

# Total domination in cubic Knödel graphs

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## Abstract

A subset  $D$  of vertices of a graph  $G$  is a *dominating set* if for each  $u \in V(G) \setminus D$ ,  $u$  is adjacent to some vertex  $v \in D$ . The *domination number*,  $\gamma(G)$  of  $G$ , is the minimum cardinality of a dominating set of  $G$ . A set  $D \subseteq V(G)$  is a *total dominating set* if for each  $u \in V(G)$ ,  $u$  is adjacent to some vertex  $v \in D$ . The *total domination number*,  $\gamma_t(G)$  of  $G$ , is the minimum cardinality of a total dominating set of  $G$ . For an even integer  $n \geq 2$  and  $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ , a *Knödel graph*  $W_{\Delta, n}$  is a  $\Delta$ -regular bipartite graph of even order  $n$ , with vertices  $(i, j)$ , for  $i = 1, 2$  and  $0 \leq j \leq n/2 - 1$ , where for every  $j$ ,  $0 \leq j \leq n/2 - 1$ , there is an edge between vertex  $(1, j)$  and every vertex  $(2, (j + 2^k - 1) \bmod (n/2))$ , for  $k = 0, 1, \dots, \Delta - 1$ . In this paper, we determine the total domination number in 3-regular Knödel graphs  $W_{3, n}$ .

**Keywords:** Knödel graph, domination number, total domination number, Pigeonhole Principle. **Mathematics Subject Classification [2010]:** 05C69, 05C30

## 1 introduction

For graph theory notation and terminology not given here, we refer to [8]. Let  $G = (V, E)$  denote a simple graph of order  $n = |V(G)|$  and size  $m = |E(G)|$ . Two vertices  $u, v \in V(G)$  are *adjacent* if  $uv \in E(G)$ . The *open neighborhood* of a vertex  $u \in V(G)$  is denoted by  $N(u) = \{v \in V(G) | uv \in E(G)\}$  and for a vertex set  $S \subseteq V(G)$ ,  $N(S) = \bigcup_{u \in S} N(u)$ . The cardinality of  $N(u)$  is called the *degree* of  $u$  and is denoted by  $\deg(u)$ , (or  $\deg_G(u)$  to refer it to  $G$ ). The *maximum degree* and *minimum degree* among all vertices in  $G$  are denoted by  $\Delta(G)$  and  $\delta(G)$ , respectively. A graph  $G$  is a *bipartite graph* if its vertex set can partition to two disjoint sets  $X$  and  $Y$  such that each edge in  $E(G)$  connects a vertex in  $X$  with a vertex in  $Y$ . A set  $D \subseteq V(G)$  is a *dominating set* if for each  $u \in V(G) \setminus D$ ,  $u$  is adjacent to some vertex  $v \in D$ . The *domination number*,  $\gamma(G)$  of  $G$ , is the minimum cardinality of a dominating set of  $G$ . A set  $D \subseteq V(G)$  is a *total dominating set* if

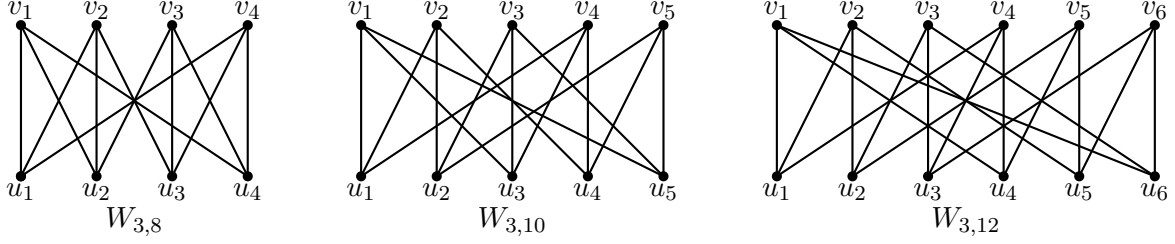


Figure 1: New labeling of Knödel graphs  $W_{3,8}$ ,  $W_{3,10}$  and  $W_{3,12}$ .

for each  $u \in V(G)$ ,  $u$  is adjacent to some vertex  $v \in D$ . The *total domination number*,  $\gamma_t(G)$  of  $G$ , is the minimum cardinality of a total dominating set of  $G$ . The concept of domination theory is a widely studied concept in graph theory and for a comprehensive study see, for example [8, 9].

An interesting family of graphs namely *Knödel graphs* have been introduced about 1975 [10], and have been studied seriously by some authors since 2001, see for example [2, 4, 5, 6]. For an even integer  $n \geq 2$  and  $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ , a *Knödel graph*  $W_{\Delta,n}$  is a  $\Delta$ -regular bipartite graph of even order  $n$ , with vertices  $(i, j)$ , for  $i = 1, 2$  and  $0 \leq j \leq n/2 - 1$ , where for every  $j$ ,  $0 \leq j \leq n/2 - 1$ , there is an edge between vertex  $(1, j)$  and every vertex  $(2, (j + 2^k - 1) \bmod (n/2))$ , for  $k = 0, 1, \dots, \Delta - 1$  (see [14]). Knödel graphs,  $W_{\Delta,n}$ , are one of the three important families of graphs that they have good properties in terms of broadcasting and gossiping, see for example [7]. It is worth-noting that any Knödel graph is a Cayley graph and so it is a vertex-transitive graph (see [6]).

Xueliang et. al. [14] studied the domination number in 3-regular Knödel graphs  $W_{3,n}$ . They obtained exact domination number for  $W_{3,n}$ . In this paper, we determine the total domination number in 3-regular Knödel graphs  $W_{3,n}$ . In Section 2, we prove some properties in the Knödel graphs. In Section 3, we present the total domination number in the 3-regular Knödel graphs  $W_{3,n}$ . We need the following simple observation from number theory.

**Observation 1.1.** *If  $a, b, c, d$  and  $x$  are positive integers such that  $x^a - x^b = x^c - x^d \neq 0$ , then  $a = c$  and  $b = d$ .*

## 2 Properties in the Knödel graphs

For simplicity, in this paper, we re-label the vertices of a Knödel graph as follows: we label  $(1, i)$  by  $u_{i+1}$  for each  $i = 0, 1, \dots, n/2 - 1$ , and  $(2, j)$  by  $v_{j+1}$  for  $j = 0, 1, \dots, n/2 - 1$ . Let  $U = \{u_1, u_2, \dots, u_{n/2}\}$  and  $V = \{v_1, v_2, \dots, v_{n/2}\}$ . From now on, the vertex set of each Knödel graph  $W_{\Delta,n}$  is  $U \cup V$  such that  $U$  and  $V$  are the two partite sets of the graph. If  $S$  is a set of vertices of  $W_{\Delta,n}$ , then clearly,  $S \cap U$  and  $S \cap V$  partition  $S$ ,  $|S| = |S \cap U| + |S \cap V|$ ,  $N(S \cap U) \subseteq V$  and  $N(S \cap V) \subseteq U$ . Note that two vertices  $u_i$  and  $v_j$  are adjacent if and only if  $j \in \{i + 2^0 - 1, i + 2^1 - 1, \dots, i + 2^{\Delta-1} - 1\}$ , where the addition is taken in modulo  $n/2$ . Figure 1, shows new labeling of Knödel graphs  $W_{3,8}$ ,  $W_{3,10}$  and  $W_{3,12}$ .

For any subset  $\{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$  of  $U$  with  $1 \leq i_1 < i_2 < \dots < i_k \leq \frac{n}{2}$ , we correspond a sequence based on the differences of the indices of  $u_j$ ,  $j = i_1, \dots, i_k$ , as follows.

**Definition 2.1.** For any subset  $A = \{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$  of  $U$  with  $1 \leq i_1 < i_2 < \dots < i_k \leq \frac{n}{2}$  we define a sequence  $n_1, n_2, \dots, n_k$ , namely **cyclic-sequence**, where  $n_j = i_{j+1} - i_j$  for  $1 \leq j \leq k-1$  and  $n_k = \frac{n}{2} + i_1 - i_k$ . For two vertices  $u_{i_j}, u_{i_{j'}} \in A$  we define **index-distance** of  $u_{i_j}$  and  $u_{i_{j'}}$  by  $id(u_{i_j}, u_{i_{j'}}) = \min\{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$ .

**Observation 2.2.** Let  $A = \{u_{i_1}, u_{i_2}, \dots, u_{i_k}\} \subseteq U$  be a set such that  $1 \leq i_1 < i_2 < \dots < i_k \leq \frac{n}{2}$  and let  $n_1, n_2, \dots, n_k$  be the corresponding cyclic-sequence of  $A$ . Then,

- (1)  $n_1 + n_2 + \dots + n_k = \frac{n}{2}$ .
- (2) If  $u_{i_j}, u_{i_{j'}} \in A$ , then  $id(u_{i_j}, u_{i_{j'}})$  equals to sum of some consecutive elements of the cyclic-sequence of  $A$  and  $\frac{n}{2} - id(u_{i_j}, u_{i_{j'}})$  is sum of the remaining elements of the cyclic-sequence. Furthermore,  $\{id(u_{i_j}, u_{i_{j'}}), \frac{n}{2} - id(u_{i_j}, u_{i_{j'}})\} = \{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$ .

We henceforth use the notation  $\mathcal{M}_\Delta = \{2^a - 2^b : 0 \leq b < a < \Delta\}$  for  $\Delta \geq 2$ .

**Lemma 2.3.** In the Knödel graph  $W_{\Delta, n}$  with vertex set  $U \cup V$ , for two distinct vertices  $u_i$  and  $u_j$ ,  $N(u_i) \cap N(u_j) \neq \emptyset$  if and only if  $id(u_i, u_j) \in \mathcal{M}_\Delta$  or  $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$ .

*Proof.* Since  $W_{\Delta, n}$  is vertex-transitive, for simplicity, we put  $1 = i < j \leq \frac{n}{2}$ . We have  $id(u_1, u_j) = \min\{j-1, \frac{n}{2} - (j-1)\}$  and so  $\frac{n}{2} - id(u_1, u_j) = \max\{j-1, \frac{n}{2} - (j-1)\}$ . Also, we have  $N(u_1) = \{v_1, v_2, v_4, \dots, v_{2^{\Delta-1}}\}$  and  $N(u_j) = \{v_j, v_{j+1}, v_{j+3}, \dots, v_{j+2^{\Delta-1}-1}\}$ . First assume that  $N(u_1) \cap N(u_j) \neq \emptyset$ . Let  $v_k \in N(u_1) \cap N(u_j)$ . There exist two integers  $a$  and  $b$  such that  $0 \leq a, b \leq \Delta-1$  and  $k \equiv 2^a \equiv j + 2^b - 1 \pmod{n/2}$ . Since  $1 \leq 2^a, 2^b, j \leq \frac{n}{2}$ , we have  $1 \leq j + 2^b - 1 < n$ . If  $1 \leq j + 2^b - 1 \leq \frac{n}{2}$ , then  $2^a = j + 2^b - 1$  and  $j-1 = 2^a - 2^b \in \mathcal{M}_\Delta$  and if  $\frac{n}{2} < j + 2^b - 1 < n$ , then  $2^a = j + 2^b - 1 - \frac{n}{2}$  and  $\frac{n}{2} - (j-1) = 2^b - 2^a \in \mathcal{M}_\Delta$ . Therefore, by Observation 2.2,  $id(u_i, u_j) \in \mathcal{M}_\Delta$  or  $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$ .

Conversely, suppose  $id(u_1, u_j) \in \mathcal{M}_\Delta$  or  $\frac{n}{2} - id(u_1, u_j) \in \mathcal{M}_\Delta$ . Then  $j-1 \in \mathcal{M}_\Delta$  or  $\frac{n}{2} - (j-1) \in \mathcal{M}_\Delta$ . If  $j-1 \in \mathcal{M}_\Delta$ , then we have  $j-1 = 2^a - 2^b$  for two integers  $0 \leq a, b \leq \Delta-1$ . Then  $2^a = j + 2^b - 1$  and  $v_{2^a} \in N(u_1) \cap N(u_j)$ . If  $\frac{n}{2} - (j-1) \in \mathcal{M}_\Delta$ , then we have  $\frac{n}{2} - (j-1) = 2^c - 2^d$  for two integers  $0 \leq c, d \leq \Delta-1$ . Now  $2^c = j + 2^d - 1 - \frac{n}{2} \equiv j + 2^d - 1 \pmod{n/2}$  and  $v_{2^c} \in N(u_1) \cap N(u_j)$ . Thus in each case,  $N(u_i) \cap N(u_j) \neq \emptyset$ .  $\square$

**Lemma 2.4.** In the Knödel graph  $W_{\Delta, n}$  with vertex set  $U \cup V$ , for two distinct vertices  $u_i$  and  $u_j$ ,  $|N(u_i) \cap N(u_j)| = 2$  if and only if  $id(u_i, u_j) \in \mathcal{M}_\Delta$  and  $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$ .

*Proof.* Without loss of generality, we assume that  $1 \leq i < j \leq \frac{n}{2}$ . Suppose that  $|N(u_i) \cap N(u_j)| = 2$  and  $v_k, v_{k'} \in N(u_i) \cap N(u_j)$  are two distinct vertices in  $V$ . There exist two integers  $a$  and  $b$  such that  $0 \leq a, b \leq \Delta-1$  and  $k \equiv i + 2^a - 1 \equiv j + 2^b - 1 \pmod{n/2}$ . Similarly, there exist two integers  $a'$  and  $b'$  such that  $0 \leq a', b' \leq \Delta-1$  and  $k' \equiv i + 2^{a'} - 1 \equiv j + 2^{b'} - 1 \pmod{n/2}$ . Now we have  $j-i \equiv 2^b - 2^a \equiv 2^{b'} - 2^{a'} \pmod{n/2}$ . We know that  $-\frac{n}{2} < 2^b - 2^a, 2^{b'} - 2^{a'} < \frac{n}{2}$ . If  $-\frac{n}{2} < 2^b - 2^a, 2^{b'} - 2^{a'} < 0$  or  $0 < 2^b - 2^a, 2^{b'} - 2^{a'} < \frac{n}{2}$ , then we have  $2^b - 2^a = 2^{b'} - 2^{a'} \neq 0$ . Observation 1.1 implies that  $b = b'$  and therefore  $k \equiv k' \pmod{n/2}$  and  $v_k = v_{k'}$ , a contradiction. By symmetry, we assume that  $0 < 2^b - 2^a < \frac{n}{2}$  and  $-\frac{n}{2} < 2^{b'} - 2^{a'} < 0$ . Since  $0 < j-i < \frac{n}{2}$ , we have  $j-i = 2^b - 2^a$  and  $\frac{n}{2} - (j-i) = 2^{a'} - 2^{b'}$  which implies that  $j-i \in \mathcal{M}_\Delta$  and  $\frac{n}{2} - (j-i) \in \mathcal{M}_\Delta$ . Thus by Observation 2.2,  $id(u_i, u_j) \in \mathcal{M}_\Delta$  and  $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$ .

Conversely, assume that  $id(u_i, u_j) \in \mathcal{M}_\Delta$  and  $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$  for two distinct vertices  $u_i$  and  $u_j$ . There exist two integers  $a$  and  $b$  such that  $0 \leq b < a \leq \frac{n}{2}$  and  $j - i = 2^a - 2^b$ . Also there exist two integers  $a'$  and  $b'$  such that  $0 \leq a' < b' \leq \frac{n}{2}$  and  $\frac{n}{2} - (j - i) = 2^{b'} - 2^{a'}$ . Now we have  $i + 2^a - 1 = j + 2^b - 1$  and  $i + 2^{a'} - 1 = j + 2^{b'} - 1 - \frac{n}{2} \equiv j + 2^{b'} - 1 \pmod{n/2}$ . We set  $k = i + 2^a - 1$  and  $k' = i + 2^{a'} - 1$ . Then  $v_k, v_{k'} \in N(u_i) \cap N(u_j)$  and  $|N(u_i) \cap N(u_j)| \geq 2$ . Notice that  $k \not\equiv k' \pmod{n/2}$ , since otherwise  $a = a'$  and  $2^{b'} - 2^b = \frac{n}{2}$ , a contradiction. Suppose that  $|N(u_i) \cap N(u_j)| \geq 3$ . Let  $v_k, v_{k'}, v_{k''} \in N(u_i) \cap N(u_j)$  be three distinct vertices. Similar to the first part of the proof, for  $v_k$  and  $v_{k'}$ , there exist two integers  $a''$  and  $b''$  such that  $0 \leq a'', b'' \leq \Delta - 1$  and  $k'' \equiv i + 2^{a''} - 1 \equiv j + 2^{b''} - 1 \pmod{n/2}$  and thus  $j - i \equiv 2^{a''} - 2^{b''} \pmod{n/2}$ . Since  $u_i$  and  $u_j$  are distinct, we have  $a'' \neq b''$ . If  $a'' > b''$ , then  $j - i = 2^{a''} - 2^{b''}$  and it can be seen that  $j - i = \frac{n}{2} - (2^a - 2^b) = \frac{n}{2} - (2^{a'} - 2^{b'})$  and Observation 1.1 implies that  $a = a'$  and thus  $v_k = v_{k'}$ , a contradiction. If  $a'' < b''$ , then  $j - i = \frac{n}{2} - (2^{a''} - 2^{b''})$  and it can be seen that  $j - i = 2^a - 2^b = 2^{a'} - 2^{b'}$  and Observation 1.1 implies that  $a = a'$ , a contradiction. Consequently  $|N(u_i) \cap N(u_j)| = 2$ .  $\square$

**Corollary 2.5.** (i) In the Knödel graph  $W_{\Delta, n}$  with vertex set  $U \cup V$ , for each  $1 \leq i < j \leq n/2$ ,  $|N(u_i) \cap N(u_j)| = 1$  if and only if precisely one of the values  $id(u_i, u_j)$  and  $\frac{n}{2} - id(u_i, u_j)$  belongs to  $\mathcal{M}_\Delta$ .

(ii) In the Knödel graph  $W_{\Delta, n}$ , there exist distinct vertices with two common neighbors if and only if  $n = 2^a - 2^b + 2^c - 2^d$  and  $a > b \geq 1, c > d \geq 1$ .

**Corollary 2.6.** Any three vertices in the Knödel graph  $W_{\Delta, n}$  have at most one common neighbor. Indeed, any Knödel graph is a  $K_{2,3}$ -free graph.

**Lemma 2.7.** In the Knödel graph  $W_{\Delta, n}$  with vertex set  $U \cup V$  and  $\Delta < \log_2(n/2 + 2)$ , we have:

- (i)  $|N(u_i) \cap N(u_j)| \leq 1, 1 \leq i < j \leq n/2$ .
- (ii)  $|N(u_i) \cap N(u_j)| = 1$  if and only if  $id(u_i, u_j) \in \mathcal{M}_\Delta$ .

*Proof.* (i) Suppose to the contrary that  $|N(u_i) \cap N(u_j)| > 1$ , then by Corollary 2.6 we have  $|N(u_i) \cap N(u_j)| = 2$ . Then the Lemma 2.4 implies that  $id(u_i, u_j) \in \mathcal{M}_\Delta$  and  $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$ . Thus  $id(u_i, u_j) \leq 2^{\Delta-1} - 1, \frac{n}{2} - id(u_i, u_j) \leq 2^{\Delta-1} - 1$  and  $\frac{n}{2} \leq 2^\Delta - 2$ . This inequality implies that  $\Delta \geq \log_2(n/2 + 2)$ , a contradiction. Hence  $|N(u_i) \cap N(u_j)| \leq 1$ , as desired.

(ii) Assume that  $|N(u_i) \cap N(u_j)| = 1$ . By Corollary 2.5, precisely one of the values  $id(u_i, u_j)$  and  $\frac{n}{2} - id(u_i, u_j)$  belongs to  $\mathcal{M}_\Delta$ . If  $\frac{n}{2} - id(u_i, u_j) \in \mathcal{M}_\Delta$ , then  $\frac{n}{2} - id(u_i, u_j) \leq 2^{\Delta-1} - 1$  and so  $2^\Delta - 2 - id(u_i, u_j) < 2^{\Delta-1} - 1$ . Now, we have  $2^{\Delta-1} - 1 < id(u_i, u_j)$  and so  $\frac{n}{2} - id(u_i, u_j) < id(u_i, u_j)$ , a contradiction by definition of index-distance. Therefore,  $id(u_i, u_j) \in \mathcal{M}_\Delta$ .

Conversely, Assume that  $id(u_i, u_j) \in \mathcal{M}_\Delta$ . Thus,  $id(u_i, u_j) \leq 2^{\Delta-1} - 1$  and so  $\frac{n}{2} - id(u_i, u_j) \geq \frac{n}{2} - 2^{\Delta-1} + 1 > 2^\Delta - 2 - 2^{\Delta-1} + 1 = 2^{\Delta-1} - 1$ . Therefore,  $\frac{n}{2} - id(u_i, u_j) \notin \mathcal{M}_\Delta$  and by Corollary 2.5 we have  $|N(u_i) \cap N(u_j)| = 1$ .  $\square$

**Lemma 2.8.** Let  $W_{\Delta, n}$  be a Knödel graph with vertex set  $U \cup V$ . For any non-empty subset  $A \subseteq U$ :

- (i)  $\sum_{v \in N(A)} |N(v) \cap A| = \Delta|A|$ .
- (ii) The corresponding cyclic-sequence of  $A$  has at most  $\Delta|A| - |N(A)|$  elements belonging to  $\mathcal{M}_\Delta$ .

*Proof.* Let  $A \subseteq U$  be a non-empty set.

(i) It is obvious that the induced subgraph  $H = W_{\Delta, n}[A \cup N(A)]$  is a bipartite graph

and  $|E(H)| = \sum_{u \in A} \deg_H(u) = \sum_{v \in N(A)} \deg_H(v)$ , where  $E(H)$  is the edge set of  $S$ . If  $u \in A$ , then  $\deg_H(u) = \Delta$ , and for  $v \in N(A)$  we have  $\deg_H(v) = |N(v) \cap A|$ . Thus,  $\sum_{u \in A} \deg_H(u) = \sum_{u \in A} \Delta = \Delta|A|$  and  $\sum_{v \in N(A)} \deg_H(v) = \sum_{v \in N(A)} |N(v) \cap A|$ . Consequently,  $\sum_{v \in N(A)} |N(v) \cap A| = \Delta|A|$ .

(ii) Suppose that  $A = \{u_{i_1}, u_{i_2}, \dots, u_{i_{|A|}}\}$ , where  $1 \leq i_1 < i_2 < \dots < i_{|A|} \leq \frac{n}{2}$ , and let  $n_1, n_2, \dots, n_{|A|}$  be the corresponding cyclic-sequence of  $A$ . For any vertex  $v \in N(A)$ , let  $r(v) = |N(v) \cap A|$ . Let  $J = \{j : n_j \in \mathcal{M}_\Delta\}$  and  $R = \Delta|A| - |N(A)|$ . We prove that  $R \geq |J|$ . If  $R \geq |A|$ , then we have nothing to prove, since  $|J| \leq |A|$ . Assume that  $R < |A|$  and notice that by part (i),

$$R = \Delta|A| - |N(A)| = \sum_{v \in N(A)} |N(v) \cap A| - \sum_{v \in N(A)} 1 = \sum_{v \in N(A)} [r(v) - 1].$$

If  $\{v \in N(A) : r(v) \geq 2\} = \emptyset$ , then  $R = 0$  and  $J = \emptyset$ , and so  $R \geq |J|$ . Thus assume that  $\{v \in N(A) : r(v) \geq 2\} \neq \emptyset$ . Then  $R = \sum_{\substack{v \in N(A) \\ r(v) \geq 2}} [r(v) - 1]$ .

Assume that there exists  $v' \in N(A)$  such that  $r(v') = |A|$ . Then  $R = r(v') - 1 + \sum_{\substack{v \in N(A) \\ r(v) \geq 2 \\ v \neq v'}} [r(v) - 1] =$

$|A| - 1 + \sum_{\substack{v \in N(A) \\ r(v) \geq 2 \\ v \neq v'}} [r(v) - 1]$ . Since  $R < |A|$ , we obtain that  $\sum_{\substack{v \in N(A) \\ r(v) \geq 2 \\ v \neq v'}} [r(v) - 1] = 0$ ,  $R = |A| - 1$ , and for

each  $v \in N(A) \setminus \{v'\}$  we have  $r(v) = 1$ . Since  $W_{\Delta, n}$  is vertex transitive, without loss of generality, we assume that  $v' = v_{n/2}$ .

According to the definition of a Knodel graph, there exist integers  $0 \leq a_{|A|} < a_{|A|-1} < \dots < a_2 < a_1 \leq \Delta - 1$  such that  $i_j = \frac{n}{2} - 2^{a_j} + 1$  for each  $1 \leq j \leq |A|$ . Moreover,  $n_j = i_{j+1} - i_j = 2^{a_{i_j}} - 2^{a_{i_{j+1}}} \in \mathcal{M}_\Delta$  for each  $1 \leq j \leq |A| - 1$ . Evidently,  $i_{|A|} - i_{|A|-1} = n_1 + n_2 + \dots + n_{|A|-1} = 2^{a_{|A|-1}} - 2^{a_{|A|}} \in \mathcal{M}_\Delta$  and  $n_{|A|} = n/2 - (i_{|A|} - i_{|A|-1})$ . We show that  $n_{|A|} \notin \mathcal{M}_\Delta$ . Suppose to the contrary that  $n_{|A|} \in \mathcal{M}_\Delta$ . Since  $n_{|A|} = \frac{n}{2} - (i_{|A|} - i_{|A|-1}) \in \mathcal{M}_\Delta$  and  $i_{|A|} - i_{|A|-1} \in \mathcal{M}_\Delta$ , by Observation 2.2,  $id(u_{i_1}, u_{i_{|A|}}) \in \mathcal{M}_\Delta$  and  $\frac{n}{2} - id(u_{i_1}, u_{i_{|A|}}) \in \mathcal{M}_\Delta$ , and by Lemma 2.4,  $|N(u_{i_1}) \cap N(u_{i_{|A|}})| = 2$ . Now there exists  $v'' \neq v_{n/2}$  such that  $v'' \in N(u_{i_1}) \cap N(u_{i_{|A|}})$  and  $r(v'') \geq 2$ , a contradiction. Therefore,  $n_{|A|} \notin \mathcal{M}_\Delta$ . Since  $n_j \in \mathcal{M}_\Delta$  for each  $1 \leq j \leq |A| - 1$ , we obtain that  $|J| = |A| - 1 = R$ . Thus there are at most  $R = |A| - 1$  elements of the cyclic sequence of  $A$  which belong to  $\mathcal{M}_\Delta$ .

Next assume that  $r(v) < |A|$  for any  $v \in N(A)$ . Let  $X_v = \{j : a_{i_j}, a_{i_{j+1}} \in N(A) \cap A\}$ . We prove that  $J \subseteq \bigcup_{v \in N(A)} X_v$ . Let  $j \in J$ . Then  $n_j = i_{j+1} - i_j \in \mathcal{M}_\Delta$ . By Observation 2.2,  $n_j = i_{j+1} - i_j \in \{id(a_{i_j}, a_{i_{j+1}}), \frac{n}{2} - id(a_{i_j}, a_{i_{j+1}})\}$  and by Lemma 2.3,  $|N(a_{i_j}) \cap N(a_{i_{j+1}})| \geq 1$ . Let  $v \in N(a_{i_j}) \cap N(a_{i_{j+1}})$ . Then  $a_{i_j}, a_{i_{j+1}} \in N(v) \cap A$ . Therefore  $j \in X_v$  and  $j \in \bigcup_{v \in N(A)} X_v$  that implies  $J \subseteq \bigcup_{v \in N(A)} X_v$ . Then  $|J| \leq \left| \bigcup_{v \in N(A)} X_v \right|$ . Observe that  $X_v = \{j : a_{i_j} \in N(v) \cap A, a_{i_{j+1}} \notin N(v) \cap A\}$ , and  $|\{j : a_{i_j} \in N(v) \cap A\}| = |N(v) \cap A| = r(v)$ . Since  $N(v) \cap A \not\subseteq A$ , we have  $\{j : a_{i_j} \in N(v) \cap A, a_{i_{j+1}} \notin N(v) \cap A\} \neq \emptyset$ . Therefore  $|X_v| \leq r(v) - 1$ . Consequently,  $|J| \leq \left| \bigcup_{v \in N(A)} X_v \right| \leq \sum_{v \in N(A)} |X_v| \leq \sum_{v \in N(A)} [r(v) - 1]$ .  $\square$

We remark that one can define the cyclic-sequence and index-distance for any subset of  $V$  in a similar way, and thus the Observation 2.2, Lemmas 2.3 and 2.4 and corollaries 2.5 and 2.6 are valid for cyclic-sequence and index-distance on subsets of  $V$  as well.

### 3 Total domination number of 3-regular Knödel graphs

We are now ready to determine the total domination number of  $W_{3,n}$ . Clearly  $n \geq 8$  is an even integer by the definition of  $W_{3,n}$ .

**Theorem 3.1.** *For each even integer  $n \geq 8$ ,  $\gamma_t(W_{3,n}) = 4\lceil \frac{n}{10} \rceil - \begin{cases} 0 & n \equiv 0, 6, 8 \pmod{10} \\ 2 & n \equiv 2, 4 \pmod{10} \end{cases}$ .*

*Proof.* We divide the proof into five cases depending on  $n$ .

**Case 1:**  $n \equiv 0 \pmod{10}$ . Let  $n = 10t$ , where  $t \geq 1$ . Then the set  $D_1 = \{u_{5k+b}, v_{5k+b} : k = 0, 1, \dots, t-1; b = 1, 2\}$  is a total dominating set for  $W_{3,n}$  and thus  $\gamma_t(W_{3,n}) \leq |D_1| = 4t = 4\lceil \frac{n}{10} \rceil$ . We show that  $\gamma_t(W_{3,n}) = 4t$ . Suppose to the contrary, that  $\gamma_t(W_{3,n}) < 4t$ . Let  $D$  be a total dominating set with  $4t - 1$  elements. Then by the Pigeonhole Principle either  $|D \cap U| \leq 2t - 1$  or  $|D \cap V| \leq 2t - 1$ . Without loss of generality, assume that  $|D \cap U| \leq 2t - 1$ . Let  $|D \cap U| = 2t - 1 - a$ , where  $a \geq 0$ . Then  $|D \cap V| = 2t + a$ . Observe that  $D \cap U$  dominates at most  $3|D \cap U| = 6t - 3 - 3a$  vertices of  $V$ , and so  $6t - 3 - 3a \geq 5t = |V|$ , since  $D \cap U$  dominates  $V$ . Clearly the inequality  $6t - 3 - 3a \geq 5t$  does not hold if  $t \in \{1, 2\}$ , and thus this contradiction implies that  $\gamma_t(W_{3,n}) = 4t = 4\lceil \frac{n}{10} \rceil$  for  $t = 1, 2$ . From here on, assume that  $t \geq 3$ . Thus there are at most  $(6t - 3 - 3a) - 5t = t - 3 - 3a$  vertices of  $V$  that are dominated by at least two vertices of  $D \cap U$ . In the other words, by Lemma 2.8 at most  $t - 3 - 3a$  elements of the cyclic-sequence of  $D \cap U$  belong to  $\mathcal{M}_3 = \{1, 2, 3\}$ . Furthermore, at least  $(2t - 1 - a) - (t - 3 - 3a) = t + 2 + 2a$  elements of the cyclic-sequence of  $D \cap U$  are greater than 3 (do not belong to  $\mathcal{M}_3$  by Lemma 2.3). Then by Observation 2.2,  $5t = \sum_{i=1}^{2t-1} n_i \geq 4(t + 2 + 2a) + (t - 3 - 3a) = 5t + 5 + 5a$ , a contradiction. Therefore,  $\gamma_t(W_{3,n}) = 4t = 4\lceil \frac{n}{10} \rceil$ .

**Case 2:**  $n \equiv 2 \pmod{10}$ . Let  $n = 10t + 2$ , where  $t \geq 1$ . Then the set  $D_2 = \{u_{5k+b}, v_{5k+b} : k = 0, 1, \dots, t-1; b = 1, 2\} \cup \{u_{5t+1}, v_{5t+1}\}$  is a total dominating set for  $W_{3,n}$  and thus  $\gamma_t(W_{3,n}) \leq |D_2| = 4t + 2 = 4\lceil \frac{n}{10} \rceil - 2$ . We show that  $\gamma_t(W_{3,n}) = 4t + 2$ . Suppose to the contrary, that  $\gamma_t(W_{3,n}) < 4t + 2$ . Let  $D$  be a total dominating set with  $4t + 1$  elements. Then by the Pigeonhole Principle either  $|D \cap U| \leq 2t$  or  $|D \cap V| \leq 2t$ . Without loss of generality, assume that  $|D \cap U| \leq 2t$ . Let  $|D \cap U| = 2t - a$ , where  $a \geq 0$ . Then  $|D \cap V| = 2t + 1 + a$ . Observe that  $D \cap U$  dominates at most  $6t - 3a$  vertices of  $V$  and so  $6t - 3a \geq 5t + 1 = |V|$ , since  $D \cap U$  dominates  $V$ . Then there are at most  $(6t - 3a) - (5t + 1) = t - 1 - 3a$  vertices of  $V$  that are dominated by at least two vertices of  $D \cap U$ . By Lemma 2.8, at most  $t - 1 - 3a$  elements of the cyclic-sequence of  $D \cap U$  belong to  $\mathcal{M}_3$ . Furthermore, at least  $(2t - a) - (t - 1 - 3a) = t + 1 + 2a$  elements of the cyclic-sequence of  $D \cap U$  are greater than 3 (do not belong to  $\mathcal{M}_3$  by Lemma 2.3). Then by Observation 2.2,  $5t + 1 = \sum_{i=1}^{2t-a} n_i \geq 4(t + 1 + 2a) + (t - 1 - 3a) = 5t + 3 + 5a$ , a contradiction. Therefore,  $\gamma_t(W_{3,n}) = 4t + 2 = 4\lceil \frac{n}{10} \rceil - 2$ .

**Case 3:**  $n \equiv 4 \pmod{10}$ . Let  $n = 10t + 4$ , where  $t \geq 1$ . Then the set  $D_3 = \{u_{5k+b}, v_{5k+b} : k = 0, 1, \dots, t-1; b = 1, 2\} \cup \{u_{5t+1}, v_{5t+2}\}$  is a total dominating set for  $W_{3,n}$  and thus  $\gamma_t(W_{3,n}) \leq |D_3| = 4t + 2 = 4\lceil \frac{n}{10} \rceil - 2$ . We show that  $\gamma_t(W_{3,n}) = 4t + 2$ . Suppose to the contrary, that  $\gamma_t(W_{3,n}) < 4t + 2$ . Let  $D$  be a total dominating set with  $4t + 1$  elements. Then by the Pigeonhole Principle either  $|D \cap U| \leq 2t$  or  $|D \cap V| \leq 2t$ . Without loss of generality, assume that  $|D \cap U| \leq 2t$ . Let  $|D \cap U| = 2t - a$ , where  $a \geq 0$ . Then  $|D \cap V| = 2t + 1 + a$ . Observe that  $D \cap U$  dominates at most  $6t - 3a$  vertices of  $V$ , and so  $6t - 3a \geq 5t + 2 = |V|$ , since  $D \cap U$  dominates  $V$ . Clearly the inequality  $6t - 3a \geq 5t + 2$  does not hold if  $t = 1$ , and thus this contradiction implies that  $\gamma_t(W_{3,n}) = 4t + 2 = 4\lceil \frac{n}{10} \rceil - 2$  for  $t = 1$ . From here on, assume that  $t \geq 2$ . Then there are at most  $(6t - 3a) - (5t + 2) = t - 2 - 3a$  vertices of  $V$  that are dominated by at least two vertices of  $D \cap U$ . By Lemma 2.8, at most  $t - 2 - 3a$  elements of the cyclic-sequence of  $D \cap U$  belong to  $\mathcal{M}_3$ . Furthermore, at least  $(2t - a) - (t - 2 - 3a) = t + 2 + 2a$  elements of the cyclic-sequence of  $D \cap U$  are greater than 3 (do not belong to  $\mathcal{M}_3$  by Lemma 2.3). Then by Observation 2.2,  $5t + 2 = \sum_{i=1}^{2t-a} n_i \geq 4(t + 2 + 2a) + (t - 2 - 3a) = 5t + 6 + 5a$ , a contradiction. Therefore,  $\gamma_t(W_{3,n}) = 4t + 2 = 4\lceil \frac{n}{10} \rceil - 2$ .

**Case 4:**  $n \equiv 6 \pmod{10}$ . Let  $n = 10t + 6$ , where  $t \geq 1$ . Then the set  $D_4 = \{u_{5k+b}, v_{5k+b} : k = 0, 1, \dots, t-1; b = 1, 2\} \cup \{u_{5t+1}, v_{5t+1}, u_{5t+2}, v_{5t+3}\}$  is a total dominating set for  $W_{3,n}$  and thus  $\gamma_t(W_{3,n}) \leq |D_4| = 4t + 4 = 4\lceil \frac{n}{10} \rceil$ . We show that  $\gamma_t(W_{3,n}) = 4t + 4$ . Suppose to the contrary, that  $\gamma_t(W_{3,n}) < 4t + 4$ . Let  $D$  be a total dominating set with  $4t + 3$  elements. Then by the Pigeonhole Principle either  $|D \cap U| \leq 2t + 1$  or  $|D \cap V| \leq 2t + 1$ . Without loss of generality, assume that  $|D \cap U| \leq 2t + 1$ . Let  $|D \cap U| = 2t + 1 - a$ , where  $a \geq 0$ . Then  $|D \cap V| = 2t + 2 + a$ . Observe that  $D \cap U$  dominates at most  $6t + 3 - 3a$  vertices of  $V$ , and so  $6t + 3 - 3a \geq 5t + 3 = |V|$ , since  $D \cap U$  dominates  $V$ . Then there are at most  $(6t + 3 - 3a) - (5t + 3) = t - 3a$  vertices of  $V$  that are dominated by at least two vertices of  $D \cap U$ . By Lemma 2.8, at most  $t - 3a$  elements of the cyclic-sequence of  $D \cap U$  belong to  $\mathcal{M}_3$ . Furthermore, at least  $(2t + 1 - a) - (t - 3a) = t + 1 + 2a$  elements of the cyclic-sequence of  $D \cap U$  are greater than 3 (do not belong to  $\mathcal{M}_3$  by Lemma 2.3). Then by Observation 2.2,  $5t + 3 = \sum_{i=1}^{2t+1-a} n_i \geq 4(t + 2 + 2a) + (t - 3a) = 5t + 8 + 5a$ , a contradiction. Therefore,  $\gamma_t(W_{3,n}) = 4t + 4 = 4\lceil \frac{n}{10} \rceil$ .

**Case 5:**  $n \equiv 8 \pmod{10}$ . Let  $n = 10t + 8$ , where  $t \geq 0$ . Then the set  $D_5 = \{u_{5k+b}, v_{5k+b} : k = 0, 1, \dots, t-1; b = 1, 2\} \cup \{u_{5t+1}, v_{5t+2}, u_{5t+3}, v_{5t+4}\}$  is a total dominating set for  $W_{3,n}$  and thus  $\gamma_t(W_{3,n}) \leq |D_5| = 4t + 4 = 4\lceil \frac{n}{10} \rceil$ . We show that  $\gamma_t(W_{3,n}) = 4t + 4$ . Suppose to the contrary, that  $\gamma_t(W_{3,n}) < 4t + 4$ . Let  $D$  be a total dominating set with  $4t + 3$  elements. Then by the Pigeonhole Principle either  $|D \cap U| \leq 2t + 1$  or  $|D \cap V| \leq 2t + 1$ . Without loss of generality, assume that  $|D \cap U| \leq 2t + 1$ . Let  $|D \cap U| = 2t + 1 - a$  and  $a \geq 0$ . Then  $|D \cap V| = 2t + 2 + a$ . Observe that  $D \cap U$  dominates at most  $6t + 3 - 3a$  vertices of  $V$ , and so  $6t + 3 - 3a \geq 5t + 4 = |V|$ , since  $D \cap U$  dominates  $V$ . Then there are at most  $(6t + 3 - 3a) - (5t + 4) = t - 1 - 3a$  vertices of  $V$  that are dominated by at least two vertices of  $D \cap U$ . By Lemma 2.8, at most  $t - 1 - 3a$  elements of the cyclic-sequence of  $D \cap U$  belong to  $\mathcal{M}_3$ . Furthermore, at least  $(2t + 1 - a) - (t - 1 - 3a) = t + 2 + 2a$  elements of the cyclic-sequence of  $D \cap U$  are greater than 3 (do not belong to  $\mathcal{M}_3$  by Lemma 2.3). Then by Observation 2.2,  $5t + 4 = \sum_{i=1}^{2t+1-a} n_i \geq 4(t + 2 + 2a) + (t - 1 - 3a) = 5t + 7 + 5a$ , a contradiction. Therefore  $\gamma_t(W_{3,n}) = 4t + 4 = 4\lceil \frac{n}{10} \rceil$ .  $\square$

## 4 Conclusion

Domination number and total domination number of the 3-regular Knödel graphs have already been determined. Determining other variations of domination (such as connected domination number, independent domination number, etc.) on these graphs seems of sufficient interest. Moreover, determining the domination variants of the  $k$ -regular Knödel graphs for  $k \geq 4$  are still open.

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