Complex symmetric weighted composition operators

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Abstract

In this paper we find all complex symmetric weighted composition operators with special conjugations. Then we give spectral properties of these complex symmetric weighted composition operators.

[1]

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space, denoted $H^2(\mathbb{D}) = H^2$, is the set of all analytic functions f on $\mathbb{D}$, satisfying the norm condition

$$\|f\|^2 = \lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

The space $H^\infty(\mathbb{D}) = H^\infty$ consists of all the functions that are analytic and bounded on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Let φ be an analytic map from the open unit disk $\mathbb{D}$ into itself. The operator that takes the analytic map f to $f \circ \varphi$ is a composition operator and is denoted by C_φ. A natural generalization of a composition operator is an operator that takes f to $\psi \cdot f \circ \varphi$, where ψ is a fixed analytic map on $\mathbb{D}$. This operator is aptly named a weighted composition operator and is usually denoted by $C_{\psi,\varphi}$. More precisely, if z is in the unit disk then $(C_{\psi,\varphi} f)(z) = \psi(z)f(\varphi(z))$.

The automorphisms of $\mathbb{D}$, that is, the one-to-one analytic maps of the disk onto itself, are just the functions $\varphi(z) = \lambda\frac{a-z}{1-\overline{a}z}$, where $|\lambda| = 1$ and $|a| < 1$.

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Let P denote the orthogonal projection of $L^2(\partial\mathbb{D})$ onto H^2. For each $b \in L^\infty(\partial\mathbb{D})$, the Toeplitz operator T_b acts on H^2 by $T_b(f) = P(bf)$.

In [5], Cowen obtained an adjoint formula of a composition operator whose symbol is a linear-fractional self-map of $\mathbb{D}$; for $\varphi(z) = \frac{az+b}{cz+d}$ which is a linear-fractional self-map of $\mathbb{D}$, he showed that $C_\varphi^* = T_g C_{\sigma_\varphi} T_h^*$, where σ_φ, g and h are the Cowen auxiliary functions given by $\sigma_\varphi(z) := (\overline{a}z - \overline{c})/(-\overline{b}z + \overline{d})$, $g(z) := (-\overline{b}z + \overline{d})^{-1}$ and $h(z) := (cz + d)$. We can see that $\sigma_\varphi(z) = \frac{1}{\overline{\varphi^{-1}(\frac{1}{\overline{z}})}}$, so σ_φ maps $\mathbb{D}$ into itself. It follows that $\varphi(a) = a$ if and only if $\sigma_\varphi(\frac{1}{\overline{a}}) = \frac{1}{\overline{a}}$, where $a \in \mathbb{C}$. Note that g and h are in H^∞. If $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma_\varphi(\eta) = \zeta$. We know that φ is an automorphism if and only if σ_φ is, and in this case $\sigma_\varphi = \varphi^{-1}$. we will write σ for σ_φ except when confusion could arise. From now on, unless otherwise stated, we assume that σ, h and g are given as above.

A bounded operator T on a complex Hilbert space H is said to be a complex symmetric operator if there exists a conjugation C (an isometric, antilinear involution) such that $CT^*C = T$. The complex symmetric operators class was initially addressed by Garcia and Putinar (see [9] and [10]) and includes the normal operators, Hankel operators and Volterra integration operators. Invoking [11, Theorem 2], any composition operator with an involutive automorphism symbol is complex symmetric. In [18], Bourdon et al. showed that among the automorphisms of $\mathbb{D}$, only the elliptic ones may introduce complex symmetric operators. Moreover, they proved that for φ, not the rotation and involutive automorphism, which is elliptic automorphism of order q that $4 \leq q \leq \infty$, C_φ is not complex symmetric. In this paper we use the symbol J for the special conjugation that $(Jf)(z) = \overline{f(\overline{z})}$ for each analytic function f. In [7] and [14], all J-symmetric weighted composition operators were characterized. Recently in [17] Narayan et al. have found complex symmetric composition operators whose symbols are linear-fractional, but not an automorphism.

In the second section of this paper, first we find all unitary weighted composition operators which are J-symmetric. Then we consider the special conjugations which are the products of these unitary weighted composition operators and the conjugation J. Next in Theorem 2.5, we obtain complex symmetric weighted composition operators with these conjugations. In addition, in Theorem 2.7, we characterize all complex symmetric weighted composition operators which are isometries.

In the third section, we provide a characterization of φ, when φ has the form which was stated in [7, Proposition 2.9] and [14, Theorem 3.3]. Finally, we obtain the spectrum and spectral radius of some complex symmetic weighted composition operators that study in the second section.

2 Weighted composition operators

Suppose that $\varphi(z) = \frac{az+b}{1-cz}$ is a linear-fractional self-map of $\mathbb{D}$. If φ is written as $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$, then it is not hard to see that $c = b = a_0$. We use this fact frequently in this paper.

An operator T is said to be unitary if $T^*T = TT^* = I$. In the following proposition, we find all unitary weighted composition operators $C_{\psi,\varphi}$ which are J-symmetric.

Proposition 2.1. *The weighted composition operator $C_{\psi,\varphi}$ is unitary and J-symmetric if and only if either $\psi(z) = c\frac{(1-|p|^2)^{1/2}}{1-\overline{p}z}$ and $\varphi(z) = \frac{\overline{p}}{p}\frac{p-z}{1-\overline{p}z}$, where $p \in \mathbb{D} - \{0\}$ and $|c| = 1$ or $\psi \equiv \mu$ and $\varphi(z) = \lambda z$, when $|\mu| = |\lambda| = 1$.*

Proof. Let $C_{\psi,\varphi}$ be unitary and J-symmetric. By [2, Theorem 6],

$$\varphi(z) = \lambda\frac{p-z}{1-\overline{p}z} \tag{1}$$

and

$$\psi(z) = c\frac{(1-|p|^2)^{1/2}}{1-\overline{p}z}, \tag{2}$$

where $|\lambda| = |c| = 1$. First suppose that $p \neq 0$. We can see that $\varphi(0) = \lambda p$ and $\varphi'(0) = \lambda(|p|^2 - 1)$. Let $\widetilde{\varphi}(z) = \lambda p + \frac{\lambda(|p|^2-1)z}{1-\lambda pz}$. It is not hard to see that $\widetilde{\varphi} \equiv \varphi$ if and only if $\lambda = \frac{\overline{p}}{p}$. Invoking [14, Theorem 3.3] (see also [7, Proposition 2.9]), the conclusion follows for $p \neq 0$. Letting $p = 0$ in Equations (1) and (2), we get ψ is a constant function and $\varphi(z) = -\lambda z$, when $|\lambda| = 1$.

Conversely, suppose that either $\varphi(z) = \frac{\overline{p}}{p}\frac{p-z}{1-\overline{p}z}$ and $\psi(z) = c\frac{(1-|p|^2)^{1/2}}{1-\overline{p}z}$ or ψ is a constant function and $\varphi(z) = \lambda z$. In this both cases, by [2, Theorem 6], $C_{\psi,\varphi}$ is unitary. Rewriting φ, we see that $\varphi(z) = \frac{\overline{p}}{p}\frac{p-z}{1-\overline{p}z} = \overline{p} + \frac{\frac{\overline{p}}{p}(|p|^2-1)z}{1-\overline{p}z}$. Again [7, Proposition 2.9] and [14, Theorem 3.3] imply that in these both cases the weighted composition operators $C_{\psi,\varphi}$ are J-symmetric. □

From now on, unless otherwise stated, we assume that $\varphi_p(z) = \frac{\overline{p}}{p}\frac{p-z}{1-\overline{p}z}$, where $p \in \mathbb{D} - \{0\}$ and $\psi_p(z) = c\frac{(1-|p|^2)^{1/2}}{1-\overline{p}z}$, when $p \in \mathbb{D}$ and $|c| = 1$. The proof of the next lemma is left to the reader.

Lemma 2.2. *If U is a unitary and complex symmetric operator with conjugation C, then UC is a conjugation.*

Let C be a conjugation. Then $CJ = W$ is a unitary operator and W is both C-symmetric and J-symmetric (see [8, Lemma 3.2]). We have $C = WJ$. Then all conjugations can be considered as a product of a J-symmetric unitary operator W and the conjugation J.

Proposition 2.3. *Suppose that U is unitary and complex symmetric with conjugation WJ, where W is unitary. Then an operator A is WJ-symmetric if and only if UA is UWJ-symmetric.*

Proof. Suppose that A is WJ-symmetric. Invoking Lemma 2.2, UWJ is a conjugation. We have

$$UWJ(UA)^*UWJ = UWJA^*U^*UWJ = UA,$$

so UA is complex symmetric with conjugation UWJ.

Conversely, suppose that UA is UWJ-symmetric. We see that

$$WJA^*WJ = U^*UWJ(UA)^*UWJ = U^*UA = A.$$

Hence, A is WJ-symmetric. □

Assume that an operator B is UJ-symmetric. Let $W = I$ in Proposition 2.3. By Proposition 2.3, U^*B is J-symmetric. We have $B = UU^*B$. It shows that every complex symmetric operator can be written as a product of a unitary J-symmetric operator and a J-symmetric operator. Since recently a lot of J-symmetric operators have been found, this idea may be useful in order to obtain complex symmetric operators more.

In the following example, we find a complex symmetric Toeplitz operator T_f with $|f| = 1$ on $\partial\mathbb{D}$ (see [15, Corollary 2.2]).

Example 2.4. Suppose that $p \in (-1, 1)$ is a real number. By [18, Lemma 2.1] and [18, Lemma 2.2], U_{φ_p} is J-symmetric, when U_{φ_p} is the unitary part in the polar decomposition of C_{φ_p} (note that in this case φ_p is an involutive automorphism). By the proof of [16, Lemma 4.7], $U_{\varphi_p} = C_{\varphi_p}T_{\frac{|1-pz|}{(1-|p|^2)^{1/2}}}$. Proposition 2.1 implies that C_{ψ_p,φ_p} is J-symmetric. Invoking Proposition 2.3, $C_{\psi_p,\varphi_p}C_{\varphi_p}T_{\frac{|1-pz|}{(1-|p|^2)^{1/2}}}$ is $C_{\psi_p,\varphi_p}J$-symmetric. Then $T_{1/(1-pz)}T_{|1-pz|}$ is symmetric. Thus, $(T_{1/(1-pz)}T_{|1-pz|})^{-1} = T_{1/|1-pz|}T_{p-z} =$

$T_{\frac{p-z}{|1-pz|}}$ is symmetric with conjugation $C_{\psi_p,\varphi_p}J$.

In the following theorem, we find all complex symmetric weighted composition operators with conjugations UJ that U is unitary and J-symmetric weighted composition operator which was stated in Proposition 2.1.

Theorem 2.5. *Let $a_0 \in \mathbb{D}$ and $a_1, b \in \mathbb{C}$. Suppose that $\psi(z) = \frac{b}{1-a_0 z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$ that φ is an analytic self-map of $\mathbb{D}$.*
(1) For $p \neq 0$, the weighted composition operator $C_{\widetilde{\psi},\widetilde{\varphi}}$ is complex symmetric with conjugation $C_{\psi_p,\varphi_p}J$ if and only if $\widetilde{\psi} = \psi_p \cdot \psi \circ \varphi_p$ and $\widetilde{\varphi} = \varphi \circ \varphi_p$ for some φ and ψ.
(2) For $|\lambda| = 1$, the weighted composition operator $C_{\widetilde{\psi},\widetilde{\varphi}}$ is complex symmetric with conjugation $C_{\lambda z}J$ if and only if $\widetilde{\psi} = \psi(\lambda z)$ and $\widetilde{\varphi}(z) = \varphi(\lambda z)$ for some φ and ψ.

Proof. (1) Let $\widetilde{\psi} = \psi_p \cdot \psi \circ \varphi_p$ and $\widetilde{\varphi} = \varphi \circ \varphi_p$ for some φ and ψ. Then $C_{\widetilde{\psi},\widetilde{\varphi}} = T_{\psi_p \cdot \psi \circ \varphi_p} C_{\varphi \circ \varphi_p} = C_{\psi_p,\varphi_p} C_{\psi,\varphi}$. Since C_{ψ_p,φ_p} and $C_{\psi,\varphi}$ are J-symmetric (see [7, Proposition 2.9] and [14, Theorem 3.3]), by Proposition 2.3, $C_{\widetilde{\psi},\widetilde{\varphi}}$ is $C_{\psi_p,\varphi_p}J$-symmetric.

Conversely, suppose that $C_{\widetilde{\psi},\widetilde{\varphi}}$ is complex symmetric with conjugation $C_{\psi_p,\varphi_p}J$. By Proposition 2.3, $C^*_{\psi_p,\varphi_p} C_{\widetilde{\psi},\widetilde{\varphi}}$ is J-symmetric. The Cowen adjoint formula shows that $C^*_{\psi_p,\varphi_p}$ is also a weighted composition operator. Then $C^*_{\psi_p,\varphi_p} C_{\widetilde{\psi},\widetilde{\varphi}}$ is a weighted composition operator which was defined in [7, Proposition 2.9] and [14, Theorem 3.3]. Then there is a J-symmetric weighted composition operator $C_{\psi,\varphi}$ that $C_{\widetilde{\psi},\widetilde{\varphi}} = C_{\psi_p,\varphi_p} C_{\psi,\varphi}$.

(2) By the same idea which was stated in the proof of Part (1), the conclusion follows. □

Proposition 2.6. *Suppose that T is a bounded operator on a Hilbert space H. Then T is a complex symmetric operator and an isometry if and only if T is unitary.*

Proof. Suppose that T is complex symmetric with conjugation C. Since T is an isometry, for each $f \in H^2$,

$$\|T^* C f\| = \|C T^* C f\| = \|T f\| = \|f\|.$$

Then $\|T^* f\| = \|T^* C(Cf)\| = \|Cf\| = \|f\|$, and so T^* is an isometry. We infer that T is a unitary operator from [4, Proposition 2.17, p. 35] and [4, Proposition 2.18, p. 35].

Conversely, it is obvious. $\square$

In Theorem 2.7, we show that a weighted composition operator which is both a complex symmetric operator and an isometry is unitary; moreover, we find all conjugations for unitary weighted composition operators.

Theorem 2.7. *A weighted composition operator $C_{\psi,\varphi}$ is both an isometry and a complex symmetric operator if and only if $\varphi(z) = \lambda\frac{p-z}{1-\overline{p}z}$ and $\psi \equiv \psi_p$, where $|\lambda| = 1$ and $p \in \mathbb{D}$. Furthermore, if $p \neq 0$, then the conjugation for $C_{\psi,\varphi}$ is $C_{\psi_p,\varphi_p}J$.*

Proof. By Proposition 2.6 and [2, Theorem 6], the first part is obvious. According to [14, Theorem 3.3] and [7, Proposition 2.9], $C_{\gamma z}$ is J-symmetric, when $|\gamma| = 1$. Let $\gamma = \lambda\frac{p}{\overline{p}}$. We obtain $C_{\psi,\varphi} = C_{\psi_p,\varphi_p}C_{\gamma z}$. By Propositions 2.1 and 2.3, we complete the proof. $\square$

3 Spectral theory

Recall that a nontrivial automorphism φ of $\mathbb{D}$ (i.e., φ is not the identity function of $\mathbb{D}$) is called elliptic if φ has a fixed point in $\mathbb{D}$ and the other fixed point is in the complement of the closed disk.

We say that φ has a finite angular derivative at $\zeta \in \partial\mathbb{D}$ if the nontangential limit $\varphi(\zeta)$ exists, has modulus 1, and $\varphi'(\zeta) = \angle\lim_{z\to\zeta}\frac{\varphi(z)-\varphi(\zeta)}{z-\zeta}$ exists and finite. Let $\varphi_0 = I$ and $\varphi_n = \varphi\circ\varphi\circ...\circ\varphi$ denote the n-th iterate of φ. If φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is a holomorphic self-map of $\mathbb{D}$, then there is a unique point w in $\overline{\mathbb{D}}$ so that the iterates φ_n of φ tend to w uniformly on compact subsets of $\mathbb{D}$ (see [6, Theorem 2.51]). The point w will be referred to as the Denjoy-Wolff point of φ. We know that the Denjoy-Wolff point of φ can be described as the unique fixed point of φ in $\overline{\mathbb{D}}$ with $|\varphi'(w)| \leq 1$.

Suppose that φ, not an automorphism, is a linear-fractional self-map of $\mathbb{D}$ with a fixed point on $\partial\mathbb{D}$. Then φ satisfies one of the following

(a) φ is hyperbolic with one fixed point $\zeta \in \partial\mathbb{D}$ and the other fixed point outside the closed unit disk. Let $T(z) = \frac{\zeta+z}{\zeta-z}$. Then $\phi(z) = (T\circ\varphi\circ T^{-1})(z) = rz + t$, where $r = 1/\varphi'(\zeta)$ (note that $r > 1$) and $\mathrm{Re}(t) \geq 0$ (and $\mathrm{Re}(t) = 0$ if and only if φ is an automorphism; moreover, in this case both fixed points of φ lie on $\partial\mathbb{D}$). We call t the translation number of φ. Then we obtain

$$\varphi(z) = \frac{(1+r-t)z + (r+t-1)\zeta}{(r-t-1)\overline{\zeta}z + (1+r+t)}. \tag{3}$$

(b) φ is hyperbolic with one fixed point, w, inside the unit disk, and the other fixed point ζ, on the unit circle. It is not hard to see that φ is hyperbolic with this type (Denjoy-Wolff point of φ is in $\mathbb{D}$) if and only if the Cowen auxiliary function σ_φ is hyperbolic under the condition (a). Hence in this case

$$\varphi(z) = \frac{(1+r-\overline{t})z - (r-\overline{t}-1)\zeta}{1+r+\overline{t} - (r+\overline{t}-1)\overline{\zeta}z}, \tag{4}$$

where t is the translation number of σ_φ and $r = \frac{1}{\sigma'_\varphi(\zeta)}$. Note that in this case, since φ has a Denjoy-Wolff point in $\mathbb{D}$, φ is not an automorphism. Hence σ_φ is not automorphism, so $\mathrm{Re}(t) > 0$.

(c) φ is parabolic with only one fixed point $\zeta \in \partial\mathbb{D}$. Let $T(z) = \frac{\zeta+z}{\zeta-z}$. Then $\phi(z) = (T \circ \varphi \circ T^{-1})(z) = z + t$, where $\mathrm{Re}(t) \geq 0$. Let us call t the translation number of φ. Note $\mathrm{Re}(t) = 0$ if and only if φ is an automorphism. In [19, p. 3] Shapiro showed that among the linear-fractional self-map of $\mathbb{D}$ fixing $\zeta \in \partial\mathbb{D}$, the parabolic ones are characterized by $\varphi'(\zeta) = 1$. We see that in this case

$$\varphi(z) = \frac{(2-t)z + t\zeta}{2 + t - t\overline{\zeta}z}. \tag{5}$$

Suppose that φ_1 and φ_2 are parabolic with the same fixed point. It is not hard to see that $\varphi_1 \circ \varphi_2$ is also parabolic. We use this fact in the proof of Theorem 3.5.

Lemma 3.1. *Suppose that φ is hyperbolic with fixed point $\zeta \in \partial\mathbb{D}$. If φ is written as*

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}, \tag{6}$$

when $a_0 \in \mathbb{D}$ and $a_1 \in \mathbb{C}$, then φ is an automorphism.

Proof. First suppose that φ is hyperbolic with Denjoy-Wolff point $\zeta \in \partial\mathbb{D}$. Then by Equation (3), we have

$$\varphi(z) = \frac{\frac{1+r-t}{1+r+t}z + \frac{(r+t-1)\zeta}{1+r+t}}{\frac{(r-t-1)\overline{\zeta}}{1+r+t}z + 1}.$$

If φ is written as in Equation (6), then $\frac{-r+t+1}{1+r+t}\overline{\zeta} = \frac{r+t-1}{1+r+t}\zeta$. Therefore, $-r\overline{\zeta} + t\overline{\zeta} + \overline{\zeta} = r\zeta + t\zeta - \zeta$ and so $(\mathrm{Re}(\zeta))(1-r) = t(\mathrm{Im}(\zeta))i$. It shows that t

is pure imaginary and the result follows. Now suppose that φ is hyperbolic with Denjoy-Wolff $w \in \mathbb{D}$ and fixed point $\zeta \in \partial\mathbb{D}$. By Equation (4), assume that

$$\varphi(z) = \frac{\frac{(1+r-\overline{t})}{1+r+\overline{t}}z - \frac{r-\overline{t}-1}{1+r+\overline{t}}\zeta}{1 - \frac{(r+\overline{t}-1)\overline{\zeta}z}{1+r+\overline{t}}}.$$

Since φ is as in Equation (6), $\frac{(r+\overline{t}-1)\overline{\zeta}}{1+r+\overline{t}} = -\frac{(r-\overline{t}-1)\zeta}{1+r+\overline{t}}$. Then $(\mathrm{Re}(\zeta))(r-1) = \overline{t}(\mathrm{Im}(\zeta))i$. It follows that t is pure imaginary which is a contradiction. □

Lemma 3.2. *Suppose that φ is parabolic with fixed point $\zeta \in \partial\mathbb{D}$. Then φ is as in Equation (6) if and only if $\zeta = 1$ or $\zeta = -1$.*

Proof. Suppose that φ is written as in Equation (6) and by Equation (5), $\varphi(z) = \frac{\frac{(2-t)z}{2+t} + \frac{t\zeta}{2+t}}{-\frac{t\overline{\zeta}}{2+t}z+1}$. If $t = 0$, then $\varphi(z) = z$ and φ is not parabolic. Then we assume that $t \neq 0$. Since φ is as in Equation (6), $\overline{\zeta} = \zeta$. It shows that $\zeta = 1$ or $\zeta = -1$.

Conversely, it is obvious. □

Note that by the proof of the pervious lemma, we see that if φ is parabolic with Denjoy-Wolff point 1 which is written as in Equation (5), then

$$\varphi(z) = \frac{\frac{2-t}{2+t}z + \frac{t}{2+t}}{\frac{-t}{t+2}z + 1}. \qquad (7)$$

Lemma 3.3. *Suppose that $p \in \mathbb{D} - \{0\}$. If 1 is the fixed point of φ_p, then φ_p is a parabolic automorphism. Moreover, if -1 is the fixed point of φ_p, then φ_p is a hyperbolic automorphism with Denjoy-Wolff point -1.*

Proof. Assume that $\varphi_p(1) = 1$. Note that $p = |p|\cos(\theta) + |p|\sin(\theta)i$, where $\theta = \mathrm{Arg}(p)$. Then $\overline{p}(p-1) = p(1-\overline{p})$ and so $|p|^2 = \mathrm{Re}(p) = |p|\cos(\theta)$. Hence $|p| = \cos(\theta)$. Since $\frac{\overline{p}}{p} = e^{-2\theta i}$, [19, Exercise 4, p. 7] implies that φ_p is parabolic. Now suppose that $\varphi_p(-1) = -1$. Then $\overline{p}(p+1) = -p(1+\overline{p})$ and so $|p|^2 = -\mathrm{Re}(p) = -|p|\cos(\theta)$. Hence $|p| = -\cos(\theta)$. Again by [19, Exercise 4, p. 7], φ_p is a hyperbolic automorphism. Now we show that -1 is the Denjoy-Wolff point of φ_p. We have $\varphi_p'(-1) = \frac{\overline{p}}{p}\frac{|p|^2-1}{(1+\overline{p})^2}$. We know that $p = |p|\cos(\theta) + |p|\sin(\theta)i = -\cos^2(\theta) - \sin(\theta)\cos(\theta)i$. Therefore, $1 + \overline{p} = 1 - \cos^2(\theta) + \sin(\theta)\cos(\theta)i = \sin^2(\theta) + \sin(\theta)\cos(\theta)i$. We obtain $|\varphi_p'(-1)|^2 = |\frac{|p|^2-1}{(1+\overline{p})^2}|^2 = |\frac{\cos^2(\theta)-1}{(\sin^2(\theta)+\sin(\theta)\cos(\theta)i)^2}|^2 = \frac{\sin^4(\theta)}{\sin^4(\theta)+\sin^2(\theta)\cos^2(\theta)} < 1$.

Then $|\varphi_p'(-1)| < 1$ and it follows that -1 is the Denjoy-Wolff point of φ_p. $\Box$

In the rest of this paper, we suppose that $C_{\psi,\varphi}$ is J-symmetric and φ and ψ were represented in Theorem 2.5. From now on, unless otherwise stated, we assume that $C_{\widetilde{\psi},\widetilde{\varphi}}$ is weighted composition operator which was given in the first part of Theorem 2.5. Suppose that T is a bounded operator on a Hilbert space H. Through this paper, the spectrum of T and the spectral radius of T are denoted by $\sigma(T)$ and $r(T)$, respectively.

Theorem 3.4. *If $C_{\widetilde{\psi},\widetilde{\varphi}}$ is compact or power compact, then $r(C_{\widetilde{\psi},\widetilde{\varphi}}) = |\psi_p(w)\psi(\varphi_p(w))|$ and $\sigma(C_{\widetilde{\psi},\widetilde{\varphi}}) = \{\psi_p(w)\psi(\varphi_p(w))(\widetilde{\varphi}'(w))^m : m = 0, 1, ...\} \cup \{0\}$, where w is the Denjoy-Wolff point of $\widetilde{\varphi} = \varphi \circ \varphi_p$.*

Proof. If $\varphi \circ \varphi_p$ is compact or power compact, then it is easy to see that $\varphi \circ \varphi_p$ has a Denjoy-Wolff point $w \in \mathbb{D}$. There is an integer n such that $C^n_{\widetilde{\psi},\widetilde{\varphi}} = C_{\widetilde{\psi}\cdot\widetilde{\psi}\circ\widetilde{\varphi}...\widetilde{\psi}\circ\widetilde{\varphi}_{n-1},\widetilde{\varphi}_n}$ is compact. By the Spectral Mapping Theorem, $\sigma_e(C_{\widetilde{\psi},\widetilde{\varphi}}) = \{0\}$. Moreover, by [12, Theorem 1], $\sigma(C^n_{\widetilde{\psi},\widetilde{\varphi}}) = \{(\widetilde{\psi}(w))^n(\widetilde{\varphi}'(w))^{mn} : m = 0, 1, ...\}$ and it follows from the Spectral Mapping Theorem that all elements of $\sigma(C_{\widetilde{\psi},\widetilde{\varphi}})$ are in $\partial\sigma(C_{\widetilde{\psi},\widetilde{\varphi}})$. Then by [4, Proposition 6.7, p. 210] and [4, Proposition 4.4, p. 359], $\sigma(C_{\widetilde{\psi},\widetilde{\varphi}}) = \sigma_e(C_{\widetilde{\psi},\widetilde{\varphi}}) \cup \sigma_p(C_{\widetilde{\psi},\widetilde{\varphi}})$. The result follows by [7, Proposition 2.6] and [14, Theorem 4.3]. $\Box$

Now suppose that $\widetilde{\varphi} = \varphi \circ \varphi_p$ is not power compact and it is not an automorphism. Then either
(i) $\varphi(\zeta) = \zeta$ and $\varphi_p(\zeta) = \zeta$, where $\zeta \in \partial\mathbb{D}$
or
(ii) $\varphi(\zeta) = \eta$ and $\varphi_p(\eta) = \zeta$, where $\zeta, \eta \in \partial\mathbb{D}$ and $\zeta \neq \eta$.

By Lemmas 3.1 and 3.2, we see that if φ and φ_p satisfy the conditions of Part (i), then either $\zeta = 1$ or $\zeta = -1$. In the following theorem, we find $r(C_{\widetilde{\psi},\widetilde{\varphi}})$, when φ and φ_p satisfy the conditions of Part (i) and $\widetilde{\varphi}$ is parabolic.

Theorem 3.5. *Suppose that $\widetilde{\varphi}$ is not an automorphism. If φ and φ_P fix 1, then φ, φ_p and $\widetilde{\varphi}$ are parabolic, $r(C_{\widetilde{\psi},\widetilde{\varphi}}) = |\frac{\psi(0)(1-|p|^2)^{1/2}(2+t)}{2(1-\overline{p})}|$ and $\sigma(C_{\widetilde{\psi},\widetilde{\varphi}}) = \{\frac{\psi(0)(1-|p|^2)^{1/2}(2+t)}{2(1-\overline{p})}e^{-b(t+\widetilde{t})} : b \geq 0\} \cup \{0\}$, where t and $\widetilde{t}$ are the translation number of φ and φ_p, respectively.*

Proof. Since φ_p is an automorphism and $\varphi \circ \varphi_p$ is not an automorphism, φ is not an automorphism. Invoking Lemmas 3.1 and 3.2, φ must be parabolic with fixed point 1. By Lemma 3.3, φ_p must be parabolic with fixed point 1. Hence $\widetilde{\varphi} = \varphi \circ \varphi_p$ is also parabolic. Since φ is parabolic with fixed point 1 and $C_{\psi,\varphi}$ is J-symmetric, by Equation (7), we find that $\psi(z) = \frac{\psi(0)}{1-\frac{t}{2+t}z}$. Then by [1, Theorem 4.7], $r(C_{\widetilde{\psi},\widetilde{\varphi}}) = |\widetilde{\psi}(1)| = |\psi_p(1)\cdot\psi(1)| = |\frac{\psi(0)(1-|p|^2)^{1/2}(2+t)}{2(1-\overline{p})}|$ and $\sigma(C_{\widetilde{\psi},\widetilde{\varphi}}) = \{\frac{\psi(0)(1-|p|^2)^{1/2}(2+t)}{2(1-\overline{p})}e^{-b(t+\widetilde{t})} : b \geq 0\} \cup \{0\}$.□

In the next theorem, we find $r(C_{\widetilde{\psi},\widetilde{\varphi}})$, when $\widetilde{\varphi}$ is a hyperbolic non-automorphism with Denjoy-Wolff point -1.

Theorem 3.6. *Suppose that $\widetilde{\varphi}$ is not an automorphism. If φ and φ_p fix -1, then $\widetilde{\varphi}$ is hyperbolic with Denjoy-Wolff point -1, φ is parabolic with translation number t, $r(C_{\widetilde{\psi},\widetilde{\varphi}}) = |\frac{2+t}{2(1+\overline{p})}(\frac{1-|p|^2}{\varphi_p'(-1)})^{1/2}|$ and $\sigma(C_{\widetilde{\psi},\widetilde{\varphi}}) = \{z : |z| \leq r(C_{\widetilde{\psi},\widetilde{\varphi}})\}$.*

Proof. It is easy to see that -1 is the fixed point of $\varphi \circ \varphi_p$. We see that $(\varphi \circ \varphi_p)'(-1) = \varphi'(\varphi_p(-1)) \cdot \varphi_p'(-1)$. Since by Lemmas 3.1 and 3.2, φ is parabolic, $\varphi'(-1) = 1$ and so $(\widetilde{\varphi})'(-1) = \varphi_p'(-1)$. Lemma 3.3 shows that $|(\varphi \circ \varphi_p)'(-1)| < 1$ and -1 is the Denjoy-Wolff point of $\widetilde{\varphi}$. Then $\widetilde{\varphi}$ must be a hyperbolic non-automorphism with Denjoy-Wolff point -1. From Equation (5), $\varphi(z) = \frac{(2-t)z-t}{tz+2+t} = \frac{\frac{2-t}{2+t}z-\frac{t}{2+t}}{1+\frac{t}{2+t}z}$, where t is the translation number of φ. Since $C_{\psi,\varphi}$ is J-symmetric, it is easy to see that $\psi(z) = \frac{1}{1+\frac{t}{2+t}z}$. Invoking [1, Theorem 4.5], $r(C_{\widetilde{\psi},\widetilde{\varphi}}) = |\frac{\psi(-1)}{1+\overline{p}}(\frac{1-|p|^2}{\varphi_p'(-1)})^{1/2}| = |\frac{2+t}{2(1+\overline{p})}(\frac{1-|p|^2}{\varphi_p'(-1)})^{1/2}|$ and $\sigma(C_{\widetilde{\psi},\widetilde{\varphi}}) = \{z : |z| \leq r(C_{\widetilde{\psi},\widetilde{\varphi}})\}$. □

Note that although every complex symmetric composition C_φ must have a fixed point in $\mathbb{D}$ (see [3, Proposition 2.1]), Theorems 3.5 and 3.6 showed that there are complex symmetric weighted composition operators $C_{\widetilde{\psi},\widetilde{\varphi}}$ that $\widetilde{\varphi}$ has no fixed point in $\mathbb{D}$.

In [14, Theorem 4.6] Jung et al. obtained an inequality for $r(C_{\psi,\varphi})$, when $C_{\psi,\varphi}$ is J-symmetric. In the next corollary, we find the spectral radius of J-symmetric weighted composition operators, when φ is not an automorphism. Note that for an automorphism φ, the spectral radius of $C_{\psi,\varphi}$ was found in [13].

Corollary 3.7. *Assume that φ is not an automorphism. Let w be the Denjoy-Wolff point of φ.*

(a) If $w \in \mathbb{D}$, then $r(C_{\psi,\varphi}) = |\psi(w)|$.
(b) If $w \in \partial\mathbb{D}$, then φ is parabolic and w is either 1 or -1. Moreover, if $w = 1$, then $r(C_{\psi,\varphi}) = \left|\frac{\psi(0)(2+t)}{2}\right|$ and if $w = -1$, then $r(C_{\psi,\varphi}) = \left|\frac{\psi(0)(2+t)}{2+2t}\right|$, where t is the translation number of φ.

Proof. (a) If $w \in \mathbb{D}$, then by Lemmas 3.1 and 3.2, φ has no fixed point in $\partial\mathbb{D}$. Then C_φ is compact or power compact. The result follows by the Spectral Mapping Theorem, [14, Theorem 4.3] and [7, Proposition 2.6].
(b) Assume that φ has a Denjoy-Wolff point $w \in \partial\mathbb{D}$. As we saw in Lemmas 3.1 and 3.2, φ must be parabolic with fixed point 1 or -1. By [1, Theorem 4.7] and the similar idea which was stated in the proof of Theorem 3.5, we see that if w=1, then $r(C_{\psi,\varphi}) = |\psi(1)| = \left|\frac{\psi(0)(2+t)}{2}\right|$ and if $w = -1$, then $r(C_{\psi,\varphi}) = |\psi(-1)| = \left|\frac{\psi(0)(2+t)}{2+2t}\right|$. □

It is not hard to see that for $|\lambda| = 1$, $C_{\varphi(\lambda z)}$ is not power compact if and only if there is $\zeta \in \partial\mathbb{D}$ such that $\varphi(\lambda\zeta) = \zeta$. In the following proposition, we find the spectrum of power compact weighted composition operator $C_{\widetilde{\psi},\widetilde{\varphi}}$, when $C_{\widetilde{\psi},\widetilde{\varphi}}$ was given in the second part of Theorem 2.5.

Proposition 3.8. *Assume that φ is not an automorphism. Suppose that $C_{\widetilde{\psi},\widetilde{\varphi}}$ is as in the second part of Theorem 2.5. Let λ be constant and $|\lambda| = 1$. If for each $\zeta \in \partial\mathbb{D}$, $\varphi(\lambda\zeta) \neq \zeta$, then $\sigma(C_{\widetilde{\psi},\widetilde{\varphi}}) = \{\psi(\lambda w)(\lambda\varphi'(\lambda w))^n : n = 0, 1, ...\}$ and $r(C_{\widetilde{\psi},\widetilde{\varphi}}) = |\psi(\lambda w)|$, when $w \in \mathbb{D}$ and $\varphi(\lambda w) = w$.*

Proof. One may easily see that in this case $C_{\widetilde{\psi},\widetilde{\varphi}}$ is power compact and the result follows by the similar idea which was stated in the proof of Theorem 3.4. □

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