(\mathcal{M}_{\beta}), w \notin K, \beta(w) \neq 0$. Assume that $(\mathcal{M}_{\beta} - wI)g = (z - w)g - c_g\beta = h$, where $g, h \in \mathcal{C}(K, \sigma), c_g = (zg)_{\infty}$. Then $g = \frac{h + c_g\beta}{z - w}$. Since g, h, β are analytic outside K and $\beta(w) \neq 0$ we conclude that $c_g = -g(w)/\beta(w)$ and so

$$(\mathcal{M}_{\beta} - wI)^{-1}g(z) = \frac{g(z) - \frac{\beta(z)}{\beta(w)}g(w)}{z - w}.$$

Thus,

$$\left(V(\mathcal{L} - wI)^{-1}u\right)(z) = \left((\mathcal{M}_{\beta} - wI)^{-1}Vu\right)(z)$$

for any $w \notin K \ w \in \rho(\mathcal{M}_{\beta}) \cap \rho(\mathcal{L}), \ \beta(w) \neq 0$. Since there are infinitely many such w, we conclude that $V\mathcal{L} = \mathcal{M}_{\beta}V$.

Clearly, $\beta(\infty) = 1$ and so $\beta \notin \mathcal{C}(K, \sigma)$. Also,

$$\frac{\beta(z) - \beta(\lambda)}{z - \lambda} = \int \frac{a(\zeta)b^{-1}(\zeta)}{(\zeta - \lambda)(\zeta - z)} d\sigma(\zeta) \in \mathcal{C}(K, \sigma)$$

for any $\lambda \in \mathbb{C} \setminus K$.

Let us prove the converse. Assume that β satisfy conditions (i)–(iii). Then, for some fixed $\lambda \in \mathbb{C} \setminus K$, there exists $u \in L^2(\sigma)$ such that

$$\beta(z) = \beta(\lambda) + (z - \lambda) \int \frac{u(\zeta)}{\zeta - z} d\sigma(\zeta) = \beta(\lambda) - \int u(\zeta) d\sigma(\zeta) + \int \frac{(\zeta - \lambda)u(\zeta)}{\zeta - z} d\sigma(\zeta).$$

From (iii) it follows that $\beta(\lambda) - \int u(\zeta) d\sigma(\zeta) = 1$. It remains to take any $b \neq 0$ σ -a.e. and to put $\nu = |b|^{-2}\sigma$ and $a(\zeta) = (\zeta - \lambda)u(\zeta)b(\zeta)$. Then β is of the form (3.1) and so it appears in the model for a rank one perturbation of multiplication by z in $L^2(\nu)$.

3.2. Preliminaries on Cauchy transforms and spaces $\mathcal{H}(T, A, \mu)$. To apply the functional model to the study of completeness one needs to have good growth estimates for the Cauchy transforms of (discrete) planar measures. Powerfull tools for this were developed in [7]. The following results from [7] will be extensively used in what follows.

We say that $\Omega \subset \mathbb{C}$ is a set of zero area density if

$$\lim_{R \to \infty} \frac{m_2(\Omega \cap D(0, R))}{R^2} = 0,$$

where m_2 denotes the area Lebesgue measure in \mathbb{C} . Clearly, a union of two sets of zero density has zero density, a fact that we will constantly use.

The first statement shows that the Cauchy transform of a finite measure ν behaves asymptotically as $\nu(\mathbb{C})z^{-1}$ when $|z| \to \infty$. This is trivial for measures with compact support. The same is true in general up to a set of zero density.

Lemma 3.2. ([7, Proof of Lemma 4.3]) Let ν be a finite complex Borel measure in \mathbb{C} . Then, for any $\varepsilon > 0$, there exists a set Ω of zero area density such that

$$\left|\int_{\mathbb{C}} \frac{d\nu(\xi)}{z-\xi} - \frac{\nu(\mathbb{C})}{z}\right| < \frac{\varepsilon}{|z|}, \qquad z \in \mathbb{C} \setminus \Omega.$$

The following variant of this statement will be often useful: if $\sum_n |t_n^{-1}d_n| < \infty$, then

(3.2)
$$\sum_{n} \frac{d_n}{z - t_n} = o(1), \qquad |z| \to \infty, \ z \in \mathbb{C} \setminus \Omega,$$

for some set Ω of zero area density. This follows from Lemma 3.2 and the formula

$$\sum_{n} \frac{d_n}{z - t_n} = -\sum_{n} \frac{d_n}{t_n} + \sum_{n} \frac{d_n}{t_n(z - t_n)}.$$

Let us mention one simple situation where we can conclude that the Cauchy transform has the natural asymptotics along some rays.

Lemma 3.3. Let ν be a finite complex Borel measure such that for some $\theta_0 \in [0, 2\pi)$ and $\delta > 0$ we have supp $\nu \cap \{re^{i\theta} : r \ge 0, |\theta - \theta_0| \le \delta\} = \emptyset$. Then

$$\int_{\mathbb{C}} \frac{d\nu(\xi)}{z-\xi} = \frac{\nu(\mathbb{C})}{z} + o\left(\frac{1}{z}\right), \qquad z = re^{i\theta_0}, \ r \to \infty.$$

Proof. Since dist $(z, \operatorname{supp} \nu) \asymp |z|, z = re^{i\theta_0}$, we have

$$\left| \int_{\mathbb{C}} \frac{d\nu(\xi)}{z-\xi} - \frac{\nu(\mathbb{C})}{z} \right| \le \frac{1}{|z|} \int_{\mathbb{C}} \frac{|\xi|d|\nu|(\xi)}{|z-\xi|} = o\left(\frac{1}{z}\right), \qquad z = re^{i\theta_0}, \ r \to \infty,$$

by the Dominated Convergence theorem.

We also will need the following extension of the Liouville theorem. This result which is due to A. Borichev appeared in [7, Lemma 4.2] (in a slightly more general form). We include a short proof to make the exposition self-contained.

Theorem 3.4. If an entire function f of finite order is bounded on $\mathbb{C} \setminus \Omega$ for some set Ω of zero area density, then f is a constant.

Proof. We prove an equivalent statement: If f is an entire function of finite order and $|f(z)| \to 0$ as $|z| \to \infty$, $z \notin \Omega$, for some set Ω of zero area density, then $f \equiv 0$.

Assume that f is non-zero. Then $\varphi(z) = \log |f(z)|$ is subharmonic. Since f takes arbitrarily large values, we may assume without loss of generality that $\varphi(0) = 1$. At the same time $\varphi < 0$ on $\mathbb{C} \setminus \Omega$ for some set Ω of zero density. Let W(R) be the connected component of the open set $\{z : \varphi(z) > 0\} \cap D(0, R)$ which contains the point 0 and let $S(R) = \{|z| = R\} \cap \partial W(R)$. Denote by $\sigma(R)$ the total length of the arcs in S(R).

Denote by $\omega_{W(R)}(0, E)$ the harmonic measure at 0 of $E \subset \partial W(R)$. By the Ahlfors– Carleman estimate [17, Ch. IV, Th. 6.2], there exists $r_0 > 0$ such that for $R > r_0$,

$$\omega_{W(R)}(0, S(R)) \le C \exp\bigg(-\pi \int_{r_0}^R \frac{dr}{\sigma(r)}\bigg),$$

where C > 0 is some absolute constant. Recall that f is of finite order. So, for some $C_1, N > 0, \varphi(z) \leq C_1 R^N, |z| = R$. Since $\varphi = 0$ on $\partial W(R) \setminus \{|z| = R\}$ and $\varphi(z) \leq R^N, |z| = R$, we conclude by the "Two Constants Theorem" that

$$\varphi(0) \le C_1 R^N \omega_{W(R)}(0, S(R)) \le C C_1 R^N \exp\bigg(-\pi \int_{r_0}^R \frac{dr}{\sigma(r)}\bigg).$$

It remains to show that since the set $\{z : \varphi(z) > 0\}$ has zero density, we have $\sigma(r) = o(r)$ on "most of the circles". This can be formalized as follows. Given $\varepsilon > 0$, for any sufficiently large k (say, $k \ge n_0$) there exists $E_k \subset [2^k, 2^{k+1}]$ with $|E_k| > 2^{k-1}$ such that for $r \in E_k$ we have $\sigma(r) < \varepsilon r$. Otherwise,

$$m_2(W(R) \cap \{2^k < |z| < 2^{k+1}\}) = \int_{2^k}^{2^{k+1}} \sigma(r) dr \ge \varepsilon 2^{2k-1},$$

a contradiction to the fact that Ω has zero area density.

Now, for $2^{n+1} \leq R \leq 2^{n+2}$ we have for some constant $C_2 > 0$,

$$\varphi(0) \leq C_2 R^N \exp\left(-\pi \sum_{k=n_0}^n \int_{E_k} \frac{dr}{\sigma(r)}\right)$$
$$\leq C_2 R^N \exp\left(-\pi \sum_{k=n_0}^n \int_{E_k} \frac{dr}{\varepsilon r}\right)$$
$$\leq C_2 R^N \exp\left(-\frac{\pi}{4\varepsilon}(n-n_0)\right)$$
$$\leq C_2 R^N \exp\left(-\frac{\pi}{4\varepsilon}(\log R - n_0)\right).$$

If ε is sufficiently small and R is sufficiently large, we conclude that $\varphi(0) < 1$, a contradiction.

In [1] the following properties of functions in the spaces $\mathcal{H}(T, A, \mu)$ were discussed.

Lemma 3.5. ([1, Lemma 2.5]) Let A be an entire function of order ρ with the zero set T. Then for any $\varepsilon > 0$ there exists a set $E \subset (0, \infty)$ of zero linear density (i.e., $|E \cap (0, R)| = o(R)$, $R \to \infty$, where |e| denotes one-dimensional Lebesgue measure of $e \subset \mathbb{R}$) such that for any entire function $f \in \mathcal{H}(T, A, \mu)$,

(3.3)
$$|f(z)| \lesssim |z|^{\rho+1+\varepsilon} |A(z)|, \qquad |z| \notin E.$$

In particular, if A is of order ρ and of type \varkappa , then any element of $\mathcal{H}(T, A, \mu)$ is of order at most ρ and of type at most \varkappa with respect to this order.

Lemma 3.6. If $f \in \mathcal{H}(T, A, \mu)$, then

$$||f||_{\mathcal{H}(T,A,\mu)}^{2} = \sum_{n} \frac{|f(t_{n})|^{2}}{|A'(t_{n})|^{2}\mu_{n}}$$

and there exists a set Ω of zero area density such that

$$|f(z)| = o(|A(z)|), \qquad |z| \to \infty, \quad z \in \mathbb{C} \setminus \Omega.$$

Proof. Note that for $f = A \sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n} \in \mathcal{H}(T, A, \mu)$ we have $f(t_n) = A'(t_n) c_n \mu_n^{1/2}$ and, by definition, $\|f\|_{\mathcal{H}(T,A,\mu)} = \|\{c_n\}\|_{\ell^2}$.

To prove that |f(z)| = o(|A(z)|) recall that $\sum_n (1 + |t_n|)^{-2} \mu_n < \infty$ whence $\sum_n |c_n t_n^{-1}| \mu_n^{1/2} < \infty$ and the estimate follows from (3.2).

Using the above estimates one can state various criteria for the inclusion of f into $\mathcal{H}(T, A, \mu)$.

Theorem 3.7. ([1, Theorem 2.6]) Let $\mathcal{H}(T, A, \mu)$ be a Cauchy-de Branges space and let A be of finite order. Then an entire function f is in $\mathcal{H}(T, A, \mu)$ if and only if the following three conditions hold:

(i)
$$\sum_{n} \frac{|f(t_n)|^2}{|A'(t_n)|^2 \mu_n} < \infty;$$

- (ii) there exists a set $E \subset (0, \infty)$ of zero linear density and N > 0 such that $|f(z)| \le |z|^N |A(z)|, |z| \notin E$;
- (iii) there exists a set Ω of positive area density such that $|f(z)| = o(|A(z)|), |z| \to \infty, z \in \Omega.$

In many cases one can relax the conditions (ii)–(iii) and require the estimates on a smaller set.

3.3. **Proof of Theorem 2.9.** Let $\nu = \sum_{n} \nu_n \delta_{s_n}$, $s_n \in \mathbb{C}$, $s_n \neq 0$, $s_n \to 0$, and $t_n = s_n^{-1}$. Let $a, b \in L^2(\nu)$, $b_n \neq 0$ for any n and $\sigma = |b|^2 \nu$. By Theorem 3.1 the perturbation $\mathcal{L} = \mathcal{A} + a \otimes b$ is unitary equivalent to the model operator \mathcal{M}_β on $\mathcal{C}(K, \sigma)$ where β is given by (3.1). Now we put

$$\mu_n = |t_n|^2 \sigma_n = |t_n|^2 |b_n|^2 \nu_n$$

Fix an entire function A with zero set $\{t_n\}$ and A(0) = 1, and consider the space $\mathcal{H}(T, A, \mu)$.

Put $G(z) = A(z)\beta(z^{-1})$. Then we have

(3.4)

$$G(z) = A(z)\beta(z^{-1}) = A(z)\left(1 + z\sum_{n} \frac{a_{n}b_{n}t_{n}\nu_{n}}{z - t_{n}}\right)$$

$$= A(z)\left(1 + \sum_{n} a_{n}\bar{b}_{n}t_{n}^{2}\nu_{n}\left(\frac{1}{z - t_{n}} + \frac{1}{t_{n}}\right)\right).$$

It is easy to see that $G \in Assoc (T, A, \mu)$ and G(0) = 1.

We will show that the mapping $U : f \mapsto A(z)z^{-1}f(z^{-1})$ maps $\mathcal{C}(K, \sigma)$ unitarily onto $\mathcal{H}(T, A, \mu)$ and realizes a unitary equivalence between \mathcal{M}_{β} and \mathcal{T}_{G} .

Let $f(z) = \sum_{n \frac{u_n \sigma_n}{z - s_n}}$, $u = (u_n) \in L^2(\sigma)$ and $g(z) = A(z)z^{-1}f(z^{-1})$. Since

$$z^{-1}f(z^{-1}) = \sum_{n} \frac{u_n t_n \sigma_n}{t_n - z} = \sum_{n} \frac{u_n t_n |t_n|^{-1} \sigma_n^{1/2} \mu_n^{1/2}}{t_n - z},$$

we conclude that $g \in \mathcal{H}(T, A, \mu)$ and

$$||g||_{\mathcal{H}(T,A,\mu)}^{2} = \sum_{n} |u_{n}|^{2} \sigma_{n} = ||f||_{\mathcal{C}(K,\sigma)}^{2}.$$

Note also that $(zf)_{\infty} = \sum_{n} u_n \sigma_n = (Uf)(0)$. Then we have

$$(U\mathcal{M}_{\beta}f)(z) = A(z) \left(z^{-2} f(z^{-1}) - (zf)_{\infty} z^{-1} \beta(z^{-1}) \right)$$
$$= \frac{(Uf)(z) - (Uf)(0)G(z)}{z} = (\mathcal{T}_{G}Uf)(z).$$

Thus, \mathcal{L} is unitary equivalent to \mathcal{T}_G .

Finally, it is easy to see that any function $G \in \text{Assoc}(T, A, \mu)$ with G(0) = 1 admits representation (3.4) for some a, b, ν such that $a, b \in L^2(\nu)$ and $\mu_n = |b_n|^2 |t_n|^2 \nu_n$. Therefore, the function β such that $G(z) = A(z)\beta(z^{-1})$ satisfies conditions (i)–(iii) of Theorem 3.1. Thus, any such function G appears in the model of some rank one perturbation of the compact normal operator with the spectral measure ν .

In what follows we will often use the following simple observation.

Lemma 3.8. The function G given by (3.4) belongs to $\mathcal{H}(T, A, \mu)$ if and only if $a \in zL^2(\nu)$ (*i.e.*, $\sum_n |a_n|^2 |t_n|^2 \nu_n < \infty$) and

$$(3.5) 1 + \sum_{n} a_n \bar{b}_n t_n \nu_n = 0$$

Proof. Assume that $G \in \mathcal{H}(T, A, \mu)$. Note that $G(t_n) = A'(t_n)t_n^2 a_n \bar{b}_n \nu_n$. Then, by Lemma 3.6 and the fact that $\mu_n = |t_n|^2 |b_n|^2 \nu_n$,

$$\sum_{n} \frac{|G(t_n)|^2}{|A'(t_n)|^2 \mu_n} = \sum_{n} |a_n|^2 |t_n|^2 \nu_n < \infty.$$

Now the series $\sum_{n} |a_n b_n t_n| \nu_n$ converges and we may write

(3.6)
$$G(z) = A(z) \left(1 + \sum_{n} a_n \bar{b}_n t_n \nu_n + \sum_{n} \frac{a_n \bar{b}_n t_n^2 \nu_n}{z - t_n} \right).$$

Inclusion $A(z) \sum_{n} \frac{a_n \bar{b}_n t_n^2 \nu_n}{z - t_n} \in \mathcal{H}(T, A, \mu)$ also follows from the condition $\sum_{n} |a_n|^2 |t_n|^2 \nu_n < \infty$. Since, $A \notin \mathcal{H}(T, A, \mu)$ we conclude that the coefficient given by the left-hand side of (3.5) is zero.

The converse statement follows immediately from (3.6).

It is easy to describe the point spectrum and eigenfunctions of the model operator \mathcal{T}_G .

Lemma 3.9. $\eta \neq 0$ is an eigenvalue for \mathcal{T}_G if and only if $\lambda = \eta^{-1}$ is a zero of G. The corresponding eigenvector of \mathcal{T}_G is given by $\frac{G}{z-\lambda}$ while the reproducing kernel k_{λ} of $\mathcal{H}(T, A, \mu)$ is the eigenvector of \mathcal{T}_G^* corresponding to $\bar{\eta}$. *Proof.* We have $\mathcal{T}_G f = \eta f$ if and only if $f = c \frac{G}{1-\eta z} \in \mathcal{H}(T, A, \mu)$. Since f is entire, this is equivalent to $G(\lambda) = 0$, $\lambda = \eta^{-1}$. Note that 0 is an eigenvalue of \mathcal{T}_G if and only if $G \in \mathcal{H}(T, A, \mu)$.

Now, if $\lambda \neq 0$ and $G(\lambda) = 0$, then we have for any $f \in \mathcal{H}(T, A, \mu)$,

$$(f, \mathcal{T}_G^* k_\lambda) = (\mathcal{T}_G f, k_\lambda) = \frac{f(z) - f(0)G(z)}{z} \Big|_{z=\lambda} = \frac{f(\lambda)}{\lambda} = \frac{1}{\lambda} (f, k_\lambda),$$

whence $\mathcal{T}_G^* k_\lambda = \bar{\eta} k_\lambda$.

Remark 3.10. If G has a zero λ of multiplicity m > 1, then the corresponding root subspace for \mathcal{T}_G is spanned by $\frac{G}{(z-\lambda)^j}$, $j = 1, \ldots, m$. Similarly, one can find root vectors for \mathcal{T}_G^* which are essentially the reproducing kernels for the derivatives (see [10] for details). To avoid uninteresting technicalities we assume in what follows that all zeros of G are simple.

Thus, the properties of rank one perturbations of normal compact operators may be translated via the functional model to the geometric properties of systems of reproducing kernels in $\mathcal{H}(T, A, \mu)$, e.g.:

- \mathcal{L}^* is complete if and only if the system $\{k_\lambda\}_{\lambda \in \mathcal{Z}_G}$ is complete in $\mathcal{H}(T, A, \mu)$ (equivalently, \mathcal{Z}_G is a *uniqueness set* for $\mathcal{H}(T, A, \mu)$);
- \mathcal{L} is complete if and only if the biorthogonal system $\left\{\frac{G}{z-\lambda}\right\}_{\lambda\in\mathcal{Z}_G}$ is complete;
- \mathcal{L} admits spectral synthesis if and only if the system $\{k_{\lambda}\}_{\lambda \in \mathcal{Z}_{G}}$ is hereditarily complete (see the definition in Section 5).

Uniqueness sets in de Branges spaces (and in a more general setting of model subspaces of the Hardy space) were studied in [3, 16, 31]. Completeness of systems biorthogonal to systems of reproducing kernels was considered in [4], while in [5, 6] a more or less complete understanding of hereditary completeness in de Branges spaces was achieved.

4. Completeness of \mathcal{L} and \mathcal{L}^*

In this section we prove Theorems 2.1 and 2.2 on completeness of rank one perturbations and their adjoints. First we remark that the condition $b_n \neq 0$ (which is required in the functional model) does not lead to a loss of generality.

Lemma 4.1. Let \mathcal{A} be a compact normal operator with simple spectrum $\{s_n\}_{n\in\mathbb{N}}$, i.e., multiplication by z in $H = L^2(\nu)$, $\nu = \sum_n \nu_n \delta_{s_n}$. Let $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$. Then we can write $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where \mathcal{A}_j is multiplication by z in $H_j = L^2(\nu|_{\{s_n\}_{n\in\mathbb{N}_j}})$. Now let $\mathcal{L} = \mathcal{A} + a \otimes b$. Assume that with respect to decomposition $H = H_1 \oplus H_2$ we have $a = a_1 \oplus a_2$, $b = b_1 \oplus 0$. Consider the operator $\mathcal{L}_1 = \mathcal{A}_1 + a_1 \otimes b_1$ on H_1 . Then

- (i) if \mathcal{L}_1 is complete, then \mathcal{L} is complete;
- (ii) if \mathcal{L}_1^* is complete, then \mathcal{L}^* is complete.

Proof. For $u = u_1 \oplus u_2$ we have

$$\mathcal{L}u = \left(\mathcal{A}_1u_1 + (u_1, b_1)a_1\right) \oplus \left(\mathcal{A}_2u_2 + (u_1, b_1)a_2\right).$$

Denote by $(e_m)_{m \in N_1}$ the standard orthogonal basis of H_2 , $(e_m)_n = 0$, $m \neq n$, and $(e_m)_m = 1$. It is clear that $0 \oplus e_m$ is an eigenvector of \mathcal{L} corresponding to the eigenvalue s_m , $m \in N_2$. Denote by (f_k) the eigenvectors of \mathcal{L}_1 corresponding to the eigenvalues λ_k . For simplicity we assume that all eigenvalues are simple and also that $\{\lambda_k\} \cap \{s_m\}_{m \in N_2} = \emptyset$. If u is an eigenvector of \mathcal{L} and $u_1 \neq 0$, then $u_1 = f_k$ for some k. In this case for u_2 we have the equation:

$$\mathcal{A}_2 u_2 + (u_1, b_1) a_2 = \lambda_k u_2,$$

whence $u_2 = -(u_1, b_1)(\mathcal{A}_2 - \lambda_k I)^{-1}a_2$. In the case when $\lambda_k = s_m$ for some *m* we have a root vector instead of an eigenvector. We omit the details.

Thus, the eigenvectors of \mathcal{L} are of the form $\{0 \oplus e_m\} \cup \{f_k \oplus g_k\}$ for some $g_k \in H_2$. It is now obvious that this system is complete in H if and only if the system $\{f_k\}$ is complete in H_1 .

The statement for the adjoint operator is based on similar straightforward computations. Indeed,

$$\mathcal{L}^*u=ig(\mathcal{A}_1^*u_1+((u_1,a_1)+(u_2,a_2))b_1ig)\oplus\mathcal{A}_2^*u_2.$$

Let f_k^* be the eigenvectors of \mathcal{L}_1^* . Clearly, $u_1 \oplus 0$ is an eigenvector of \mathcal{L}^* if and only if $u_1 = f_k^*$ for some k. If $u_2 \neq 0$ and $\mathcal{L}^* u = \lambda u$, then $\lambda = \bar{s}_m, u_2 = e_m$ for some m and u_1 can be found from the equation $(\mathcal{L}_1^* - \bar{s}_m I)u_1 = -(e_m, a_2)b_1$. If $s_m \neq \lambda_k$ for any k, then there exists a unique vector h_m such that $h_m \oplus e_m$ is an eigenvector of \mathcal{L}^* (in the case $s_m = \lambda_k$ there is a root vector). Again, it is obvious that if the system $\{f_k^*\}$ is complete in H_1 , then the system $\{f_k^* \oplus 0\} \cup \{h_m \oplus e_m\}$ is complete in H. Note that the converse is not so clear.

4.1. **Proof of Theorem 2.1.** By Lemma 4.1 we may assume without loss of generality that $b_n \neq 0$ for any n. Then we can construct the functional model for \mathcal{L} in $\mathcal{H}(T, A, \mu)$. In view of the symmetry of conditions, it is sufficient to prove the theorem for the adjoint operator \mathcal{L}^* . In this case the eigenvectors are given by reproducing kernels $k_{\lambda}, \lambda \in \mathbb{Z}_G$.

Assume first that (2.1) holds, that is, the first moment is nonzero. If the system $\{k_{\lambda}\}_{\lambda \in \mathbb{Z}_G}$ is incomplete, then there exists a nonzero function $f \in \mathcal{H}(T, A, \mu)$ which vanish on \mathbb{Z}_G and

20

so f = GU for some entire function U. By Lemma 3.6 we have |G(z)U(z)| = o(|A(z)|), $|z| \to \infty$, outside a zero density set Ω_1 . On the other hand,

$$G(z) = A(z) \left(1 + \sum_{n} a_n \overline{b}_n t_n \nu_n + \sum_{n} \frac{a_n \overline{b}_n t_n^2 \nu_n}{z - t_n} \right).$$

The last sum in brackets can be estimated by (3.2) and we conclude that $|G(z)| \gtrsim |A(z)|$, $z \in \mathbb{C} \setminus \Omega_2$, for some Ω_2 of zero density. Hence, |U(z)| = o(1) when $|z| \to \infty$, $z \in \mathbb{C} \setminus \Omega$, for some set Ω of zero density.

Recall that all elements of $\mathcal{H}(T, A, \mu)$ are of finite order not exceeding the order of A. Hence, U is of finite order. By Theorem 3.4 $U \equiv 0$.

Now we assume that conditions (2.2) are satisfied, i.e., the moment with number N + 1 is the first nonzero moment. Using the elementary formula

$$\frac{1}{z-t_n} = \frac{1}{z} + \frac{t_n}{z^2} + \dots + \frac{t_n^{m-1}}{z^m} + \frac{t_n^m}{z^m(z-t_n)}$$

we get

$$G(z) = \frac{A(z)}{z^{N-1}} \sum_{n} \frac{a_n \bar{b}_n t_n^{N+1} \nu_n}{z - t_n}$$

whence, by Lemma 3.2, $|G(z)| \gtrsim |z|^{-N} |A(z)|$, $z \notin \Omega$, for some Ω of zero density. If $f = GU \in \mathcal{H}(T, A, \mu)$ then, arguing as above, we conclude that $|U(z)| = o(|z|^N)$ as $|z| \to \infty$ outside a set of zero density and so U is a polynomial of degree at most N - 1. Thus, the orthogonal complement to $\{k_{\lambda}\}_{\lambda \in \mathcal{Z}_{G}}$ is contained in $\mathcal{P}_{N-1}G$ and so dim $(\mathcal{E}(\mathcal{L}^{*}))^{\perp} \leq N$.

It remains to show that if $a \notin zL^2(\nu)$, then \mathcal{L}^* is complete. Indeed, by Lemma 3.8, $G \notin \mathcal{H}(T, A, \mu)$. Therefore $GU \notin \mathcal{H}(T, A, \mu)$ for any polynomial U. Thus, the orthogonal complement to $\{k_{\lambda}\}_{\lambda \in \mathcal{Z}_G}$ is trivial.

4.2. Parametrization of the orthogonal complement to a system biorthogonal to a system of reproducing kernels. Let $\{k_{\lambda}\}_{\lambda \in \Lambda}$ be a minimal system in $\mathcal{H}(T, A, \mu)$. We assume that $\{\lambda_n\} \cap T = \emptyset$. Let G be an entire function which vanishes on Λ and such that $\frac{G}{z-\lambda} \in \mathcal{H}(T, A, \mu)$. Such function exists due to minimality of the system $\{k_{\lambda}\}_{\lambda \in \Lambda}$; it is possible that $G \notin \mathcal{H}(T, A, \mu)$, but $G \in \operatorname{Assoc}(T, A, \mu)$. Then it is clear that the system $\{\frac{G(z)}{G'(\lambda)(z-\lambda)}\}_{\lambda \in \Lambda}$ is biorthogonal to $\{k_{\lambda}\}_{\lambda \in \Lambda}$. The following parametrization of the orthogonal complement to the biorthogonal system was suggested in [4].

Assume that $h(z) = A(z) \sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n} \in \mathcal{H}(T, A, \mu)$ is orthogonal to the system $\left\{\frac{G(z)}{z - \lambda}\right\}_{\lambda \in \Lambda}$. Note that for $g, h \in \mathcal{H}(T, A, \mu)$ one has $(g, h) = \sum_{n} \frac{g(t_n)\overline{h(t_n)}}{|A'(t_n)|^2 \mu_n}$ whence

$$\sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(t_n-\lambda)} = 0, \qquad \lambda \in \Lambda.$$

Therefore, the entire function $A(z) \sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z-t_n)}$ vanishes on Λ and we can write

(4.1)
$$A(z)\sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z-t_n)} = G(z)S(z)$$

for some entire function S. Note that, conversely, for any entire function S which satisfies equation (4.1) with some sequence $(c_n) \in \ell^2$, the function $h(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z - t_n}$ belongs to $\mathcal{H}(T, A, \mu)$ and is orthogonal to the system $\left\{\frac{G(z)}{z - \lambda}\right\}_{\lambda \in \Lambda}$. We denote the class of all functions S of the form (4.1) by S. Note that S is a linear space.

Comparing the values at t_n we see that $G(t_n)S(t_n) = G(t_n)\bar{c}_n\mu_n^{-1/2}$, whence $S(t_n) = \bar{c}_n\mu_n^{-1/2}$ and so $\sum_n |S(t_n)|^2\mu_n < \infty$. When equipped with the norm $||S||^2 = \sum_n |S(t_n)|^2\mu_n$ the space S becomes a Hilbert space and the mapping

$$S \mapsto A(z) \sum_{n} \frac{\overline{S(t_n)}\mu_n}{z - t_n}$$

is a unitary map from \mathcal{S} onto the orthogonal complement to $\left\{\frac{G(z)}{z-\lambda}\right\}_{\lambda\in\Lambda}$ in $\mathcal{H}(T,A,\mu)$.

We will need the following result from [4]:

Lemma 4.2. [4, Lemma 2.3] If $S \in S$, then $\frac{S(z)-S(w)}{z-w} \in S$ for any $w \in \mathbb{C}$.

Proof. Let $S \in \mathcal{S}$ and let $\lambda_0 \in \Lambda$. Then we have

$$\frac{G(z)S(z)}{A(z)} = \sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z-t_n)}, \qquad \frac{G(z)}{A(z)} = (z-\lambda_0)\sum_{n} \frac{G(t_n)}{A'(t_n)(t_n-\lambda_0)(z-t_n)}.$$

From this it is easy to show that

$$\frac{1}{z-w}\left(\frac{G(z)S(z)}{A(z)} - \frac{G(w)S(w)}{A(w)}\right) \quad \text{and} \quad \frac{S(w)}{z-w}\left(\frac{G(w)}{A(w)} - \frac{G(z)}{A(z)}\right)$$

have required representations.

22

		l
		L
		L
_	_	

4.3. **Proof of Theorem 2.2.** We will need the following lemma.

Lemma 4.3. Let d_n be such that

$$\sum_n \frac{|d_n|}{|t_n|} < \infty \qquad and \qquad \sum_n \frac{|d_n|^2}{|t_n|^{2N} \mu_n} < \infty$$

for some $N \in \mathbb{N}_0$. Then $f(z) = A(z) \sum_n \frac{d_n}{z - t_n} \in \mathcal{P}_N \mathcal{H}(T, A, \mu)$.

Proof. Since

(4.2)
$$\frac{1}{z-t_n} = -\frac{1}{t_n} - \frac{z}{t_n^2} - \dots - \frac{z^{N-1}}{t_n^N} + \frac{z^N}{t_n^N(z-t_n)},$$

we have

$$f(z) = A(z)P(z) + z^{N}A(z)\sum_{n} \frac{d_{n}}{t_{n}^{N}(z - t_{n})},$$

where P is a polynomial of degree at most N-1. By the hypothesis $A(z) \sum_{n t_n^N(z-t_n)} \in A(z)$ $\mathcal{H}(T, A, \mu)$ Also, $A \in \operatorname{Assoc}(T, A, \mu)$ and so $AP \in \mathcal{P}_N \mathcal{H}(T, A, \mu)$.

Proof of Theorem 2.2. By Lemma 4.1 we may assume that $a_n, b_n \neq 0$ for any n. Note that $\mathcal{L}^* = \mathcal{A}^* + b \otimes a$ also is a rank one perturbation of a normal operator. By the symmetry, we can prove the following statement which is equivalent to Theorem 2.2:

If \mathcal{L}^* is complete and

$$\sum_n \frac{|a_n|^2}{|b_n|^2 |t_n|^{2N}} < \infty,$$

then dim $(\mathcal{E}(\mathcal{L}))^{\perp} \leq N$. If, moreover, $b \notin zL^2(\nu)$, then \mathcal{L} is complete.

Consider the functional model for \mathcal{L} in the Cauchy-de Branges space $\mathcal{H} = \mathcal{H}(T, A, \mu)$. Recall that the eigenvectors of \mathcal{L} are of the form $\frac{G}{z-\lambda}$, $\lambda \in \mathcal{Z}_G$, where

$$G(z) = A(z) \left(1 + z \sum_{n} \frac{a_n \bar{b}_n t_n \nu_n}{z - t_n} \right),$$

while the eigenfunctions of \mathcal{L}^* are the reproducing kernels k_{λ} , $\lambda \in \mathcal{Z}_G$. By the discussion in the Subsection 4.2, if $f(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z - t_n}$ is orthogonal to $\left\{\frac{G}{z-\lambda}\right\}_{\lambda\in\mathcal{Z}_G}$, then there exists an entire function S from the corresponding space S such that

$$G(z)S(z) = A(z)\sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z-t_n)} = A(z)\sum_{n} \frac{d_n}{z-t_n}.$$

Note that the coefficients d_n satisfy

(4.3)
$$|d_n| = \left| \frac{G(t_n)c_n}{A'(t_n)\mu_n^{1/2}} \right| = \left| \frac{a_n b_n t_n^2 \nu_n c_n}{\mu_n^{1/2}} \right| = |a_n t_n \nu_n^{1/2} c_n|.$$

Hence,

(4.4)
$$\sum_{n} \frac{|d_n|^2}{|t_n|^{2N} \mu_n} \lesssim \sum_{n} \frac{|a_n|^2}{|b_n|^2 |t_n|^{2N}} < \infty,$$

and $GS \in \mathcal{P}_N \mathcal{H}(T, A, \mu)$ by Lemma 4.3.

If S has at least N zeros $z_1, \ldots z_N$ counting multiplicities, then

$$(z-z_1)^{-1}\dots(z-z_N)^{-1}G(z)S(z)\in\mathcal{H}(T,A,\mu).$$

This contradicts the fact that \mathcal{L} is complete and so the set \mathcal{Z}_G is a uniqueness set for $\mathcal{H}(T, A, \mu)$. Thus, $S = Q_1 e^{Q_2}$ where Q_1 is a polynomial of degree at most N - 1 and Q_2 is some entire function.

By Lemma 4.2, $\frac{S(z)-S(w)}{z-w} \in S$ for any $S \in S$. If $S = Q_1 e^{Q_2}$ and $Q_2 \neq const$, then there exists $w \in \mathbb{C}$ such that $\frac{S(z)-S(w)}{z-w}$ have infinitely many zeros and, repeating the above argument, we again come to a contradiction. We conclude that $S \subset \mathcal{P}_{N-1}$ and so $\dim (\mathcal{E}(\mathcal{L}))^{\perp} \leq N$.

Note that $S(t_n) = \bar{c}_n \mu_n^{-1/2}$ and so

$$\sum_{n} |S(t_n)|^2 |b_n|^2 |t_n|^2 \nu_n < \infty.$$

If S is a polynomial and $b \notin zL^2(\nu)$, then $S \equiv 0$ and so \mathcal{L} is complete.

Remark 4.4. For the case of selfadjoint operators a result similar to Theorem 2.2 was proved in [10] with pointwise (in place of integral) domination and for T with finite convergence exponent. It is obvious that in this case each of the conditions $|a_n|^2 \nu_n \gtrsim |t_n|^{-N}$ or $|b_n| \leq |t_n|^N |a_n|$ of [10] implies condition (2.3).

5. Spectral synthesis

5.1. Hereditary complete systems. The spectral synthesis for an operator is equivalent to a certain "strong completeness" property of its root vectors. Let $\{x_n\}_{n\in N}$ be a complete and minimal system in a separable Hilbert space H and let $\{\tilde{x}_n\}_{n\in N}$ be its biorthogonal system. The system $\{x_n\}$ is said to be *hereditarily complete* (or to be a *strong Markushevich basis*, or to admit *spectral synthesis*) if $x \in \overline{\text{Span}}\{(x, \tilde{x}_n)x_n\}$ for any $x \in H$. In other

words, any x can be approximated by partial sums of its Fourier series with respect to the biorthogonal pair $\{x_n\}, \{\tilde{x}_n\}$.

An equivalent definition of a hereditarily complete system is that for any partition $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, of the index set N, the *mixed system*

$${x_n}_{n \in N_1} \cup {\tilde{x}_n}_{n \in N_2}$$

is complete in H.

By a theorem of A. Markus [32, Theorem 4.1], a compact operator with complete set of root vectors $\{x_n\}$ admit the spectral synthesis if and only if the system $\{x_n\}$ is hereditarily complete. For a survey of hereditary completeness and its relations to operator theory we refer to [34, Chapter 4] (see also [6] and references therein).

5.2. Hereditary completeness for systems of reproducing kernels. Now let \mathcal{L} be a rank one perturbation of a compact normal operator and let \mathcal{T}_G be its functional model in a space $\mathcal{H}(T, A, \mu)$. Now the possibility of spectral synthesis for \mathcal{L} reduces to the hereditary completeness of the system $\{\frac{G}{z-\lambda}\}_{\lambda\in\mathcal{Z}_G}$ (equivalently, $\{k_\lambda\}_{\lambda\in\mathcal{Z}_G}$), that is, completeness of all mixed systems. A method for the study of hereditary completeness of systems of reproducing kernels in de Branges spaces was developed and successfully applied in [5, 6]. In particular, in [5] a long-standing problem of the spectral synthesis for exponential systems was solved. In [10] these results were used to study spectral synthesis for rank one perturbations of compact selfadjoint operators.

We will see that these methods apply to the Cauchy–de Branges spaces as well. Let $\{k_{\lambda}\}_{\lambda\in\Lambda}$ be a complete and minimal system of reproducing kernels and let $\{\frac{G(z)}{z-\lambda}\}_{\lambda\in\Lambda}$ be its biorthogonal system. Here G is the unique (up to multiplication by a constant) function in Assoc (T, A, μ) such that $\mathcal{Z}_G = \Lambda$. For the partition $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, consider the corresponding mixed system

(5.1)
$$\mathcal{K}(\Lambda_1, \Lambda_2) := \{k_\lambda\}_{\lambda \in \Lambda_1} \cup \left\{\frac{G}{z - \lambda}\right\}_{\lambda \in \Lambda_2}$$

We assume that $\Lambda \cap T = \emptyset$. This is not a restriction since both properties of being hereditarily complete or to be a Riesz basis of reproducing kernels in $\mathcal{H}(T, A, \mu)$ are stable under small perturbations of points.

One can parametrize the orthogonal complement to the system (5.1) similarly to Subsection 4.2. Choose two functions G_1 and G_2 such that $G = G_1G_2$, $\mathcal{Z}_{G_1} = \Lambda_1$, $\mathcal{Z}_{G_2} = \Lambda_2$. Assume that $f(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z - t_n} \in \mathcal{H}(T, A, \mu)$ is orthogonal to the system (5.1). The fact that $f \perp \{k_\lambda\}_{\lambda \in \Lambda_1}$ is equivalent to $f = G_1S_1$ for some entire function S_1 . As in

Subsection 4.2, the orthogonality $f \perp \left\{\frac{G(z)}{z-\lambda}\right\}_{\lambda \in \Lambda_2}$ can be rewritten as

$$\sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(t_n-\lambda)} = 0, \qquad \lambda \in \Lambda_2,$$

and so the entire function $A(z) \sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z-t_n)}$ is divisible by G_2 , We conclude that $f(z) = A(z) \sum_{n} \frac{c_n \mu_n^{1/2}}{z-t_n} \in \mathcal{H}(T, A, \mu)$ is orthogonal to the system (5.1)

We conclude that $f(z) = A(z) \sum_{n} \frac{c_n \mu_n^{-z}}{z - t_n} \in \mathcal{H}(T, A, \mu)$ is orthogonal to the system (5.1) if and only if there exist two entire functions S_1, S_2 such that we have two interpolation formulas:

(5.2)
$$A(z)\sum_{n} \frac{c_{n}\mu_{n}^{1/2}}{z-t_{n}} = G_{1}(z)S_{1}(z),$$
$$A(z)\sum_{n} \frac{G(t_{n})\bar{c}_{n}}{A'(t_{n})\mu_{n}^{1/2}(z-t_{n})} = G_{2}(z)S_{2}(z).$$

Conversely, if there exist two entire functions S_1, S_2 satisfying (5.2) for some $(c_n) \in \ell^2$, then $f(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z - t_n}$ is orthogonal to the system (5.1). We denote by S_{12} the set of all pairs (S_1, S_2) satisfying (5.1), this set parametrizes the orthogonal complement to (5.1). Note that if (S_1, S_2) and $(\tilde{S}_1, \tilde{S}_2)$ are in S_{12} , then $(S_1 + \tilde{S}_1, S_2 + \tilde{S}_2) \in S_{12}$. However, S_{12} is not a linear space: for $(S_1, S_2) \in S_{12}$ and $\alpha \in \mathbb{C}$, we have $(\alpha S_1, \bar{\alpha} S_2) \in S_{12}$.

Comparing the values at t_n , we get

$$A'(t_n)c_n\mu_n^{1/2} = G_1(t_n)S_1(t_n), \qquad G(t_n)\bar{c}_n\mu_n^{-1/2} = G_2(t_n)S_2(t_n),$$

whence

$$S_1(t_n)S_2(t_n) = |c_n|^2 A'(t_n).$$

Hence, if we put $S = S_1 S_2$, we see that the entire functions S and $A \sum_{n=1}^{\infty} \frac{|c_n|^2}{z-t_n}$ coincide on T. Thus, there exists an entire function R such that

$$S(z) = A(z) \left(\sum_{n} \frac{|c_n|^2}{z - t_n} + R(z)\right).$$

This representation will play the key role in the proofs of Theorems 2.3–2.5. Note that in the case when A is of finite order all functions in the space $\mathcal{H}(T, A, \mu)$ (for any admissible measure μ) are of finite order by Lemma 3.5. In particular, G, G_1S_1 , G_2S_2 are of finite order and so R is of finite order.

The following observation also will be useful. Let $(\tilde{S}_1, \tilde{S}_2)$ be another element of S_{12} corresponding to a function $g(z) = A(z) \sum_n \frac{d_n \mu_n^{1/2}}{z - t_n}$ orthogonal to $\mathcal{K}(\Lambda_1, \Lambda_2)$. Then, analogously,

$$A'(t_n)d_n\mu_n^{1/2} = G_1(t_n)\tilde{S}_1(t_n), \qquad G(t_n)\bar{d}_n\mu_n^{-1/2} = G_2(t_n)\tilde{S}_2(t_n),$$

and so $S_1(t_n)\tilde{S}_2(t_n) = c_n \bar{d}_n A'(t_n), \ \tilde{S}_1(t_n)S_2(t_n) = \bar{c}_n d_n A'(t_n)$, whence

$$\frac{S_1(z)\tilde{S}_2(z)}{A(z)} = \sum_n \frac{c_n \bar{d}_n}{z - t_n} + U(z), \qquad \frac{\tilde{S}_1(z)S_2(z)}{A(z)} = \sum_n \frac{\bar{c}_n d_n}{z - t_n} + V(z),$$

for some entire functions U and V.

5.3. **Proof of Theorem 2.3.** Assume that $b_n \neq 0$ for any n and $b \in zL^2(\nu)$. Then for μ defined by $\mu_n = |b_n|^2 |t_n|^2 \nu_n$ we have $\sum_n \mu_n < \infty$.

Let \mathcal{T}_G be the model operator in $\mathcal{H}(T, A, \mu)$ unitarily equivalent to \mathcal{L} . We need to show that \mathcal{T}_G admits the spectral synthesis, i.e., that any mixed system $\mathcal{K}(\Lambda_1, \Lambda_2)$ (where $\Lambda_1 \cup \Lambda_2 = \Lambda = \mathcal{Z}_G$) is complete in $\mathcal{H}(T, A, \mu)$. Note that the systems $\{k_\lambda\}_{\lambda \in \Lambda}$ and $\{\frac{G(z)}{z-\lambda}\}_{\lambda \in \Lambda}$ are complete by Theorem 2.1.

As in Theorem 2.1, we have

$$G(z) = A(z) \left(1 + \sum_{n} a_n \overline{b}_n t_n \nu_n + \sum_{n} \frac{a_n \overline{b}_n t_n^2 \nu_n}{z - t_n} \right),$$

whence, by (3.2), $|G(z)| \simeq |A(z)|$, $z \in \mathbb{C} \setminus \Omega_1$ for some Ω_1 of zero density. Assume that $f(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z - t_n} \in \mathcal{H}(T, A, \mu)$ is orthogonal to $\mathcal{K}(\Lambda_1, \Lambda_2)$ and let $(S_1, S_2) \in \mathcal{S}_{12}$ be the corresponding entire functions for which (5.2) holds. Multiplying the equations in (5.2) we get

(5.3)
$$\frac{G(z)}{A(z)} \left(\sum_{n} \frac{|c_n|^2}{z - t_n} + R(z) \right) = \left(\sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n} \right) \cdot \left(\sum_{n} \frac{G(t_n) \bar{c}_n}{A'(t_n) \mu_n^{1/2} (z - t_n)} \right),$$

which can be rewritten as

(5.4)
$$\frac{G(z)}{A(z)}R(z) = \left(\sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n}\right) \cdot \left(\sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z - t_n)}\right) - \frac{G(z)}{A(z)}\sum_{n} \frac{|c_n|^2}{z - t_n}$$

By (3.2), the right-hand side in (5.4) is o(1) as $|z| \to \infty$ outside a set of zero density. Hence, $|R(z)| = o(1), |z| \to \infty, z \notin \Omega_2, \Omega_2$ of zero density. Since R is of finite order (see Subsection 5.2), $R \equiv 0$ by Theorem 3.4. Thus, we have

(5.5)
$$\frac{G(z)}{A(z)} \sum_{n} \frac{|c_n|^2}{z - t_n} = \left(\sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n}\right) \cdot \left(\sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z - t_n)}\right).$$

By Lemma 3.2 and the fact that $|G(z)| \simeq |A(z)|$ outside a set of zero density, the modulus of the left-hand side is $\gtrsim |z|^{-1}$ outside a set of zero density. The second term in the

right-hand side of (5.5) is o(1) as $|z| \to \infty$ outside a set of zero density. Finally, note that $(c_n \mu_n^{1/2}) \in \ell^1$ since $\sum_n \mu_n < \infty$. Hence, by Lemma 3.2,

$$\sum_{n} \left| \frac{c_n \mu_n^{1/2}}{z - t_n} \right| \lesssim \frac{1}{|z|}$$

outside a set of zero density. Thus, the right-hand side of (5.5) is $o(|z|^{-1})$ as $|z| \to \infty$ outside a set of zero density, a contradiction.

5.4. **Proof of Theorem 2.4.** We will prove the equivalent result for the model operator \mathcal{T}_G in $\mathcal{H}(T, A, \mu)$. Let \mathcal{M} be \mathcal{T}_G invariant and let $\mathcal{E}(\mathcal{M}, \mathcal{T}_G) = \overline{\text{Span}} \{ \frac{G}{z-\lambda} : \lambda \in \Lambda_2 \}$, where $\Lambda_2 \subset \Lambda = \mathcal{Z}_G$. Then it is not difficult to show that $\mathcal{M}^{\perp} \supset \{k_{\lambda} : \lambda \in \Lambda_1 = \Lambda \setminus \Lambda_2\}$ (see, e.g., the proof of Lemma 4.2 in [32]). Hence,

$$\overline{\operatorname{Span}}\left\{\frac{G}{z-\lambda}:\ \lambda\in\Lambda_2\right\}\subset\mathcal{M}\subset\left(\overline{\operatorname{Span}}\left\{k_\lambda:\ \lambda\in\Lambda_1\right\}\right)^{\perp}.$$

Thus, dim $\mathcal{M} \ominus \mathcal{E}(\mathcal{M}, \mathcal{T}_G)$ does not exceed the dimension of the orthogonal complement to the mixed system $\mathcal{K}(\Lambda_1, \Lambda_2) = \{k_\lambda\}_{\lambda \in \Lambda_1} \cup \{\frac{G(z)}{z-\lambda}\}_{\lambda \in \Lambda_2}$. It remains to show that this dimension admits an estimate depending on N only.

Step 1. Let $f(z) = A(z) \sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n} \in \mathcal{H}(T, A, \mu)$ be orthogonal to $\mathcal{K}(\Lambda_1, \Lambda_2)$ and let $(S_1, S_2) \in \mathcal{S}_{12}$ be the corresponding entire functions for which (5.2) holds. As in the proof of Theorem 2.3, multiplying the equations in (5.2) we get (5.3)–(5.4). By (3.2), the right-hand side in (5.4) is o(1) as $|z| \to \infty$, $z \notin \Omega_1$ for some set Ω_1 of zero density. We use the fact that $|G| \leq |A|$ outside a set of zero density.

We consider the case $N \ge 1$, the case N = 0 is analogous. As in the proof of Theorem 2.1,

$$G(z) = \frac{A(z)}{z^{N-1}} \sum_{n} \frac{a_n \bar{b}_n t_n^{N+1} \nu_n}{z - t_n},$$

whence, by Lemma 3.2, $|G(z)| \gtrsim |z|^{-N} |A(z)|$, $z \notin \Omega_2$, for some Ω_2 of zero density. Thus, $|R(z)| = o(|z|^N)$ as $|z| \to \infty$, $z \notin \Omega$, Ω of zero density. Recall that R is of finite order. Then, by Theorem 3.4, R is a polynomial of degree at most N - 1.

Step 2. Let $f(z) = A(z) \sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n}$ and $g(z) = A(z) \sum_{n} \frac{d_n \mu_n^{1/2}}{z - t_n}$ be two mutually orthogonal functions in $(\mathcal{K}(\Lambda_1, \Lambda_2))^{\perp}$ and let corresponding elements (S_1, S_2) and $(\tilde{S}_1, \tilde{S}_2)$ of \mathcal{S}_{12} satisfy

$$\frac{S_1(z)S_2(z)}{A(z)} = \sum_n \frac{|c_n|^2}{z - t_n} + R(z), \qquad \frac{\tilde{S}_1(z)\tilde{S}_2(z)}{A(z)} = \sum_n \frac{|d_n|^2}{z - t_n} + \tilde{R}(z),$$

$$\frac{S_1(z)\tilde{S}_2(z)}{A(z)} = \sum_n \frac{c_n \bar{d}_n}{z - t_n}, \qquad \frac{\tilde{S}_1(z)S_2(z)}{A(z)} = \sum_n \frac{\bar{c}_n d_n}{z - t_n},$$

i.e. in the "cross-products" $S_1 \tilde{S}_2$ and $\tilde{S}_1 S_2$ there are no polynomial terms. Since f and g are orthogonal in $\mathcal{H}(T, A, \mu)$, we have $\sum_n c_n \bar{d}_n = 0$. Hence, by Lemma 3.2,

$$\frac{S_1(z)\tilde{S}_2(z)}{A(z)} = o\left(\frac{1}{|z|}\right), \qquad \frac{\tilde{S}_1(z)S_2(z)}{A(z)} = o\left(\frac{1}{|z|}\right), \qquad |z| \to \infty, \quad z \notin \Omega_1,$$

where Ω_1 is a set of zero density. On the other hand, by the same Lemma 3.2,

$$\left|\frac{S_1(z)S_2(z)}{A(z)}\right| \gtrsim \frac{1}{|z|}, \qquad \left|\frac{S_1(z)S_2(z)}{A(z)}\right| \gtrsim \frac{1}{|z|}, \qquad z \notin \Omega_2,$$

for a set Ω_2 of zero density. Note that, e.g., $\frac{S_1(z)S_2(z)}{A(z)}$ has even a larger estimate from below if R is a nonzero polynomial. Combined together, these estimates obviously lead to a contradiction.

Step 3. In what follows we will denote by C the set of all functions of the form $\sum_{n} \frac{d_n}{z-t_n}$, where $(d_n) \in \ell^1$.

Assume that dim $(\mathcal{K}(\Lambda_1, \Lambda_2))^{\perp} > (N+1)^2$ and choose in $(\mathcal{K}(\Lambda_1, \Lambda_2))^{\perp}$ an orthogonal system $\{f_j\}_{j=1}^m \cup \{g_k\}_{k=1}^{N+1}$ where m > N(N+1). Let (S_1^j, S_2^j) and $(\tilde{S}_1^k, \tilde{S}_2^k)$ be the elements of \mathcal{S}_{12} corresponding to f_j and g_k , respectively (see Subsection 5.2). If

$$f_j(z) = A(z) \sum_n \frac{c_n^j \mu_n^{1/2}}{z - t_n}, \qquad g_k(z) = A(z) \sum_n \frac{d_n^k \mu_n^{1/2}}{z - t_n},$$

then there exist polynomials U_{jk} of degree at most N-1 such that

$$\frac{S_1^j(z)\tilde{S}_2^k(z)}{A(z)} = \sum_n \frac{c_n^j \bar{d}_n^k}{z - t_n} + U_{jk}(z).$$

Since $m > (N + 1) \dim \mathcal{P}_{N-1}$, there exist $\{\alpha_j\}_{j=1}^m$ such that $\sum_{j=1}^m \alpha_j U_{jk} \equiv 0$ for any $k = 1, \ldots N + 1$. Put $f = \sum_{j=1}^m \alpha_j f_j$ and let (S_1, S_2) be the corresponding element of \mathcal{S}_{12} . Then we have $\frac{S_1 \tilde{S}_2^k}{A} \in \mathcal{C}$ for any k. On the other hand, there exist polynomials $V_k \in \mathcal{P}_{N-1}$ such that

$$\frac{\tilde{S}_1^k S_2}{A} - V_k \in \mathcal{C}.$$

Choose $\{\beta_k\}_{k=1}^{N+1}$ such that $\sum_{k=1}^{N+1} \beta_k V_k \equiv 1$ and put $g = \sum_{k=1}^{N+1} \beta_k g_k$. If we denote by $(\tilde{S}_1, \tilde{S}_2)$ the element of S_{12} corresponding to g, then we have both $\frac{S_1\tilde{S}_2}{A} \in \mathcal{C}$ and $\frac{\tilde{S}_1S_2}{A} \in \mathcal{C}$, and, moreover, (f, g) = 0. As we have already seen in Step 2, this leads to a contradiction. This shows that

$$\dim \mathcal{M} \ominus \mathcal{E}(\mathcal{M}, \mathcal{T}_G) = \dim \left(\mathcal{K}(\Lambda_1, \Lambda_2) \right)^{\perp} \leq (N+1)^2.$$

29

Theorem 2.4 is proved.

5.5. Proof of Theorem 2.5. We will need the following technical lemma.

Lemma 5.1. Let $T = \{t_n\}$ be power separated with power M, i.e., T satisfies (2.4). Let $d_n^{(j)}$, $j = 1, \ldots 4$, be such that $d_n^{(1)} d_n^{(2)} = d_n^{(3)} d_n^{(4)}$ for any n and, for some $N \in \mathbb{N}_0$,

$$\sum_{n} \frac{|d_n^{(j)}|}{|t_n|} < \infty, \quad j = 1, \dots, 4, \qquad and \qquad \sum_{n} \frac{|d_n^{(j)}|^2}{|t_n|^{2N}\mu_n} < \infty, \quad j = 1, 3.$$

Then

$$f(z) = A(z) \left(\sum_{n} \frac{d_n^{(1)}}{z - t_n} \cdot \sum_{n} \frac{d_n^{(2)}}{z - t_n} - \sum_{n} \frac{d_n^{(3)}}{z - t_n} \cdot \sum_{n} \frac{d_n^{(4)}}{z - t_n} \right)$$

belongs to $\mathcal{P}_{M+N}\mathcal{H}(T, A, \mu)$.

Proof. Using formula (4.2) we can rewrite

$$\sum_{n} \frac{d_n^{(j)}}{z - t_n} = P_j(z) + z^{M+1} \sum_{n} \frac{p_n^{(j)}}{z - t_n}, \qquad j = 2, 4,$$

where P_j is the polynomial of degree at most M and $p_n^{(j)} = d_n^{(j)} t_n^{-M-1}$ satisfies $\sum_n |t_n|^M |p_n^{(j)}| < \infty$. By Lemma 4.3,

$$A(z)P_2(z)\sum_n \frac{d_n^{(1)}}{z-t_n}, \quad A(z)P_4(z)\sum_n \frac{d_n^{(3)}}{z-t_n} \in \mathcal{P}_{M+N}\mathcal{H}(T,A,\mu)$$

It remains to show that

$$A(z)\left(\sum_{n}\frac{d_n^{(1)}}{z-t_n}\cdot\sum_{n}\frac{p_n^{(2)}}{z-t_n}-\sum_{n}\frac{d_n^{(3)}}{z-t_n}\cdot\sum_{n}\frac{p_n^{(4)}}{z-t_n}\right)\in\mathcal{P}_N\mathcal{H}(T,A,\mu).$$

Note that by condition $d_n^{(1)}d_n^{(2)} = d_n^{(3)}d_n^{(4)}$ the coefficient at $(z - t_n)^{-2}$ is zero. We have

$$\sum_{n} \frac{d_n^{(1)}}{z - t_n} \cdot \sum_{m} \frac{p_m^{(2)}}{z - t_m} = \frac{d_n^{(1)} p_n^{(2)}}{(z - t_n)^2} + \sum_{n} \left(\sum_{m \neq n} \frac{p_m^{(2)}}{t_n - t_m}\right) \frac{d_n^{(1)}}{z - t_n}.$$

It follows from power separation that $|t_n - t_m| \gtrsim |t_m|^{-M}$, $n \neq m$, and so $\left|\sum_{m \neq n} \frac{p_m^{(2)}}{t_n - t_m}\right| \lesssim 1$. Now the inclusion

$$A(z)\sum_{n}\left(\sum_{m\neq n}\frac{p_m^{(2)}}{t_n-t_m}\right)\frac{d_n^{(1)}}{z-t_n}\in\mathcal{P}_N\mathcal{H}(T,A,\mu)$$

follows from Lemma 4.3.

30

- 6	_	_	_	

Proof of Theorem 2.5. By the symmetry, we can prove the spectral synthesis up to a finite defect for the operator \mathcal{L}^* in place of \mathcal{L} . So we interchange the roles of a and b and assume that \mathcal{L}^* is complete,

$$\sum_{n} \frac{|a_n|^2}{|b_n|^2 |t_n|^{2N}} < \infty, \qquad |b_n|^2 \nu_n \gtrsim |t_n|^{-2N-2}$$

We show that the adjoint model operator \mathcal{T}_{G}^{*} in $\mathcal{H}(T, A, \mu)$ admits spectral synthesis up to a finite defect. Assume that $f(z) = A(z) \sum_{n} \frac{c_{n}\mu_{n}^{1/2}}{z-t_{n}} \in \mathcal{H}(T, A, \mu)$ is orthogonal to the mixed system $\mathcal{K}(\Lambda_{1}, \Lambda_{2})$ defined by (5.1), and let $(S_{1}, S_{2}) \in \mathcal{S}_{12}$ be the corresponding entire functions for which (5.2) holds. Multiplying the equations in (5.2) we get equality (5.3) which can be rewritten as follows:

$$G(z)R(z) = A(z) \left(\sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z-t_n)} \cdot \sum_{n} \frac{c_n\mu_n^{1/2}}{z-t_n} - \frac{G(z)}{A(z)}\sum_{n} \frac{|c_n|^2}{z-t_n} \right).$$

Note that

$$\frac{G(z)}{A(z)} = 1 + z \sum_{n} \frac{a_n \overline{b}_n t_n \nu_n}{z - t_n},$$

Put

$$d_n^{(1)} = \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}}, \quad d_n^{(2)} = c_n\mu_n^{1/2}, \quad d_n^{(3)} = a_n\bar{b}_nt_n\nu_n, \quad d_n^{(4)} = |c_n|^2.$$

Then $d_n^{(1)}$, $d_n^{(2)}$, $d_n^{(3)}$ and $d_n^{(4)}$ satisfy conditions of Lemma 5.1. The fact that $\sum_n |d_n^{(1)}|^2 |t_n|^{-2N} \mu_n^{-1} < \infty$ follows from (4.3)–(4.4) in the proof of Theorem 2.2, while it is clear that

$$\sum_{n} |d_n^{(3)}|^2 \mu_n^{-1} = \sum_{n} |a_n|^2 \nu_n < \infty.$$

Now we may write

$$G(z)R(z) = -A(z)\sum_{n} \frac{|c_n|^2}{z - t_n} + A(z) \left(\sum_{n} \frac{d_n^{(1)}}{z - t_n} \cdot \sum_{n} \frac{d_n^{(2)}}{z - t_n} - \sum_{n} \frac{d_n^{(3)}}{z - t_n} \cdot \sum_{n} \frac{d_n^{(4)}}{z - t_n}\right).$$

By Lemma 4.3, $A(z) \sum_{n} \frac{|c_n|^2}{z-t_n} \in \mathcal{P}_N \mathcal{H}(T, A, \mu)$. Indeed,

$$\sum_{n} \frac{|c_n|^4}{|t_n|^{2N} \mu_n} = \sum_{n} \frac{|c_n|^4}{|t_n|^{2N+2} |b_n|^2 \nu_n} \lesssim \sum_{n} |c_n|^4 < \infty$$

by (2.5). Hence, by Lemma 5.1, $GR \in \mathcal{P}_{M+N}\mathcal{H}(T, A, \mu)$.

Recall that \mathcal{T}_G^* is complete and so there is no nonzero function of the form $GU \in \mathcal{H}(T, A, \mu)$. If R has at least M + N zeros counting multiplicities, then, dividing R by a polynomial P of degree M + N we have $GRP^{-1} \in \mathcal{H}(T, A, \mu)$, a contradiction. Thus, $R = Pe^Q$, where P is a polynomial of degree at most M + N - 1 and Q is some polynomial

(recall that R is of finite order). Assume that $Q \neq const$. Since $G \in Assoc (T, A, \mu)$, the function $G(z)\frac{R(z)-R(w)}{z-w}$ belongs to $\mathcal{P}_{M+N}\mathcal{H}(T, A, \mu)$ and has infinitely many zeros for all w except at most 1, a contradiction. Thus, we conclude that $Q \equiv const$ and so R is a polynomial of degree at most M + N - 1.

The rest of the proof is the same as the proof of Theorem 2.4. If we assume that $\dim (\mathcal{K}(\Lambda_1, \Lambda_2))^{\perp} > (M + N + 1)^2$, then there exist two mutually orthogonal functions f, g in $(\mathcal{K}(\Lambda_1, \Lambda_2))^{\perp}$ with the properties as in Step 2 of the proof of Theorem 2.4 which again leads to a contradiction.

6. Counterexamples

In this section we prove Theorem 2.6. The following observation is trivial, but leads to a substantial simplification of the construction compared to [6, 10]. It says that it is sufficient to construct counterexamples on an arbitrarily sparse part of the spectrum.

Lemma 6.1. Let \mathcal{A} be a compact normal cyclic operator and let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ with respect to decomposition $H = H_1 \oplus H_2$. Let $K \in \mathbb{N} \cup \{\infty\}$. Assume that either there exists a rank one perturbation \mathcal{L}_1 of \mathcal{A}_1 such that \mathcal{L}_1 is complete and dim $(\mathcal{E}(\mathcal{L}_1^*))^{\perp} = K$, or \mathcal{L}_1 and \mathcal{L}_1^* are complete, but dim $\mathcal{M}_1 \oplus \mathcal{E}(\mathcal{M}_1, \mathcal{L}_1) = K$ for some \mathcal{L}_1 -invariant subspace \mathcal{M}_1 . Then for the operator $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{A}_2$ we have, respectively, that \mathcal{L} is complete, but dim $(\mathcal{E}(\mathcal{L}^*))^{\perp} = K$, or \mathcal{L} and \mathcal{L}^* are complete, but dim $\mathcal{M} \oplus \mathcal{E}(\mathcal{M}, \mathcal{L}) = K$ for some \mathcal{L} -invariant subspace \mathcal{M} .

Proof. The proof is obvious. Consider, e.g., the statement about synthesis. Assume that there exists a \mathcal{L}_1 -invariant subspace \mathcal{M}_1 such that dim $(\mathcal{M}_1 \ominus \mathcal{E}(\mathcal{M}_1, \mathcal{L}_1)) = K$. Then $\mathcal{M} = \mathcal{M}_1 \oplus \{0\}$ is \mathcal{L} -invariant and $\mathcal{E}(\mathcal{M}, \mathcal{L}) = \mathcal{E}(\mathcal{M}_1, \mathcal{L}_1) \oplus \{0\}$.

It is clear that in Lemma 6.1 \mathcal{L} is a rank one perturbation of \mathcal{A} . Thus we see that it is sufficient to construct examples for the restriction of \mathcal{A} to any invariant (with respect to \mathcal{A} and \mathcal{A}^*) subspace. We will choose $H_1 = L^2(\nu_1)$ where ν_1 is the restriction of the initial measure μ to an infinite, but sparse part of the spectrum $\{s_n\}$. By sparseness we will mean (Hadamard-type) *lacunarity* of the inverse spectrum $\{t_n\}, t_n = s_n^{-1}$:

(6.1)
$$\inf_{n} \left| \frac{t_{n+1}}{t_n} \right| > 1.$$

In view of Lemma 6.1 in our examples below we can always assume that \mathcal{A} is a compact normal operator with spectrum $\{s_n\}$ such that $\{t_n\}$ is a lacunary sequence.

6.1. Proof of Theorem 2.6: biorthogonal systems with finite defect. Let $N \in \mathbb{N}$. We will prove the following statement:

Let $T = \{t_n\}$ be a lacunary sequence satisfying (6.1). Then there exists a space $\mathcal{H}(T, A, \mu)$ and a function $G \in \operatorname{Assoc}(T, A, \mu) \setminus \mathcal{H}(T, A, \mu)$ with simple zeros such that for $\Lambda = \mathcal{Z}_G$ the system $\{k_\lambda\}_{\lambda \in \Lambda}$ is complete in $\mathcal{H}(T, A, \mu)$, but

$$\dim\left(\mathcal{H}(T,A,\mu)\ominus\overline{\operatorname{Span}}\left\{\frac{G}{z-\lambda}:\lambda\in\Lambda\right\}\right)=N.$$

Thus, the model operator \mathcal{T}_G is incomplete with defect N, while its biorthogonal is complete. Then $\mathcal{L} = \mathcal{T}_G^*$, the adjoint to the model operator, will be a rank one perturbation of a compact normal operator with spectrum $\{\bar{s}_n\}$, $s_n = t_n^{-1}$, and its adjoint will be incomplete with defect N.

Let $A(z) = \prod \left(1 - \frac{1}{t_n}\right)$. Put $\mu_n = |t_n|^{-2N}$. Let $\tilde{t}_n = t_n + \frac{1}{2}$, $\tilde{T} = {\tilde{t}_n}$ and $\tilde{A}(z) = \prod (1 - z/\tilde{t}_n)$. Then, by the standard estimates of infinite products with lacunary zeros, for $z \in \mathbb{C} \setminus T$,

(6.2)
$$\left|\frac{\tilde{A}(z)}{A(z)}\right| \asymp \left|\frac{\operatorname{dist}(z,\tilde{T})}{\operatorname{dist}(z,T)}\right|, \quad \left|\frac{\tilde{A}(t_n)}{A'(t_n)}\right| \asymp 1, \quad t_n \in T.$$

We will also use the following simple observation: if $\sum_n |d_n| < \infty$, then

(6.3)
$$\left|\sum_{n} \frac{d_n}{z - t_n}\right| = o(1), \quad \left|\sum_{n} \frac{t_n d_n}{z - t_n}\right| = o(|z|), \quad |z| \to \infty, \quad \text{dist} (z, T) \ge 1.$$

In particular, |f(z)| = o(|zA(z)|) for any $f \in \mathcal{H}(T, A, \mu)$ when $|z| \to \infty$, dist $(z, T) \ge 1$. Let P be a polynomial of degree N such that $\mathcal{Z}_P \subset \mathcal{Z}_{\tilde{A}}$. Put $G = \tilde{A}/P$, $\Lambda = \mathcal{Z}_G$.

Step 1: $G \in Assoc (T, A, \mu) \setminus \mathcal{H}(T, A, \mu)$. We have

$$\frac{|G(t_n)|^2}{|A'(t_n)|^2\mu_n} = \frac{|\tilde{A}(t_n)|^2}{|A'(t_n)|^2|P(t_n)|^2\mu_n} \approx 1$$

whence $G \notin \mathcal{H}(T, A, \mu)$ by Lemma 3.6. On the other hand, $\sum_{n} \frac{|G(t_n)|^2}{|A'(t_n)|^2 |t_n|^2 \mu_n} < \infty$. Let us show that $\frac{G}{z-\lambda} \in \mathcal{H}(T, A, \mu)$ for any $\lambda \in \Lambda$. We need to show that

$$\frac{G(z)}{(z-\lambda)A(z)} = \sum_{n} \frac{G(t_n)}{(t_n-\lambda)A'(t_n)(z-t_n)}.$$

The following standard argument will be repeated several times in the constructions below. Put

$$H(z) = \frac{G(z)}{(z-\lambda)A(z)} - \sum_{n} \frac{G(t_n)}{(t_n-\lambda)A'(t_n)(z-t_n)}.$$

The poles cancel and so H is an entire function. We have $|G(z)| \simeq |P(z)|^{-1}|A(z)|$, dist $(z,T) \ge 1$. Also, by (6.3), the last series is o(1) when $|z| \to \infty$ and dist $(z,T) \ge 1$. Hence, $|H(z)| \to 0$, $|z| \to \infty$, and so $H \equiv 0$. We conclude that $\frac{G}{z-\lambda} \in \mathcal{H}(T, A, \mu)$.

Step 2: $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is complete in $\mathcal{H}(T, A, \mu)$.

Assume that $GU \in \mathcal{H}(T, A, \mu)$ for some entire U. Then, $G(z)U(z) = A(z)\sum_{n} \frac{d_n \mu_n^{1/2}}{z-t-n}$ for some $(d_n) \in \ell^2$ and so, by (6.3), |G(z)U(z)/A(z)| = o(|z|) when $|z| \to \infty$, dist $(z, T) \ge 1$. Also,

$$\left|\frac{G(z)U(z)}{A(z)}\right| = \left|\frac{\tilde{A}(z)U(z)}{A(z)P(z)}\right| \asymp \left|\frac{U(z)}{P(z)}\right|, \quad \text{dist}(z,T) \ge 1.$$

We conclude that U is a polynomial.

Since $GU \in \mathcal{H}(T, A, \mu)$ we have

$$\sum_{n} |U(t_n)|^2 \asymp \sum_{n} \frac{|G(t_n)U(t_n)|^2}{|A'(t_n)|^2 \mu_n} < \infty,$$

and so $U \equiv 0$.

Step 3: dim
$$\left\{\frac{G(z)}{z-\lambda} : \lambda \in \Lambda\right\}^{\perp} = N.$$

We use parametrization of the orthogonal complement to a system of the form $\left\{\frac{G}{z-\lambda}\right\}_{\lambda\in\Lambda}$ introduced in Subsection 4.2. It is parametrized by the space S which consists of entire functions S such that

(6.4)
$$S(z)G(z) = A(z)\sum_{n} \frac{\bar{c}_n G(t_n)}{A'(t_n)\mu_n^{1/2}(z-t_n)}$$

where $(c_n) \in \ell^2$. Assume first that $S \in S$. Then it follows from representation (6.4) that $S(z)G(z) = A(z)\sum_n \frac{d_n}{z-t_n}$, where $\sum_n |t_n|^{-1}|d_n| < \infty$, and so, by (6.3),

$$\left|\frac{S(z)\tilde{A}(z)}{A(z)P(z)}\right| = \left|\frac{S(z)G(z)}{A(z)}\right| = o(|z|), \quad \text{dist}(z,T) \ge 1.$$

It follows from the estimates (6.2) that S is a polynomial. Also we have $c_n = \overline{S(t_n)} \mu_n^{1/2}$, and so $\sum_n |S(t_n)|^2 \mu_n < \infty$. Since $\mu_n = |t_n|^{-2N}$ we conclude that degree of S does not exceed N-1.

Conversely, let $S \in \mathcal{P}_{N-1}$ and put $c_n = \overline{S(t_n)} \mu_n^{1/2}$. We need to show that (6.4) holds, that is, there is the interpolation formula

(6.5)
$$\frac{S(z)G(z)}{A(z)} = \sum_{n} \frac{S(t_n)G(t_n)}{A'(t_n)(z-t_n)}.$$

We argue as in Step 1. The residues at the points t_n coincide and so the difference between the left-hand side and the right-hand side of (6.5) is an entire function. We have

$$\left|\frac{S(z)G(z)}{A(z)}\right| \lesssim \frac{1}{|z|}, \qquad \left|\sum_{n} \frac{S(t_n)G(t_n)}{A'(t_n)(z-t_n)}\right| = o(1), \qquad |z| \to \infty, \quad \text{dist} \, (z,T) \ge 1,$$

and so (6.5) holds. Thus, $S = \mathcal{P}_{N-1}$ and dim $\left\{\frac{G}{z-\lambda} : \lambda \in \Lambda\right\}^{\perp} = \dim S = N.$

6.2. Proof of Theorem 2.6: biorthogonal systems with infinite defect. As in Subsection 6.1, for a lacunary sequence T, we construct a measure μ and a function $G \in$ Assoc $(T, A, \mu) \setminus \mathcal{H}(T, A, \mu)$ such that for $\Lambda = \mathcal{Z}_G$ the system $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is complete in $\mathcal{H}(T, A, \mu)$, but

$$\dim\left(\mathcal{H}(T,A,\mu)\ominus\overline{\operatorname{Span}}\left\{\frac{G}{z-\lambda}:\lambda\in\Lambda\right\}\right)=\infty.$$

The following lemma from [6] will be crucial here. One can also use a simpler, but less sharp result of [10, Lemma 7.1]. As usual, for a sequence Γ , we denote by n_{Γ} its counting function: $n_{\Gamma}(r) = \#\{\gamma \in \Gamma : |\gamma| < r\}$. For an entire function f we write n_f in place of $n_{\mathcal{Z}(f)}$.

Lemma 6.2. ([6, Lemma 9.2]) Let Γ be a lacunary sequence and let f be an entire function of zero exponential type such that

$$\int_0^R \frac{n_f(r)}{r} dr = o\left(\int_0^R \frac{n_\Gamma(r)}{r} dr\right), \qquad R \to \infty$$

If $\{f(\gamma)\}_{\gamma\in\Gamma} \in \ell^{\infty}$, then f is a constant.

Step 1: Construction of G.

Choose an infinite $T_0 \subset T$ such that $n_{T_0} = o(n_{T \setminus T_0})$. As in the previous subsection let $\tilde{T} = {\tilde{t}_n}, \tilde{t}_n = t_n + \frac{1}{2}$, and let

$$\tilde{A}(z) = \prod_{n} \left(1 - \frac{1}{\tilde{t}_n}\right), \qquad U(z) = \prod_{t_n \in T_0} \left(1 - \frac{1}{\tilde{t}_n}\right)$$

Define the measure $\mu = \sum_{n} \mu_n \delta_{t_n}$ by

$$\mu_n = \begin{cases} 1, & t_n \in T_0, \\ |U'(t_n)|^{-2}, & t_n \in T \setminus T_0. \end{cases}$$

Finally, put $G = \tilde{A}/U$. Then we have

$$\frac{G(z)}{A(z)} = \sum_{n} \frac{\tilde{A}(t_n)}{A'(t_n)U(t_n)(z - t_n)},$$
$$\frac{G(z)}{(z - \lambda)A(z)} = \sum_{n} \frac{\tilde{A}(t_n)}{A'(t_n)U(t_n)(t_n - \lambda)(z - t_n)}, \qquad \lambda \in \Lambda = \mathcal{Z}_G$$

Convergence of the above series follows from estimate (6.2) and the fact that $|U(t_n)| \gtrsim |t_n|^N$ for any N > 0. The interpolation formulas follow by the same arguments as in Subsection 6.1, Step 1. We also have

$$\sum_{n} \frac{|G(t_n)|^2}{|A'(t_n)|^2 \mu_n} \asymp \sum_{n} \frac{1}{|U(t_n)|^2 \mu_n} = \infty,$$

since $|U(t_n)|^2 \mu_n = 1$, $t_n \in T \setminus T_0$. However, for any $\lambda \in \Lambda = \mathcal{Z}_G$,

$$\sum_{n} \frac{|G(t_n)|^2}{|t_n - \lambda|^2 |A'(t_n)|^2 \mu_n} \asymp \sum_{t_n \in T_0} \frac{1}{|t_n - \lambda|^2 |U(t_n)|^2} + \sum_{t_n \in T \setminus T_0} \frac{1}{|t_n - \lambda|^2} < \infty.$$

Thus, $G \notin \mathcal{H}(T, A, \mu)$, but $\frac{G(z)}{z-\lambda} \in \mathcal{H}(T, A, \mu), \lambda \in \Lambda$.

Step 2: $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is complete in $\mathcal{H}(T, A, \mu)$.

Assume that the system $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is not complete and so there is an entire function V such that $GV \in \mathcal{H}(T, A, \mu)$. We have, by (6.3),

$$\left|\frac{V(z)}{U(z)}\right| = \left|\frac{G(z)V(z)}{\tilde{A}(z)}\right| \asymp \left|\frac{G(z)V(z)}{A(z)}\right| \lesssim |z|, \quad \text{dist} (z,T) \ge 1.$$

Hence, $\log M_V(r) \leq \log M_U(r)$ (where $M_f(r) = \max_{|z|=r} |f(z)|$) and so, by the classical Iensen formula,

$$\int_0^R \frac{n_V(r)}{r} dr \lesssim \int_0^R \frac{n_{T_0}(r)}{r} dr$$

On the other hand,

$$\sum_{n} \frac{|V(t_n)|^2}{|U(t_n)|^2 \mu_n} \asymp \sum_{n} \frac{|G(t_n)V(t_n)|^2}{|A'(t_n)|^2 \mu_n} < \infty,$$

whence $\sum_{t_n \in T \setminus T_0} |V(t_n)|^2 < \infty$. Since $n_{T_0} = o(n_{T \setminus T_0}), V \equiv 0$ by Lemma 6.2.

Step 3: dim $\left\{\frac{G(z)}{z-\lambda} : \lambda \in \Lambda\right\}^{\perp} = \infty$.

Construct an entire function

$$S(z) = \prod_{t_n \in T_0} \left(1 - \frac{z}{t_n + \varepsilon_n} \right),$$

where $\varepsilon_n \neq 0$ are chosen to be so small that $\sum_{t_n \in T_0} |S(t_n)|^2 < \infty$. Then, similarly to (6.2), we have $|S(t_n)| \simeq |U(t_n)|$, $t_n \in T \setminus T_0$. Let P be an arbitrary nonconstant polynomial such that $\mathcal{Z}_P \subset \mathcal{Z}_S$. Then

$$\sum_{n} \left| \frac{G(t_n)S(t_n)}{A'(t_n)P(t_n)} \right| < \infty$$

and

$$\sum_{n} \left| \frac{S(t_n)}{P(t_n)} \right|^2 \mu_n = \sum_{t_n \in T_0} \left| \frac{S(t_n)}{P(t_n)} \right|^2 + \sum_{t_n \in T \setminus T_0} \frac{|S(t_n)|^2}{|P(t_n)|^2 |U(t_n)|^2} < \infty.$$

Put $c_n = \overline{S(t_n)} \mu_n^{1/2} / \overline{P(t_n)}$. Then, by the argument used in Subsection 6.1, Step 1, we have the interpolation formula

$$\frac{G(z)S(z)}{P(z)A(z)} = \sum_{n} \frac{G(t_n)S(t_n)}{A'(t_n)P(t_n)(z-t_n)} = \sum_{n} \frac{G(t_n)\bar{c}_n}{A'(t_n)\mu_n^{1/2}(z-t_n)}$$

Hence, by Subsection 4.2, $f(z) = A(z) \sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n}$ is orthogonal to $\left\{\frac{G(z)}{z - \lambda}\right\}_{\lambda \in \Lambda}$ and S/P belongs to the corresponding space S. Choose a sequence P_j of polynomials such that P_j is a polynomials of degree j and $Z_{P_j} \subset Z_S$. Clearly, the system S/P_j is linearly independent in S whence dim $\left\{\frac{G(z)}{z - \lambda} : \lambda \in \Lambda\right\}^{\perp} = \infty$.

6.3. Proof of Theorem 2.6: complete perturbations without synthesis. Let $N \in \mathbb{N} \cup \{\infty\}$. By the results of Subsections 6.1 and 6.2, for any lacunary spectrum, there is an example of a complete system of reproducing kernels whose biorthogonal system is incomplete with defect N.

Now let T be lacunary and let $T = T_1 \cup T_2$ so that $n_{T_2}(r) = o(n_{T_1}(r)), r \to \infty$. Let $A_2(z) = \prod_{t_n \in T_2} (1 - z/t_n)$. Assume that we have chosen a measure $\mu^{(2)} = \sum_{t_n \in T_2} \mu_n^{(2)} \delta_{t_n}$ and a function G_2 such that

- G_2 is a canonical product with lacunary zeros;
- $G_2 \in \operatorname{Assoc}(T_2, A_2, \mu^{(2)}) \setminus \mathcal{H}(T_2, A_2, \mu^{(2)});$
- for $\Lambda_2 = \mathcal{Z}_{G_2}$ the system $\{k_{\lambda}^{(2)}\}_{\lambda \in \Lambda_2}$ is complete in $\mathcal{H}(T_2, A_2, \mu^{(2)});$
- dim $\left(\mathcal{H}(T_2, A_2, \mu^{(2)}) \ominus \overline{\operatorname{Span}}\left\{\frac{G_2(z)}{z-\lambda} : \lambda \in \Lambda_2\right\}\right) = N.$

Here $k_{\lambda}^{(2)}$ denote the reproducing kernel in the space $\mathcal{H}(T_2, A_2, \mu^{(2)})$. Thus, there exist N linearly independent functions of the form $f(z) = A_2(z) \sum_{t_n \in T_2} \frac{c_n(\mu_n^{(2)})^{1/2}}{z - t_n}$ such that

(6.6)
$$A_2(z) \sum_{t_n \in T_2} \frac{G_2(t_n)\bar{c}_n}{A_2'(t_n)(\mu_n^{(2)})^{1/2}(z-t_n)} = G_2(z)S_2(z)$$

for some entire function S_2 .

37

Put $\mu_n = 1$, $t_n \in T_1$, and $\mu_n = \mu_n^{(2)}$, $t_n \in T_2$, and consider the corresponding Cauchyde Branges space $\mathcal{H}(T, A, \mu)$, $A = A_1A_2$. Define $c_n = 0$, $t_n \in T_1$. Then, multiplying the formula for f and (6.6) by A_1 , we get

$$A_1(z)f(z) = A_1(z)A_2(z)\sum_{t_n\in T}\frac{c_n\mu_n^{1/2}}{z-t_n},$$

$$G_2(z)A_1(z)S_2(z) = A_1(z)A_2(z)\sum_{t_n\in T}\frac{G_2(t_n)A_1(t_n)\bar{c}_n}{(A_1A_2)'(t_n)\mu_n^{1/2}(z-t_n)}$$

By the discussion in Subsection 5.2, these equations are equivalent to the fact that the function $A_1 f$ is orthogonal to the mixed system

(6.7)
$$\{k_{\lambda}\}_{\lambda\in T_1} \cup \left\{\frac{A_1(z)G_2(z)}{z-\lambda}\right\}_{\lambda\in\Lambda_2}$$

in $\mathcal{H}(T, A, \mu)$ (here we denote by k_{λ} the reproducing kernel of the space $\mathcal{H}(T, A, \mu)$. Moreover, if some function $g \in \mathcal{H}(T, A, \mu)$ is orthogonal to the system (6.7), then $g(t_n) = 0$, $t_n \in T_1$. Hence, $g = A_1 f$ where $f(z) = A_2(z) \sum_{t_n \in T_2} \frac{c_n \mu_n^{1/2}}{z - t_n} \in \mathcal{H}(T_2, A_2, \mu^{(2)})$. Writing the equation for the orthogonality of $A_1 f$ to $\left\{\frac{A_1(z)G_2(z)}{z - \lambda}\right\}_{\lambda \in \Lambda_2}$, we get (6.6) with some entire S_2 and so f is orthogonal to $\left\{\frac{G_2(z)}{z - \lambda}\right\}_{\lambda \in \Lambda_2}$ in $\mathcal{H}(T_2, A_2, \mu^{(2)})$. Thus, the codimension of the system (6.7) in $\mathcal{H}(T, A, \mu)$ equals the codimension of the system $\left\{\frac{G_2(z)}{z - \lambda}\right\}_{\lambda \in \Lambda_2}$ in $\mathcal{H}(T_2, A_2, \mu^{(2)})$, that is, N.

Let us show that the system $\{k_{\lambda}\}_{\lambda \in T_1 \cup \Lambda_2}$ is complete in $\mathcal{H}(T, A, \mu)$. Assume that $A_1G_2V \in \mathcal{H}(T, A, \mu)$ for some entire function V. Then there exists a sequence $(d_n) \in \ell^2$ such that

$$A_1(z)G_2(z)V(z) = A_1(z)A_2(z)\sum_n \frac{d_n\mu_n^{1/2}}{z-t_n},$$

and so $d_n = 0, t_n \in T_1$. Hence,

$$G_2(z)V(z) = A_2(z) \sum_{t_n \in T_2} \frac{d_n \mu_n^{1/2}}{z - t_n} \in \mathcal{H}(T_2, A_2, \mu^{(2)}),$$

a contradiction to completeness of $\{k_{\lambda}^{(2)}\}_{\lambda \in \Lambda_2}$ in $\mathcal{H}(T_2, A_2, \mu^{(2)})$.

It remains to show that the biorthogonal system $\left\{\frac{A_1(z)G_2(z)}{z-\lambda}\right\}_{\lambda\in T_1\cup\Lambda_2}$ is complete in $\mathcal{H}(T, A, \mu)$. Assume that there exists $(c_n) \in \ell^2$ such that the function $g(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z-t_n}$ is orthogonal to the above system. Then, as in Subsection 4.2, we have, for $\lambda \in \Lambda_2$,

$$0 = \left(\frac{A_1 G_2}{z - \lambda}, g\right) = \sum_n \frac{A_1(t_n) G_2(t_n) \bar{c}_n}{A'(t_n) \mu_n^{1/2}(t_n - \lambda)} = \sum_{t_n \in T_2} \frac{G_2(t_n) \bar{c}_n}{A'_2(t_n) \mu_n^{1/2}(t_n - \lambda)},$$

and so there exists an entire function S_2 such that

$$A_2(z) \sum_{t_n \in T_2} \frac{G_2(t_n)\bar{c}_n}{A_2'(t_n)\mu_n^{1/2}(z-t_n)} = G_2(z)S_2(z)$$

On the other hand, if $\lambda = t_m \in T_1$, then

$$0 = \left(\frac{A_1 G_2}{z - t_m}, g\right) = \sum_{t_n \in T_2} \frac{A_1(t_n) G_2(t_n) \bar{c}_n}{A'(t_n) \mu_n^{1/2}(t_n - t_m)} + \frac{A'_1(t_m) G_2(t_m) \bar{c}_m}{A'(t_m) \mu_m^{1/2}}$$
$$= \sum_{t_n \in T_2} \frac{G_2(t_n) \bar{c}_n}{A'_2(t_n) \mu_n^{1/2}(t_n - t_m)} + \frac{G_2(t_m) \bar{c}_m}{A_2(t_m) \mu_m^{1/2}}.$$

We conclude that

$$\frac{G_2(t_m)\bar{c}_m}{A_2(t_m)\mu_m^{1/2}} = \frac{G_2(t_m)S_2(t_m)}{A_2(t_m)}, \qquad t_m \in T_1,$$

and so $S_2(t_m) = \bar{c}_m$ (recall that $\mu_m = 1$).

By estimates (6.3), $|G_2(z)S_2(z)/A_2(z)| \leq |z|$, dist $(z, T_2) \geq 1$. Since G_2 is lacunary product we conclude that $\log M_{S_2}(r) \leq \log M_{A_2}(r)$, whence

$$\int_0^R \frac{n_{S_2}(r)}{r} dr \lesssim \int_0^R \frac{n_{T_2}(r)}{r} dr$$

At the same time $n_{T_2}(r) = o(n_{T_1}(r)), r \to \infty$, and $\{S_2(t_n)\}_{t_n \in T_1} \in \ell^2$. Hence, by Lemma 6.2, $S_2 \equiv 0$ and so the system $\{\frac{A_1(z)G_2(z)}{z-\lambda}\}_{\lambda \in T_1 \cup \Lambda_2}$ is complete. This completes the proof of Theorem 2.6.

7. Proof of ordering theorems

7.1. Nearly invariant and division-invariant subspaces. Let \mathcal{H} be a reproducing kernel Hilbert space of functions analytic in some domain D. A closed subspace \mathcal{H}_0 of \mathcal{H} is said to be *nearly invariant* if there is $w_0 \in D$ such that $\frac{f(z)}{z-w_0} \in \mathcal{H}_0$ whenever $f \in \mathcal{H}_0$ and $f(w_0) = 0$. This notion goes back to the work of Hitt [22] and Sarason [37]. Usually it is assumed that $w_0 = 0$. It is known that nearly invariance is equivalent to a stronger division invariance property. Denote by $\mathcal{Z}(\mathcal{H}_0)$ the set of common zeros for \mathcal{H}_0 , that is, the set of $w \in D$ such that f(w) = 0 for any $f \in \mathcal{H}_0$. Then the division invariance means that for any $w \in D \setminus \mathcal{Z}(\mathcal{H}_0)$,

$$f \in \mathcal{H}_0, \quad f(w) = 0 \implies \frac{f(z)}{z - w} \in \mathcal{H}_0.$$

In the context of Hardy spaces in general domains the equivalence of nearly invariance and division invariance is shown in [2, Proposition 5.1]; a similar argument works for general spaces of analytic functions.

Proposition 7.1. Let \mathcal{H}_0 be a nearly invariant subspace of some reproducing kernel Hilbert space \mathcal{H} of functions analytic in some domain D. Then, for any $w \in D \setminus \mathcal{Z}(\mathcal{H}_0)$ and any $f \in \mathcal{H}_0$ such that f(w) = 0, we have $\frac{f(z)}{z-w} \in \mathcal{H}_0$.

Proof. We show that the set of w in $D \setminus \mathcal{Z}(\mathcal{H}_0)$ satisfying the conclusions of the proposition is both open and closed in $D \setminus \mathcal{Z}(\mathcal{H}_0)$.

First of all note that, for any $w \in D \setminus \mathcal{Z}(\mathcal{H}_0)$, the operator $f \mapsto \frac{f(z)}{z-w}$ is bounded from the subspace $\{f \in \mathcal{H} : f(w) = 0\}$ to \mathcal{H} by the Closed Graph Theorem. Let f(w) = 0 and write $\frac{f(z)}{z-w} = g_w + h_w$, where $g_w \in \mathcal{H}_0$, $h_w \perp \mathcal{H}_0$. Fix some function $f_0 \in \mathcal{H}_0$ such that $f_0(w_0) = 1$. Then, by a simple computation,

$$\frac{f - f(w_0)f_0}{z - w_0} = g_w + h_w + (w_0 - w) \left(\frac{g_w - g_w(w_0)f_0}{z - w_0} + \frac{h_w - h_w(w_0)f_0}{z - w_0}\right) \in \mathcal{H}_0.$$

Since g_w and $\frac{g_w - g_w(w_0)f_0}{z - w_0}$ are in \mathcal{H}_0 , we conclude that

$$h_w = (w - w_0) \mathcal{P}_{\mathcal{H}_0^{\perp}} \left(\frac{h_w - h_w(w_0) f_0}{z - w_0} \right),$$

where $P_{\mathcal{H}_0^{\perp}}$ denotes the orthogonal projection. This is impossible when w is sufficiently close to w_0 unless $h_w = 0$, since $\left\|\frac{h_w - h_w(w_0)f_0}{z - w_0}\right\| \leq C \|h_w\|$ for some constant C independent on w. We used the fact that \mathcal{H} is a reproducing kernel Hilbert space to estimate $h_w(w_0)$ by $\|h_w\|$. Thus, we have seen that if we can divide in \mathcal{H}_0 by $z - w_0$ then we can divide by z - w with w close to w_0 .

Let us show that the set of w satisfying the conclusion of the proposition is closed. Let $w \in D \setminus \mathcal{Z}(\mathcal{H}_0)$ and assume that $w_n \to w$ is a sequence such that one can divide by $z - w_n$ in \mathcal{H}_0 . Fix some function $g \in \mathcal{H}_0$ such that $g(w) \neq 0$. We may assume that $g(w_n) \neq 0$ for all n. Then all operators T_{w_n} , where

$$T_w f = \frac{f - \frac{f(w)}{g(w)}g}{z - w},$$

are bounded. An easy computation leads to a resolvent-type identity

$$T_{w_n} - T_w = (w_n - w)T_{w_n}T_w.$$

Therefore $||T_{w_n}|| \leq ||T_w|| + |w_n - w| \cdot ||T_{w_n}|| \cdot ||T_w||$, whence $\sup_n ||T_{w_n}|| < \infty$. It follows that $||T_{w_n} - T_w|| \to 0, n \to \infty$. Since $T_{w_n}f \in \mathcal{H}_0$ for any $f \in \mathcal{H}_0$, we conclude that $T_w\mathcal{H}_0 \subset \mathcal{H}_0$.

7.2. Reduction to an ordering theorems for nearly invariant subspaces. We show that Theorems 2.7 and 2.8 are equivalent to an ordering theorem for nearly invariant subspaces. Let $\mathcal{L} = \mathcal{A} + a \otimes b$ be a rank one perturbation of a compact normal operator with simple spectrum $\{s_n\}$. Passing to the model operator \mathcal{T}_G in the Cauchy–de Branges space $\mathcal{H}(T, A, \mu), T = \{t_n\} = \{s_n\}^{-1}$, the problem becomes equivalent to the following. Let \mathcal{M} be an invariant subspace of \mathcal{T}_G . Put $\Lambda = \mathcal{Z}_G$ and let Λ_2 be the set of those $\lambda \in \Lambda$ for which $\frac{G}{z-\lambda} \in \mathcal{M}$. Then

(7.1)
$$\overline{\operatorname{Span}}\left\{\frac{G}{z-\lambda}:\ \lambda\in\Lambda_2\right\}\subset\mathcal{M}\subset\left(\overline{\operatorname{Span}}\left\{k_\lambda:\ \lambda\in\Lambda_1\right\}\right)^{\perp},$$

where $\Lambda_1 = \Lambda \setminus \Lambda_2$. Since $\frac{G}{z-\lambda} \in \mathcal{M}$ if and only if $\lambda \in \Lambda_2$, we conclude that the set of common zeros $\mathcal{Z}(\mathcal{M})$ coincides with Λ_1 .

Recall that G(0) = 1 and so $0 \notin \Lambda$. Since \mathcal{M} is \mathcal{T}_G -invariant, we have, in particular,

$$F \in \mathcal{M}, \ F(0) = 0 \implies \frac{F(z)}{z} \in \mathcal{M},$$

that is, \mathcal{M} is nearly invariant and, hence, division-invariant by Lemma 7.1.

Now Theorems 2.7 and 2.8 follow from the following ordering theorem.

Theorem 7.2. Let $\mathcal{H} = \mathcal{H}(T, A, \mu)$ be a de Branges-Cauchy space and assume that one of the following conditions holds:

- (i) $T \subset \mathbb{R}$ and $|t_n| \to \infty$ as $|n| \to \infty$;
- (ii) T satisfies one of the conditions \mathbf{Z} , $\boldsymbol{\Pi}$ or \mathbf{A}_{γ} .

Let $G \in \operatorname{Assoc}(T, A, \mu)$ and $\Lambda = \mathcal{Z}_G$. Assume that $\{k_\lambda\}_{\lambda \in \Lambda}$ is a complete and minimal system of reproducing kernels in \mathcal{H} and let $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$. Then the set of all nearly invariant subspaces \mathcal{M} satisfying (7.1) is totally ordered by inclusion.

The proof of this theorem is based on the ideas of L. de Branges and is very similar to the proof of [14, Theorem 35] or [1, Theorems 1.3, 1.4]. However, our hypothesis is somewhat different and we cannot just refer to the above results. Therefore, and in order to make the exposition more self-contained, we present below the proof of Theorem 7.2.

7.3. Preliminaries on the Smirnov class and Krein's theorem. Here we will use some basic notions of Hardy spaces theory and Nevanlinna inner-outer factorization (see, e.g., [24]). Recall that a function f analytic in \mathbb{C}^+ is said to be of *bounded type* in \mathbb{C}^+ , if f = g/h for some functions g, h analytic and bounded in \mathbb{C}^+ . If, moreover, h can be taken to be outer, we say that f belongs to the *Smirnov class* in \mathbb{C}^+ .

Assume now that $T \subset \mathbb{C}^-$ and consider a discrete Cauchy transform

(7.2)
$$f(z) = \sum_{n} \frac{c_n}{t_n - z}, \qquad \sum_{n} \frac{|c_n|}{|t_n|} < \infty.$$

Then f is in the Smirnov class in \mathbb{C}^+ . This follows from the fact that Im f > 0 in \mathbb{C}^+ if $c_n > 0$. Any function with positive imaginary part belongs to the Smirnov class, as well as a sum of two Smirnov class functions.

In particular, a discrete Cauchy transform with real poles is a function of Smirnov class both in the upper half-plane \mathbb{C}^+ and in the lower half-plane \mathbb{C}^- . Therefore, in the case $T \subset \mathbb{R}$, the function f/A is of Smirnov class in \mathbb{C}^+ and \mathbb{C}^- for any $f \in \mathcal{H}(T, A, \mu)$ and, thus, zeros of f satisfy the Blaschke condition in \mathbb{C}^+ and \mathbb{C}^- .

A classical result by M.G. Krein says that if f is an entire function which is in the Smirnov class both in \mathbb{C}^+ and in \mathbb{C}^- , then f is of zero exponential type. For different approaches to this result see [21, Part II, Chapter 1], [28, Lecture 16] or [14].

In what follows we will need the following (very standard) observation that the generating function of a complete and minimal system must have a maximal growth with respect to order 1.

Lemma 7.3. Let $\mathcal{H} = \mathcal{H}(T, A, \mu)$ be a de Branges-Cauchy space such that $T \subset \{-r \leq \text{Im } z \leq r\}$ and either r = 0 (i.e., $T \subset \mathbb{R}$) or r > 0 and T has finite convergence exponent. Let $G \in \text{Assoc}(T, A, \mu)$ and $\Lambda = \mathcal{Z}_G$. Assume that the system $\{k_\lambda\}_{\lambda \in \Lambda}$ is complete and minimal in \mathcal{H} . Then the inner-outer factorizations for G/A in \mathbb{C}^+ + ir and \mathbb{C}^- - ir are, respectively, $G/A = \mathcal{O}B$ and $G/A = \widetilde{\mathcal{O}}\widetilde{B}$, where $\mathcal{O}, \widetilde{\mathcal{O}}$ are the corresponding outer functions and B, \widetilde{B} are some Blaschke products.

Proof. We prove the factorization in $\mathbb{C}^+ + ir$, the case of $\mathbb{C}^- - ir$ is analogous. Since $\frac{G}{z-\lambda} \in \mathcal{H}$ for any $\lambda \in \Lambda$, we conclude that $\frac{G}{A(z-\lambda)}$ is a discrete Cauchy transform of the form (7.2) and so is in the Smirnov class in \mathbb{C}^+ . Hence, $G/A = \mathcal{O}Be^{iaz}$ with $a \geq 0$ (since G/A is meromorphic in \mathbb{C} , the singular inner factor reduces to e^{iaz}). Assume that a > 0. Put

(7.3)
$$H(z) = \frac{G(z)(e^{-iaz/2} - 1)}{zA(z)} - \sum_{n} \frac{G(t_n)(e^{-iat_n/2} - 1)}{t_n A'(t_n)(z - t_n)}.$$

Since $G \in Assoc(T, A, \mu)$ and the sequence $\{e^{-iat_n/2}\}$ is bounded, the above Cauchy transform converges absolutely. The residues at t_n coincide and so H is an entire function.

Consider first the case r = 0. Then G/A is in the Smirnov class in \mathbb{C}^+ and \mathbb{C}^- . Moreover, since a > 0, the function $Ge^{-iaz/2}/A = \mathcal{O}Be^{iaz/2}$ also is in the Smirnov class in \mathbb{C}^+ and \mathbb{C}^- . Since the discrete Cauchy transform in (7.3) also is in the Smirnov class in \mathbb{C}^+ and in \mathbb{C}^- , we conclude that H is of zero exponential type by Krein's theorem. Recall that any Smirnov class function u in \mathbb{C}^+ satisfies

(7.4)
$$\log^+ |u(re^{i\theta})| = o(r), \qquad r \to \infty,$$

for any fixed $\theta \in (0, \pi)$ (as usual, $\log^+ t = \max(\log t, 0)$). Hence, $Ge^{-iaz/2}/A$ tends to zero along the imaginary axis when $y \to \pm \infty$. Thus, $|H(iy)| \to 0$, $|y| \to \pm \infty$, and we conclude that $H \equiv 0$. Now we have

$$\frac{G(z)(e^{-iaz/2}-1)}{z} = A(z)\sum_{n} \frac{G(t_n)(e^{-iat_n/2}-1)}{t_n A'(t_n)(z-t_n)},$$

whence $\frac{G(e^{-iaz/2}-1)}{z} \in \mathcal{H}$, a contradiction to the fact that the system $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is complete in \mathcal{H} .

If r > 0, we argue similarly to show that $Ge^{-iaz/2}/A$ (and, hence, H) is of Smirnov class in $\mathbb{C}^+ + ir$ and $\mathbb{C}^- - ir$. Since we also know that A is of finite order, we conclude by Lemma 3.5 that H is of finite order. Combining this with (7.4) we see that H is of zero exponential type by the standard Phragmén–Lindelöf principle and, hence, $H \equiv 0$. The end of the proof is the same as for r = 0.

7.4. Proof of Theorem 7.2. In what follows we put

$$\nu = \sum_{n} \frac{\delta_{t_n}}{|A'(t_n)|^2 \mu_n}.$$

By Lemma 3.6, the embedding $\mathcal{H}(T, A, \mu)$ into $L^2(\nu)$ is isometric and so we can compute the norm and inner product in $\mathcal{H}(T, A, \mu)$ as the integral with respect to ν .

Assume that \mathcal{M}_1 and \mathcal{M}_2 are two subspaces satisfying (7.1) and neither $\mathcal{M}_1 \subset \mathcal{M}_2$ nor $\mathcal{M}_2 \subset \mathcal{M}_1$. Choose nonzero functions $F_1, F_2 \in \mathcal{H}$ such that $F_1 \perp \mathcal{M}_2$ but F_1 is not orthogonal to \mathcal{M}_1 , while $F_2 \perp \mathcal{M}_1$ but F_2 is not orthogonal to \mathcal{M}_2 .

We fix two functions G_1 and G_2 such that $G = G_1G_2$, $\Lambda_1 = \mathbb{Z}_{G_1}$, $\Lambda_2 = \mathbb{Z}_{G_2}$. Note that if \mathcal{M} satisfies (7.1), then any function in \mathcal{M} is of the form G_1F for some entire F.

Now let $G_1F \in \mathcal{M}_1$ and $G_1H \in \mathcal{M}_2$. Define two functions

$$f(w) = \left\langle G_1 \frac{F - \frac{F(w)}{H(w)}H}{z - w}, F_1 \right\rangle_{\mathcal{H}(T,A,\mu)} = \int G_1(z) \frac{F(z) - \frac{F(w)}{H(w)}H(z)}{z - w} \overline{F_1(z)}d\nu(z),$$
$$h(w) = \left\langle G_1 \frac{H - \frac{H(w)}{F(w)}F}{z - w}, F_2 \right\rangle_{\mathcal{H}(T,A,\mu)} = \int G_1(z) \frac{H(z) - \frac{H(w)}{F(w)}F(z)}{z - w} \overline{F_2(z)}d\nu(z).$$

The functions f and h are well-defined and analytic on the sets $\{w : H(w) \neq 0\}$ and $\{w : F(w) \neq 0\}$, respectively.

Step 1: f and h are entire functions, f does not depend on the choice of H and h does not depend on the choice of F.

Let \tilde{f} be a function associated in a similar way to another function $G_1 \tilde{H} \in \mathcal{M}_2$,

$$\tilde{f}(w) = \int G_1(z) \frac{F(z) - \frac{F(w)}{\tilde{H}(w)}\tilde{H}(z)}{z - w} \overline{F_1(z)}d\nu(z).$$

Then, for $\tilde{F}(w) \neq 0$ and $\tilde{H}(w) \neq 0$, we have

$$\tilde{f}(w) - f(w) = \frac{F(w)}{H(w)\tilde{H}(w)} \int G_1(z) \frac{\tilde{H}(w)H(z) - H(w)\tilde{H}(z)}{z - w} \overline{F_1(z)} d\nu(z) = 0,$$

since

(7.5)
$$G_1 \frac{\tilde{H}(w)H - H(w)\tilde{H}}{z - w} \in \mathcal{M}_2.$$

This inclusion is obvious when $w \notin \Lambda_1$ since \mathcal{M}_2 is division-invariant; since the norm in $\mathcal{H}(T, A, \mu)$ coincides with the norm in $L^2(\nu)$, by continuity we obtain the inclusion (7.5) for $w \in \Lambda_1$ as well.

For any w we can choose H such that $H(w) \neq 0$ and so we can extend f analytically to any point w. Thus, f and h are entire functions.

Step 2: f and h are of zero exponential type.

Case
$$T \subset \mathbb{R}$$
. We have

$$f(w) = \int G_1(z) \frac{F(z)\overline{F_1(z)}}{z - w} d\nu(z) - \frac{F(w)}{H(w)} \int G_1(z) \frac{H(z)\overline{F_1(z)}}{z - w} d\nu(z)$$

Since $G_1F, G_1H \in \mathcal{H}(T, A, \mu)$ and the points t_n are real, the functions G_1F/A and G_1H/A are in the Smirnov class in \mathbb{C}^+ and \mathbb{C}^- by discussion in Subsection 7.3. The function f does not depend on the choice of H, and we can take $H = \frac{G_2}{z-\lambda}$ for some $\lambda \in \Lambda_1$. Then, by Lemma 7.3, G_1H/A has no exponential factor in its canonical factorization in \mathbb{C}^+ and we conclude that F/H = u/B for some Smirnov class function u and some Blaschke product B in \mathbb{C}^+ . The discrete Cauchy transforms

$$\int \frac{G_1(z)F(z)\overline{F_1(z)}}{z-w}d\nu(z), \qquad \int \frac{G_1(z)H(z)\overline{F_1(z)}}{z-w}d\nu(z),$$

are also in Smirnov class in \mathbb{C}^+ (as functions of w), and so f is in the Smirnov class in \mathbb{C}^+ . Similarly, f is in the Smirnov class in \mathbb{C}^- . By Krein's theorem f is of zero exponential type.

Case II. We assume without loss of generality that $T \subset \{-r \leq \text{Im } z \leq r\}$. Similarly to the above case we have that f is of the Smirnov class in $\mathbb{C}^+ + ir$ and $\mathbb{C}^- - ir$. By the

hypothesis T has a finite convergence exponent and so A has finite order. By Lemma 3.5 the functions G_1F , G_1H ,

$$A(w)\int \frac{G_1(z)F(z)\overline{F_1(z)}}{z-w}d\nu(z), \qquad A(w)\int \frac{G_1(z)H(z)\overline{F_1(z)}}{z-w}d\nu(z)$$

are all of finite order, whence f is of finite order. Since any function u in the Smirnov class in \mathbb{C}^+ satisfies $\log^+ |u(re^{i\theta})| = o(r), r \to \infty$, for any $\theta \in (0, \pi)$, the standard Phragmén– Lindelöf principle implies that f is of zero exponential type.

Case Z or A_{γ}. Here we can refer to [1, Theorem 1.4, Corollary 3.1]: If *T* satisfies **Z** or **A**_{γ} and an entire function *f* satisfies $f = u_1 + u_2 \frac{u_3}{u_4}$ where u_j are discrete Cauchy transforms with poles in *T*, then *f* is of zero exponential type.

Similarly, h is of zero exponential type.

Step 3: Either f or h is identically zero.

Given w such that $F(w) \neq 0$, $H(w) \neq 0$, we have

(7.6)
$$|f(w)| \leq \left| \int \frac{G_1(z)F(z)\overline{F_1(z)}}{z-w} d\nu(z) \right| + \left| \frac{F(w)}{H(w)} \right| \cdot \left| \int \frac{G_1(z)H(z)\overline{F_1(z)}}{z-w} d\nu(z) \right|,$$
$$|h(w)| \leq \left| \int \frac{G_1(z)H(z)\overline{F_2(z)}}{z-w} d\nu(z) \right| + \left| \frac{H(w)}{F(w)} \right| \cdot \left| \int \frac{G_1(z)F(z)\overline{F_2(z)}}{z-w} d\nu(z) \right|.$$

If $T \subset \mathbb{R}$, then by a rough estimate of the Cauchy transforms, we have

$$|f(w)| \lesssim \frac{1}{|\operatorname{Im} w|} \left(1 + \left| \frac{F(w)}{H(w)} \right| \right), \qquad |h(w)| \lesssim \frac{1}{|\operatorname{Im} w|} \left(1 + \left| \frac{H(w)}{F(w)} \right| \right).$$

From this we conclude that

$$\min\left(|f(w)|, |h(w)|\right) \lesssim \frac{1}{|\operatorname{Im} w|}$$

Applying a well-known and deep result by de Branges [14, Lemma 8], we conclude that either f or h is zero.

In the cases \mathbf{Z} , $\mathbf{\Pi}$ or \mathbf{A}_{γ} we argue as in the proof of [1, Theorem 1.3]. To estimate the Cauchy transforms in (7.6) we use Lemma 3.5: there exists M > 0 and a set $E \subset (0, \infty)$ of zero linear density such that

$$|f(w)| \lesssim |w|^M \left(1 + \left| \frac{F(w)}{H(w)} \right| \right), \qquad |h(w)| \lesssim |w|^M \left(1 + \left| \frac{H(w)}{F(w)} \right| \right), \qquad |w| \notin E.$$

We conclude that

$$\min\left(|f(w)|, |h(w)|\right) \lesssim |w|^M, \qquad |w| \notin E.$$

Since E has zero linear density, we can choose a sequence $R_j \to \infty$ such that $R_j \notin E$ and $R_{j+1}/R_j \leq 2$. Applying the maximum principle to the annuli $R_j \leq |z| \leq R_{j+1}$, we conclude that

$$\min\left(|f(w)|, |h(w)|\right) \lesssim |w|^M, \qquad |w| \ge 1.$$

Since both f and h are of zero exponential type, a small variation of [14, Lemma 8] gives that either f or h is a polynomial.

Assume that f is a nonzero polynomial. Now we define a line Γ which is separated from T (at infinity). If $T \subset \{-r \leq \text{Im } z \leq r\}$, put $\Gamma = i\mathbb{R}$. If T is contained in some angle of the size $\pi\gamma$, $\gamma < 1$ (say, $\{0 \leq \arg z \leq \pi\gamma\}$), put $\Gamma = \{\rho e^{i(\pi\gamma+\delta)} : \rho \in \mathbb{R}\}$, where $0 < \delta < \pi(1-\gamma)$. In each of these cases, by Lemma 3.3, we have

(7.7)
$$\left|\int \frac{G_1(z)F(z)\overline{F_1(z)}}{z-w}d\nu(z)\right| + \left|\int \frac{G_1(z)H(z)\overline{F_1(z)}}{z-w}d\nu(z)\right| = O\left(\frac{1}{|w|}\right),$$

when $|w| \to \infty$, $w \in \Gamma$. Hence, from (7.6), $|F(w)/H(w)| \to \infty$ as $|w| \to \infty$, $w \in \Gamma$, and so

$$\begin{aligned} |h(w)| &\leq \left| \int \frac{G_1(z)H(z)\overline{F_2(z)}}{z-w} d\nu(z) \right| \\ &+ \left| \frac{H(w)}{F(w)} \right| \cdot \left| \int \frac{G_1(z)F(z)\overline{F_2(z)}}{z-w} d\nu(z) \right| = O\left(\frac{1}{|w|}\right), \qquad w \in \Gamma. \end{aligned}$$

Using the fact that h is of zero exponential type, we conclude that $h \equiv 0$.

In the case \mathbb{Z} we have no information about location of the points t_n . However, by Lemma 3.2, there exists a set Ω of zero area density such that (7.7) holds when $w \notin \Omega$. Hence, $|F(w)/H(w)| \to \infty$ as $|w| \to \infty$, $w \notin \Omega$, and so

$$|h(w)| \leq \left| \int \frac{G_1(z)H(z)\overline{F_2(z)}}{z-w} d\nu(z) \right| + \left| \frac{H(w)}{F(w)} \right| \cdot \left| \int \frac{G_1(z)F(z)\overline{F_2(z)}}{z-w} d\nu(z) \right| = O\left(\frac{1}{|w|}\right), \qquad w \notin \Omega \cup \widetilde{\Omega},$$

where $\widetilde{\Omega}$ is another set of zero area density (here we again applied Lemma 3.2). Thus, h tends to zero outside a set of zero density and so $h \equiv 0$ by Theorem 3.4.

Step 4: End of the proof.

Without loss of generality, let $f \equiv 0$. Then

(7.8)
$$\frac{F(w)}{H(w)} \int \frac{G_1(z)H(z)\overline{F_1(z)}}{z-w} d\nu(z) = \int \frac{G_1(z)F(z)\overline{F_1(z)}}{z-w} d\nu(z)$$

for any $G_1F \in \mathcal{M}_1, G_1H \in \mathcal{M}_2$.

Recall that F_1 is not orthogonal to \mathcal{M}_1 and so we can choose $G_1F \in \mathcal{M}_1$ such that $\langle G_1F, F_1 \rangle_{\mathcal{H}(T,A,\mu)} = \int G_1F\overline{F_1}d\nu \neq 0$. Assume first that $T \subset \{-r \leq \text{Im } z \leq r\}$ or Tsatisfies \mathbf{A}_{γ} and let Γ be the line constructed in Step 3. We now compare the asymptotics of the right-hand side and on the left-hand side of (7.8) on Γ . By Lemma 3.3, we have

(7.9)
$$\left| \int \frac{G_1(z)F(z)\overline{F_1(z)}}{z-w} d\nu(z) \right| \gtrsim \frac{1}{|w|}$$

when $|w| \to \infty$, $w \in \Gamma$. Since $G_1H \perp F_1$ for any $G_1H \in \mathcal{M}_2$, we have (again by Lemma 3.3)

(7.10)
$$\left| \int \frac{G_1(z)H(z)\overline{F_1(z)}}{z-w} d\nu(z) \right| = o\left(\frac{1}{|w|}\right),$$

when $|w| \to \infty$, $w \in \Gamma$. We conclude that $|F(w)/H(w)| \to \infty$ when $|w| \to \infty$, $w \in \Gamma$. Applying this fact and Lemma 3.3 to h we conclude from (7.6) that $|h(w)| \to 0$ when $|w| \to \infty$, $w \in \Gamma$. Since h is of zero exponential type, we have $h \equiv 0$.

Thus, we have

(7.11)
$$\frac{H(w)}{F(w)} \int \frac{G_1(z)F(z)\overline{F_2(z)}}{z-w} d\nu(z) = \int \frac{G_1(z)H(z)\overline{F_2(z)}}{z-w} d\nu(z)$$

and we may repeat the above argument. Choose $G_1H \in \mathcal{M}_2$ such that $\langle G_1H, F_2 \rangle_{\mathcal{H}(T,A,\mu)} = \int G_1 H \overline{F}_2 d\nu \neq 0$. Then, by Lemma 3.3, the modulus of the right-hand side in (7.11) is $\geq |w|^{-1}$, while the left-hand side is $o(|w|^{-1})$ when $|w| \to \infty$, $w \in \Gamma$. This contradiction proves Theorem 7.2 in the cases when $T \subset \{-r \leq \text{Im } z \leq r\}$ (in particular, $T \subset \mathbb{R}$) or T satisfies \mathbf{A}_{γ} .

The proof for the case \mathbb{Z} is similar but instead of asymptotics along a line we consider the asymptotics outside a set of zero density. Let F be chosen as above. By Lemma 3.2, we have (7.9) and (7.10) when $|w| \to \infty$, $w \notin \Omega$, for some set Ω of zero density. Applying this fact and Lemma 3.2 to h we conclude that $|h(w)| \to 0$ outside a set of zero density and so $h \equiv 0$ by Theorem 3.4. Then we obtain (7.11) and, repeating the argument once again, find that the modulus of the right-hand side in (7.11) is $\gtrsim |w|^{-1}$, while the left-hand side is $o(|w|^{-1})$ when $|w| \to \infty$ outside a set of zero density. Case \mathbb{Z} of Theorem 7.2 is also proved.

8. VOLTERRA RANK ONE PERTURBATIONS

In this section we discuss the following problem: for which compact normal operators \mathcal{A} there exists a rank one perturbation \mathcal{L} which is a Volterra operator? In the case when \mathcal{A} is

a compact selfadjoint operator this question was answered in [9] in terms of the so-called Krein class of entire functions.

Recall that an entire function F is said to be in the *Krein class* if

- F is real on \mathbb{R} and has simple real zeros t_n (we assume for simplicity that $t_n \neq 0$);
- for some integer $k \ge 0$, we have $\sum_n \frac{1}{|t_n|^{k+1}|F'(t_n)|} < \infty$;
- there exists a polynomial R such that

(8.1)
$$\frac{1}{F(z)} = R(z) + \sum_{n} \frac{1}{F'(t_n)} \cdot \left(\frac{1}{z - t_n} + \frac{1}{t_n} + \dots + \frac{z^{k-1}}{t_n^k}\right).$$

This class was introduced by M.G. Krein who showed that in this case F is necessarily of the Cartwright class and, in particular, of exponential type (see [26, Theorem 5] or [28, Lecture 16]). Indeed, F is of bounded type in \mathbb{C}^+ and \mathbb{C}^- , whence (by yet another result of Krein, see Subsection 7.3) F is of exponential type. Some further refinements of this result are due to A.G. Bakan and V.B. Sherstyukov (see, e.g., [38] and references therein).

One can extend the definition of the Krein class to the case when $T = \{t_n\} \subset \mathbb{C}$ no longer is assumed to be real. If F satisfies all other conditions above, we say that F belongs to the generalized Krein class. However, one cannot extend Krein's theorem to this situation: a function in the generalized Krein class can have arbitrarily large order. Indeed, for a Krein class function F one can write (8.1) with z^m in place of z, where $m \in \mathbb{N}$. Then, expanding $(z^m - t_n)^{-1}$ as a sum of simple fractions one can show that $\tilde{F}(z) = F(z^m)$ is in the generalized Krein class.

In what follows we will say that an entire function F with simple zeros in the set $T = \{t_n\} \subset \mathbb{C} \setminus \{0\}$ belongs to the (generalized) Krein class \mathcal{K}_1 if

(8.2)
$$\frac{1}{F(z)} = \frac{1}{F(0)} + \sum_{n} \frac{1}{F'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n} \right), \qquad \sum_{n} \frac{1}{|t_n|^2 |F'(t_n)|} < \infty.$$

Then the main result of [9] can be stated as follows:

Theorem 8.1. ([9]) Let \mathcal{A} be a compact selfadjoint operator with simple spectrum $\{s_n\}$, $s_n \neq 0$, and let $t_n = s_n^{-1}$. Then the following are equivalent:

- (i) There exists a rank one perturbation \mathcal{L} of \mathcal{A} which is a Volterra operator;
- (ii) There exists a function $F \in \mathcal{K}_1$ such that the zero set of F coincides with $\{t_n\}$.

In the case when \mathcal{A} satisfies condition (i) above, we say that the spectrum $\{s_n\}$ (or $\{t_n\}$) is *removable*. An unexpected (and rather counterintuitive) consequence of Theorem 8.1 is that adding a finite number of points to the spectrum helps it to become removable, while deleting a finite number of points from a removable spectrum may make it nonremovable.

A similar description of spectra removable by a rank one perturbation holds true for compact normal operators.

Theorem 8.2. Let \mathcal{A} be a compact normal operator with simple spectrum $\{s_n\}$, $s_n \neq 0$, and let $t_n = s_n^{-1}$. Then the following are equivalent:

(i) There exists a rank one perturbation \mathcal{L} of \mathcal{A} which is a Volterra operator;

(ii) There exist a Cauchy-de Branges space $\mathcal{H}(T, A, \mu)$, $T = \{t_n\}$, and an entire function $G \in \operatorname{Assoc}(T, A, \mu)$ which does not vanish in \mathbb{C} ;

(iii) There exists an entire function $A \in \mathcal{K}_1$ such that the $\mathcal{Z}_A = T$.

Proof. (i) \Longrightarrow (ii): Assume that \mathcal{L} is a Volterra perturbation. By the functional model of Theorem 2.9 there exist a Cauchy–de Branges space $\mathcal{H}(T, A, \mu)$ and an entire function $G \in \operatorname{Assoc}(T, A, \mu), G(0) = 1$, such that \mathcal{L} is unitarily equivalent to \mathcal{T}_G . Since \mathcal{T}_G is a Volterra operator, we conclude that $G \neq 0$ (otherwise, any $\lambda \in \mathcal{Z}_G$ is an eigenvalue for \mathcal{T}_G).

(ii) \Longrightarrow (iii): Let G be a nonvanishing function in Assoc (T, A, μ) . Then $G = e^H$ for some entire function H. It is clear that $e^{-H}\mathcal{H}(T, A, \mu) = \mathcal{H}(T, e^{-H}A, \mu)$ and $1 = e^{-H}G \in$ Assoc $(T, e^{-H}A, \mu)$. Therefore, changing A we may assume that $1 \in$ Assoc (T, A, μ) .

Now let $g(z) = A(z) \sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n}$ be an arbitrary function in $\mathcal{H}(T, A, \mu)$ such that g(0) = 1. Then $\frac{1 - g(z)}{z} \in \mathcal{H}(T, A, \mu)$ and so

$$1 = A(z) \left(\sum_{n} \frac{c_n \mu_n^{1/2}}{z - t_n} + z \sum_{n} \frac{d_n \mu_n^{1/2}}{z - t_n} \right)$$

for some $(d_n) \in \ell^2$. We conclude that

$$\frac{1}{A(z)} = -\sum_{n} \frac{c_n \mu_n^{1/2}}{t_n} + \sum_{n} (c_n + d_n t_n) \mu_n^{1/2} \left(\frac{1}{z - t_n} + \frac{1}{t_n}\right).$$

It follows that $A'(t_n)^{-1} = (c_n + d_n t_n) \mu_n^{1/2}$ and so A satisfies (8.2). Thus, $A \in \mathcal{K}_1$.

(iii) \Longrightarrow (i): Let A satisfy (8.2). Similarly to the above computations, one can show that $\frac{1-g(z)}{z} \in \mathcal{H}(T, A, \mu)$ for any $g \in \mathcal{H}(T, A, \mu)$ such that g(0) = 1. Thus, $G \equiv 1 \in$ Assoc (T, A, μ) . By Theorem 2.9 there exists a rank one perturbation \mathcal{L} which is unitarily equivalent to \mathcal{T}_G . Since $G \neq 0$, \mathcal{T}_G obviously is a Volterra operator.

Remark 8.3. In view of the role of the Krein class in the description of removable spectra, it is a natural problem to extend Krein's theorem to generalized Krein class imposing some conditions on T. V.B. Sherstykov [38] showed that the conclusion of Krein's theorem

(i.e., F is of exponential type) remains true if F is of finite order, satisfies (8.1) and T is contained in some strip. Recently it was shown in [1] that if T is contained in some angle of size $\pi\gamma$, $\gamma \in (0, 1)$ and F is a function of order strictly less than $1/\gamma$ satisfying (8.1), then F is of zero exponential type. Moreover, this growth restriction is sharp: for any $\gamma \in (0, 1)$ there exists F in the generalized Krein class with zeros in an angle of size $\pi\gamma$ and of order exactly $1/\gamma$.

Example 8.4. It is clear that the class \mathcal{K}_1 is stable under multiplication by polynomials. Thus, adding a finite set to T does not change the property to be removable by a rank one perturbation. However, deleting a finite number of points may turn a removable spectrum into a nonremovable one. Thus, removable spectra have a certain rigidity.

Let us give some concrete examples (the first two appeared in [9]):

(i) Let $T = \left\{ \pi \left(n + \frac{1}{2} \right) \right\}_{n \in \mathbb{Z}}$ and $A(z) = \cos \pi z$. Then $A \in \mathcal{K}_1$ and T is removable. However, $T \setminus \{t_n\}$ is nonremovable for any $t_n \in T$, since for $F(z) = A(z)/(z - t_n)$ one has $|F'(t_m)| \simeq |t_m|^{-1}$, $m \neq n$, and the series $\sum_{m \neq n} |F'(t_m)|^{-1} t_m^{-2}$ diverges.

(ii) Let $T = \left\{ \pi \left(n + \frac{1}{2} \right)^2 \right\}_{n \in \mathbb{N}_0}$ and $A(z) = \cos \pi \sqrt{z}$. Then T is removable, but it becomes nonremovable after deleting any of its elements.

(iii) Let $A(z) = \cos(\pi z^k)$, $k \in \mathbb{N}$. Then it is not difficult to show that $A \in \mathcal{K}_1$ and so its zero set $T = \{ |n + \frac{1}{2}|^{1/k} e^{\pi i j/k} \}_{n \in \mathbb{N}_0, 0 \le j \le 2k-1}$ is removable. However, $T \setminus \{t_n\}$ is nonremovable for any $t_n \in T$.

References

- E. Abakumov, A. Baranov, Yu. Belov, Krein-type theorems and ordered structure of Cauchy-de Branges spaces, preprint.
- [2] A. Aleman, S. Richter, Simply invariant subspaces of H² of some multiply connected regions, Integr Equat. Oper Theory 24 (1996), 2, 127–155.
- [3] A.D. Baranov, Completeness and Riesz bases of reproducing kernels in model subspaces, Int. Math. Res. Notices 2006, Article ID 81530.
- [4] A. Baranov, Yu. Belov, Systems of reproducing kernels and their biorthogonal: completeness or incompleteness? Int. Math. Res. Notices 2011, 22, 5076–5108.
- [5] A. Baranov, Yu. Belov, A. Borichev, Hereditary completeness for systems of exponentials and reproducing kernels, Adv. Math. 235 (2013), 525–554.
- [6] A. Baranov, Yu. Belov, A. Borichev, Spectral synthesis in de Branges spaces, Geom. Funct. Anal. (GAFA) 25 (2015), 2, 417–452.
- [7] A. Baranov, Yu. Belov, A. Borichev, Summability properties of Gabor expansions, J. Funct. Anal. 274 (2018), 9, 2532–2552.

- [8] A. Baranov, Yu. Belov, A. Borichev, D. Yakubovich, Recent developments in spectral synthesis for exponential systems and for non-self-adjoint operators, Recent Trends in Analysis (Proceedings of the Conference in Honor of Nikolai Nikolski, Bordeaux, 2011), Theta Foundation, Bucharest, 2013, pp. 17–34.
- [9] A. Baranov, D. Yakubovich, One-dimensional perturbations of unbounded selfadjoint operators with empty spectrum, J. Math. Anal. Appl. 424 (2015), 2, 1404–1424.
- [10] A. Baranov, D. Yakubovich, Completeness and spectral synthesis of nonselfadjoint onedimensional perturbations of selfadjoint operators, Adv. Math. 302 (2016), 740–798.
- [11] A. Baranov, D. Yakubovich, Completeness of rank one perturbations of normal operators with lacunary spectrum, J. Spectral Theory 8 (2018), 1, 1–32.
- [12] Yu. Belov, Complementability of exponential systems, C. R. Math. Acad. Sci. Paris 353 (2015), 215–218.
- [13] Yu. Belov, T. Mengestie, K. Seip, Discrete Hilbert transforms on sparse sequences, Proc. Lond. Math. Soc. 103 (2011), 3, 73–105.
- [14] L. de Branges, Hilbert Spaces of Entire Functions, Prentice–Hall, Englewood Cliffs, 1968.
- [15] D. Deckard, C. Foiaş, C. Pearcy, Compact operators with root vectors that span, Proc. Amer. Math. Soc. 76, (1979), 1, 101–106.
- [16] E. Fricain, Complétude des noyaux reproduisants dans les espaces modèles, Ann. Inst. Fourier (Grenoble) 52 (2002), 2, 661–686.
- [17] J.B. Garnett, D.E. Marshall, *Harmonic Measure*, Cambridge University Press, 2005.
- [18] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Amer. Math. Soc., Providence, R.I., 1969.
- [19] G. M. Gubreev, A. A. Tarasenko, Spectral decomposition of model operators in de Branges spaces, Mat. Sb. 201 (2010), 11, 41–76; English transl.: Sb. Math. 201 (2010), 11, 1599–1634.
- [20] H. Hamburger, Über die Zerlegung des Hilbertschen Raumes durch vollstetige lineare Transformationen, Math. Nachr. 4 (1951), 56–69.
- [21] V. Havin, B. Jöricke, The Uncertainty Principle in Harmonic Analysis, Springer-Verlag, Berlin, 1994.
- [22] D. Hitt, Invariant subspaces of H^2 of an annulus, Pacific J. Math. 134 (1988), 1, 101–120.
- [23] V. V. Kapustin, One-dimensional perturbations of singular unitary operators, Zapiski Nauchn. Sem. POMI 232 (1996), 118–122; English transl.: J. Math. Sci. (New York) 92 (1998), 1, 3619–3621.
- [24] P. Koosis, Introduction to H^p spaces. Cambridge Univ. Press, Cambridge, 1980.
- [25] P. Koosis, The Logarithmic Integral I, Cambridge Stud. Adv. Math. 12, 1988.
- [26] M. Krein, A contribution to the theory of entire functions of exponential type, Izv. Akad. Nauk SSSR Ser. Mat. 11 (1947), 4, 309–326.
- [27] B. Ya. Levin, Distribution of Zeros of Entire Functions, GITTL, Moscow, 1956; English translation: Amer. Math. Soc., Providence, 1964; revised edition: Amer. Math. Soc., 1980.
- [28] B. Ya. Levin, Lectures on Entire Functions, Transl. Math. Monogr. Vol. 150, AMS, Providence, RI, 1996.

- [29] A. A. Lunyov, M. M. Malamud, On the completeness and Riesz basis property of root subspaces of boundary value problems for first order systems and applications, J. Spectr. Theory 5 (2015), 1, 17–70.
- [30] A. A. Lunyov, M. M. Malamud, On spectral synthesis for dissipative Dirac type operators, Integr. Equat. Oper. Theory 80 (2014), 1, 79–106.
- [31] N. Makarov, A. Poltoratski, Meromorphic inner functions, Toeplitz kernels, and the uncertainty principle, in Perspectives in Analysis, Springer Verlag, Berlin, 2005, 185–252.
- [32] A. Markus, The problem of spectral synthesis for operators with point spectrum, Izv. Akad. Nauk SSSR 34 (1970), 3, 662–688; English transl.: Math. USSR-Izv. 4 (1970), 3, 670–696.
- [33] N. Nikolski, Complete extensions of Volterra operators, Izv. Akad. Nauk SSSR 33 (1969), 6, 1349–1353 (Russian); English transl.: Math. USSR-Izv. 3 (1969), 6, 1271–1276.
- [34] N. Nikolski, S. Khruschev, A function model and some problems in spectral function theory, Trudy Math. Inst. Steklov 176 (1987), 97–210; English transl. in Proc. Steklov. Inst. Math. 3 (1988), 101–214.
- [35] C. Remling, Schrödinger operators and de Branges spaces, J. Funct. Anal. 196 (2002), 2, 323–394.
- [36] R. Romanov, Canonical Systems and de Branges Spaces, arXiv:1408.6022.
- [37] D. Sarason, Nearly invariant subspaces of the backward shift, Oper. Theory Adv. Appl., vol. 35, Birkhäuser, Basel, 1988, pp. 481-493.
- [38] V. B. Sherstyukov, Expanding the reciprocal of an entire function with zeros in a strip in a Krein series, Sb. Math., 202 (2011), 12, 1853–1871.
- [39] J. Wermer, On invariant subspaces of normal operators, Proc. Amer. Math. Soc. 3 (1952) 2, 270–277.
- [40] J. Wolff, Sur les series $\sum \frac{A_k}{z-\alpha_k}$, C. R. Acad. Sci. Paris **173** (1921), 1327–1328.

ANTON BARANOV,

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, ST. PETERS-BURG, RUSSIA,

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, ST. PETERSBURG, RUSSIA, anton.d.baranov@gmail.com