BRAUER GROUP OF THE MODULI SPACES OF STABLE VECTOR BUNDLES OF FIXED DETERMINANT OVER A SMOOTH CURVE

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ABSTRACT. Let X be an irreducible smooth projective curve, defined over an algebraically closed field k, of genus at least three and L a line bundle on X. Let $\mathcal{M}_X(r, L)$ be the moduli space of stable vector bundles on X of rank r and determinant L with $r \geq 2$. We prove that the Brauer group $\mathrm{Br}(\mathcal{M}_X(r, L))$ is cyclic of order g.c.d.$(r, \mathrm{degree}(L))$. We also prove that $\mathrm{Br}(\mathcal{M}_X(r, L))$ is generated by the class of the projective bundle obtained by restricting the universal projective bundle. These results were proved earlier in [BBGN] under the assumption that $k = \mathbb{C}$.

1. Introduction

Let X be a compact connected Riemann surface of genus g, with $g \geq 3$. Fix a holomorphic line bundle L over X and also fix an integer $r \geq 2$. Let $\mathcal{M}_X(r, L)$ denote the moduli space of stable vector bundles on X of rank r and determinant L, which is a smooth quasiprojective complex variety of dimension $(r^2 - 1)(g - 1)$. There is a Poincaré vector bundle over $X \times \mathcal{M}_X(r, L)$ if and only if r and degree(L) are coprime [Ra]. When r and degree(L) are coprime, any two Poincaré vector bundle over $X \times \mathcal{M}_X(r, L)$ differ by tensoring with a line bundle pulled back from $\mathcal{M}_X(r, L)$. Hence the projectivized Poincaré bundle in unique. Even when r and degree(L) are not coprime, there is a unique projective Poincaré bundle over $X \times \mathcal{M}_X(r, L)$, although it is not a projectivization of a vector bundle.

In [BBGN] it was proved that the Brauer group of $\mathcal{M}_X(r, L)$ is cyclic of order g.c.d.$(r, \mathrm{degree}(L))$. As mentioned above, there is a universal projective bundle $\mathcal{P}$ on $X \times \mathcal{M}_X(r, L)$. Fixing a point $x \in X$, let $\mathcal{P}_x$ be the projective bundle on $\mathcal{M}_X(r, L)$ obtained by restricting $\mathcal{P}$ to $\{x\} \times \mathcal{M}_X(r, L)$. In [BBGN] it was also shown that the Brauer group $\mathrm{Br}(\mathcal{M}_X(r, L))$ is generated by the class of $\mathcal{P}_x$.

Our aim here is to prove these results for all algebraically closed fields; see Theorem 2.3.

The computation in [BBGN] crucially uses the calculation of the Picard group of $\mathcal{M}_X(r, L)$. It may be mentioned that the assumption in [DN] that the characteristic of the base field is zero is used in the computation of the Picard group of the moduli space $\mathcal{M}_X(r, L)$. In particular, the Reynolds' operators, which play a crucial role in the computation, are valid only in characteristic zero. A recent theorem of Hoffmann shows that the Picard group of the moduli space does not depend on the base field [Hof]. The proof of Theorem 2.3 follows the strategy of [BBGN]; some details not given in [BBGN] are given here.

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2. Universal projective bundle and Brauer group

Let k be an algebraically closed field. Let X be an irreducible smooth projective curve, defined over k, of genus g, with $g \geq 3$. Fix an integer $r \geq 2$ and also fix a line bundle L over X. The degree of L will be denoted by d. Let $\mathcal{M}_X(r, d)$ be the moduli space of stable vector bundles on X of rank r and degree d. Consider the morphism

$$\phi : \mathcal{M}_X(r, d) \longrightarrow \text{Pic}^d(X), \quad E \longmapsto \bigwedge^r E.$$

Let

$$\mathcal{M}_X = \mathcal{M}_X(r, L) := \phi^{-1}(L)$$

be the fiber of ϕ over the point $L \in \text{Pic}^d(X)$. This moduli space $\mathcal{M}_X$ is canonically identified with the following two moduli spaces:

(1) moduli space of pairs of the form (E, ξ), where E is a stable vector bundle over X of rank r, and $\xi : \bigwedge^r E \longrightarrow L$ is an isomorphism, and
(2) the moduli space of stable vector bundles E on X of rank r such that $\bigwedge^r E$ is isomorphic to L

(see [Hof, p. 1308, Proposition 2.1]).

It is known that there is a universal projective bundle

$$\mathcal{P} \longrightarrow X \times \mathcal{M}_X \tag{2.1}$$

([Ra], [Ne]). This follows from the construction of the moduli space and the fact that the global automorphisms of a stable vector bundle are nonzero constant scalar multiplications; this projective bundle $\mathcal{P}$ is described in the proof of Proposition 2.1. Fix a closed point $x \in X$. Let

$$\mathcal{P}_x := \mathcal{P}|_{\{x\}\times\mathcal{M}_X} \xrightarrow{f} \mathcal{M}_X \tag{2.2}$$

be the restriction of $\mathcal{P}$ to $\{x\} \times \mathcal{M}_X$.

For any quasiprojective variety Y defined over the field k, the Brauer group $\text{Br}(Y)$ of Y is defined to be the Morita equivalence classes of Azumaya algebras over the variety Y. It is known that this Brauer group $\text{Br}(Y)$ coincides with the equivalence classes of all principal PGL_k–bundles over Y, where two principal PGL_k–bundles P and Q are identified if there are two vector bundles V_1 and V_2 over Y satisfying the condition that the two principal PGL_k–bundles $P \otimes \mathbb{P}(V_1)$ and $Q \otimes \mathbb{P}(V_2)$ are isomorphic. The addition of two projective bundles P and Q in the Brauer group $\text{Br}(Y)$ is defined to be the equivalence class of the projective bundle $P \otimes Q$. The inverse of a projective bundle P in $\text{Br}(Y)$ is the equivalence class of the dual projective bundle P^*. (See [Gr1], [Gr2], [Gr3], [Mi], [Ga] for properties of Brauer groups.) The cohomological Brauer group $\text{Br}'(Y)$ of the variety Y is the torsion part of the étale cohomology group $H^2_{et}(Y, \mathbb{G}_m)$. There is a natural injective homomorphism $\text{Br}(Y) \longrightarrow \text{Br}'(Y)$ which is in fact an isomorphism by a theorem of Gabber [dJ], [Hoo].

Proposition 2.1. *The Brauer group* $\text{Br}(\mathcal{M}_X)$ *is generated by the class* $cl(\mathcal{P}_x) \in \text{Br}(\mathcal{M}_X)$ *of the projective bundle* $\mathcal{P}_x$ *defined in* (2.2).

Proof. Given any line bundle L_0 on X, the morphism

$$\mathcal{M}_X = \mathcal{M}_X(r, L) \longrightarrow \mathcal{M}_X(r, L\otimes L_0^r), \quad E \longmapsto E\otimes L_0$$

is an isomorphism. The natural isomorphism of $\mathbb{P}(E\otimes L_0)$ with $\mathbb{P}(E)$ produces an isomorphism between the universal projective bundles over $X\times\mathcal{M}_X(r, L)$ and $X\times\mathcal{M}_X(r, L\otimes L_0^r)$. Therefore, after tensoring with a line bundle L_0 of sufficiently large degree, we may assume that

$$\frac{d}{r} > 2g-1\,.$$

Let $\overline{\mathcal{M}}_X$ denote the moduli space of semistable vector bundles E on X of rank r with $\bigwedge^r E = L$.

The cotangent bundle of X will be denoted by K_X. For any vector bundle $E\in\overline{\mathcal{M}}_X$ and any point $y\in X$,

$$H^1(Y,\, E\otimes\mathcal{O}_X(-y)) = H^0(Y,\, E^*\otimes K_X\otimes\mathcal{O}_X(y))^* = 0$$

because degree$(E^*\otimes K_X\otimes\mathcal{O}_X(y)) < 0$ and $E^*\otimes K_X\otimes\mathcal{O}_X(y)$ is semistable. So from the long exact sequence of cohomologies associated to the short exact sequence

$$0\longrightarrow E\otimes\mathcal{O}_X(-y)\longrightarrow E\longrightarrow E_y\longrightarrow 0$$

it follows that the evaluation homomorphism $H^0(X,\, E)\longrightarrow E_y$ is surjective; hence E is generated by its global sections.

Take any $E\in\overline{\mathcal{M}}_X$. Since the vector bundle E is generated by its global sections, there is a short exact sequence

$$0\longrightarrow \mathcal{O}_X^{\oplus(r-1)}\longrightarrow E\longrightarrow \bigwedge^r E = L\longrightarrow 0\,. \tag{2.3}$$

This short exact sequence does not split because E is semistable and degree$(L) > 0$. All such nontrivial extensions are parameterized by

$$\mathbb{P}(H^1(X, \mathrm{Hom}(L, \mathcal{O}_X^{\oplus(r-1)}))^*) = \mathbb{P}((H^1(X, L^*)^*\otimes_k k^{\oplus(r-1)}) = \mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*)\,.$$

The standard action of $\mathrm{GL}(r-1,k)$ on $k^{\oplus(r-1)}$ produces an action of $\mathrm{GL}(r-1,k)$ on the projective space $\mathbb{P}((H^1(X, L^*)^{\oplus(r-1)})^*)$. The moduli space $\overline{\mathcal{M}}_X$ is the geometric invariant theoretic quotient

$$\mathbb{P}((H^1(X, L^*)^{r-1})^*)/\!\!/\mathrm{GL}(r-1,k) = \mathbb{P}((H^1(X, L^*)^{r-1})^*)/\!\!/\mathrm{PGL}(r-1,k) = \overline{\mathcal{M}}_X$$

(see [Ne], [DN]).

The tautological line bundle $\mathcal{O}_{\mathbb{P}((H^1(X,L^*)^{r-1})^*)}(1)$ on $\mathbb{P}((H^1(X, L^*)^{r-1})^*)$ will be denoted by $\mathcal{L}_0$. Let

$$p_1 : X\times\mathbb{P}((H^1(X, L^*)^{r-1})^*)\longrightarrow X\,,\ \ p_2 : X\times\mathbb{P}((H^1(X, L^*)^{r-1})^*)\longrightarrow \mathbb{P}((H^1(X, L^*)^{r-1})^*)$$

be the natural projections. There is a universal extension over $X\times\mathbb{P}((H^1(X, L^*)^{r-1})^*)$

$$0\longrightarrow (p_1^*\mathcal{O}_X^{r-1})\otimes p_2^*\mathcal{L}_0\longrightarrow \mathcal{E}\longrightarrow p_1^*L\longrightarrow 0\,. \tag{2.4}$$

Let $\mathcal{U} \subset \mathbb{P}((H^1(X, L^*)^{r-1})^*)$ be the subset defined by all points $t \in \mathbb{P}((H^1(X, L^*)^{r-1})^*)$ such that the vector bundle $\mathcal{E}|_{X\times\{t\}}$ is stable. This subset $\mathcal{U}$ is nonempty Zariski open. Let

$$\theta : \mathcal{U} \longrightarrow \mathcal{U}/\!\!/\mathrm{PGL}(r-1, k) = \mathcal{M}_X \tag{2.5}$$

be the quotient map. Consider the action of $\mathrm{PGL}(r-1, k)$ on $X \times \mathbb{P}((H^1(X, L^*)^{r-1})^*)$ given by the trivial action on X and the above action of $\mathbb{P}((H^1(X, L^*)^{r-1})^*)$. This action lifts to an action of $\mathrm{PGL}(r-1, k)$ on $\mathbb{P}(\mathcal{E})$. The corresponding geometric invariant theoretic quotient

$$\mathcal{P} := (\mathbb{P}(\mathcal{E})|_{X\times\mathcal{U}})/\!\!/\mathrm{PGL}(r-1, k)$$

is the universal projective bundle on $X \times \mathcal{M}_X$ (see (2.1)).

Consider the map

$$F := \mathrm{Id}_X \times f : X \times \mathcal{P}_x \longrightarrow X \times \mathcal{M}_X\,, \tag{2.6}$$

where f is the projection in (2.2). We will construct a vector bundle

$$\mathcal{V} \longrightarrow X \times \mathcal{P}_x$$

with the property that $\mathbb{P}(\mathcal{V}) = F^*\mathcal{P}$.

Let $\mathcal{E}_x := \mathcal{E}|_{\{x\}\times\mathcal{U}} \longrightarrow \mathcal{U}$ be the vector bundle obtained by restricting $\mathcal{E}$ in (2.4) to $\{x\} \times \mathcal{U}$. Let

$$\mathcal{Q} := \mathbb{P}(\mathcal{E}_x) \stackrel{\beta'}{\longrightarrow} \mathcal{U}$$

be the corresponding projective bundle. Define

$$\beta := \mathrm{Id}_X \times \beta' : X \times \mathcal{Q} \longrightarrow X \times \mathcal{U}\,,$$

and consider the pulled back vector bundle

$$\widetilde{\mathcal{E}} := (\beta^*\mathcal{E}) \otimes (q_2^*\mathcal{O}_{\mathcal{Q}}(-1)) \longrightarrow X \times \mathcal{Q}\,,$$

where $q_2 : X \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is the natural projection, and

$$\mathcal{O}_{\mathcal{Q}}(1) \longrightarrow \mathbb{P}(\mathcal{E}_x) = \mathcal{Q}$$

is the tautological line bundle. For the natural action of $\mathrm{GL}(r-1, k)$ on $\widetilde{\mathcal{E}}$, the center $\mathbb{G}_m$ of $\mathrm{GL}(r-1, k)$ acts trivially on $\widetilde{\mathcal{E}}$. Consequently, the geometric invariant theoretic quotient

$$\mathcal{V} := \widetilde{\mathcal{E}}/\!\!/\mathrm{GL}(r-1, k) \longrightarrow X \times (\mathcal{Q}/\!\!/\mathrm{GL}(r-1, k)) = X \times \mathcal{P}_x$$

is a vector bundle. It is straight-forward to check that

- $\mathbb{P}(\mathcal{V}) = F^*\mathcal{P}$, where F is the map in (2.6), and
- for each point $y \in \mathcal{P}_x$, the vector bundle $\mathcal{V}|_{X\times\{y\}}$ on X lies in the isomorphism class of vector bundles associated to the point $f(y) \in \mathcal{M}_X$, where f is defined in (2.2).

Let $B : X\times\mathcal{P}_x \longrightarrow \mathcal{P}_x$ be the natural projection. Consider the direct image $B_*\mathcal{V} \longrightarrow \mathcal{P}_x$. Let $Z \subset \mathcal{P}_x$ be a nonempty Zariski open subset such that the restriction $(B_*\mathcal{V})|_Z$ is a trivial vector bundle. Fix a trivialization of $(B_*\mathcal{V})|_Z$. Take a point $y_0 \in Z$ and choose $r-1$ linearly independent sections

$$s_1, \cdots, s_{r-1} \in H^0(X \times \{y_0\}, \mathcal{V}|_{X\times\{y_0\}})$$

such that the coherent subsheaf of $\mathcal{V}|_{X\times\{y_0\}}$ generated by $s_1, \cdots, s_{r-1}$ is a subbundle of $\mathcal{V}|_{X\times\{y_0\}}$ of rank $r-1$; we note that from (2.3) it follows immediately that such $r-1$

linearly independent sections exist. Extend each s_i to a section $\widetilde{s}_i$ of $\mathcal{V}|_{X\times Z}$ using the above trivialization of $(B_*\mathcal{V})|_Z$. There is a Zariski open subset $Z' \subset Z$ containing y_0 such that the coherent subsheaf of $\mathcal{V}|_{X\times Z}$ generated by $\widetilde{s}_1, \cdots, \widetilde{s}_{r-1}$ is a subbundle of $\mathcal{V}|_{X\times Z'}$ of rank $r-1$. Note that this subbundle over $X\times Z'$ is trivial and a trivialization is given by the images of $\widetilde{s}_1, \cdots, \widetilde{s}_{r-1}$. Therefore, on $X\times Z'$, we have a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_{X\times Z'}^{\oplus(r-1)} \longrightarrow \mathcal{V}|_{X\times Z'} \longrightarrow L' \longrightarrow 0,$$

where L' is a line bundle on $X\times Z'$. Considering the top exterior products it follows that for each point $y\in Z'$, the restriction $L'|_{X\times\{y\}}$ is isomorphic to the line bundle L. Now from the seesaw theorem (see [Mu, p. 51, Corollary 6]) it follows that there is a line bundle L'' on Z' such that the line bundle $L'\otimes B^*L''$ on $X\times Z'$ is isomorphic to the pullback of L to $X\times Z'$. We may trivialize L'' over suitable nonempty Zariski open subsets of Z'. Therefore, it follows that there is a nonempty Zariski open subset

$$\iota : \mathcal{W} \hookrightarrow Z' \subset \mathcal{P}_x \tag{2.7}$$

such that the restriction $L'|_{X\times\mathcal{W}}$ is isomorphic to the pullback of L to $X\times\mathcal{W}$.

Consequently, there is a morphism

$$\varphi : \mathcal{W} \longrightarrow \mathcal{U} \subset \mathbb{P}((H^1(X,L^*)^{r-1})^*)$$

such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{W} & \stackrel{\varphi}{\longrightarrow} & \mathcal{U} \\ \big\downarrow \iota & & \big\downarrow \theta \\ \mathcal{P}_x & \stackrel{f}{\longrightarrow} & \mathcal{M}_X \end{array} \tag{2.8}$$

where θ, ι and f are the morphisms in (2.5), (2.7) and (2.2) respectively.

The codimension of the complement

$$\mathbb{P}((H^1(X,L^*)^{r-1})^*)\setminus\mathcal{U} \subset \mathbb{P}((H^1(X,L^*)^{r-1})^*)$$

is at least two. To prove this, note that $\mathrm{Pic}(\mathcal{U}) = \mathbb{Z}$ [DN, p. 89, Proposition 7.13] (here we need the assumption that $g \geq 3$); this immediately implies that the codimension of the complement $\mathcal{U}^c$ is at least two. Since the Brauer group of a projective space is zero, in view of this codimension estimate, it follows from the "Cohomological purity" [Mi, p. 241, Theorem VI.5.1] (it also follows from [Gr1, p. 292–293]) that

$$\mathrm{Br}(\mathcal{U}) = 0. \tag{2.9}$$

From the commutativity of (2.8) we conclude that the pullback homomorphism

$$\iota^*\circ f^* = (f\circ\iota)^* : \mathrm{Br}(\mathcal{M}_X) \longrightarrow \mathrm{Br}(\mathcal{W})$$

coincides with the homomorphism $\varphi^*\circ\theta^* : \mathrm{Br}(\mathcal{M}_X) \longrightarrow \mathrm{Br}(\mathcal{W})$. On the other hand, from (2.9) we know that $\varphi^*\circ\theta^* = 0$. Hence

$$\iota^*\circ f^* = 0.$$

On the other hand, the homomorphism

$$\iota^* : \mathrm{Br}(\mathcal{P}_x) \longrightarrow \mathrm{Br}(\mathcal{W})$$

is injective because $\mathcal{W}$ is a Zariski open dense subset of $\mathcal{P}_x$ (see [Mi, p. 142, Theorem 2.5]). Consequently,

$$f^* : \mathrm{Br}(\mathcal{M}_X) \longrightarrow \mathrm{Br}(\mathcal{P}_x)$$

is the zero homomorphism. On the other hand, the kernel of the above homomorphism f^* is generated by the class of $\mathcal{P}_x$ [Ga, p. 193, Theorem 2]. Therefore, we conclude that $\mathrm{Br}(\mathcal{M}_X)$ is generated by the class of $\mathcal{P}_x$. This completes the proof. □

We will denote the integer g.c.d.(r, d) by δ.

Lemma 2.2. *The order of the Brauer class $cl(\mathcal{P}_x) \in \mathrm{Br}(\mathcal{M}_X)$ is δ.*

Proof. Let $\widetilde{\mathcal{M}}_X = \widetilde{\mathcal{M}}_X(r, L)$ be the moduli stack of pairs of the form (E, ξ), where E is a stable vector bundle over X of rank r and $\xi : \bigwedge^r E \longrightarrow L$ is an isomorphism. Let μ_r denote the kernel of the homomorphism

$$\mathbb{G}_m \longrightarrow \mathbb{G}_m\,, \quad z \longmapsto z^r\,.$$

The natural morphism

$$\gamma : \widetilde{\mathcal{M}}_X \longrightarrow \mathcal{M}_X$$

makes $\widetilde{\mathcal{M}}_X$ a μ_r–gerbe over $\mathcal{M}_X$. Let

$$c_0 \in H^2(\mathcal{M}_X,\, \mu_r)$$

be the class of this μ_r–gerbe. The Brauer class

$$cl(\mathcal{P}_x) \in \mathrm{Br}(\mathcal{M}_X) = H^2(\mathcal{M}_X,\, \mathbb{G}_m)$$

coincides with the image of c_0 under the homomorphism

$$\eta : H^2(\mathcal{M}_X,\, \mu_r) \longrightarrow H^2(\mathcal{M}_X,\, \mathbb{G}_m)$$

given by the inclusion of μ_r in $\mathbb{G}_m$.

There is a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathcal{M}_X) \xrightarrow{\gamma^*} \mathrm{Pic}(\widetilde{\mathcal{M}}_X) \xrightarrow{\nu} \mathrm{Hom}(\mu_r,\, \mathbb{G}_m) = \mathbb{Z}/r\mathbb{Z} \xrightarrow{\alpha} \mathrm{Br}(\mathcal{M}_X) \tag{2.10}$$

[BH, p. 232, Lemma 4.4], where ν sends a line bundle on the μ_r–gerbe $\widetilde{\mathcal{M}}_X$ to the weight associated to the action of μ_r on it. The image

$$\alpha(1) \in \mathrm{Br}(\mathcal{M}_X) = H^2(\mathcal{M}_X,\, \mathbb{G}_m)$$

coincides with the image of the class of the μ_r–gerbe $\widetilde{\mathcal{M}}_X$ under the above homomorphism η. From this it follows that

$$\alpha(1) = cl(\mathcal{P}_x) \tag{2.11}$$

because $cl(\mathcal{P}_x)$ also coincides with the image of the class of the μ_r–gerbe $\widetilde{\mathcal{M}}_X$ under the above homomorphism η.

From [Hof, p. 1311, Lemma 3.6], [Hof, p. 1311, Lemma 3.3] and [Hof, p. 1310, Theorem 3.1] it follows that

$$\mathbb{Z}/\text{image}(\nu) \,=\, \mathbb{Z}/\delta\mathbb{Z}\,,$$

where ν is the homomorphism in (2.10). Therefore, from (2.10) we conclude that the order of $\alpha(1) \,\in\, \text{Br}(\mathcal{M}_X)$ is δ. Now the lemma follows from (2.11). □

Combining Proposition 2.1 and Lemma 2.2 we have:

Theorem 2.3. *The Brauer group* $\text{Br}(\mathcal{M}_X)$ *is cyclic of order* $\delta \,=\, \text{g.c.d.}(r, d)$. *The group* $\text{Br}(\mathcal{M}_X)$ *is generated by the class* $cl(\mathcal{P}_x) \,\in\, \text{Br}(\mathcal{M}_X)$ *of the projective bundle* $\mathcal{P}_x$ *defined in* (2.2).

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