About some exponential inequalities related to the sinc function

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Marija Rašajski, Tatjana Lutovac, Branko Malešević *

School of Electrical Engineering, University of Belgrade,
Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia

Abstract. In this paper we prove some exponential inequalities involving the sinc function. We analyze and prove inequalities with constant exponents as well as inequalities with certain polynomial exponents. Also, we establish intervals in which these inequalities hold.

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1 Introduction

Inequalities related to the sinc function, i.e. $\operatorname{sinc} x = \dfrac{\sin x}{x}$ $\big(x \neq 0\big)$, occur in many fields of mathematics and engineering [2], [6], [7], [9], [10], [13], [15] such as Fourier analysis and its applications, information theory, radio transmission, optics, signal processing, sound recording, etc.

The following inequalities are proved in [8]:

$$\cos^2\frac{x}{2} \leq \frac{\sin x}{x} \leq \cos^3\frac{x}{3} \leq \frac{2+\cos x}{3} \tag{1}$$

for every $x \in (0,\pi)$.

In [1], the authors considered possible refinements of the inequality (1) by a real analytic function $\varphi_a(x) = \left(\dfrac{\sin x}{x}\right)^a$, for $x \in (0,\pi)$ and parameter $a \in \mathbb{R}$, and proposed and proved the following inequalities:

*Corresponding author.
Emails:
Marija Rašajski <marija.rasajski@etf.rs>, *Tatjana Lutovac* <tatjana.lutovac@etf.rs>,
Branko Malešević <branko.malesevic@etf.rs>

Statement 1 ([1], Theorem 10) *The following inequalities hold true, for every $x\in(0,\pi)$ and $a\in\left(1,\dfrac{3}{2}\right)$:*

$$\cos^2\frac{x}{2} \le \left(\frac{\sin x}{x}\right)^a \le \frac{\sin x}{x}. \tag{2}$$

In the paper [1], based on the analysis of the sign of the analytic function

$$F_a(x) = \left(\frac{\sin x}{x}\right)^a - \cos^2\frac{x}{2}$$

in the right neighborhood of zero, the corresponding inequalities for parameter values $a \ge \dfrac{3}{2}$ are discussed.

In this paper, in subsection 3.1, using the power series expansions and the application of the Wu-Debnath theorem, we prove that the inequality (2) holds for $a=\dfrac{3}{2}$. At the same time, this proof represents another proof of Statement 1. Also, we analyze the cases $a\in\left(\dfrac{3}{2},2\right)$ and $a\ge 2$ and we prove the corresponding inequalities.

In subsection 3.2 we introduce and prove a new double-sided inequality of similar type involving polynomial exponents.

Finally, in subsection 3.3, we establish a relation between the cases of the constant and of the polynomial exponent.

2 Preliminaries

In this section we review some results that we use in our study.

In accordance with [11], the following expansions hold:

$$\ln\frac{\sin x}{x} = -\sum_{k=1}^{\infty}\frac{2^{2k-1}|\mathbf{B}_{2k}|}{k(2k)!}x^{2k}, \qquad (0<x<\pi), \tag{3}$$

$$\ln\cos x = -\sum_{k=1}^{\infty}\frac{2^{2k-1}(2^{2k}-1)|\mathbf{B}_{2k}|}{k(2k)!}x^{2k}, \qquad (-\pi/2<x<\pi/2), \tag{4}$$

where $\mathbf{B}_i$ $(i\in\mathbb{N})$ are Bernoulli's numbers.

The following theorem proved by Wu and Debnath in [5], is used in our proofs.

Theorem WD. ([5], Theorem 2) *Suppose that $f(x)$ is a real function on (a,b), and that n is a positive integer such that $f^{(k)}(a+), f^{(k)}(b-)$, $(k\in\{0,1,2,\dots,n\})$ exist.*

(i) *Supposing that $(-1)^{(n)}f^{(n)}(x)$ is increasing on (a,b), then for all $x\in(a,b)$ the following inequality holds:*

$$\begin{gathered}\sum_{k=0}^{n-1}\frac{f^{(k)}(b-)}{k!}(x-b)^k + \frac{1}{(a-b)^n}\left(f(a+) - \sum_{k=0}^{n-1}\frac{(a-b)^k f^{(k)}(b-)}{k!}\right)(x-b)^n \\ < f(x) < \sum_{k=0}^{n}\frac{f^{(k)}(b-)}{k!}(x-b)^k.\end{gathered} \tag{5}$$

Furthermore, if $(-1)^n f^{(n)}(x)$ is decreasing on (a,b), then the reversed inequality of (5) holds.

(ii) *Supposing that* $f^{(n)}(x)$ *is increasing on* (a,b)*, then for all* $x\in(a,b)$ *the following inequality also holds:*

$$\begin{aligned}(6)\qquad &\sum_{k=0}^{n}\frac{f^{(k)}(a+)}{k!}(x-a)^k < f(x) <\\ &< \sum_{k=0}^{n-1}\frac{f^{(k)}(a+)}{k!}(x-a)^k + \frac{1}{(b-a)^n}\left(f(b-)-\sum_{k=0}^{n-1}\frac{(b-a)^k f^{(k)}(a+)}{k!}\right)(x-a)^n.\end{aligned}$$

Furthermore, if $f^{(n)}(x)$ *is decreasing on* (a,b)*, then the reversed inequality of* (6) *holds.*

Remark 1 *Note that inequalities* (5) *and* (6) *hold for* $n\in\mathbb{N}$ *as well as for* $n=0$*. Here, and throughout this paper, a sum where the upper bound of the summation is lower than the lower bound of the summation, is understood to be zero.*

The following Theorem, which is a consequence of Theorem WD, was proved in [14].

Theorem 2 ([14], Theorem 1) *Let the function* $f:(a,b)\longrightarrow\mathbb{R}$ *have the following power series expansion:*

$$(7)\qquad f(x)=\sum_{k=0}^{\infty}c_k(x-a)^k$$

for $x\in(a,b)$*, where the sequence of coefficients* $\{c_k\}_{k\in\mathbb{N}_0}$ *has a finite number of non-positive members and their indices are in the set* $J=\{j_0,\dots,j_\ell\}$*.*

Then, for the function

$$(8)\qquad F(x)=f(x)-\sum_{i=0}^{\ell}c_{j_i}(x-a)^{j_i}=\sum_{k\in\mathbb{N}_0\setminus J}c_k(x-a)^k,$$

and the sequence $\{C_k\}_{k\in\mathbb{N}_0}$ *of the non-negative coefficients defined by:*

$$(9)\qquad C_k=\begin{cases}c_k & : \; c_k>0,\\ 0 & : \; c_k\le 0;\end{cases}$$

holds that:

$$(10)\qquad F(x)=\sum_{k=0}^{\infty}C_k(x-a)^k,$$

for every $x\in(a,b)$*.*

It is also $F^{(k)}(a+)=k!\,C_k$ *and the following inequalities hold:*

$$\begin{aligned}(11)\qquad &\sum_{k=0}^{n}C_k(x-a)^k < F(x) <\\ &< \sum_{k=0}^{n-1}C_k(x-a)^k+\frac{1}{(b-a)^n}\left(F(b-)-\sum_{k=0}^{n-1}C_k(b-a)^k\right)(x-a)^n,\end{aligned}$$

for every $x\in(a,b)$ *and* $n\in\mathbb{N}_0$*, i.e.*

(12)

$$\sum_{k=0}^{m} C_k(x-a)^k + \sum_{i=0}^{\ell} c_{j_i}(x-a)^{j_i} \; < \; f(x) \; <$$

$$< \sum_{k=0}^{m-1} C_k(x-a)^k + \sum_{i=0}^{\ell} c_{j_i}(x-a)^{j_i} + \frac{(x-a)^m}{(b-a)^m}\left(f(b-) - \sum_{k=0}^{m-1} C_k(b-a)^k - \sum_{i=0}^{\ell} c_{j_i}(b-a)^{j_i}\right)$$

for every $x \in (a,b)$ *and* $m > max\{j_0, \ldots, j_\ell\}$.

3 Main results

3.1 Inequalities with constants in the exponents

First, we consider a connection between the number of zeros of a real analytic function and some properties of its derivatives. It is well known that the zeros of a non-constant analytic function are isolated [4], see also [3] and [12].

We prove the following assertion:

Theorem 3 *Let* $f : (0,c) \longrightarrow \mathbb{R}$ *be real analytic function such that* $f^{(k)}(x) > 0$ *for* $x \in (0,c)$ *and* $k = m, m+1, \ldots$, *(for some* $m \in \mathbb{N}$*).*

If the following conditions hold:

1) *there is a right neighbourhood of zero in which the following inequalities hold true:* $f(x) < 0$, $f'(x) < 0$, $\ldots$, $f^{(m-1)}(x) < 0$,

and

2) $f(c_-) > 0, f'(c_-) > 0, \ldots, f^{(m-1)}(c_-) > 0$,

then there exists exactly one zero $x_0 \in (0,c)$ *of the function* f.

Proof. As $f^{(m)}(x) > 0$ for $x \in (0,c)$, it follows that $f^{(m-1)}(x)$ is monotonically increasing function for $x \in (0,c)$. Based on conditions 1) and 2), we conclude that there exists exactly one zero $x_{m-1} \in (0,c)$ of the function $f^{(m-1)}(x)$. Next, we can conclude that function $f^{(m-2)}(x)$ is monotonically decreasing for $x \in (0, x_{m-1})$ and monotonically increasing for $x \in (x_{m-1}, c)$. It is clear that function $f^{(m-2)}(x)$ has exactly one minimum in the interval $(0,c)$ at point x_{m-1} and $f^{(m-2)}(x_{m-1}) < 0$. On the basis of conditions 2), it follows that function $f^{(m-2)}(x)$ has exactly one root x_{m-2} on the interval $(0,c)$ and $x_{m-2} \in (x_{m-1}, c)$.

By repeating the described procedure, we get the assertion given in the theorem. □

Let us consider the family of functions

(13)
$$f_a(x) = a \ln \frac{\sin x}{x} - 2 \ln \cos \frac{x}{2},$$

for $x \in (0,\pi)$ and parameter $a \in (1, +\infty)$.

Obviously, the following equivalence is true:

(14)
$$a_1 < a \Longleftrightarrow f_a(x) < f_{a_1}(x),$$

for $a, a_1 > 1$ and $x \in (0,\pi)$.

Thus:

$$\frac{3}{2} < a \Longleftrightarrow f_a(x) < f_{\frac{3}{2}}(x), \qquad \text{for } x \in (0,\pi). \tag{15}$$

Based on the power series expansions (3) and (4), we have:

$$f_a(x) = \sum_{k=1}^{\infty} E_k\, x^{2k} \tag{16}$$

for $a > 1$ and $x \in (0, \pi)$, where

$$E_k = \frac{\big((2-a)\,4^k - 2\big)|\mathbf{B}_{2k}|}{2k \cdot (2k)!} \quad (k \in \mathbb{N}). \tag{17}$$

For $a = \frac{3}{2}$, it is true that $E_1 = 0$ and $E_k > 0$ for $k = 2, 3, \ldots$. Thus, from (16), we have

$$f_{\frac{3}{2}}(x) > 0 \qquad \text{for } x \in (0,\pi),$$

and consequently the following theorem holds:

Theorem 4 *The following inequalities hold true, for every $x \in (0, \pi)$:*

$$\cos^2\frac{x}{2} \le \left(\frac{\sin x}{x}\right)^{\frac{3}{2}} \le \frac{\sin x}{x}.$$

As the inequality

$$\left(\frac{\sin x}{x}\right)^{\frac{3}{2}} \le \left(\frac{\sin x}{x}\right)^{a}$$

holds for $x \in (0, \pi)$ and $a \in \left(1, \frac{3}{2}\right]$, the previous theorem can be thought of as a new proof of Statement 1.

Consider now the family of functions $f_a(x) = a \ln \frac{\sin x}{x} - 2 \ln \cos \frac{x}{2}$, for $x \in (0, \pi)$ and parameter $a > \frac{3}{2}$.

It easy to check that for the sequence $\{\alpha_k\}_{k \in \mathbb{N}}$ where

$$\alpha_k = 2 - \frac{2}{4^k}. \tag{18}$$

the following equivalences are true:

$$\begin{aligned} &a = \alpha_k \iff E_k = 0, \\ &a \in (\alpha_k, \alpha_{k+1}) \iff (\forall i \in \{1, 2, \ldots, k\})\, E_i < 0 \ \wedge\ (\forall i > k)\, E_i > 0. \end{aligned} \tag{19}$$

Let us now consider the function $\mathfrak{m} \colon \left[\frac{3}{2}, 2\right) \longrightarrow \mathbb{N}_0$ defined by:

$$\mathfrak{m}(a) = k \quad \text{if and only if} \quad a \in (\alpha_k, \alpha_{k+1}]\,. \tag{20}$$

It is not difficult to check that $\lim\limits_{a \to 2_-} \mathfrak{m}(a) = +\infty$, while for a fixed $a \in \left(\frac{3}{2}, 2\right)$ the number of negative elements of the sequence $\{E_k\}_{k \in \mathbb{N}}$ is $\mathfrak{m}(a)$ and their indices are in the set $\{1, \ldots, \mathfrak{m}(a)\}$. For this reason, we distinguish two cases $a \in \left(\frac{3}{2}, 2\right)$ or $a \ge 2$.

As for the parameter $a = 2$ and $x \in (0, \pi)$ we have:

$$\left(\frac{\sin x}{x}\right)^2 \le \cos^2\frac{x}{2} \iff \sin^2\frac{x}{2} \le \left(\frac{x}{2}\right)^2,$$

while for $a > 2$ and $x \in (0, \pi)$ we have:

$$\left(\frac{\sin x}{x}\right)^a \le \left(\frac{\sin x}{x}\right)^2.$$

Hence, we have proved the following theorem:

Theorem 5 *For every $a \ge 2$ and every $x \in (0, \pi)$ the following inequality holds true*:

$$\left(\frac{\sin x}{x}\right)^a \le \cos^2\frac{x}{2}. \tag{21}$$

Consider now the case when the parameter $a \in \left(\frac{3}{2}, 2\right)$. As noted above, for any fixed $a \in \left(\frac{3}{2}, 2\right)$ there is a finite number of negative coefficients in the power series expansion (17), so it is possible to apply Theorem 2.

According to Theorem 2, the following inequalities hold:

$$\begin{aligned} &\sum_{k=\mathfrak{m}(a)+1}^{n} E_k x^k + \sum_{i=0}^{\mathfrak{m}(a)-1} E_i x^i < \\ &< f_a(x) < \\ &< \left(f_a(c-) - \sum_{k=\mathfrak{m}(a)+1}^{n-1} E_k c^k - \sum_{i=0}^{\mathfrak{m}(a)-1} E_i c^i\right)\frac{x^n}{c^n} + \sum_{k=\mathfrak{m}(a)+1}^{n-1} E_k x^k + \sum_{i=0}^{\mathfrak{m}(a)-1} E_i x^i, \end{aligned} \tag{22}$$

for every $x \in (0, c)$, $c \in (0, \pi)$, $n > \mathfrak{m}(a) + 1$ and $a \in \left(\frac{3}{2}, 2\right)$.

The family of functions $f_a(x)$, for $x \in (0, \pi)$ and $a \in \left(\frac{3}{2}, 2\right)$, satisfy the conditions 1) and 2) of Theorem 3, as we prove in the following Lemma:

Lemma 1 *Consider the family of functions $f_a(x) = a \ln\frac{\sin x}{x} - 2\ln\cos\frac{x}{2}$ for $x \in (0, \pi)$ and parameter $a \in \left(\frac{3}{2}, 2\right)$. Let $m = \mathfrak{m}(a)$, where $\mathfrak{m}(a)$ is defined as in* (20).

Then, it is true that $\frac{d^k}{dx^k} f_a(x) > 0$ for $k = m, m+1, \ldots$ and $x \in (0, \pi)$, and the following assertions hold true:

1) *There is a right neighbourhood of zero in which the following inequalities hold true*: $f_a(x) < 0,\ \frac{d}{dx} f_a(x) < 0,\ \ldots,\ \frac{d^{m-1}}{dx^{m-1}} f_a(x) < 0,$

2) $f_a(\pi_-) > 0,\ \frac{d}{dx} f_a(\pi_-) > 0, \ldots, \frac{d^{m-1}}{dx^{m-1}} f_a(\pi_-) > 0.$

Proof. Let us recall that for any fixed $a \in \left(\frac{3}{2}, 2\right)$ there is a finite number of negative coefficients in the power series expansion (17). Also, we have:

$$\left(\frac{d}{dx} f_a\right)(x) = a\left(\cot x - \frac{1}{x}\right) + \tan\frac{x}{2}.$$

For the derivations of the function $f_a(x)$ in the left neighborhood of π, it is enough to observe the following:

$$\left(\frac{d}{dx} f_a\right)(\pi - x) = a\left(-\cot x - \frac{1}{\pi - x}\right) + \cot\frac{x}{2} = \frac{2-a}{x} - \frac{a}{\pi} + \left(a\left(\frac{1}{3} - \frac{1}{\pi^2}\right) - \frac{1}{6}\right)x + \ldots .$$

From this, the conclusions of the lemma can be directly derived. □

Thus, for every $a \in \left(\frac{3}{2}, 2\right)$, the corresponding function $f_a(x) = a \ln\frac{\sin x}{x} - 2\ln\cos\frac{x}{2}$ has exactly one zero on the interval $(0, \pi)$. Let us denote it by x_a.

The following Theorem is a direct consequence of these considerations.

Theorem 6 *For every* $a \in \left(\frac{3}{2}, 2\right)$, *and every* $x \in (0, x_a]$, *where* $0 < x_a < \pi$, *the following inequality holds true:*

$$\left(\frac{\sin x}{x}\right)^a \le \cos^2\frac{x}{2}. \tag{23}$$

For the selected discrete values of $a \in \left(\frac{3}{2}, 2\right)$, the zeros x_a of the corresponding functions $f_a(x)$ are shown in Table 1. Although the values x_a can be obtained by any numerical method, the following remark can also be used to locate them.

Remark 7 *For a fixed* $a \in \left(\frac{3}{2}, 2\right)$, *select* $n > \mathfrak{m}(a) + 1$ *and consider inequalities* (22). *Denote the corresponding polynomials on the left-hand side and the right-hand side of* (22) *by* $P_L(x)$ *and* $P_R(x)$, *respectively. These polynomials are of negative sign in a right neighborhood of zero (see* [12], *Theorem* 2.5.), *and they have positive leading coefficients. Then, the root* x_a *of the equation* $f_a(x) = 0$ *is always localized between the smallest positive root of the equation* $P_L(x) = 0$ *and the smallest positive root of the equation* $P_R(x) = 0$.

3.2 Inequalities with the polynomial exponents

In this subsection we propose and prove a new double-sided inequality involving the sinc function with polynomial exponents.

To be more specific, we find two polynomials of the second degree which, when placed in the exponent of the sinc function, give an upper and a lower bound for $\cos^2\frac{x}{2}$.

Theorem 8 *For every* $x \in (0, 3.1)$ *the following double-sided inequality holds*:

$$\left(\frac{\sin x}{x}\right)^{p_1(x)} < \cos^2\frac{x}{2} < \left(\frac{\sin x}{x}\right)^{p_2(x)}, \tag{24}$$

where $p_1(x) = \frac{3}{2} + \frac{x^2}{2\pi^2}$ *and* $p_2(x) = \frac{3}{2} + \frac{x^2}{80}$.

Proof. Consider the equivalent form of the inequality (24):

$$p_1(x)\ln\frac{\sin x}{x} < 2\ln\cos\frac{x}{2} < p_2(x)\ln\frac{\sin x}{x}.$$

Now, let us introduce the following notation:

$$G_i(x) = p_i(x)\ln\frac{\sin x}{x} - 2\ln\cos\frac{x}{2},$$

for $i = 1, 2$.

Based on the Theorem WD, from (3) we obtain:

$$\begin{aligned}&-\sum_{k=1}^{m-1}\frac{2^{2k-1}|B_{2k}|}{k(2k)!}x^{2k} + \left(\frac{1}{c}\right)^{2m}\left(\ln\frac{\sin c}{c} - \sum_{k=1}^{m-1}\frac{2^{2k-1}|B_{2k}|}{k(2k)!}c^{2k}\right)x^{2m} < \\ &< \ln\frac{\sin x}{x} < -\sum_{k=1}^{n}\frac{2^{2k-1}|B_{2k}|}{k(2k)!}x^{2k},\end{aligned} \tag{25}$$

for $x\in(0,\pi)$ where $n, m\in\mathbb{N}$, $m, n \geq 2$.

Based on the Theorem WD, from (4) we obtain:

$$\begin{aligned}&-\sum_{k=1}^{m-1}\frac{2^{2k-1}(2^{2k}-1)|B_{2k}|}{k(2k)!}x^{2k} + \left(\frac{1}{c}\right)^{2m}\left(\ln\cos c - \sum_{k=1}^{m-1}\frac{2^{2k-1}(2^{2k}-1)|B_{2k}|}{k(2k)!}c^{2k}\right)x^{2m} < \\ &< \ln\cos x < -\sum_{k=1}^{n}\frac{2^{2k-1}(2^{2k}-1)|B_{2k}|}{k(2k)!}x^{2k},\end{aligned} \tag{26}$$

for $x\in(0,c)$ and where $0<c<\dfrac{\pi}{2}$, $n, m\in\mathbb{N}$, $m, n \geq 2$, i.e.

$$\begin{aligned}&\sum_{k=1}^{n}\frac{(2^{2k}-1)|B_{2k}|}{2k(2k)!}x^{2k} < -\ln\cos\frac{x}{2} < \\ &<\sum_{k=1}^{m-1}\frac{(2^{2k}-1)|B_{2k}|}{2k(2k)!}x^{2k} - \left(\frac{2}{c}\right)^{2m}\left(\ln\cos\frac{c}{2} - \sum_{k=1}^{m-1}\frac{(2^{2k}-1)|B_{2k}|}{2k(2k)!}c^{2k}\right)x^{2m},\end{aligned} \tag{27}$$

for $x\in(0,c)$ and $0<c<\pi$, $n, m\in\mathbb{N}$, $m, n \geq 2$.

Now, let us introduce the notation:

$$\begin{aligned}H_1(x, m_1, n_1, c_1) &= -p_1(x)\sum_{k=1}^{m_1-1}\tfrac{2^{2k-1}|B_{2k}|}{k(2k)!}x^{2k} - \\ &-2\left(-\sum_{k=1}^{m_1-1}\tfrac{\left(2^{2k}-1\right)|B_{2k}|}{2k(2k)!}x^{2k} + \tfrac{1}{c_1^{2m_1}}\left(\ln\tfrac{c_1}{2} + \sum_{k=1}^{n_1-1}\tfrac{\left(2^{2k}-1\right)|B_{2k}|}{2k(2k)!}c_1^{2k}\right)x^{2m_1}\right),\end{aligned}$$

for $m_1, n_1 \in \mathbb{N}$, $m_1, n_1 \geq 2$, $c_1 \in (0, \pi)$, and $x \in (0, c_1)$.

$$\begin{aligned}H_2(x, m_2, n_2, c_2) = p_2(x)&\left(-\sum_{k=1}^{m_2-1}\tfrac{2^{2k-1}|B_{2k}|}{k(2k)!}x^{2k} + \tfrac{1}{c_2^{2m_2}}\left(\ln\tfrac{\sin c_2}{c_2} + \sum_{k=1}^{m_2-1}\tfrac{2^{2k-1}|B_{2k}|}{k(2k)!}c_2^{2k}\right)x^{2m_2}\right) + \\ &+2\sum_{k=1}^{n_2}\tfrac{\left(2^{2k}-1\right)|B_{2k}|}{2k(2k)!}x^{2k},\end{aligned}$$

for $m_2, n_2 \in \mathbb{N}$, $m_2, n_2 \geq 2$, $c_2 \in (0, \pi)$, and $x \in (0, c_2)$.

Based on the inequalities (25) and (27) the following holds true:

$$G_1(x) < H_1(x, m_1, n_1, c_1),$$

$$G_2(x) > H_2(x, m_2, n_2, c_2),$$

for $m_1, n_1, m_2, n_2 \in \mathbb{N}$ and $c_1, c_2 \in (0, \pi)$.

For $c_1 = c_2 = 3.1$, $m_1 = 25$ and $n_1 = 10$ and for $m_2 = 13$ and $n_2 = 27$, it is easy to prove that $H_1(x, m_1, n_1, c_1) < 0$ and $H_2(x, m_2, n_2, c_2) > 0$, for every $x \in (0, c_1)$.

Hence, we conclude that $G_1(x) < 0$ and $G_2(x) > 0$ for every $x \in (0, 3.1)$, and the double-sided inequality (24) holds. □

Remark 9 *Note that this method can be used to prove that the inequality* (24) *of Theorem* 8 *holds on any interval* $(0, c)$ *where* $c \in (0, \pi)$*, but the degrees of the polynomials* H_1 *and* H_2 *get larger as* c *approaches* π.

3.3 Constant vs. polynomial exponents

Let us observe the inequalities in Theorem 6 and Theorem 8, inequality (24), containing constants and polynomials in the exponents, respectively.

A question of establishing a relation between these functions, with different types of exponents, comes up naturally. The following theorem addresses this question.

Theorem 10 *For every* $a \in \left(\frac{3}{2}, 2\right)$ *and every* $x \in (0, m_a)$*, where* $m_a = \sqrt{2\pi^2\left(a - \frac{3}{2}\right)}$*, the following double-sided inequality holds*:

(28)
$$\left(\frac{\sin x}{x}\right)^a < \left(\frac{\sin x}{x}\right)^{\frac{3}{2} + \frac{x^2}{2\pi^2}} < \cos^2 \frac{x}{2}.$$

Proof. Let $a = \frac{3}{2} + \varepsilon$, $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $x > 0$. Then:

$$\left(\tfrac{3}{2} + \tfrac{x^2}{2\pi^2}\right) \ln \tfrac{\sin x}{x} - a \ln \tfrac{\sin x}{x} = \left(\tfrac{3}{2} + \tfrac{x^2}{2\pi^2}\right) \ln \tfrac{\sin x}{x} - \left(\tfrac{3}{2} + \varepsilon\right) \ln \tfrac{\sin x}{x} =$$

$$= \left(\tfrac{x^2}{2\pi^2} - \varepsilon\right) \ln \tfrac{\sin x}{x} = \tfrac{1}{2\pi^2}\left(x - \sqrt{2\pi^2\varepsilon}\right)\left(x + \sqrt{2\pi^2\varepsilon}\right) \ln \tfrac{\sin x}{x}.$$

Now we have:

$$x \in \left(0, \sqrt{2\pi^2\varepsilon}\right) \iff \left(\frac{3}{2} + \alpha x^2\right) \ln \frac{\sin x}{x} > \left(\frac{3}{2} + \varepsilon\right) \ln \frac{\sin x}{x} \iff \left(\frac{\sin x}{x}\right)^{\frac{3}{2} + \frac{x^2}{2\pi^2}} > \left(\frac{\sin x}{x}\right)^{\frac{3}{2} + \varepsilon}.$$

Hence, applying Theorem 8, the double-sided inequality (28) holds for every $a \in \left(\frac{3}{2}, 2\right)$ and every $x \in (0, m_a)$. □

Now, in Table 1, we show the values x_a and m_a for some specified $a \in \left(\dfrac{3}{2}, 2\right)$:

a	1.501	1.502	1.503	1.504	1.505	1.506	1.507	1.508	1.509	1.510
x_a	0.282...	0.398...	0.487...	0.561...	0.626...	0.685...	0.738...	0.788...	0.834...	0.878...
m_a	0.140...	0.198...	0.243...	0.280...	0.314...	0.344...	0.371...	0.397...	0.421...	0.444...

a	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59	1.60	1.65
x_a	1.220...	1.468...	1.666...	1.831...	1.973...	2.096...	2.205...	2.302...	2.302...	2.302...
m_a	0.628...	0.769...	0.888...	0.993...	1.088...	1.175...	1.256...	1.256...	1.256...	1.256...

a	1.70	1.75	1.80	1.85	1.90	1.92	1.94	1.96	1.98	1.9999
x_a	2.911...	3.034...	3.103...	3.133...	3.141...	3.141...	3.141...	3.141...	3.141...	3.141...
m_a	1.986...	2.221...	2.433...	2.628...	2.809...	2.879...	2.947...	3.013...	3.087...	3.141...

Table 1

Remark 11 *Note that Theorem* 10 *represents another proof of the following assertion from* [1]:

$$\big(\forall a \in (3/2, 2)\big)\big(\exists \delta > 0\big)\big(\forall x \in (0, \delta)\big)\left(\frac{\sin x}{x}\right)^a < \cos^2 \frac{x}{2}.$$

4 Conclusion

In this paper, using the power series expansions and the application of the Wu-Debnath theorem, we proved that the inequality (2) holds for $a = \dfrac{3}{2}$. At the same time, this proof represents a new short proof of Statement 1.

We analyzed the cases $a \in \left(\dfrac{3}{2}, 2\right)$ and $a \geq 2$ and we prove the corresponding inequalities. We introduced and prove a new double-sided inequality of similar type involving polynomial exponents. Also, we established a relation between the cases of the constant and of the polynomial exponent.

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