

A FOOTNOTE TO THE CRISIS IN CONTEMPORARY MATHEMATICS

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ABSTRACT. We examine the preparation and context of the paper “The Crisis in Contemporary Mathematics” by Errett Bishop, published 1975 in *Historia Mathematica*. Bishop tried to moderate the differences between Hilbert and Brouwer with respect to the interpretation of logical connectives and quantifiers. He also commented on Robinson’s *Non-standard Analysis*, fearing that it might lead to what he referred to as ‘a debasement of meaning.’ The ‘debasement’ comment can already be found in a draft version of Bishop’s lecture, but not in the audio file of the actual lecture of 1974. We elucidate the context of the ‘debasement’ comment and its relation to Bishop’s position vis-a-vis the Law of Excluded Middle.

Keywords: Constructive mathematics; Robinson’s framework; infinitesimal analysis.

1. INTRODUCTION

We will compare three extant versions of Errett Bishop’s 1974 lecture entitled “The crisis in contemporary mathematics.” Errett Bishop (1928–1983) delivered a plenary lecture in the session on the Foundations of Mathematics of the Workshop on the Evolution of Modern Mathematics organized by the American Academy of Arts and Sciences (AAAS) in 1974.

Three versions of the lecture are extant. The first one is a 2-page initial draft of the lecture [Bishop 1974a]. The second is an audio recording [Bishop 1974b] of the lecture delivered on 9 august 1974. The third is the published version of the lecture in *Historia Mathematica* [Bishop 1975].

2. THE THREE VERSIONS

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2.1. The draft version. The draft of the lecture already sets out its main theme, namely the Brouwer–Hilbert differences on the meaning of logical connectives and quantifiers:

What I am recommending, and I do not know whether the possibility occurred [sic] to Hilbert, is that we accept Brouwer’s definitions of “or”, “there exists”, and all the other connectives and quantifiers, without damaging the paradise that Hilbert wished to preserve. [Bishop 1974a, p. 1]

There follows a paragraph concerning the work of Abraham Robinson [Robinson 1966] on infinitesimal analysis¹ and of H. Jerome Keisler [Keisler 1971] on infinitesimal calculus:

A more recent attempt at mathematics by formal finesse is non-standard analysis. I gather that it has met with some degree of success, whether at the expense of giving significantly less meaningful proofs I do not know. My interest in non-standard analysis is that attempts are being made to introduce it into calculus courses. It is difficult to believe that debasement of meaning could be carried so far. [Bishop 1974a, p. 2].

Bishop goes on to discuss recursive function theory, and then comments on applications in science:

The reason that mathematics is so successful in the physical sciences is not clear. To Hermann Weyl, the utility of mathematics extended even to that part of mathematics that was not inherently computational. Although I hesitate to disagree with such an authority, my own impression is that the opposite is true. It would be interesting and worthwhile to settle this point. [Bishop 1974a, p. 2].

2.2. The published version. Bishop’s lecture was published in *Historia Mathematica* in 1975 as part of the proceedings of the AAAS workshop. The 11-page published version [Bishop 1975] contains an expanded discussion of Brouwer–Hilbert disagreements over connectives and quantifiers, followed by a constructive analysis of the classical result that a function of bounded variation is differentiable almost everywhere. Bishop’s conclusion, echoing the corresponding remarks in the draft version, is the following:

¹For a historical analysis of the genesis of Robinson’s theory see [Dauben 1995].

In a way, the imaginary dialogue that I presented here might be regarded as a historical investigation if you believe as I do that it shows how two titanic figures such as these might have reached an accommodation that would have changed the course of mathematics in a profound way, had they spoken to each other with less emotion and more concern for understanding each other. Instead, Hilbert tried to show that it was all right to neglect computational meaning, because it could ultimately be recovered by an elaborate formal analysis of the techniques of proof. This artificial program failed.² [Bishop 1975, p. 513].

Immediately following, in the published version, is the ‘debasement’ passage on Robinson and Keisler already found in the draft version (see Section 2.1). Bishop goes on to make some comments on recursive function theory identical to those found in the draft version, and concludes:

That is all I want to say about pure mathematics. I would like to consider next another very interesting question that has occupied many people: what does the constructivist point of view entail for the applications of mathematics to physics? My own feeling is that the only reason mathematics is applicable is because of its inherent constructive content. By making that constructive content explicit, you can only make mathematics more applicable, Hermann Weyl seems to have had an opposite opinion. For him, the utility of mathematics extended even to that part of mathematics that was not inherently computational. I hesitate to disagree with Weyl, but I do. It is a very serious subject for investigation; it would be interesting and worthwhile to settle this point. [Bishop 1975, p. 514]

Following the published version of Bishop’s lecture in [Bishop 1975] is an extended exchange among mathematicians and philosophers present at the workshop, containing a number of reactions to Bishop’s ideas by

²Bishop’s negative appraisal notwithstanding, the proof mining program spearheaded by Ulrich Kohlenbach (see e.g., [Kohlenbach 2008]) has been successful at extracting computational content from proofs in classical mathematics, going as far as improving known results in the literature (classical or constructive). The proof mining program is the archetype of an “elaborate formal analysis of the techniques of proof” which has successfully produced a plethora of natural results.

Moore, Kline, Mackey, Birkhoff, Freudenthal, Dieudonné, Abhyankar, Kahane, and Dreben. Their responses don't include any reaction to Bishop's 'debasement' comments on Robinson and Keisler and it will soon become clear why.

2.3. The audio version. The recorded lecture lasted 47 minutes (including an introduction by Birkhoff; not including responses from the audience). At minutes 43 and 44 one finds the following comments by Bishop:

In a way the imaginary dialog I presented here might be regarded as a historical investigation if you believe as I do that it shows how these titanic figures may have reached an accommodation that would have changed the course of mathematics in a somewhat profound way if they had spoken to each other at a less emotional level. Now that's all I want to say about pure mathematics. I want to read a little bit if I still have it. I don't have it, well, I'll try to remember it then. [Bishop 1974b, minute 43]

Note that Bishop *first* declared that “that's all I want to say about pure mathematics” and *then* searched unsuccessfully for his notes to present the next segment of his lecture. This indicates that the misplaced notes did not deal with pure mathematics but rather with applications to physics, which as he said he delivered from memory. *Immediately following* the above in the audio version is the following comment:

One very interesting question... is what this would mean, what this would entail for the applications of mathematics to physics... My own feeling is that the only reason mathematics is applicable is because of its inherent constructive content. And by making that constructive context explicit we can only make mathematics more applicable. [Bishop 1974b, minute 44]

The passage at minute 43 closely parallels the discussion of the Brouwer–Hilbert differences found in both [Bishop 1974a] and [Bishop 1975]. The passage at minute 44 closely parallels the discussion of applications in both [Bishop 1974a] and [Bishop 1975].

However, the intermediate 'debasement' passage that appears in identical form in the preliminary draft version [Bishop 1974a] and the published version [Bishop 1975] is *absent* from the audio recording of the lecture. Thus, Bishop never made those comments in the actual

lecture, though he was apparently planning to present them according to [Bishop 1974a], and ultimately did publish them in [Bishop 1975].

One can only speculate concerning the reasons that may have led Bishop to suppress the ‘debasement’ comment when faced with an actual audience on 9 august 1974, or what he meant exactly when he declared, at the exact spot of the omission, that “that’s all I want to say about pure mathematics.” However, a reader of the published version who may have been surprised or disappointed not to find any reaction to the ‘debasement’ comment on the part of the audience that included a number of logicians (see end of Section 2.2), will now have an explanation for their silence.

3. RECEPTION AND MEANING

The reception of Bishop’s program among mainstream mathematicians has ranged from lukewarm to sceptical. Thus, Jean Dieudonné wrote:

L’Auteur expose une défense des points de vue de L. E. J. Brouwer (dont il se déclare le disciple) sur la “signification” des mathématiques et les tabous qui en résulteraient si ces points de vue étaient adoptés. Le terme de “crise” qu’il emploie ne semble guère justifié, car un tel mot désigne un conflit ayant un certain caractère d’acuité, et l’Auteur reconnaît que pour la très grande majorité des mathématiciens les questions soulevées par Brouwer n’ont pas d’intérêt.³ [Dieudonné 1975]

A small group of followers has successfully pursued Bishop’s program of what he defined as constructive mathematics. The most notable of his disciples are Douglas Bridges and Fred Richman who have published widely in constructive mathematics; see e.g., [Bridges–Richman 1987]. There have also been attempts to bridge a perceived gap between constructive mathematics and Robinson’s framework for infinitesimals; see e.g., [Schuster et al. 2001].

What Bishop meant by ‘meaning’ is well known: meaningful mathematics has computational content. Namely, according to Bishop to say a mathematical object exists, is to provide a construction for it.

³Translation: “The Author presents a defense of the viewpoints of L. E. J. Brouwer (of whom he professes to be the disciple) on the ‘meaning’ of mathematics and the taboos that would result should these viewpoints be adopted. The term ‘crisis’ he employs hardly seems justified since such a word refers to a conflict possessing a certain acuteness, and the Author acknowledges that for the very great majority of mathematicians the issues raised by Brouwer have no interest.”

The other logical symbols have similar computational interpretations, i.e., essentially as in intuitionistic logic as pioneered by Brouwer and Heyting; see [Heyting 1956]. In particular Bishop’s program called for redevelopment of mathematics based on a computational interpretation in which the integers constitute the foundation of everything:⁴

Everything attaches itself to *number*, and every mathematical statement ultimately expresses the fact that if we perform certain computations within the set of positive *integers*, we shall get certain results. (A constructivist manifesto in [Bishop 1967], emphasis ours)

According to Bishop’s reading, the Law of Excluded Middle (LEM) is then meaningless as it does not have any computational content,⁵ and is therefore rejected. LEM is the crucial ingredient in a typical proof by contradiction. Such proofs are ubiquitous in modern mathematics based on classical logic. Many mathematicians do recognize that a proof by contradiction sometimes lacks to deliver constructive content in the sense that the entity whose existence is proved in this way often lacks explicit description. Classically trained mathematicians can also relate sympathetically to the normative sentiment that *an existence proof is a construction, not the impossibility of non-existence*.

In this light, Bishop’s use of the phrase “debasement of meaning” should be interpreted as referring to an allegedly fundamental and irreparable absence of numerical content, the latter sketched in the above quote.

It seems very difficult to construct e.g., infinitesimals in terms of ordinary integers (but see [Borovik et al. 2012]), i.e., it seems the former cannot be reduced to the latter in any way acceptable to Bishop. Hence, the very core of Robinson’s framework for infinitesimal analysis deals with objects (seemingly) unacceptable⁶ to Bishop, which is what presumably led him to the ‘debasement’ comment.

⁴Bishop recognized Brouwer’s criticism of classical mathematics, but ultimately deemed intuitionism to be an unsatisfactory answer. Bishop felt the same way about (constructive) recursive mathematics. Bishop actually formulated an informal framework which produces results acceptable in intuitionism, constructive recursive mathematics, and classical mathematics.

⁵The idea of LEM as computationally meaningless is analyzed in [Katz et al. 2014] in the context of a (weak) Brouwerian counterexample to the Extreme Value Theorem.

⁶However, Erik Palmgren and others have established constructive NSA; see e.g., [Palmgren 1998].

Thus, the thrust of Bishop’s critique of Robinson’s framework consisted in alleging that the presence of ideal objects (in particular infinitesimals) in Robinson’s framework entails the absence of meaning (i.e., computational content). Similar sentiments have been expressed by Alain Connes in [Connes et al. 2001].⁷

Recently the Bishop–Connes critique has been challenged. Namely, the presence of ideal objects (in particular infinitesimals) in Robinson’s framework arguably yields the ubiquitous presence of computational content; see e.g., [Sanders 2017], [Sanders 2018].

For instance, the various nonstandard definitions (involving the relation \approx of infinite proximity) of continuity, differentiability, Riemann integration, etc., are actually stand-ins for Bishop’s constructive definitions involving moduli. More generally, in Robinson’s classical framework for infinitesimal analysis, the quantifier “there exists a standard object” has a meaning akin to Bishop’s constructive existential quantifier, while “there exists an object” has no computational content.

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