# METRIZABLE QUOTIENTS OF $C_p$-SPACES

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Abstract. The famous Rosenthal-Lacey theorem asserts that for each infinite compact set $K$ the Banach space $C(K)$ admits a quotient which is either a copy of $c$ or $\ell_2$. What is the case when the uniform topology of $C(K)$ is replaced by the pointwise topology? Is it true that $C_p(X)$ always has an infinite-dimensional separable (or better metrizable) quotient? In this paper we prove that for a Tychonoff space $X$ the function space $C_p(X)$ has an infinite-dimensional metrizable quotient if $X$ either contains an infinite discrete $C^*$-embedded subspace or else $X$ has a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets such that for every $n$ the space $K_n$ contains two disjoint topological copies of $K_{n+1}$. Applying the latter result, we show that under $\diamondsuit$ there exists a zero-dimensional Efimov space $K$ whose function space $C_p(K)$ has an infinite-dimensional metrizable quotient. These two theorems essentially improve earlier results of Kąkol and Sliwa on infinite-dimensional separable quotients of $C_p$-spaces.

## 1. Introduction

Let $X$ be a Tychonoff space. By $C_p(X)$ and $C_c(X)$ we denote the space of real-valued continuous functions on $X$ endowed with the pointwise and the compact-open topology, respectively.

The classic Rosenthal-Lacey theorem, see [21], [15], and [19], asserts that the Banach space $C(K)$ of continuous real-valued maps on an infinite compact space $K$ has a quotient isomorphic to $c$ or $\ell_2$.

This theorem motivates the following natural question (first discussed in [18]):

**Problem 1** (Kąkol, Sliwa). *Does $C_p(K)$ admit an infinite-dimensional separable quotient for any infinite compact space $K$?*

In particular, does $C_p(\beta\mathbb{N})$ admit an infinite-dimensional separable quotient (shortly $SQ$)? Our main theorem of [18, Theorem 4] showed that $C_p(K)$ has $SQ$ for any compact space $K$ containing a copy of $\beta\mathbb{N}$. Consequently, this theorem reduces Problem 1 to the case when $K$ is an *Efimov space* (i.e. $K$ is an infinite compact space that contains neither a non-trivial convergent sequence nor a copy of $\beta\mathbb{N}$). Although, it is unknown if Efimov spaces exist in ZFC (see [5], [6], [7], [8], [10], [11], [12], [14]) we showed in [18] that under $\diamondsuit$ for some Efimov spaces $K$ the function space $C_p(K)$ has $SQ$.

On the other hand, in [17] it was shown that $C_p(K)$ has an *infinite-dimensional separable quotient algebra* if and only if $K$ contains an infinite countable closed subset. Hence $C_p(\beta\mathbb{N})$ lacks infinite-dimensional separable quotient algebras.

Clearly Problem 1 is motivated by Rosenthal-Lacey theorem, but one can provide more specific motivations. Indeed, although it is unknown whether $C_c(X)$ or $C_p(X)$ always has $SQ$, some partial results are known: If $X$ is of pointwise countable type, then $C_c(X)$ has a quotient isomorphic to either $\mathbb{R}^{\mathbb{N}}$ or $c$ or $\ell_2$, see [17, Corollary 22]. Also $C_c(X)$ has $SQ$ provided $C_c(X)$

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is barrelled, [16]. Recall that all first countable spaces and all locally compact spaces are of pointwise countable type, see [9].

In [17, Corollary 11] we proved that for a fixed Tychonoff space $X$, if $C_p(X)$ has $SQ$, then also $C_c(X)$ has $SQ$. Conversely, if $C_c(X)$ has $SQ$ and for every infinite compact $K \subset X$ the space $C_p(K)$ has $SQ$, then $C_p(X)$ has also $SQ$. Indeed, two cases are possible.

(1) *Every compact subset of $X$ is finite.* Then $C_p(X) = C_c(X)$ is barrelled and [16] applies to get that $C_p(X)$ has $SQ$.
(2) *$X$ contains an infinite compact subset $K$.* Then $C_p(K)$ has $SQ$ (by assumption). Since the restriction map $f \to f|K$, $f \in C(X)$ is a continuous open surjection from $C_p(X)$ onto $C_p(K)$, the desired conclusion holds.

Let $X$ be a Tychonoff space. First observe that each metrizable quotient of $C_p(X)$ is separable. Indeed, this follows from the separability of metizable spaces of countable cellularity and the fact that $C_p(X)$ has countable cellularity, being a dense subspace of $\mathbb{R}^X$, see [2].

Although Problem 1 is still left open, but being motivated by the above mentioned partial results one can formulate the following much stronger version of thi question.

**Problem 2.** *For which Tychonoff spaces $X$ the function space $C_p(X)$ admits a metrizable infinite-dimensional quotient?*

This problem has a simple solution for Tychonoff spaces which are not pseudocompact. We recall that a Tychonoff space $X$ is *pseudocompact* if each continuous real-valued function on $X$ is bounded.

**Proposition 1.** *For a Tychonoff space $X$ the following conditions are equivalent:*

(1) *$X$ is not pseudocompact;*
(2) *$C_p(X)$ has a subspace, isomorphic to $\mathbb{R}^{\mathbb{N}}$;*
(3) *$C_p(X)$ has a quotient space, isomorphic to $\mathbb{R}^{\mathbb{N}}$;*
(4) *$C_p(X)$ admits a linear continuous map onto $\mathbb{R}^{\mathbb{N}}$.*

*Proof.* The implication (1) $\Rightarrow$ (2) is proved in Theorem 14 of [17] and (2) $\Rightarrow$ (3) follows from the complementability of $\mathbb{R}^{\mathbb{N}}$ in any locally convex space containing it, see [20, Corollary 2.6.5]. The implication (3) $\Rightarrow$ (4) is trivial. To see that (4) $\Rightarrow$ (1), observe that for a pseudocompact space $X$ the function space $C_p(X)$ is $\sigma$-bounded, which means that it can be written as the countable union $C_p(X) = \bigcup_{n=1}^{\infty}\{f \in C_p(X) : \sup_{x\in X}|f(x)| \le n\}$ of bounded subsets. Then the image of $C_p(X)$ under any linear continuous operator also is $\sigma$-bounded. On the other hand, the Baire Theorem ensures that the space $\mathbb{R}^{\mathbb{N}}$ is not $\sigma$-bounded. □

The main results of paper are the following two theorems giving two partial answers to Problem 2.

**Theorem 1.** *If a pseudocompact Tychonoff space $X$ contains an infinite discrete $C^*$-embedded subspace $D$, then the function space $C_p(X)$ has an infinite-dimensional metrizable quotient. More pricesely, for any sequence $(F_n)_{n=1}^{\infty}$ of non-empty, finite and pairwise disjoint subsets of $D$ with $\lim_n |F_n| = \infty$ and the linear subspace*

$$Z = \bigcap_{n=1}^{\infty}\{f \in C_p(X) : \sum_{x\in F_n} f(x) = 0\}$$

*the quotient space $C_p(X)/Z$ is isomorphic to the subspace $\ell_\infty = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sup_n |x_n| < \infty\}$ of $\mathbb{R}^{\mathbb{N}}$.*

A subspace $A$ of a topological space $X$ is called *$C^*$-embedded* if each bounded continuous function $f : A \to \mathbb{R}$ has a continuous extension $\bar{f} : X \to \mathbb{R}$.

If a Tychonoff space $X$ is compact, then $X$ contains an infinite discrete $C^*$-embedded subspace if and only if $X$ contains a copy of $\beta\mathbb{N}$. On the other hand, the space $\omega_1$ is pseudocompact noncompact which does not contain $C^*$-embedded infinite discrete subspaces. Moreover, the space $\Lambda := \beta\mathbb{R}\backslash(\beta\mathbb{N}\backslash\mathbb{N})$ discussed in [13, 6P, p.97] is pseudocompact, noncompact and contains $\mathbb{N}$ as a closed discrete $C^*$-embedded set.

**Corollary 1.** *For any infinite discrete space $D$ the space $C_p(\beta D)$ has a quotient space, isomorphic to the subspace $\ell_\infty$ of $\mathbb{R}^\mathbb{N}$.*

Theorem 1 and Proposition 1 yield immediately

**Corollary 2.** *For any Tychonoff space $X$ containing a $C^*$-embedded infinite discrete subspace, the function space $C_p(X)$ has an infinite-dimensional metrizable quotient, isomorphic to $\mathbb{R}^\mathbb{N}$ or $\ell_\infty$.*

Besides the subspace $\ell_\infty$ of $\mathbb{R}^\mathbb{N}$, the following corollary of Theorem 1 involves also the subspace $c_0 := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N} : \lim_{n\to\infty} x_n = 0\}$ of $\mathbb{R}^\mathbb{N}$.

**Corollary 3.** *If a compact Hausdorff space $X$ is not Efimov, then its function space $C_p(X)$ has a quotient space, isomorphic to the subspaces $\ell_\infty$ or $c_0$ in $\mathbb{R}^\mathbb{N}$.*

*Proof.* The space $X$, being non-Efimov, contains either an infinite converging sequence or a copy of $\beta\mathbb{N}$. In the latter case $X$ contains an infinite discrete $C^*$-embedded subspace and Theorem 1 implies that $C_p(X)$ has a quotient space, isomorphic to $\ell_\infty \subset \mathbb{R}^\mathbb{N}$. If $X$ contains a sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise distinct points that converges to a point $x \in X$, then for the compact subset $K := \{x\} \cup \{x_n\}_{n\in\mathbb{N}}$ of $X$ the function space $C_p(K)$ is isomorphic to $c_0 \subset \mathbb{R}^\mathbb{N}$ and is complemented in $C_p(X)$, see [1, Theorem 1, p.130, Proposition 2, p.128]. □

We prove also the following theorem which will be applied for Example 1 below.

**Theorem 2.** *For a Tychonoff space $X$ the space $C_p(X)$ has a metrizable infinite-dimensional quotient if there exists a sequence $(K_n)_{n\in\omega}$ of non-empty compact subsets of $X$ such that for every $n \in \omega$ the compact set $K_n$ contains two disjoint topological copies of $K_{n+1}$.*

We do not know if in Theorem 2 the obtained quotient is isomorphic to $\ell_\infty$ or $c_0$. Example 1 below provides an Efimov space $K$ (under $\diamondsuit$) for which Theorem 2 applies.

**Example 1.** *Under $\diamondsuit$ there exists a Efimov space $K$ whose function space $C_p(K)$ has a metrizable infinite-dimensional quotient.*

*Proof.* De la Vega [4, Theorem 3.22] (we refer also to [3]) constructed under $\diamondsuit$ a compact zero-dimensional hereditary separable space $K$ (hence not containing a copy of $\beta\mathbb{N}$) such that:

(i) $K$ does not contain non-trivial convergent sequences.
(ii) $K$ has a base of clopen pairwise hemeomorphic sets.

It is easy to see that $K$ admits a sequence $(K_n)$ of infinite compact subsets such that each $K_n$ contains two disjoint subsets homeomorphic to $K_{n+1}$; therefore by Theorem 2 the space $C_p(K)$ has the desired property. □

As the space $K$ from Example 1 does not contain $\beta\mathbb{N}$, the assumption of Theorem 1 is not satisfied. Note that in [18, Example 17] we provided (again under $\diamondsuit$) an example of an Efimov space $K$ for which Theorem 2 cannot be applied.

## 2. Proof of Theorem 1

Let $D$ be an infinite discrete $C^*$-embedded subspace of a pseudocompact Tychonoff space $X$. Choose any sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty, finite and pairwise disjoint subsets of $D$ with $\lim_{n\to\infty}|F_n|=\infty$.

For every $n\in\mathbb{N}$ consider the probability measure $\mu_n=\frac{1}{|F_n|}\sum_{x\in F_n}\delta_x$, where $\delta_x$ is the Dirac measure concentrated at $x$.

The pseudocompactness of the space $X$ guarantees that the linear continuous operator

$$T:C_p(X)\to\ell_\infty\subset\mathbb{R}^{\mathbb{N}},\ \ T:f\mapsto(\mu_n(f))_{n\in\mathbb{N}}$$

is well-defined.

We claim that the operator $T$ is open. Given a neighborhood $U\subset C_p(X)$ of zero, we need to check that $T(U)$ is a neighborhood of zero in $\ell_\infty$. We can assume that $U$ is of the basic form

$$U:=\{f\in C_p(X):\max_{x\in E}|f(x)|<\varepsilon\}$$

for some finite set $E\subset X$ and some $\varepsilon>0$.

Choose a number $m\in\mathbb{N}$ such that $\inf_{k>m}|F_k|\ge 2(|E|+1)$. We claim that $T(U)$ contains the open neighborhood

$$V:=\{(y_k)_{k=1}^\infty\in\ell_\infty:\max_{k\le m}|x_k|<\varepsilon\}$$

of zero in $\ell_\infty\subset\mathbb{R}^{\mathbb{N}}$.

Fix any sequence $(y_k)_{k=1}^\infty\in V$. Choose any partition $\mathcal{P}$ of the set $D\setminus\bigcup_{k=1}^m F_k$ into $|\mathcal{P}|=|E|+1$ pairwise disjoint sets such that for every $P\in\mathcal{P}$ and $k>m$ the intersection $P\cap F_k$ has cardinality

$$|P\cap F_k|\ge\frac{|F_k|}{|E|+1}-1\ge 1.$$

Since the discrete subspace $D$ is $C^*$-embedded in $X$, the sets in the partition $\mathcal{P}$ have pairwise disjoint closures in $X$. Taking into account that $|\mathcal{P}|>|E|$, we can find a set $P\in\mathcal{P}$ whose closure $\bar{P}$ is disjoint with $E$.

Consider the function $f:S\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}y_k & \text{if } x\in F_k \text{ for some } k\le m;\\ y_k\cdot\frac{|F_k|}{|F_k\cap P|} & \text{if } x\in P\cap F_k \text{ for some } k>m;\\ 0 & \text{otherwise.}\end{cases}$$

The function $f$ is bounded because $\sup_{k\in\mathbb{N}}|y_k|<\infty$ and

$$\sup_{k>m}\frac{|F_k|}{|F_k\cap P|}\le\sup_{k>m}\frac{|F_k|}{\frac{|F_k|}{|E|+1}-1}=\sup_{k>m}\frac{1}{\frac{1}{|E|+1}-\frac{1}{|F_k|}}\le\frac{1}{\frac{1}{|E|+1}-\frac{1}{2(|E|+1)}}=2(|E|+1)<\infty.$$

As the space $D$ is discrete and $C^*$-embedded into $X$, the bounded function $f$ has a continuous extension $\bar{f}:X\to\mathbb{R}$. Since the space $X$ is Tychonoff, there exists a continuous function $\lambda:X\to[0,1]$ such that $\lambda(\bar{D})=\{1\}$ and $\lambda(x)=0$ for all $x\in E\setminus\bar{D}$. Replacing $\bar{f}$ by the product $\bar{f}\cdot\lambda$, we can assume that $\bar{f}(x)=0$ for all $x\in E\cap\bar{D}$. We claim that $\bar{f}\in U$.

Given any $x\in E$ we should prove that $|\bar{f}(x)|<\varepsilon$. This is clear if $x\notin\bar{D}$. If $x\in F_k$ for some $k\le m$, then $|\bar{f}(x)|=|y_k|<\varepsilon$ as $y\in V$. If $x\in\bar{D}\setminus\bigcup_{k=1}^m F_k$, then $x\in\bar{Q}$ for some set

$Q \in \mathcal{P} \setminus \{P\}$. The definition of the function $f$ ensures that $f|Q \equiv 0$ and then $|\bar{f}(x)| = 0 < \varepsilon$. This completes the proof of the inclusion $\bar{f} \in U$.

The definition of the function $f$ ensures that $\mu_k(\bar{f}) = \mu_k(f) = y_k$ for all $k \in \mathbb{N}$. So, $(y_k)_{k=1}^{\infty} = T(f) \in T(U)$ and $V \subset T(U)$. This completes the proof of the openness of the operator $T : C_p(X) \to \ell_\infty \subset \mathbb{R}^{\mathbb{N}}$. Since the kernel of the open operator $T$ equals to

$$Z = \bigcap_{n=1}^{\infty} \{f \in C_p(X) : \mu_n(f) = 0\},$$

the quotient space $C_p(X)/Z$ is isomorphic to the subspace $T(X) = \ell_\infty$ of $\mathbb{R}^{\mathbb{N}}$. □

## 3. Proof of Theorem 2

Let $X$ be a Tychonoff space and $(K_n)_{n\in\omega}$ be a sequence of compact subsets of $X$ such that for every $n \in \omega$ there are two embeddings

$$\ddot{e}_{n,0}, \ddot{e}_{n,1} : K_{n+1} \to K_n$$

such that $\ddot{e}_{n,0}(K_{n+1}) \cap \ddot{e}_{n,1}(K_{n+1}) = \emptyset$. Replacing the sequence $(K_n)_{n\in\omega}$ by the sequence $(K_{2n})_{n\in\omega}$, if necessary, we can assume that for any $n \in \omega$ the set $K_n \setminus (\ddot{e}_{n,0}(K_{n+1}) \cup \ddot{e}_{n,1}(K_{n+1}))$ is not empty and hence contains a point $\dot{x}_n$.

Let $2^{<n} := \bigcup_{k<n} 2^k$ and $2^{\le n} := \bigcup_{k\le n} 2^k$ for $n \in \omega$, where $2^k$ is the family of all binary sequences of length $k$, that is

$$2^0 = \{\emptyset\},\ 2^1 = \{(0), (1)\},\ 2^2 = \{(0,0), (0,1), (1,0), (1,1)\}, \quad \text{and so on.}$$

For a binary sequence $s = (s_0, \dots, s_{n-1}) \in 2^n$ by $|s|$ we denote the length $n$ of the sequence $s$; for $s = \emptyset \in 2^0$ we put $|s| = 0$.

Let $s\hat{\ }i = (i)$ if $s = \emptyset \in 2^0$, $i \in \{0,1\}$, and $s\hat{\ }i = (s_0, \dots, s_{n-1}, i)$ if $s = (s_0, \dots, s_{n-1}) \in 2^n$, $1 \le n < \omega$, $i \in \{0,1\}$. Similarly we define $s\hat{\ }p \in 2^{|s|+|p|}$ for all $s, p \in 2^{<\omega}$.

Consider the family of embeddings

$$\big(e_s : K_{|s|} \to X\big)_{s\in 2^{<\omega}}$$

defined by the recursive formula: $e_\emptyset : K_0 \to X$ is the identity embedding of $K_0$ into $X$ and

$$e_{s\hat{\ }i} = e_s \circ \ddot{e}_{|s|,i} \text{ for } s \in 2^{<\omega} \text{ and } i \in \{0,1\}.$$

For every $s \in 2^{<\omega}$, let $K_s := e_s(K_{|s|})$ and $x_s := e_s(\dot{x}_{|s|}) \in K_s$.

If $n \in \omega, i \in \{0,1\}, s \in 2^n$ and $t = s\hat{\ }i \in 2^{n+1}$, then $K_t \subset K_s$; indeed,

$$K_t = K_{s\hat{\ }i} = e_{s\hat{\ }i}(K_{n+1}) = e_s(\ddot{e}_{n,i}(K_{n+1})) \subset e_s(K_n) = K_s.$$

If $n \in \omega$ and $s, t \in 2^n$ with $s \ne t$, then $K_s \cap K_t = \emptyset$. Indeed, for $n = 0$ it is obvious. Assume that it is true for some $n \in \omega$. Let $s, t \in 2^n$ and $i, j \in \{0,1\}$ with $s\hat{\ }i \ne t\hat{\ }j$. If $s \ne t$, then

$$K_{s\hat{\ }i} \cap K_{t\hat{\ }j} \subset K_s \cap K_t = \emptyset.$$

If $s = t$, then $i \ne j$, so $K_{s\hat{\ }i} \cap K_{t\hat{\ }j} = K_{s\hat{\ }i} \cap K_{s\hat{\ }j} = e_s(\ddot{e}_{n,i}(K_{n+1})) \cap e_s(\ddot{e}_{n,j}(K_{n+1})) = \emptyset$ since $e_s$ is injective and $\ddot{e}_{n,i}(K_{n+1}) \cap \ddot{e}_{n,j}(K_{n+1}) = \emptyset$. Thus for all $u, v \in 2^{n+1}$ with $u \ne v$ we have $K_u \cap K_v = \emptyset$.

If $n \in \omega, i \in \{0,1\}$ and $t \in 2^n$, then

$$x_t = e_t(\dot{x}_n) \in e_t(K_n \setminus \ddot{e}_{n,i}(K_{n+1})) = e_t(K_n) \setminus e_t(\ddot{e}_{n,i}(K_{n+1})) = K_t \setminus e_{t\hat{\ }i}(K_{n+1}) = K_t \setminus K_{t\hat{\ }i}.$$

It follows that $x_t \notin K_s$ if $t, s \in 2^{<\omega}$ with $|t| < |s|$.

If $s,t,p \in 2^{<\omega}$ with $|s|=|t|$, then $e_s^{-1}\circ e_{s\hat{\ }p} = e_t^{-1}\circ e_{t\hat{\ }p}$ and

$$e_s^{-1}(x_{s\hat{\ }p}) = e_s^{-1}(e_{s\hat{\ }p}(\dot{x}_{|s|+|p|})) = e_t^{-1}(e_{t\hat{\ }p}(\dot{x}_{|t|+|p|})) = e_t^{-1}(x_{t\hat{\ }p}).$$

For every $n\in\omega$ let

$$\mu_n = 2^{-n}\sum_{s\in 2^n}\delta_{x_s}$$

be the uniformly distributed probability measure with the finite support $\{x_s : s\in 2^n\}$. Let

$$Z = \bigcap_{n\in\omega}\{f\in C_p(X) : \mu_n(f)=0\}.$$

The set $\{\mu_n : n\in\omega\}$ is linearly independent in the dual of the space $C_p(X)$, since $x_s \neq x_t$ for $s,t\in 2^{<\omega}$ with $s\neq t$. Thus the quotient space $C_p(X)/Z$ is infinite-dimensional.

We prove that this quotient space is metrizable.

Let $X_0 = \{x_s : s\in 2^{<\omega}\}$ and $K_n' = \bigcup_{s\in 2^n} K_s$ for $n\in\omega$. Clearly, $(K_n')$ is a decreasing sequence of compact subsets of $K_0$ and $X_0\setminus K_n' \subset \{x_s : s\in 2^{<n}\}$ for any $n\in\omega$. The subset

$$K = X_0 \cup \bigcap_{n\in\omega} K_n'$$

of $X$ is compact. Indeed, $\overline{X_0} = \overline{(X_0\setminus K_n')}\cup\overline{(X_0\cap K_n')} \subset X_0\cup K_n'$ for any $n\in\omega$, so $\overline{X_0}\subset X_0\cup\bigcap_{n\in\omega}K_n'$. Thus

$$\overline{K} = \overline{X_0}\cup\bigcap_{n\in\omega}K_n' = X_0\cup\bigcap_{n\in\omega}K_n' = K\subset K_0,$$

so $K$ is compact.

Consider the quotient map $q : C_p(X)\to C_p(X)/Z$. For every $n\in\omega$ consider the open neighborhood

$$V_n := \{f\in C_p(X) : |f(x_s)| < 2^{-n} \text{ for each } s\in 2^{<n}\}$$

of zero in $C_p(X)$. We claim that $\big(q(V_n)\big)_{n\in\omega}$ is a neighborhood basis at zero in the quotient space $C_p(X)/Z$.

Given any neighborhood $U\subset C_p(X)$ of zero, we need to find $n\in\omega$ such that $V_n+Z\subset U+Z$. Without loss of generality we can assume that $U$ is of basic form

$$U = \{f\in C_p(X) : \max_{x\in E}|f(x)| < \varepsilon\}$$

for some $\varepsilon>0$ and some finite set $E\subset X$. Choose $n\in\omega$ so large that

(1) $2^{-n}<\varepsilon$;
(2) $2^{1-n}|E|<1$;
(3) $E\cap X_0\subset\{x_s\}_{s\in 2^{<n}}$;

We claim that $V_n\subset U+Z$. Given any function $f\in V_n$, we should write it as $f=u+\zeta$ where $u\in U$ and $\zeta\in Z$.

Consider the set $S=\{s\in 2^n : E\cap K_s\neq\emptyset\}$. The condition (2) ensures that $|S|\le|E|\le 2^{n-1}$. So, we can find an injective map $\xi : S\to 2^n\setminus S$. Now define the function $\zeta_0 : K\to\mathbb{R}$ by the formula

$$\zeta_0(x) = \begin{cases} f(x) & x\in K_s\cap K \text{ for some } s\in S;\\ -f\circ e_s\circ e_{\xi(s)}^{-1}(x) & \text{if } x\in K_{\xi(s)}\cap K \text{ for some } s\in S;\\ 0 & \text{otherwise.}\end{cases}$$

The function $\zeta_0$ is continuous. Indeed, let $B_s = \{x_s\}$ for $s \in 2^{<n}$ and $B_s = K_s \cap K$ for $s \in 2^n$. Then $K = \bigcup_{s\in 2^{\leq n}} B_s$ and the sets $B_s, s \in 2^{\leq n}$, are compact and pairwise disjoint, so they are open and closed subsets of $K$. Since $\zeta_0|B_s$ is continuous for any $s \in 2^{\leq n}$, then $\zeta_0$ is continuous. As $K$ is compact and $E$ is finite, there exists $\zeta \in C_p(X)$ with $\zeta|K = \zeta_0$ and $\zeta(x) = f(x)$ for all $x \in E \setminus K$.

We prove that $\zeta \in Z$.

Let $m \in \omega$. If $m < n$ and $t \in 2^m$, then $\zeta(x_t) = 0$, so $\mu_m(\zeta) = 0$ for $m < n$.

If $m \geq n$, then

$$\begin{aligned}\sum_{t\in 2^m} \zeta(x_t) &= \sum_{p\in 2^{m-n}} \sum_{s\in 2^n} \zeta(x_{s\hat{}p}) = \sum_{p\in 2^{m-n}} \Big( \sum_{s\in S} \zeta(x_{s\hat{}p}) + \sum_{s\in S} \zeta(x_{\xi(s)\hat{}p}) \Big) = \\ &= \sum_{p\in 2^{m-n}} \Big( \sum_{s\in S} f(x_{s\hat{}p}) - \sum_{s\in S} f\big(e_s(e^{-1}_{\xi(s)}(x_{\xi(s)\hat{}p}))\big) \Big) = \\ &= \sum_{p\in 2^{m-n}} \Big( \sum_{s\in S} f(x_{s\hat{}p}) - \sum_{s\in S} f\big(e_s(e^{-1}_{s}(x_{s\hat{}p}))\big) \Big) = \\ &= \sum_{p\in 2^{m-n}} \Big( \sum_{s\in S} f(x_{s\hat{}p}) - \sum_{s\in S} f(x_{s\hat{}p}) \Big) = 0,\end{aligned}$$

so $\mu_m(\zeta) = 0$ for $m \geq n$. Thus $\zeta \in Z$.

Finally we prove that $f - \zeta \in U$. For $x \in E\setminus K$ we have $|f(x)-\zeta(x)| = 0 < \varepsilon$. Let $x \in E\cap K$. Then $x = x_t$ for some $t \in 2^{<n}$ or $x \in K_s$ for some $s \in S$. In the first case we have

$$|f(x) - \zeta(x)| = |f(x) - 0| < 2^{-n} < \varepsilon;$$

in the second case we get

$$|f(x) - \zeta(x)| = |f(x) - f(x)| = 0 < \varepsilon.$$

Thus $|f(x) - \zeta(x)| < \varepsilon$ for any $x \in E$, so $f - \zeta \in U$.

We have shown that the quotient space $C_p(X)/Z$ is infinite-dimensional and metrizable.

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