

A Note on Switching Conditions for the Generalized Multiplicative Connectives

Yuki Nishimuta*

Mitsuhiro Okada †

Abstract

Danos and Regnier (1989) introduced the par-switching condition for the multiplicative proof-structures and simplified the sequentialization theorem of Girard (1987) by the use of par-switching. Danos and Regnier (1989) also generalized the par-switching to a switching for n -ary connectives (an n -ary switching, in short) and showed that the “expansion” property which means that any “excluded-middle” formula has a correct proof-net in the sense of their n -ary switching. They added a remark that the sequentialization theorem does not hold with their switching. Their definition of switching for n -ary connectives is a natural generalization of the original switching for the binary connectives. However, there are many other possible definitions of switching for n -ary connectives. We give an alternative and “natural” definition of n -ary switching, and we remark that the proof of sequentialization theorem by Olivier Laurent with the par-switching works for our n -ary switching; hence that the sequentialization theorem holds for our n -ary switching. On the other hand, we remark that the “expansion” property does not hold with our switching anymore. We point out that no definition of n -ary switching satisfies both the sequentialization theorem and the “expansion” property at the same time except for the purely tensor-based (or purely par-based) connectives.

1 Introduction

The sequentialization theorem of Girard (1987) for the Multiplicative fragment of Linear Logic (MLL) says that a (graphically represented) multiplicative proof structure can be translated to a sequential MLL-proof if his “long-trip” condition is satisfied. Danos and Regnier in this Journal (1989) simplified the sequentialization theorem by introducing the notion of par-switching condition which

*Keio University, E-mail: nishimuta@abelard.flet.keio.ac.jp

†Keio University, E-mail: E-mail: mitsu@abelard.flet.keio.ac.jp

says that a resulting proof-structure from choosing one of two nodes for all par-links becomes a connected and acyclic graph. Their sequentialization theorem says that for a given proof-structure if the par-switching condition is satisfied, the proof-structure can be transformed to a sequential MLL-proof. In the same paper (1989) they generalized the usual binary multiplicative connectives, par and tensor, to n -ary connectives. They gave the way to introduce a pair of dual n -ary connectives by the notion of “orthogonality of meeting graphs”, which guarantees the main-cut elimination process of the cut between the introduction rules of the dual n -ary connectives. In addition, they chose one natural way of generalizing par-switching to a switching for n -ary connectives and they showed that (1) the “expansion property” holds if the switching condition is satisfied, that is, if $\mathcal{C}(A_1, \dots, A_n), \mathcal{C}^*(\sim A_1, \dots, \sim A_n)$ has a correct proof-net from atomic-links, where A_i is an atom, and $\sim A_i$ is its dual in the sense of the binary-dual, namely the usual negation, while \mathcal{C} and \mathcal{C}^* are new n -ary dual connectives, and that (2) the sequentialization theorem does not hold anymore with this switching definition. The purpose of this note is to remark that there is another natural way of generalizing switching condition for n -ary connectives, and the switching condition with this alternative notion of switching leads to the sequentialization theorem at the expense of the “expansion” property.

The principal formulas (of the upper-sequents) for the usual binary connectives, tensor rule and par-rule, in the typical (classical) one-sided sequent calculus can be expressed as $\vdash A_1 \vdash A_2$ for introduction rule of tensor $A_1 \otimes A_2$, while $\vdash A_1, A_2$ for that of $A_1 \wp A_2$, except for arbitrary auxiliary context formulas in MLL. The situation can be expressed as the possible partition classes $\{(1)(2)\}$ for the tensor rule and $\{(1,2)\}$ for the par rule, partition of $\{1,2\}$. The tensor has two partition classes of singleton, while par a single class of two elements. This suggests that any n -ary connective rules can be introduced with a set of partition classes; for example, if $n = 4$, one could consider a partition classes $\{(1,2)(3,4)\}$. In the case of the binary par-switching, Danos-Regnier use the selection function to choose one element from par-link $\{(1,2)\}$. When Danos-Regnier generalize this switching to a 4-ary link or rule such as $\{(1,2)(3,4)\}$, they keep this idea of the selection function and first chooses one class, either $(1,2)$ or $(3,4)$ to define the switching. There is also an alternative and natural way to define a switching; one chooses one out of each classes, namely, one out of $(1,2)$ and one out of $(3,4)$. We consider this alternative definition of switching and remark that the switching condition with such the definition implies the sequentialization theorem. We also remark that no ways of defining switching would satisfy both the sequentialization and expansion at the same time, except for the essentially original binary connectives.

Danos and Regnier’s par-switching condition is given via an association to Girard’s long-trip condition. There have been known direct proofs which imply the

sequentialization theorem from the Danos-Regnier par-switching condition [1]. As long as we know, the most well-known is Girard’s proof [4]. Olivier Laurent gave a simple and direct proof, in his unpublished note “Sequentialization of multiplicative proof nets” at 2013 (available at: <http://perso.ens-lyon.fr/olivier.laurent/seqmill.pdf>) [Accessed 1 April 2018]. The main purpose of our note is to show that Laurent’s proof of sequentialization for the usual binary MLL connectives work as it is for the generalized n -ary connectives if we take our alternative choice of the definition of switching, instead of the definition of switching which was chosen by Danos-Regnier (1989). In the course of following Laurent’s sequentialization proof, we slightly simplify some part of splitting lemma proof. The generalized connectives has been studied in Maieli (2019) and Jean-Yves Girard’s paper “transcendental syntax II: non deterministic case” at 2017, available at: <http://girard.perso.math.cnrs.fr/trsy2.pdf> [Accessed 1 April 2018], where the author remarks, among others, importance of generalized connectives. However, the issue on the switching condition which we consider in this paper are not discussed in these papers.

2 Preliminaries

We shall explain background knowledges of generalized multiplicative connectives introduced by Danos and Regnier [1]. These knowledges are contained in section 2 and 3 of [1]. Background knowledges Multiplicative Linear Logic and proof-nets are collected in Appendix.

Danos and Regnier introduced the generalized multiplicative rule satisfying the two properties of multiplicative connectives [1, p. 188]. A generalized connectives is defined by particular instances of generalized multiplicative rules.

The introduction rules of multiplicative connectives have the following two particular properties [1];

1. all the maximal subformulas of a conclusion formula occur in premises,
2. an introduction of a multiplicative connective does not need information of the contexts of the premises.

The following is the general form of an introduction rule for a multiplicative connective. This form satisfies the above two properties.

$$\frac{\vdash \Gamma_1, A_{11}, \dots, A_{1i_1} \cdots \quad \vdash \Gamma_m, A_{m1}, \dots, A_{mi_m}}{\vdash \Gamma_1, \dots, \Gamma_m, \mathcal{C}(A_1, \dots, A_n)}$$

where $ji_j \in \{1, \dots, n\}$ and \mathcal{C} is an unspecified connective.

We consider the set \mathcal{F} of partitions of a natural number n . We assume that a partition p has the form $p = \{(p_{11}, \dots, p_{1j}), \dots, (p_{k1}, \dots, p_{km_j})\}$, $p_{ij} \in \{1, \dots, n\}$, $p \in \mathcal{F}$. For each class i , a class $(p_{i1}, \dots, p_{im_i})$ corresponds to a sequent $\vdash A_{i1}, \dots, A_{im_i}$. Hence, each introduction rule of a connective \mathcal{C} corresponds to a partition p .

A set of introduction rules of a generalized connective \mathcal{C} correspond to a partition set P . $P_{\mathcal{C}}$ denotes the partition set that corresponds to a generalized connective \mathcal{C} . We regard $P_{\mathcal{C}}$ as the “right rules” of \mathcal{C} . We consider the right rules of the generalized connective \mathcal{C}^* as the left rules of \mathcal{C} , where \mathcal{C}^* is the dual connective, in the sense of Danos-Regnier’s orthogonality, which we shall explain below, of \mathcal{C} (e.g. if $\mathcal{C} = \otimes$ then $\mathcal{C}^* = \wp$). We put $\sim(\mathcal{C}(A_1, \dots, A_n)) = \mathcal{C}^*(\sim A_1, \dots, \sim A_n)$ for some \mathcal{C}^* . It is required that the dual connective \mathcal{C}^* of \mathcal{C} is defined such that the main step of the cut elimination proof holds for a pair $(\mathcal{C}, \mathcal{C}^*)$. Hence, the duality of the generalized connectives is based on cut-eliminations, and not based on logical duality. Danos and Regnier invented theory of meeting graph so as to define \mathcal{C}^* such that the cut-elimination holds. [1].

Definition 1. For any two partitions $p, q \in \mathcal{F}$, a meeting graph $\mathcal{G}(p, q)$ is a labelled graph (V_1, V_2, E) as follows;

- V_1 is the set of upper nodes: these nodes have labels such that the numbers in the same class of p correspond to the same node.
- V_2 is the set of lower nodes: these nodes have labels such that the numbers in the same class of q correspond to the same node.
- E is the set of edges connecting a node of V_1 and a node of V_2 such that there is exactly one edge between the same number of p and q

Definition 2. [1] Two partitions $p, q \in \mathcal{F}$ are orthogonal if the meeting graph $\mathcal{G}(p, q)$ is connected and acyclic. We denote $p \perp q$ if p and q are orthogonal.

Hence, $p \perp q \Leftrightarrow \mathcal{G}(p, q)$ is connected and acyclic.

Definition 3. [1] Partition sets P, Q of $\{1, \dots, n\}$ are orthogonal if for any $p \in P$ and $q \in Q$, $p \perp q$ holds. We denote $P \perp Q$ if P and Q are orthogonal. P^\perp is the maximal set which is orthogonal to P . $P^\perp = \{q \mid \forall p \in P. p \perp q\}$.

For further detailed property of a partition set, see [5], although we do not need the further properties in this paper.

Definition 4. [1] A pair of n -ary generalized connectives $(\mathcal{C}, \mathcal{C}^*)$ is a pair of non-empty finite partition sets of a natural number n $(P_{\mathcal{C}}, P_{\mathcal{C}^*})$ such that $(P_{\mathcal{C}})^\perp = P_{\mathcal{C}^*}$ and $(P_{\mathcal{C}^*})^\perp = P_{\mathcal{C}}$.

Note that \mathcal{C}^* is uniquely determined by \mathcal{C} because $(P_{\mathcal{C}})^\perp$ is the maximal set for $P_{\mathcal{C}}$. This fact means that left rules of \mathcal{C} are uniquely determined by their right rules. If the condition of a generalized connective is merely $P \perp Q$, $Q \subseteq P^\perp$, left rules of \mathcal{C} is not uniquely determined. For example, we can consider different left rules $Q_1 = \{(1, 3)(2)\}$, $Q_2 = \{(2, 3)(1)\}$, $Q_3 = \{\{(1, 3)(2)\}, \{(2, 3)(1)\}\}$ for the right rule $P = \{(1, 2)(3)\}$.

Definition 5. [1] A generalized connective \mathcal{C} is decomposable if $P_{\mathcal{C}} = P_\alpha$ for some MLL-formula α . When a connective is not decomposable, it is said non-decomposable.

Danos and Regnier gave the following example of non-decomposable connectives [1].

$$\frac{\frac{\frac{\vdash \Gamma_1, A, B \quad \vdash \Gamma_2, C, D}{\vdash \Gamma_1, \Gamma_2, \mathcal{C}(A, B, C, D)}}{\vdash \Gamma_1, A, C} \quad \vdash \Gamma_2, B, D}{\vdash \Gamma_1, \Gamma_2, \mathcal{C}(A, B, C, D)}$$

$$\frac{\frac{\frac{\vdash \Delta_1, A, D \quad \vdash \Delta_2, B}{\vdash \Delta_1, \Delta_2, \Delta_3, \mathcal{C}^*(A, B, C, D)} \quad \vdash \Delta_3, C}{\vdash \Delta_1, B, C} \quad \vdash \Delta_2, A \quad \vdash \Delta_3, D}{\vdash \Delta_1, \Delta_2, \Delta_3, \mathcal{C}^*(A, B, C, D)}$$

Jean-Yves Girard gave another examples of non-decomposable connectives in his paper “transcendental syntax II: non deterministic case” in 2017.

Generalized connectives $\mathcal{C}, \mathcal{C}^*$ satisfy the main step of the cut-elimination by the following Lemma.

Lemma 1. [1, Lemma 1] For any n -ary connectives \mathcal{C}_1 and \mathcal{C}_2 , the main cut-elimination step holds for any cut between any of the introduction rule for \mathcal{C}_1 and any of the introduction rules for \mathcal{C}_2 if and only if the corresponding partition[sets] are orthogonal.

We will extend MLL by adding generalized connectives to $\text{MLL}(\mathcal{C}_i)$ (where \mathcal{C}_i ($i = 1, \dots, n$) is generalized connectives and each one has the arity m_i). We identify the two binary generalized connectives with the tensor and par, respectively. We also identify the n -ary tensor and par connectives with the corresponding generalized connectives.

Formulas of $\text{MLL}(\mathcal{C}_i)$ are defined as follows;

$A := P | \sim P | A \otimes A | A \wp A | \mathcal{C}_i(A, \dots, A)$ for each i , where P ranges over a denumerable set of propositional variables.

The negation sign is used as the abbreviation, following Danos-Regnier's notation, although the negation symbol does not always stand for logical negation because of the appearance of the generalized connectives; $\sim(\mathcal{C}_i(A_1, \dots, A_{m_i})) = \mathcal{C}_i^*(\sim A_1, \dots, \sim A_{m_i})$, $\sim(\mathcal{C}_i^*(A_1, \dots, A_{m_i})) = \mathcal{C}_i(\sim A_1, \dots, \sim A_{m_i})$. Observe that each generalized connective \mathcal{C}_i determines its own negation by the cut-eliminationability. Here, for readability we use the symbol “ \sim ” to represent the orthogonal dual even though it does not mean the negation in the usual sense of MLL.

The set of inference rules of $\text{MLL}(\mathcal{C}_i)$ is the union of that of MLL and that of $\mathcal{C}_i (i = 1, \dots, n)$.

We also extend the definition of links and proof-structures: the node of label \mathcal{C}_i has m_i premises and one conclusion. If A_1, \dots, A_{m_i} are the labels of the premises, then the conclusion is labelled $\mathcal{C}_i(A_1, \dots, A_{m_i})$. We call this a \mathbb{C} -link. A link l is a terminal link of a proof-structure containing \mathbb{C} -links \mathcal{S} when the conclusions of l are conclusions of \mathcal{S} .

The cut-elimination theorem of $\text{MLL}(\mathcal{C}_i)$ immediately follows from the above Lemma 1.

Proposition 1. [1] *Any provable sequent in $\text{MLL}(\mathcal{C}_i)$ is provable without the cut rule.*

3 Partition switching for the sequentialization theorem

We shall introduce a new definition of switching for the generalized multiplicative connectives such that the switching condition implies the sequentialization theorem. We call our new switching the “partition switching”, because we think that our switching definition is a natural one from the view point of the underlying partition. We call Danos and Regnier's original switching for the generalized connectives “Danos-Regnier switching”, in this Section. The Danos-Regnier switching has the expansion property to be explained later. However, the sequentialization theorem for generalized connectives does not hold in general with their switching; Danos-Regnier remarks that, in general, a sequent $\vdash \mathcal{C}(A_1, \dots, A_n), \mathcal{C}^*(\sim A_1, \dots, \sim A_n)$ is not provable, and they added “Hence, we cannot expect every correct proof-net to be sequentializable [1, p.197].” Their view on the failure of the sequentialization Theorem is based on their switching

condition. Hence, the fact that a sequent with excluded middle conclusion is not provable does not necessarily imply the failure of the sequentialization Theorem. We give a new switching, the partition switching, for generalized connectives, and using our new switching, we show that the sequentialization theorem holds for generalized connectives. However, as we explain later, the expansion property does not hold for our switching.

Our definition of the partition switching is as follows.

Definition 6. Let \mathcal{S} be a proof-structure containing \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$). A partition switching I of \mathcal{S} is a family of functions indexed by partitions, $f_p : P \upharpoonright p \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{N}))$ (where P is a partition set and $p \in P$) such that $f_p(p) = \{p_{1f_p(1)}, \dots, p_{kf_p(k)}\}$ (where $p = \{(p_{11}, \dots, p_{1m_1}), \dots, (p_{k1}, \dots, p_{km_k})\}$, $p_{jl} \in \{1, \dots, n\}$, $p_{if_p(i)} \in \{p_{i1}, \dots, p_{im_i}\}$).

Definition 7. Let \mathcal{S} be a proof-structure containing \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$) and $I = \{f_1, \dots, f_r\}$ be an arbitrary partition switching. \mathcal{S}_{f_p} (where $f_p \in I$) is obtained from \mathcal{S} by deleting all edges from A_j ($j = 1, \dots, n$) to \mathcal{C}_i -node except edges corresponding to f_p . We call the induced graph \mathcal{S}_{f_p} for each f_p the correctness graph of \mathcal{S} . We define \mathcal{S}_I as follows: $\mathcal{S}_I = \bigcup_{f_p \in I} \mathcal{S}_{f_p}$.

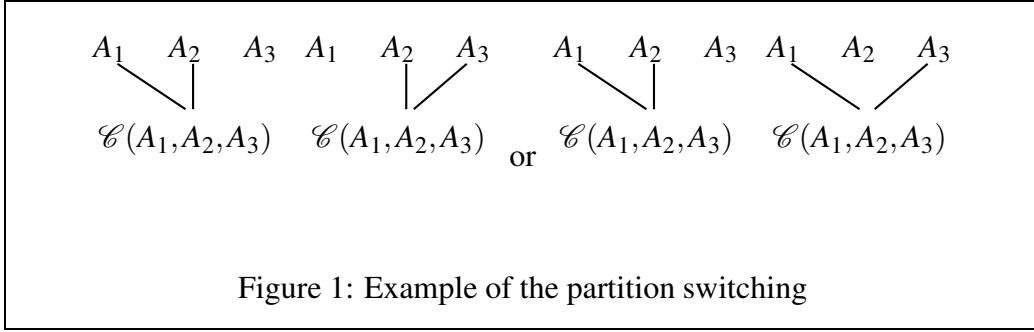
Definition 8. Let \mathcal{S} be a proof-structure containing \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$) and $I = \{f_1, \dots, f_r\}$ be an arbitrary partition switching. \mathcal{S} is correct in the sense of partition switching if and only if there exists $f_p \in I$ such that \mathcal{S}_{f_p} is connected and acyclic. A proof net in the sense of partition switching is a correct proof-structure in the sense of partition switching.

We give an example of our partition switching: $\mathcal{C}(A_1, A_2, A_3) = (A_1 \otimes A_2) \wp A_3$, $P = \{p_1, p_2\}$. For p_1 , we choose just one element for each class (e.g. a_1, a_2) and we obtain a switching $I_1 = \{a_1, a_2\}$. The case of p_2 is similar.

$$\begin{aligned} f_{p_1} : p_1 &= \{(a_1, a_3), (a_2)\} \Longrightarrow_{I_1} a_1, a_2 \\ f_{p_1} : p_1 &= \{(a_1, a_3), (a_2)\} \Longrightarrow_{I_2} a_3, a_2 \\ &\text{or} \\ f_{p_2} : p_2 &= \{(a_1), (a_2, a_3)\} \Longrightarrow_{J_1} a_1, a_2 \\ f_{p_2} : p_2 &= \{(a_1), (a_2, a_3)\} \Longrightarrow_{J_2} a_1, a_3 \end{aligned}$$

See Fig. 1 for the corresponding graphs of the above switchings.

Remark 1. The partition switching is a generalization of \wp -switching because if we restrict the partition switching to the binary case, it coincides with the usual \wp -switching. In fact, the switching for generalized connectives satisfies the following conditions when it is restricted to the binary case; (i) a switching chooses at least



one element from each class because the \otimes connective has no partition switching and $P_{\otimes} = \{(1), (2)\}$ holds: (ii) a switching chooses at most one element from each class because the \wp -connective has partition switching and $P_{\wp} = \{(1, 2)\}$ holds.

Now we show that by replacing the Danos-Regnier switching by our partition switching a proof of the sequentialization theorem works.

Theorem 1. (Sequentialization Theorem for generalized multiplicative connectives) *Let \mathcal{S} be an arbitrary proof-structure containing arbitrary \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$). If \mathcal{S} is a proof-net in the sense of the partition switching, then \mathcal{S} is sequentializable.*

We shall give a slightly modified proof of known proof for the binary connectives given by Olivier Laurent in his unpublished note “Sequentialization of multiplicative proof nets” at 2013 (available at: <http://perso.ens-lyon.fr/olivier.laurent/seqmill.pdf>) [Accessed 1 April 2018] (Laurent, 2013). Our proof of Sequentialization Theorem for generalized multiplicative connectives contains a proof of Sequentialization Theorem for the binary multiplicative connectives as the special case because we can treat the binary connectives, par and tensor, as the special case (the case $n = 2$) of generalized n -ary connectives, as remarked above. We often denote n -ary \wp as \wp^n .

Sublemma 1 (cf. (Laurent, 2013, Lemma 2)). *Let \mathcal{S} be a proof-net which does not contain any terminal \wp^k -links (for any non-zero natural number k) and which contains $n > 1$ terminal \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$). Here and in the next proof, a terminal \otimes -link is also regarded as a terminal \mathbb{C} -link. If for an arbitrary terminal \mathbb{C} -link $\mathcal{C}_i(A_{i1}, \dots, A_{im})$, $\mathcal{C}_i(A_{i1}, \dots, A_{im})$ is non-splitting, then there are some non-terminal node l such that $l \neq \mathcal{C}_i(A_{i1}, \dots, A_{im})$ and some paths p_1, \dots, p_m such that paths p_1, \dots, p_m start from $\mathcal{C}_i(A_{i1}, \dots, A_{im})$ and pass through A_{i1}, \dots, A_{im} , respectively, and meet at l .*

Proof. Let $\mathcal{C}_1(A_{11}, \dots, A_{1m})$ be one of non-splitting terminal \mathbb{C} -link and $\mathcal{C}_2(A_{21}, \dots, A_{2m})$ be an arbitrary distinct terminal \mathbb{C} -link. We obtain some path from \mathcal{C}_1 to \mathcal{C}_2 by any switching because \mathcal{S} is correct¹. We assume that a node l such that $l \neq \mathcal{C}_i(A_{i1}, \dots, A_{im})$ at which some paths p_1, \dots, p_m meet does not exist, then we show a contradiction. There are upwards paths p_1, \dots, p_m that start from $\mathcal{C}_1(A_{11}, \dots, A_{1m})$ and pass through A_{11}, \dots, A_{1m_1} , respectively. Then, at least two paths p_s, p_t ($s, t = 1, \dots, m$) has distinct targets α, β by assumption and there is no path from α to β that does not pass through $\mathcal{C}_1(A_{11}, \dots, A_{1m})$. Hence, if we remove $\mathcal{C}_1(A_{11}, \dots, A_{1m})$, \mathcal{S} is separated into two parts, the part containing α and the other containing β . It contradicts the assumption that \mathcal{C}_1 is non-splitting. Moreover, if l is a terminal \mathbb{C} -node, then \mathcal{S} contains $n + 1$ terminal \mathbb{C} -links and it contradicts the assumption. Hence l is non-terminal. □

Lemma 2 (cf. (Laurent, 2013, p.4)). (*Splitting Lemma*) *Let \mathcal{S} be a proof-net that does not contain terminal \wp^k -links and contains $n > 0$ terminal \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$). Then, at least one of terminal \mathbb{C} -links split.*

Proof. Case $n = 1$;

We assume that the terminal \mathbb{C} -links $\mathcal{C}_1(A_1, \dots, A_m)$ is non-splitting. By Sublemma 1, there exists a non-terminal node l such that $l \neq \mathcal{C}_1(A_1, \dots, A_m)$. We obtain an upward path p from $\mathcal{C}_1(A_1, \dots, A_m)$ to l in \mathcal{S}_I , where I is an arbitrary switching. There is a downward path q such that $q \neq p$ from l to $\mathcal{C}_1(A_1, \dots, A_m)$ in \mathcal{S}_I because l is non-terminal and $\mathcal{C}_1(A_1, \dots, A_m)$ is the only terminal node. We obtain a cycle $p \cdot q$ (where “ \cdot ” denotes a concatenation of two paths), which contradicts the assumption that \mathcal{S} is a proof-net.

Case $n > 1$;

The Figure 2 will be helpful to follow our proof. We assume that all the terminal \mathbb{C} -links \mathcal{C}_i 's are non-splitting and we show a contradiction. By Sublemma, for each i , there are some non-terminal node l_i and some upwards paths p_{i1}, \dots, p_{im} such that p_{i1}, \dots, p_{im} meet at l_i , paths p_{i1}, \dots, p_{im} start from $\mathcal{C}_i(A_{i1}, \dots, A_{im})$ and pass through A_{i1}, \dots, A_{im} , respectively. For all j ($j = 1, \dots, n$), there are upward paths q_j from \mathcal{C}_j to l_j in \mathcal{S}_I (where I is an arbitrary switching) by the assumption that \mathcal{S} does not contain terminal \wp^k -links and \mathcal{S} is correct. We can obtain downward paths d_j in \mathcal{S}_I (where I is an arbitrary switching) which start from l_j and end with the terminal node because l_j are non-terminal. There exists a path q such that it starts at l_s for some s ($1 \leq s \leq n$) and ends with \mathcal{C}_t for some t , where $t < s$ (say, $\mathcal{C}_1(A_{11}, \dots, A_{1m})$). This is because l_i ($i = 1, \dots, n$) are non-terminal and \mathcal{S} contains only $n > 0$ terminal links. If for an arbitrary switching J , the path q is contained in \mathcal{S}_J , we can construct the path $q_1 \cdot d_1 \cdots q_s \cdot q$.

¹We consider a proof-structure as if it is a directed graph in this proof.

But, it is a cycle, which contradicts the correctness of \mathcal{S} . If for any switching J , \mathcal{S}_J does not contain the path u , then u is separated by two parts u_1 and u_2 at some \mathbb{C} -node l' , then l' is non-terminal by the assumption. It follows that there is a path r from l' to $\mathcal{C}_i(A_{t_1}, \dots, A_{t_m})$ for some t such that $t < s$. We put $u' = u_1 \cdot r$. Without loss of generality, we assume that \mathcal{S}_J contains the path u' . We can construct the path $q_1 \cdot d_1 \cdots q_s \cdot u'$. It is a cycle, which contradicts the correctness of \mathcal{S} . \square

We return to the proof of Theorem 1.

Proof. (Proof of Sequentialization Theorem)

By induction on the number of links contained in \mathcal{S} . We prove it for the case that the terminal link is \mathbb{C} -link. If necessary, first we remove all terminal \mathcal{F}^k -links from \mathcal{S} . By Lemma 2, at least one of the terminal \mathbb{C} -links $\mathcal{C}_1(A_1, \dots, A_m)$ is splitting. Hence the removal of $\mathcal{C}_1(A_1, \dots, A_m)$ splits \mathcal{S} into several sub-proof-nets $\{\mathcal{S}_1, \dots, \mathcal{S}_i\}$. By the induction hypothesis, there are some proofs π_1, \dots, π_i such that $(\pi_i)^* = \mathcal{S}_i$ holds. We apply the \mathcal{C} -rule to π_1, \dots, π_i and obtain the proof π . \square

We showed above that Olivier Laurent's direct proof of the sequentialization theorem for the binary connectives can be adapted to the n -ary generalized connectives when we take our partition-switching as a generalization of the binary par-switching, as the above proof essentially follows Laurent's proof. At the same time, we presented a slightly simplified proof for the splitting lemma.

Figure 2 represents the $n > 1$ case of the proof above.

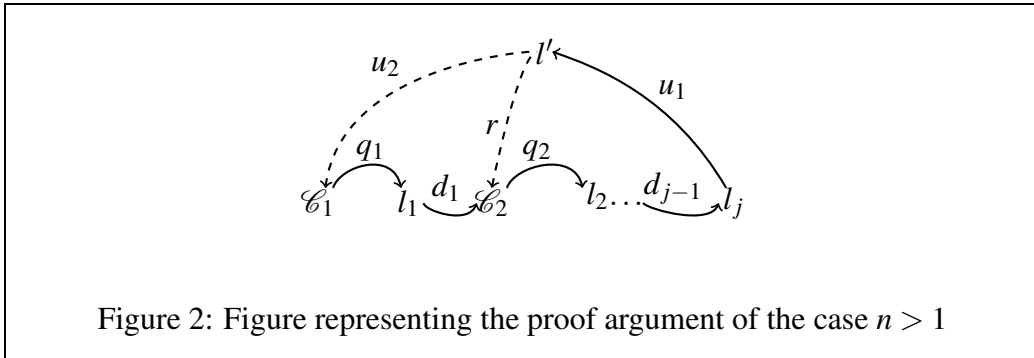


Figure 2: Figure representing the proof argument of the case $n > 1$

The reverse direction of Sequentialization Theorem also holds with our partition switching as it is in the original binary case.

Proposition 2. *Let \mathcal{S} be an arbitrary proof-structure containing arbitrary \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$). If \mathcal{S} is sequentializable, then \mathcal{S} is a proof-net in the sense of the partition switching.*

Proof. By induction on the length of a proof. It follows from the definition of the partition switching. \square

Next, we explain the Danos and Regnier switching and compare their switching with ours.

The definition of the Danos-Regnier switching is as follows. (We regard a class of a partition as a set of elements p_{ij}).

Definition 9. (*Danos-Regnier switching*[1]) Let \mathcal{S} be a proof-structure containing \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$). For an arbitrary $i \in \{1, \dots, n\}$ and $p \in P_{\mathcal{C}}$ (where $p = \{(p_{11}, \dots, p_{1j}), \dots, (p_{k1}, \dots, p_{km_k})\}$, $p_{jk} \in \{1, \dots, n\}$), a Danos-Regnier switching I is a function $f : P \rightarrow \text{Class}(P)$ (where P is a partition set) such that $f(p) = x_i$ ($x_i \in \text{Class}(p)$ and $\{p_{i1}, \dots, p_{im_i}\} = x_i$).

Definition 10. (*Correctness graph for Danos-Regnier switching*) Let \mathcal{S} be a proof-structure containing \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$) and I be an arbitrary Danos-Regnier switching. The correctness graph \mathcal{S}_I is obtained from \mathcal{S} and I as follows; for each class, connect the nodes that belong to the same class and cut off the edges from classes to the \mathcal{C}_i -node except the class $x = f(p)$ i.e. the value of I .

A proof-structures that have two nodes $\mathcal{C}(A_1, \dots, A_n), \mathcal{C}^*(\sim A_1, \dots, \sim A_n)$ for some atoms A_1, \dots, A_n is counterexample of the sequentialization theorem using Danos-Regnier switchings. It is easily shown as follows: by Fact 1, a proof-structure \mathcal{S} that has just two terminal nodes $\mathcal{C}(A_1, \dots, A_n)$ and $\mathcal{C}^*(\sim A_1, \dots, \sim A_n)$ for some A_1, \dots, A_n is correct. However, the corresponding sequent $\vdash \mathcal{C}(A_1, \dots, A_n), \mathcal{C}^*(\sim A_1, \dots, \sim A_n)$ does not necessarily have proof by Fact 2.

Fact 1. *If \mathcal{S} is a proof-structure that has just two terminal nodes $\mathcal{C}(A_1, \dots, A_n)$ and $\mathcal{C}^*(\sim A_1, \dots, \sim A_n)$ for some A_1, \dots, A_n , then \mathcal{S} is correct in the sense of Danos-Regnier switching.*

We call the terminal nodes with labels “ $\mathcal{C}, \mathcal{C}^*$ ” the excluded middle formula (with respect to \mathcal{C}).

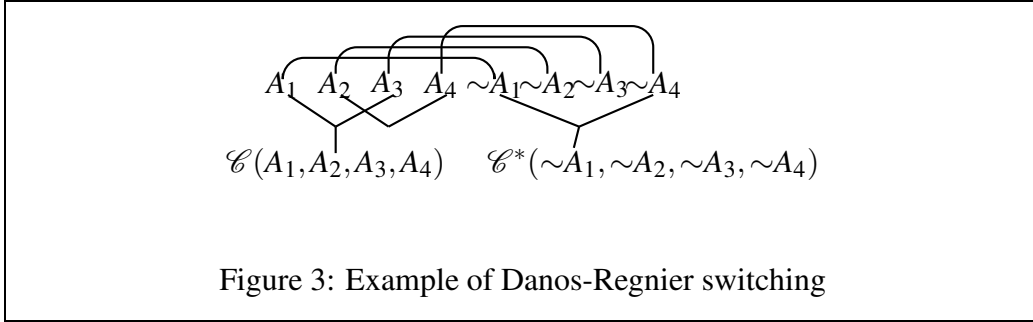
Fact 2. *A sequent $\vdash \mathcal{C}(A_1, \dots, A_n), \mathcal{C}^*(\sim A_1, \dots, \sim A_n)$ for any dual pair of generalized connectives is provable if and only if $\mathcal{C} = \otimes^k$ or $\mathcal{C} = \wp^k$.*

We note that the Facts 1 and 2 are essentially (implicitly) in Danos-Regnier [1].

Remark 2. *The Danos-Regnier switching is a generalization of \wp -switching. By considering binary case, we obtain $I_{\otimes}^1 = \{1\}$, $I_{\otimes}^2 = \{2\}$, and $I_{\wp} = \{1, 2\}$. The roles of \otimes and \wp on graphs are reversed; \otimes has the binary switching and \wp does not have it. Hence, if we use the left-one-sided sequent MLL, the sequentialization theorem hold for binary connectives.*

As we remarked in Introduction (Section 1), the Danos-Regnier switching chooses one class from a partition, while our partition switching chooses exactly one element from each class of a given partition.

The following graph (Fig. 3) is an example of a graph \mathcal{S}_1 (where \mathcal{S} is correct in the sense of Danos-Regnier switching and \mathcal{C} and \mathcal{C}^* are non-decomposable connectives of Danos-Regnier). If we delete \mathcal{C} and \mathcal{C}^* -nodes and its vertical edges, then we obtain the meeting graph $\mathcal{G}(P_{\mathcal{C}}, P_{\mathcal{C}^*})$. Conversely, if the meeting graph $\mathcal{G}(P_{\mathcal{C}}, P_{\mathcal{C}^*})$ is given, then we can choose two classes and add nodes \mathcal{C} , \mathcal{C}^* and two edges to chosen nodes. By this operation, we can obtain a correction graph. How to connect \mathbb{C} -nodes with classes is not important in the definition of Danos-Regnier switching because it is irrelevant to correctness. Hence, a meeting graph is almost the same as a correction graph in the sense of Danos-Regnier switching.



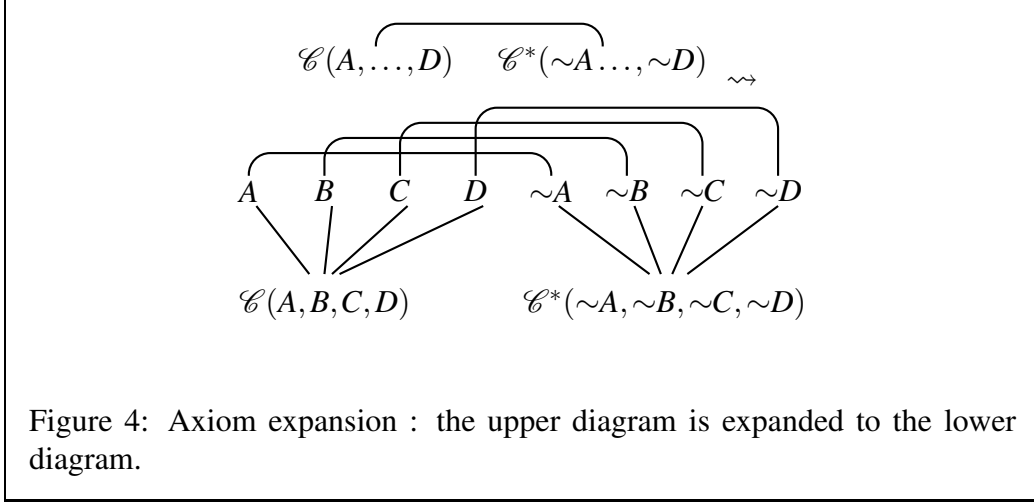
Axiom links are only allowed for atomic formulas in Definition 14. In the following part, we include axiom links for arbitrary formulas $A, \sim A$ in the definition of a link (and a proof-structure). We employ non-atomic initial axiom sequents and non-atomic initial axiom links so as to compare our switching condition and Danos and Regnier's one. When we admit axiom links and non-atomic initial sequents for any formulas, a trade-off relation which will be explained later arises.

We extend the axiom-rule and the axiom link as follows;

$$\frac{}{\vdash A, \sim A} \text{ (Ax) } \overbrace{A \quad \sim A}^{\text{ (Axiom) }} \text{ (where } A \text{ is an arbitrary formula which may include the generalized connectives.)}$$

We can show the sequentialization theorem for a proof-structure \mathcal{S} containing non-atomic axioms and generalized connectives by the same proof as Theorem 1.

Definition 11. Let \mathcal{S} be a proof-structure containing \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$) and \mathbb{C} -axiom links L_j ($1 \leq j \leq \frac{n}{2}$), (where j is a natural number). The axiom expansion is the following operation F ; $F(\mathcal{S}) = \mathcal{S}'$ where \mathcal{S}' is the same as \mathcal{S} except that each \mathbb{C} -axiom links L_j is replaced with its subformula axiom links (For example, Fig. 4).



Proposition 3. Let \mathcal{S} be a proof-net containing \mathbb{C} -links \mathcal{C}_i ($i = 1, \dots, n$) and \mathbb{C} -axiom links L_j ($1 \leq j \leq \frac{n}{2}$). the proof-structure \mathcal{S}' expanded from \mathcal{S} is correct in the sense of Danos-Regnier switching.

Proof. For any \mathbb{C} -axiom link L_j (say, L_1), \mathcal{C}_1 -node is connected to the connected and acyclic subgraph \mathcal{S}_1 . Similarly, \mathcal{C}_1^* -node is connected to the correct subgraph \mathcal{S}_2 . \mathcal{S}' consists of $\mathcal{S}_1, \mathcal{S}_2$ and the proof-structure containing two terminal links \mathcal{C}_1 and \mathcal{C}_2 . By Fact 1, \mathcal{S}' is correct. □

Their switching condition guarantees the expansion property telling that the proof-structure of any excluded middle formula satisfies their switching condition (Fact 1). The expansion property implies that any non-atomic axiom link can be expanded to the proof structure from atomic axiom links satisfying the switching condition (Proposition 3). This provides a certain connection of the dual connective \mathcal{C}^* and logical negation through the atomic negations.

We now introduce the notion of n -ary \wp -switching (n is an arbitrarily fixed non-zero natural number).

Definition 12. Let \mathcal{S} be an arbitrary proof-structure containing $\wp_j, j \in \{1, \dots, i\}$ (i is a non-zero natural number). For an arbitrary $j \in \{1, \dots, n\}$, we write the arity of \wp_j as $a(j)$. We denote the set of all n -ary \wp -links contained in a proof-structure \mathcal{S} by $\wp(\mathcal{S})$. A n -ary \wp -switching I of a proof-structure \mathcal{S} is a function $f : \wp_j \in \wp(\mathcal{S}) \rightarrow \{1, \dots, a(j)\}$. The graph \mathcal{S}_I is obtained by deleting the edges of each n -ary par link \wp_j except the number of I .

Fact 3. Let \mathcal{C} be a generalized connective such that $\mathcal{C} = \wp^n$ holds. Then, the partition switching and the \wp^n -switching coincide.

This n -ary \wp -switching is, of course, decomposable and essentially reduced to the binary connective \wp .

We summarize the trade-off relation between the sequentialization theorem and the expansion property; a proof-net in the sense of Danos-Regnier switching is not necessarily sequentializable, as we explained after Definition 16, while any proof-net in the sense of our partition switching is sequentializable. On the other hand, the expansion property does not hold for our partition switching, while it holds for Danos-Regnier's one. Note that these trade-off relation occur only when a proof-structure contains non-atomic axiom links. We formulate the trade-off relation more precisely. The next proposition says that the expansion property and the sequentialization theorem are not compatible in general.

Proposition 4. A switching $f : p \rightarrow X$ satisfies both the expansion property and the sequentialization theorem if and only if f is the n -ary \wp -switching.

Proof. (only-if part) Let f be an arbitrary switching for generalized connectives that satisfies both the excluded middle property and the sequentialization theorem. We assume that f is not n -ary \wp -switching and We show a contradiction. By the expansion property, the proof-structure \mathcal{S} containing two terminal nodes $\mathcal{C}(A, B, C) = (A \wp B) \otimes C$, $\mathcal{C}^*(\sim A, \sim B, \sim C) = (\sim A \otimes \sim B) \wp \sim C$ (where A, B and C are atoms) is correct. By the sequentialization theorem, we obtain a proof π of the sequent $\vdash \mathcal{C}(A, B, C), \mathcal{C}^*(\sim A, \sim B, \sim C)$ from \mathcal{S} . This contradicts Fact 2.

(if-part) Let I be the n -ary par switching. The proof-structure \mathcal{S} containing exactly two terminal nodes \wp^n and \otimes^n is correct. Hence the expansion property holds. By Fact 3 and Theorem 1, Sequentialization Theorem follows. □

4 Conclusions

Danos and Regnier's switching condition imply the expansion property but does not implies the sequentialization property. We gave a new switching condition

which implies the sequentialization property but does not imply the expansion property. We pointed out that no switching except for the switching of purely par (or tensor)-based connectives implies both the two properties.

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5 Appendix

Multiplicative Linear Logic MLL [3] is defined as follows ;

Definition 13. *Formulas of MLL are defined as follows;*

$$A := P | \sim P | A \otimes A | A \wp A$$

where P ranges over a denumerable set of propositional variables.

The negation sign is used as the abbreviation in the following way. Here, we sometime call $\sim A$ the dual formula of A ;

$\sim\sim P := P, \sim(A\otimes B) := \sim A\wp\sim B, \sim(A\wp B) := \sim A\otimes\sim B$, where P is an atomic formula.

The rules of sequent calculus MLL are given as Fig. 5. (We take the finite sequence of formulas in the sequent expression as a multiset, instead of ordered multiset, hence the exchange rule is not needed.):

$$\frac{}{\vdash P, \sim P} (Ax) \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, \sim A}{\vdash \Gamma, \Delta} (\text{Cut})$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A\otimes B} (\otimes) \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A\wp B} (\wp)$$

where P is an atomic formula

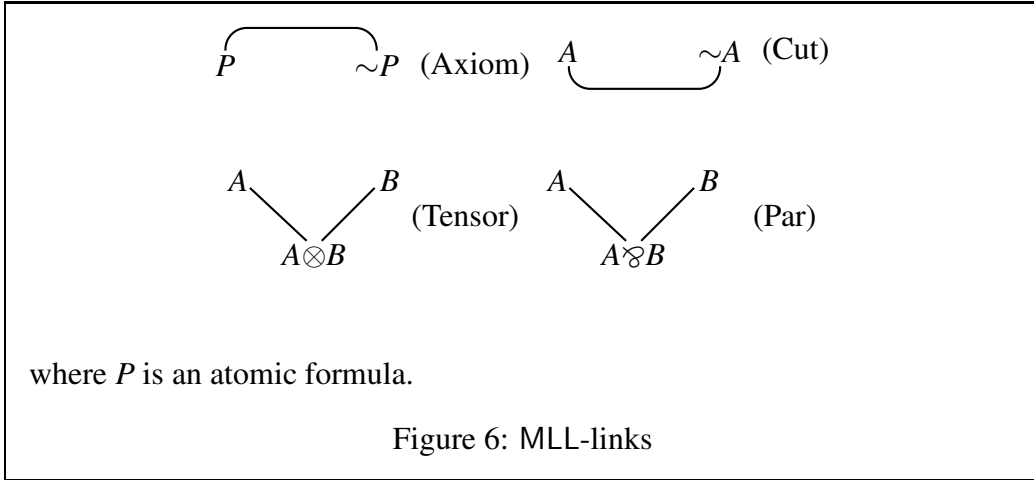
Figure 5: Inference rules of MLL

Now, we shall introduce the basic notions of graphic representations MLL-proofs.

Definition 14. MLL-links are the following four kinds of graphs (Fig. 6).

- The node of each label \otimes (resp. \wp) has two ordered premises and one conclusion. If A is the label of the first premise and B that of the second, then the conclusion is labelled $A\otimes B$ (resp. $A\wp B$);
- The node of label ax has no premise and two ordered conclusions. If the label of the first conclusion is P , the label of the second conclusion is $\sim P$, where P is an atomic formula;
- The node of label cut have two ordered premises and no conclusion. If the label of the first conclusion is A , the label of the second conclusion is $\sim A$.

We define the notion of proof-structure and of correct proof-structure or proof-net, as follows.



Definition 15. A proof-structure \mathcal{S} is a non-empty finite undirected graph whose nodes are labelled by the MLL-formulas and whose edges are labeled by the MLL-links satisfying the following condition; each formula is the premise of at most one link and the conclusion of at least one link.

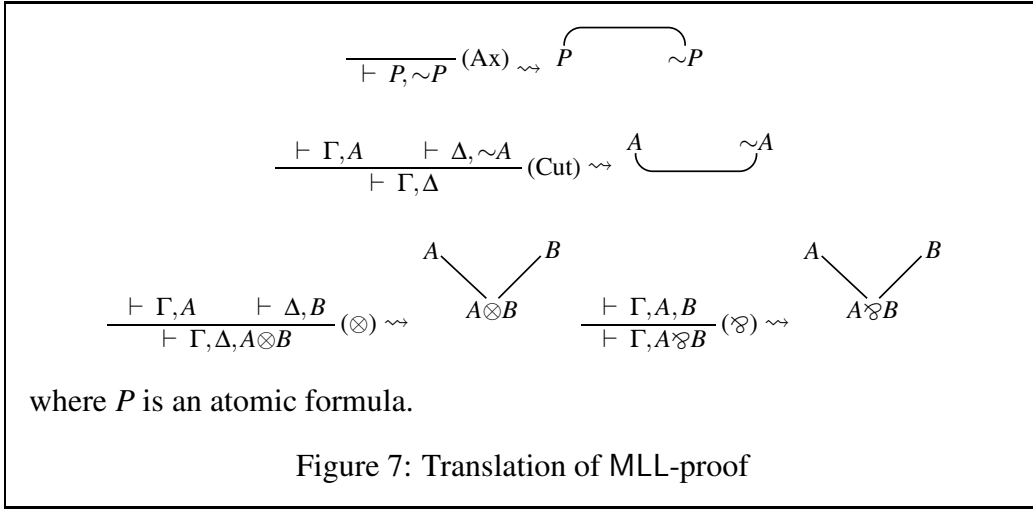
An instance of a formula in a proof structure that is not the premise of any link is called terminal.

Definition 16. We denote the set of all par links contained in a proof-structure \mathcal{S} by $\circ(\mathcal{S})$. A \circ -switching I of a proof-structure \mathcal{S} is a function $\circ(\mathcal{S}) \rightarrow \{\text{left}, \text{right}\}$.

Definition 17. (Danos and Regnier[1]) A proof-structure \mathcal{S} is correct if and only if for any switching I , the induced graph \mathcal{S}_I is connected and acyclic. A proof-net \mathcal{S} is a correct proof-structure.

The above condition is often called the switching condition. Then, one can say that a proof-structure is a proof-net when it satisfies the switching condition.

Definition 18. Let π be a proof of MLL and $\mathcal{S}(\pi)$ be the proof-structure that is obtained from π by applying the translation $\mathcal{S}(-)$ written in Fig. 7 recursively.



A proof-structure \mathcal{T} is said to be sequentializable if there is a MLL-proof π such that $\mathcal{S}(\pi) = \mathcal{T}$ holds.

Proposition 5. (Sequentialization of MLL, Girard, Danos and Regnier) [3, 1]
A proof-structure \mathcal{S} is sequentializable if and only if \mathcal{S} is a proof-net.

Girard showed the sequentialization theorem using his long trip condition [3]. After that, Danos and Regnier gave the switching condition and stated above form of the theorem [1].