Wiener Index, Hyper-wiener Index, Harary Index and Hamiltonicity of graphs

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Abstract: In this paper, we discuss the Hamiltonicity of graphs in terms of Wiener index, hyper-Wiener index and Harary index of their quasi-complement or complement. Firstly, we give some sufficient conditions for an balanced bipartite graph with given the minimum degree to be traceable and Hamiltonian, respectively. Secondly, we present some sufficient conditions for a nearly balanced bipartite graph with given the minimum degree to be traceable. Thirdly, we establish some conditions for a graph with given the minimum degree to be traceable and Hamiltonian, respectively. Finally, we provide some conditions for a k-connected graph to be Hamilton-connected and traceable for every vertex, respectively.

Keywords: Wiener index; Hyper-wiener index; Harary index; Hamiltonicity

MR Subject Classifications:

1 Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, denoted by $e(G) = |E(G)|$. The distance between two vertices v_i and v_j of G, denoted by $d_G(v_i, v_j)$, is defined as the minimum length of the paths between v_i and v_j in G. Let $N_G(v)$ be the set of vertices which are adjacent to v in G. The degree of v is denoted by $d_G(v) = |N_G(v)|$, the minimum degree of G is denoted by $\delta(G)$. Let $X \subseteq V(G)$, $G-X$ is the graph obtained from G by deleting all vertices in X. G is called *k-connected* (for $k \in \mathbb{N}$) if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$. Let $G = (X, Y; E)$ be a bipartite graph with two part sets X, Y. If $|X| = |Y|$, $G = (X, Y; E)$ is called an *balanced bipartite graph*. If $|X| = |Y| + 1$, $G = (X, Y; E)$ is called a *nearly balanced bipartite graph*. For two disjoint graphs G_1 and G_2, the union of G_1 and G_2, denoted by $G_1 + G_2$, is defined as $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$; and the join of G_1 and G_2, denoted by $G_1 \vee G_2$, is defined as $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$, and

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$E(G_1 \vee G_2) = E(G_1 + G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. Denote K_n the complete graph on n vertices, O_n the empty graph on n vertices (without edges), $K_{n,m} = O_n \vee O_m$ the complete bipartite graph with two parts having n, m vertices, respectively.

The *complement* of G is denoted by $\overline{G} = (V(\overline{G}), E(\overline{G}))$, where $V(\overline{G}) = V(G)$, $E(\overline{G}) = \{xy : x, y \in V(G), xy \notin E(G)\}$. The *quasi-complement* of $G = (X, Y; E)$ is denoted by $\widehat{G} := (X, Y; E')$, where $E' = \{xy : x \in X, y \in Y, xy \notin E\}$.

The *Wiener index* of a connected graph G, denote by $W(G)$, which is introduced by Wiener [17] in 1947, is defined to be the sum of distances between every pair of vertices in G. That is

$$W(G) = \sum_{v_i, v_j \in V(G)} d_G(v_i, v_j).$$

We denote $D_i(G) = D_G(v_i) = \sum\limits_{v_j \in V(G)} d_G(v_i, v_j)$, then

$$W(G) = \frac{1}{2} \sum_{i=1}^{n} D_i(G).$$

The hyper-Wiener index, as a generalization of the Wiener index, is traditionally denoted by $WW(G)$. The hyper-Wiener index of acyclic graphs was introduced by Milan Randić [16] in 1993 and extended to all connected graphs by Klein et al. [8]. The *hyper-Wiener index* of a connected graph G is defined as

$$WW(G) = \frac{1}{2}\Big(\sum_{v_i, v_j \in V(G)} d_G(v_i, v_j) + \sum_{v_i, v_j \in V(G)} d_G^2(v_i, v_j)\Big).$$

We denote $DD_i(G) = DD_G(v_i) = \sum\limits_{v_j \in V(G)} d_G^2(v_i, v_j)$, then

$$WW(G) = \frac{1}{4} \sum_{i=1}^{n} (D_i(G) + DD_i(G)).$$

The Harary index is also a useful topological index in chemical graph theory and has received much attention during the past decades. This index has been introduced in 1993 by Plavšić et al. [15] and by Ivanciuc et al. [7], independently. For a connected graph G, the *Harary index* of G, denoted by $H(G)$, is defined as

$$H(G) = \sum_{v_i, v_j \in V(G)} \frac{1}{d_G(v_i, v_j)}.$$

We denote $\widetilde{D}_G(v_i) = \sum\limits_{v_j \in V(G)} \frac{1}{d_G(v_i, v_j)}$, then

$$H(G) = \frac{1}{2} \sum_{i=1}^{n} \widetilde{D}_G(v_i).$$

A *Hamiltonian cycle* of the graph G is a cycle which contains all vertex of G. A *Hamiltonian path* of the graph G is a path which contains all vertex of G. The graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and is said to be *traceable* if it contains a Hamiltonian path. If every two vertices of G are connected by a Hamiltonian path, it is said to be *Hamilton-connected*. A graph G is *traceable from a vertex* x if it has a Hamiltonian x-path. A graph is *traceable from every vertex* if it contains a Hamilton path from every vertex. All these concepts belong to Hamiltonicity of graphs. The problem of deciding whether a graph has Hamiltonicity is one of the most difficult classical problems in graph theory.

Recently, some topological indices have been applied to this problem. Up to now, there are some references on the Wiener index, hyper-wiener index and Harary index conditions for a graph to be traceable, Hamiltonian, Hamilton-connected, traceable from every vertex. We refer readers to see [3, 4, 5, 6, 8, 9, 10, 11, 12, 21, 20]. Among them, Hua and Ning [6] give conditions for an balanced bipartite graph to be Hamiltonian in terms of Wiener index and Harary index. Cai et al. [4], in terms of Hyper-Wiener index, give conditions for an balanced bipartite graph to be traceable and Hamiltonian, and a k-connected graphs to be Hamiltonian, respectively. Li [11, 12] gives some conditions for a k-connected graph to be Hamiltonian in terms of Wiener index and Harary index, respectively. Yu et al. [20] give conditions for a k-connected graphs to be Hamilton-connected and traceable for every vertex and a nearly balance bipartite graph to be traceable in terms of Wiener index, Harary index and Hyper-Wiener index, respectively. Especially, Liu et al. [9] [10] give sufficient conditions for a graph to be traceable and Hamiltonian in terms of the Wiener index and Harary index of its complement, and an balanced bipartite graph to be traceable and Hamiltonian in terms of its Wiener index and Harary index of its quasi-complement, respectively. As a continuance of the these results, we also study the similar problems.

In this paper, we discuss the Hamiltonicity of graphs in terms of Wiener index, hyper-wiener index and Harary index of their quasi-complement or complement. In section 2, we present some notations and some lemmas needed in the following. In sections 3-4, we present some conditions for an balanced bipartite graph with given the minimum degree to be traceable and Hamiltion, respectively. In section 5, we give sufficient conditions for a nearly balanced bipartite graph with given the minimum degree to be traceable. In sections 6-7, we present some conditions for a graph with given the minimum degree to be traceable and Hamiltonian, respectively. In sections 8-9, we provide some conditions for a k-connected graph to be Hamilton-connected and traceable for every vertex, respectively.

2 Preliminarie

In this paper, when we mention a bipartite graph, we always fix its partite sets, e.g., $O_{n,m}$ and $O_{m,n}$ are considered as different bipartite graphs, unless $m = n$.

Let G_1, G_2 be two bipartite graphs, with the bipartition X_1, Y_1 and X_2, Y_2, respectively. We use $G_1 \sqcup G_2$ to denote the graph obtained from $G_1 + G_2$ by adding all possible edges between X_1 and Y_2 and possible edges between Y_1 and X_2. We define some classes of graphs as follows:

$$\begin{aligned}
B_n^k &= O_{k,n-k} \sqcup K_{n-k,k}(1 \leq k \leq n/2),\\
C_n^k &= O_{k,n-k} \sqcup K_{n-k-1,k}(1 \leq k \leq n/2),\\
R_n^k &= K_{k,k} + K_{n-k,n-k}(1 \leqslant k \leqslant n/2),\\
Q_n^k &= O_{k+1,n-k} \sqcup K_{n-k-1,k}(1 \leqslant k \leqslant n/2),\\
L_n^k &= K_1 \vee (K_k + K_{n-k-1}), (1 \leqslant k \leqslant (n-1)/2),\\
N_n^k &= K_k \vee (K_{n-2k} + kK_1), (1 \leqslant k \leqslant (n-1)/2),\\
\underline{L}_n^k &= K_{k+1} + K_{n-k-1}, (1 \leqslant k \leqslant (n-2)/2),\\
\underline{N}_n^k &= K_k \vee (K_{n-2k-1} + (k+1)K_1), (1 \leqslant k \leqslant (n-2)/2).
\end{aligned}$$

Note that $e(C_n^k) = n(n-k-1) + k^2$ and C_n^k is not traceable.

LEMMA 2.1 *If $\widehat{G}$ be a connected balanced bipartite graph on $2n$ vertices, then*

$$W(\widehat{G}) \leqslant 2n^3 - 3n^2 + 2n + 2(n-1)e(G).$$

Proof. Let $\widehat{G}$ be the quasi-complement of G. Then

$$\begin{aligned}
W(\widehat{G}) &= \sum_{u,v \in V(\widehat{G})} d_{\widehat{G}}(u,v)\\
&= \frac{1}{2}\sum_{i=1}^{2n} D_i(\widehat{G})\\
&\leqslant \frac{1}{2}(\sum_{i=1}^{2n}(d_{\widehat{G}}(x_i) + (2n-1)(n - d_{\widehat{G}}(x_i)) + (2n-2)(n-1)))\\
&= 4n^3 - 5n^2 + 2n - (n-1)\sum_{i=1}^{2n} d_{\widehat{G}}(x_i)\\
&= 4n^3 - 5n^2 + 2n - (n-1)\sum_{i=1}^{2n}(n - d_G(x_i))\\
&= 2n^3 - 3n^2 + 2n + (n-1)\sum_{i=1}^{2n} d_G(x_i)\\
&= 2n^3 - 3n^2 + 2n + 2(n-1)e(G).
\end{aligned}$$

This completes the proof. ■

Lemma 2.2 *Let $\widehat{G}$ be a connected balanced bipartite graph on $2n$ vertices, then*

$$WW(\widehat{G}) \leqslant 2n^4 - 5n^3 + 5n^2 - n + (2n^2 - n - 1)e(G).$$

Proof. Let $\widehat{G}$ be the quasi-complement of G. Then

$$\begin{aligned}
WW(\widehat{G}) &= \frac{1}{2}\sum_{u,v\in V(\widehat{G})}(d_{\widehat{G}}(u,v) + d^2_{\widehat{G}}(u,v)) \\
&= \frac{1}{4}\sum_{i=1}^{2n}(D_i(\widehat{G}) + DD_i(\widehat{G})) \\
&\leqslant \frac{1}{4}\sum_{i=1}^{2n}(d_{\widehat{G}}(x_i) + (2n-1)(n - d_{\widehat{G}}(x_i)) + (2n-2)(n-1)) \\
&\quad + \frac{1}{4}\sum_{i=1}^{2n}(d_{\widehat{G}}(x_i) + (2n-1)^2(n - d_{\widehat{G}}(x_i)) + (2n-2)^2(n-1)) \\
&= 4n^4 - 6n^3 + 4n^2 - n - (n^2 - \frac{1}{2}n - \frac{1}{2})\sum_{i=1}^{2n} d_{\widehat{G}}(x_i) \\
&= 4n^4 - 6n^3 + 4n^2 - n - (n^2 - \frac{1}{2}n - \frac{1}{2})\sum_{i=1}^{2n}(n - d_G(x_i)) \\
&= 2n^4 - 5n^3 + 5n^2 - n + (2n^2 - n - 1)e(G).
\end{aligned}$$

This completes the proof. ■

Lemma 2.3 *Let $\widehat{G}$ be a connected balanced bipartite graph on $2n$ vertices, then*

$$H(\widehat{G}) \geqslant \frac{4n^3 - n}{4n-2} - \frac{2n-2}{2n-1}e(G).$$

Proof. Let $\widehat{G}$ be the quasi-complement of G. Then

$$\begin{aligned}
H(\widehat{G}) &= \sum_{u,v\in V(\widehat{G})}\frac{1}{d_{\widehat{G}}(u,v)} = \frac{1}{2}\sum_{i=1}^{2n}\widetilde{D}_{\widehat{G}}(v_i) \\
&\geqslant \frac{1}{2}\sum_{i=1}^{2n}(d_{\widehat{G}}(x_i) + \frac{1}{2n-1}(n - d_{\widehat{G}}(x_i)) + \frac{n-1}{(2n-2)}) \\
&= \frac{4n^2 - n}{4n-2} + \frac{n-1}{2n-1}\sum_{i=1}^{2n} d_{\widehat{G}}(x_i)
\end{aligned}$$

$$= \frac{4n^2 - n}{4n-2} + \frac{n-1}{2n-1} \sum_{i=1}^{2n} (n - d_G(x_i))$$
$$= \frac{4n^3 - n}{4n-2} - \frac{n-1}{2n-1} \sum_{i=1}^{2n} d_G(x_i)$$
$$= \frac{4n^3 - n}{4n-2} - \frac{2n-2}{2n-1} e(G).$$

This completes the proof. ■

Lemma 2.4 *Let $\overline{G}$ be a connected graph of order n, then*

$$W(\overline{G}) \leqslant \frac{1}{2} n(n-1) + (n-2)e(G).$$

Proof.

$$W(\overline{G}) = \sum_{u,v \in V(\overline{G})} d_{\overline{G}}(u,v)$$
$$= \frac{1}{2} \sum_{i=1}^{n} D_i(\overline{G})$$
$$\leqslant \frac{1}{2} \sum_{i=1}^{n} (d_{\overline{G}}(x_i) + (n-1)(n-1-d_{\overline{G}}(x_i)))$$
$$= \frac{1}{2} n(n-1)^2 + \frac{1}{2} n(2-n) \sum_{i=1}^{n} d_{\overline{G}}(x_i)$$
$$= \frac{1}{2} n(n-1)^2 - \frac{1}{2} n(n-2) \sum_{i=1}^{n} (n-1-d_G(x_i))$$
$$= \frac{1}{2} n(n-1) + \frac{1}{2}(n-2) \sum_{i=1}^{n} d_G(x_i)$$
$$= \frac{1}{2} n(n-1) + (n-2)e(G).$$

This completes the proof. ■

Lemma 2.5 *Let $\overline{G}$ be a connected graph of order n, then*

$$WW(\overline{G}) \leqslant \frac{1}{2} n(n-1) + \frac{1}{2}(n^2 - n - 2)e(G).$$

Proof.

$$\begin{aligned}
WW(\overline{G}) &= \frac{1}{2}\sum_{u,v\in V(\overline{G})}(d_{\overline{G}}(u,v)+d^2_{\overline{G}}(u,v))\\
&= \frac{1}{4}\sum_{i=1}^{n}(D_i(\overline{G})+DD_i(\overline{G}))\\
&\leqslant \frac{1}{4}\sum_{i=1}^{n}(d_{\overline{G}}(x_i)+(n-1)(n-1-d_{\overline{G}}(x_i)))\\
&+\frac{1}{4}\sum_{i=1}^{n}(d_{\overline{G}}(x_i)+(n-1)^2(n-1-d_{\overline{G}}(x_i)))\\
&= \frac{1}{4}n^2(n-1)^2+\frac{1}{4}(2+n-n^2)\sum_{i=1}^{n}d_{\overline{G}}(x_i)\\
&= \frac{1}{4}n^2(n-1)^2+\frac{1}{4}(2+n-n^2)\sum_{i=1}^{n}(n-1-d_G(x_i))\\
&= \frac{1}{2}n(n-1)+\frac{1}{4}(n^2-n-2)\sum_{i=1}^{n}d_G(x_i)\\
&= \frac{1}{2}n(n-1)+\frac{1}{2}(n^2-n-2)e(G).
\end{aligned}$$

This completes the proof. ■

LEMMA 2.6 *Let $\overline{G}$ be a connected graph of order n, then*

$$H(\overline{G})\geqslant \frac{1}{2}(n^2-n)-\frac{n-2}{n-1}e(G).$$

Proof.

$$\begin{aligned}
H(\overline{G}) &= \sum_{u,v\in V(\overline{G})}\frac{1}{d_{\overline{G}}(u,v)}\\
&= \frac{1}{2}\sum_{i=1}^{n}\widetilde{D}_{\overline{G}}(v_i)\\
&\geqslant \frac{1}{2}\sum_{i=1}^{n}(d_{\overline{G}}(x_i)+\frac{1}{n-1}(n-1-d_{\overline{G}}(x_i)))\\
&= \frac{1}{2}n+\frac{n-2}{2(n-1)}\sum_{i=1}^{n}d_{\overline{G}}(x_i)\\
&= \frac{1}{2}(n^2-n)-\frac{n-2}{2(n-1)}\sum_{i=1}^{n}d_G(x_i)\\
&= \frac{1}{2}(n^2-n)-\frac{n-2}{n-1}e(G).
\end{aligned}$$

This completes the proof. ■

3 Traceable of balanced bipartite Graphs

LEMMA 3.1 *[14] Let k be an integer and G be an balanced bipartite graph on $2n$ vertices. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+3$ and*

$$e(G) > n(n-k-2) + (k+2)^2,$$

then G is traceable unless $G \subseteq Q_n^k$ or $k=1$, $G \subseteq R_n^1$.

THEOREM 3.2 *Let k be an integer and $\widehat{G}$ be a connected balanced bipartite graph on $2n$ vertices. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+3$, and*

$$W(\widehat{G}) > 4n^3 - (2k+9)n^2 + (2k^2+10k+14)n - 2(k+2)^2,$$

then G is traceable unless $k=1$ and $G \subseteq R_n^1$.

Proof. Since $W(\widehat{G}) > 4n^3 - (2k+9)n^2 + (2k^2+10k+14)n - 2(k+2)^2$, and by lemma 2.1, we get $e(G) > n(n-k-2) + (k+2)^2$. By lemma 3.1 we obtain that G is traceable or $G \subseteq Q_n^k$ or $k=1$, $G \subseteq R_n^1$.

If $G \subseteq Q_n^k$. Because $\widehat{G}$ is connected, and $\delta(G) \geqslant k \geqslant 1$, we get $e(G) \leq n(n-k-1) + k(k+1) - (n-k-1) - k < n(n-k-2) + (k+2)^2$, a contradiction.

If $k=1$, $G \subseteq R_n^1$. Because $\widehat{G}$ is connected, and $\delta(G) \geqslant 1$, we get $k=1$ and $G \subseteq R_n^1$.

This completes the proof. ■

THEOREM 3.3 *Let k be an integer and $\widehat{G}$ be a connected balanced bipartite graph on $2n$ vertices. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+3$, and*

$$WW(\widehat{G}) > 4n^4 - (2k+10)n^3 + (2k^2+9k+14)n^2 - (k^2+3k+3)n - (k+2)^2,$$

then G is traceable unless $k=1$ and $G \subseteq R_n^1$.

Proof. Since $WW(\widehat{G}) > 4n^4 - (2k+10)n^3 + (2k^2+9k+14)n^2 - (k^2+3k+3)n - (k+2)^2$, and by lemma 2.2 we get $e(G) > n(n-k-2) + (k+2)^2$. By lemma 3.1 we obtain that G is traceable or $G \subseteq Q_n^k$ or $k=1$, $G \subseteq R_n^1$. By the same discussion as the proof of theorem 3.2, we get the result. ■

THEOREM 3.4 *Let k be an integer and G be a connected balanced bipartite graph on $2n$ vertices. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+3$, and*

$$H(\widehat{G}) < \frac{(4k+12)n^2 - (4k^2+20k-25)n + 4k^2 + 16k + 16}{4n-2},$$

then G is traceable unless $k=1$ and $G \subseteq R_n^1$.

Proof. Since $H(\widehat{G}) < \frac{(4k+12)n^2 - (4k^2+20k-25)n + 4k^2+16k+16}{4n-2}$, and by lemma 2.3, so $e(G) > n(n-k-2) + (k+2)^2$. By lemma 3.1 we obtain that G is traceable or $G \subseteq Q_n^k$ or $k=1$, $G \subseteq R_n^1$. By the same discussion as the proof of theorem 3.2, we get the result. ■

4 Hamiltonian of balanced bipartite Graphs

Lemma 4.1 *[13] Let k be an integer and G be a balanced bipartite graph on $2n$ vertices. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+3$ and*

$$e(G) > n(n-k-1) + (k+1)^2,$$

then G is hamiltonian unless $G \subseteq B_n^k$.

Theorem 4.2 *Let k be an integer and $\widehat{G}$ be a connected balanced bipartite graph on $2n$ vertices. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+3$,*

$$W(\widehat{G}) > 4n^3 - (2k+7)n^2 + (2k+2(k+1)^2+4)n - 2(k+1)^2,$$

then G is hamiltonian.

Proof. Since $W(\widehat{G}) > 4n^3 - (2k+7)n^2 + (2k+2(k+1)^2+4)n - 2(k+1)^2$, and by lemma 2.1, thus $e(G) > n(n-k-1) + (k+1)^2$. By lemma 4.1, G is hamiltonian or $G \subseteq B_n^k$.

If $G \subseteq B_n^k$. Because $\widehat{G}$ is connected, and $\delta(G) \geqslant k \geqslant 1$, we get $e(G) \leq (n-k)^2 + nk - (n-k) - k < n(n-k-1) + (k+1)^2$, a contradiction.

This completes the proof. ■

Theorem 4.3 *Let k be an integer and $\widehat{G}$ be a connected balanced bipartite graph on $2n$ vertices. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+3$,*

$$WW(\widehat{G}) > 4n^4 - (2k+8)n^3 + (2k^2+5k+7)n^2 - (k^2+k+1)n - (k+1)^2,$$

then G is hamiltonian.

Proof. Since $WW(\widehat{G}) > 4n^4 - (2k+8)n^3 + (2k^2+5k+7)n^2 - (k^2+k+1)n - (k+1)^2$, and by lemma 2.2 we get $e(G) > n(n-k-1) + (k+1)^2$. By lemma 4.1, G is hamiltonian unless $G \subseteq B_n^k$. By the same discussion as the proof of theorem 4.2, we get the result. ■

Theorem 4.4 *Let k be an integer and $\widehat{G}$ be a connected balanced bipartite graph on $2n$ vertices. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+3$,*

$$H(\widehat{G}) < \frac{(4k+8)n^2 - (4k^2+12k+9)n + 4k^2 + 8k + 4}{4n-2},$$

then G is hamiltonian.

Proof. Since $H(\widehat{G}) < \frac{(4k+8)n^2-(4k^2+12k+9)n+4k^2+8k+4}{4n-2}$, and by lemma 2.3, so $e(G) > n(n-k-1) + (k+1)^2$. By lemma 4.1, G is hamiltonian unless $G \subseteq B_n^k$. By the same discussion as the proof of theorem 4.2, we get the result. ■

5 Traceable of nearly balanced bipartite Graphs

LEMMA 5.1 *(Yu, Fang and Fan [19]) Let G be a nearly balanced bipartite graph of order $2n-1$. If $\delta(G) \geqslant k \geqslant 1$, $n \geqslant 2k+1$, and*

$$e(G) > n(n-k-2) + (k+1)^2,$$

then G is traceable unless $G \subseteq C_n^k$.

THEOREM 5.2 *Let $\widehat{G}$ be a connected nearly balanced bipartite graph of order $2n-1$, where $n \geqslant 2k+1$, $\delta(G) \geqslant k \geqslant 1$. If*

$$W(\widehat{G}) > 4n^3 - (2k+14)n^2 + (4k + 2(k+1)^2 + 16)n - 4(k+1)^2 - 4,$$

then G is traceable.

Proof. Let $G = G[X, Y]$, where $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_{n-1}\}$.

$$\begin{aligned}
W(\widehat{G}) &= \sum_{u,v \in V(\widehat{G})} d_{\widehat{G}}(u, v) \\
&= \frac{1}{2} \sum_{i=1}^{2n-1} D_i(\widehat{G}) \\
&\leqslant \frac{1}{2}(\sum_{i=1}^{n} (d_{\widehat{G}}(x_i) + (2n-3)(n-1-d_{\widehat{G}}(x_i)) + (2n-2)(n-1))) \\
&\quad + \frac{1}{2}(\sum_{i=1}^{n-1} (d_{\widehat{G}}(y_i) + (2n-3)(n-d_{\widehat{G}}(y_i)) + (2n-4)(n-2))) \\
&= 4n^3 - 12n^2 + 12n - 4 - (n-2)(\sum_{i=1}^{n} d_{\widehat{G}}(x_i) + \sum_{i=1}^{n-1} d_{\widehat{G}}(y_i)) \\
&= 2n^3 - 6n^2 + 8n - 4 + (n-2)(\sum_{i=1}^{n} d_G(x_i) + \sum_{i=1}^{n-1} d_G(y_i)) \\
&= 2n^3 - 6n^2 + 8n - 4 + 2(n-2)e(G),
\end{aligned}$$

where $d_{\widehat{G}}(x_i) = n-1-d_G(x_i)$ and $d_{\widehat{G}}(y_i) = n - d_G(y_i)$. Because $W(\widehat{G}) > 4n^3 - (2k+14)n^2 + (4k + 2(k+1)^2 + 16)n - 4(k+1)^2 - 4$, we get $e(G) > n(n-k-2) + (k+1)^2$. By lemma 5.1, we obtain that G is traceable or $G \subseteq C_n^k$.

If $G \subseteq C_n^k$. Because $\widehat{G}$ is connected, and $\delta(G) \geqslant k \geqslant 1$, we get $e(G) \leq k^2 + n(n-k-1) - (n-k-1) - k < n(n-k-2) + (k+1)^2$, a contradiction.

This completes the proof. ■

Theorem 5.3 *Let $\widehat{G}$ be connected a nearly balanced bipartite graph of order $2n-1$, where $n \geqslant 2k+1$, $\delta(G) \geqslant k \geqslant 1$. If*

$$WW(\widehat{G}) > 4n^4 - (2k+18)n^3 + (2k^2+9k+\frac{65}{2})n^2 - (5k^2+12k+\frac{53}{2})n + 2k^2 + 4k + 8,$$

then G is traceable.

Proof. Let $G = G[X,Y]$, where $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_{n-1}\}$.

$$\begin{aligned}
WW(\widehat{G}) &= \frac{1}{2}\sum_{u,v\in V(\widehat{G})}(d_{\widehat{G}}(u,v) + d^2_{\widehat{G}}(u,v)) \\
&= \frac{1}{4}\sum_{i=1}^{2n-1}(D_i(\widehat{G}) + DD_i(\widehat{G})) \\
&\leqslant \frac{1}{4}\sum_{i=1}^{n}(d_{\widehat{G}}(x_i) + (2n-3)(n-1-d_{\widehat{G}}(x_i)) + (2n-2)(n-1)) \\
&\quad + \frac{1}{4}\sum_{i=1}^{n}(d_{\widehat{G}}(x_i) + (2n-3)^2(n-1-d_{\widehat{G}}(x_i)) + (2n-2))^2(n-1)) \\
&\quad + \frac{1}{4}\sum_{i=1}^{n-1}(d_{\widehat{G}}(y_i) + (2n-3)(n-d_{\widehat{G}}(y_i)) + (2n-4)(n-2)) \\
&\quad + \frac{1}{4}\sum_{i=1}^{n-1}(d_{\widehat{G}}(y_i) + (2n-3)^2(n-d_{\widehat{G}}(y_i)) + (2n-4))^2(n-2)) \\
&= 4n^4 - 16n^3 + \frac{51}{2}n^2 - \frac{39}{2}n + 6 + \frac{1}{4}(-4n^2+10n-4)(\sum_{i=1}^{n} d_{\widehat{G}}(x_i) + \sum_{i=1}^{n-1} d_{\widehat{G}}(y_i)) \\
&= 2n^4 - 9n^3 + \frac{37}{2}n^2 - \frac{35}{2}n + 6 + \frac{1}{4}(4n^2-10n+4)(\sum_{i=1}^{n} d_G(x_i) + \sum_{i=1}^{n-1} d_G(y_i)) \\
&= 2n^4 - 9n^3 + \frac{37}{2}n^2 - \frac{35}{2}n + 6 + (2n^2-5n+2)e(G),
\end{aligned}$$

where $d_{\widehat{G}}(x_i) = n-1-d_G(x_i)$ and $d_{\widehat{G}}(y_i) = n - d_G(y_i)$. Because $WW(\widehat{G}) > 4n^4 - (2k+18)n^3 + (2k^2+9k+\frac{65}{2})n^2 - (5k^2+12k+\frac{53}{2})n + 2k^2+4k+8$, we get $e(G) > n(n-k-2) + (k+1)^2$. By lemma 5.1, we obtain that G is traceable or $G \subseteq C_n^k$. By the same discussion as the proof of theorem 5.2, we get the result. ∎

Theorem 5.4 *Let $\widehat{G}$ be a connected nearly balanced bipartite graph of order $2n-1$, where $n \geqslant 2k+1$, $\delta(G) \geqslant k \geqslant 1$. If*

$$H(\widehat{G}) \leqslant \frac{(4k+8)n^2 - (4k^2+16k+17)n + 8k^2 + 16k + 8}{4n-6},$$

then G is traceable, unless $G \subseteq C_n^k (k \leqslant 6)$.

Proof. Let $G = G[X, Y]$, where $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_{n-1}\}$.

$$\begin{aligned}
H(\widehat{G}) &= \sum_{u,v \in V(G)} \frac{1}{d_{\widehat{G}}(u,v)} = \frac{1}{2}\sum_{i=1}^{n} \widetilde{D}_{\widehat{G}}(v_i) \\
&\geqslant \frac{1}{2}\sum_{i=1}^{n}(d_{\widehat{G}}(x_i) + \frac{1}{2n-3}(n-1-d_{\widehat{G}}(x_i)) + \frac{n-1}{(2n-2)}) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n-1}(d_{\widehat{G}}(y_i) + \frac{1}{2n-3}(n-d_{\widehat{G}}(y_i)) + \frac{n-2}{(2n-4)}) \\
&= \frac{4n^2-5n}{4n-6} + \frac{n-2}{2n-3}(\sum_{i=1}^{n} d_{\widehat{G}}(x_i) + \sum_{i=1}^{n-1} d_{\widehat{G}}(y_i)) \\
&= \frac{4n^3-8n^2+3n}{4n-6} - \frac{n-2}{2n-3}(\sum_{i=1}^{n} d_G(x_i) + \sum_{i=1}^{n-1} d_G(y_i)) \\
&= \frac{4n^3-8n^2+3n}{4n-6} - \frac{2(n-2)}{2n-3}e(G),
\end{aligned}$$

where $d_{\widehat{G}}(x_i) = n - 1 - d_G(x_i)$ and $d_{\widehat{G}}(y_i) = n - d_G(y_i)$. Because

$$H(\widehat{G}) \leqslant \frac{(4k+8)n^2 - (4k^2+16k+17)n + 8k^2 + 16k + 8}{4n-6},$$

we get $e(G) > n(n-k-2) + (k+1)^2$. By lemma 5.1, we obtain that G is traceable or $G \subseteq C_n^k$. Note $k \geqslant 7$, $H(\widehat{C_n^k}) = \frac{1}{4}(n^2 + 2kn - n - 2k^2) > \frac{(4k+8)n^3-(4k^2+24k+33)n^2+(16k^2+48k+44)n-16k^2-32k-19}{4n^2-14n+12}$. If $G \subseteq C_n^k$, then $H(\widehat{G}) > H(\widehat{C_n^k})$. We get the result. ■

6 Traceable of Graphs

Lemma 6.1 *[13] Let k be an integer and G be a graph of order $n \geqslant 6k + 10$. If $\delta(G) \geqslant k$ and $e(G) > \binom{n-k-2}{2} + (k+1)(k+2)$, then G is traceable, unless $G \subseteq \underline{L}_n^k$ or $\underline{N}_n^k$.*

Theorem 6.2 *Let k be an integer and $\overline{G}$ be a connected graph of order $n \geqslant 6k + 10$. If $\delta(G) \geqslant k$ and*

$$W(\overline{G}) > \frac{1}{2}(n^3 - (2k+6)n^2 + (3k^2+15k+19)n - 6k^2 - 22k - 20),$$

then G is traceable.

Proof. Since $W(\overline{G}) > \frac{1}{2}(n^3 - (2k+6)n^2 + (3k^2+15k+19)n - 6k^2 - 22k - 20)$, by lemma 2.4, we get $e(G) > \binom{n-k-2}{2} + (k+1)(k+2)$. By lemma 6.1, we obtain that G is traceable unless $G \subseteq \underline{L}_n^k$ or $\underline{N}_n^k$.

If $G \subseteq \underline{L}_n^k$. Note that $W(\overline{\underline{L}_n^k}) = n^2 - 2n - kn + k^2 + 2k + 1$. Then if $G \subseteq \underline{L}_n^k$, we have $W(\overline{G}) \leqslant W(\overline{\underline{L}_n^k}) < \frac{1}{2}(n^3 - (2k+6)n^2 + (3k^2+15k+19)n - 6k^2 - 22k - 20)$, a contradiction.

If $G \subseteq \underline{N_n^k}$. Note that $W(\overline{\underline{N_n^k}}) = \frac{1}{2}(2n^2 - 4n - 6kn + 5k^2 + 7k + 2)$. Then if $G \subseteq \underline{N_n^k}$, we have $W(\overline{G}) \leqslant W(\overline{\underline{N_n^k}}) < \frac{1}{2}(n^3 - (2k+6)n^2 + (3k^2 + 15k + 19)n - 6k^2 - 22k - 20)$, a contradiction.

This completes the proof. ■

Theorem 6.3 *Let k be an integer and $\overline{G}$ be a connected graph of order $n \geqslant 6k+10$. If $\delta(G) \geqslant k$ and*

$$WW(\overline{G}) > \frac{1}{4}n^4 - (\frac{1}{2}k + \frac{3}{2})n^3 + (\frac{3}{4}k^2 + \frac{13}{4}k + \frac{15}{4})n^2 - (\frac{3}{4}k^2 + \frac{7}{4}k + \frac{1}{2})n - \frac{3}{2}k^2 - \frac{11}{2}k - 5,$$

then G is traceable.

Proof. Since $WW(\overline{G}) > \frac{1}{4}n^4 - (\frac{1}{2}k + \frac{3}{2})n^3 + (\frac{3}{4}k^2 + \frac{13}{4}k + \frac{15}{4})n^2 - (\frac{3}{4}k^2 + \frac{7}{4}k + \frac{1}{2})n - \frac{3}{2}k^2 - \frac{11}{2}k - 5$, by lemma 2.5, we get $e(G) > \binom{n-k-2}{2} + (k+1)(k+2)$. By lemma 6.1, we obtain that G is traceable unless $G \subseteq \underline{L_n^k}$ or $\underline{N_n^k}$.

If $G \subseteq \underline{L_n^k}$. Note that $WW(\overline{\underline{L_n^k}}) = \frac{1}{2}(3n^2 - 7n - 4kn + 4k^2 + 8k + 4)$. Then if $G \subseteq \underline{L_n^k}$, we have $WW(\overline{G}) \leqslant WW(\overline{\underline{L_n^k}}) < \frac{1}{4}n^4 - (\frac{1}{2}k + \frac{3}{2})n^3 + (\frac{3}{4}k^2 + \frac{13}{4}k + \frac{15}{4})n^2 - (\frac{3}{4}k^2 + \frac{7}{4}k + \frac{1}{2})n - \frac{3}{2}k^2 - \frac{11}{2}k - 5$, a contradiction.

If $G \subseteq \underline{N_n^k}$. Note that $WW(\overline{\underline{N_n^k}}) = 3n^2 - 7n - 10kn + 9k^2 + 13k + 4$. Then if $G \subseteq \underline{N_n^k}$, we have $WW(\overline{G}) \leqslant WW(\overline{\underline{N_n^k}}) < \frac{1}{4}n^4 - (\frac{1}{2}k + \frac{3}{2})n^3 + (\frac{3}{4}k^2 + \frac{13}{4}k + \frac{15}{4})n^2 - (\frac{3}{4}k^2 + \frac{7}{4}k + \frac{1}{2})n - \frac{3}{2}k^2 - \frac{11}{2}k - 5$, a contradiction.

This completes the proof. ■

Theorem 6.4 *Let k be an integer and $\overline{G}$ be a connected graph of order $n \geqslant 6k+10$. If $\delta(G) \geqslant k$ and*

$$H(\overline{G}) < \frac{(2k+4)n^2 - (3k^2 + 15k + 18)n + 6k^2 + 22k + 20}{2n-2},$$

then G is traceable.

Proof. Since $H(\overline{G}) < \frac{(2k+4)n^2 - (3k^2+15k+18)n + 6k^2 + 22k + 20}{2n-2}$, by lemma 2.6, we get $e(G) > \binom{n-k-2}{2} + (k+1)(k+2)$. By lemma 6.1, we obtain that G is traceable unless $G \subseteq \underline{L_n^k}$ or $\underline{N_n^k}$.

If $G \subseteq \underline{L_n^k}$. Note that $H(\overline{\underline{L_n^k}}) = \frac{1}{4}(n^2 + n + 2kn - 2k^2 - 2)$. Then if $G \subseteq \underline{L_n^k}$, we have $H(\overline{G}) \geqslant H(\overline{\underline{L_n^k}}) > \frac{(2k+4)n^2 - (3k^2+15k+18)n + 6k^2 + 22k + 20}{2n-2}$, a contradiction.

If $G \subseteq \underline{N_n^k}$. Note that $H(\overline{\underline{N_n^k}}) = \frac{1}{4}(n^2 + n + 2kn - 2k^2 - 6k - 2)$. Then if $G \subseteq \underline{N_n^k}$, we have $H(\overline{G}) \geqslant H(\overline{\underline{N_n^k}}) > \frac{(2k+4)n^2 - (3k^2+15k+18)n + 6k^2 + 22k + 20}{2n-2}$, a contradiction.

This completes the proof. ■

7 Hamiltonian of Graphs

Lemma 7.1 *[13] Let k be an integer and G be a graph of order $n \geqslant 6k+5$. If $\delta(G) \geqslant k$ and*

$$e(G) > \binom{n-k-1}{2} + (k+1)^2.$$

then G is hamiltonian, unless $G \subseteq L_n^k$ or N_n^k.

Theorem 7.2 *Let k be an integer and $\overline{G}$ be a connected graph of order $n \geqslant 6k+5$. If $\delta(G) \geqslant k$ and*

$$W(\overline{G}) > \frac{1}{2}(n^3 - (2k+4)n^2 + (3k^2+11k+19)n - 6k^2 - 14k - 8),$$

then G is hamiltonian.

Proof. Since $W(\overline{G}) > \frac{1}{2}(n^3 - (2k+4)n^2 + (3k^2+11k+19)n - 6k^2 - 14k - 8)$, by lemma 2.4, we get $e(G) > \binom{n-k-1}{2} + (k+1)^2$. By lemma 7.1, we obtain that G is hamiltonian unless $G \subseteq L_n^k$ or N_n^k.

If $G \subseteq L_n^k$. Note that $W(\overline{L_n^k}) = n^2 - kn - 3n + k^2 + k + 2$. Then if $G \subseteq L_n^k$, we have $W(\overline{G}) \leqslant W(\overline{L_n^k}) < \frac{1}{2}(n^3 - (2k+4)n^2 + (3k^2+11k+19)n - 6k^2 - 14k - 8)$, a contradiction.

If $G \subseteq N_n^k$. Note that $W(\overline{N_n^k}) = n^2 - 3kn - n + \frac{5}{2}k^2 + \frac{3}{2}k$. Then if $G \subseteq N_n^k$, we have $W(\overline{G}) \leqslant W(\overline{N_n^k}) < \frac{1}{2}(n^3 - (2k+4)n^2 + (3k^2+11k+19)n - 6k^2 - 14k - 8)$, a contradiction.

This completes the proof. ■

Theorem 7.3 *Let k be an integer and $\overline{G}$ be a connected graph of order $n \geqslant 6k+5$. If $\delta(G) \geqslant k$ and*

$$WW(\overline{G}) > \frac{1}{4}n^4 - (\frac{1}{2}k+1)n^3 + (\frac{3}{4}k^2 + \frac{9}{4}k + \frac{3}{2})n^2 - (\frac{3}{4}k^2 + \frac{3}{4}k + \frac{1}{4})n - (\frac{3}{2}k^2 + \frac{7}{2}k + 2),$$

then G is hamiltonian.

Proof. Since $WW(\overline{G}) > \frac{1}{4}n^4 - (\frac{1}{2}k+1)n^3 + (\frac{3}{4}k^2 + \frac{9}{4}k + \frac{3}{2})n^2 - (\frac{3}{4}k^2 + \frac{3}{4}k + \frac{1}{4})n - (\frac{3}{2}k^2 + \frac{7}{2}k + 2)$, by lemma 2.5, we get $e(G) > \binom{n-k-1}{2} + (k+1)^2$. By lemma 7.1, we obtain that G is hamiltonian unless $G \subseteq L_n^k$ or N_n^k.

If $G \subseteq L_n^k$. Note that $WW(\overline{L_n^k}) = 3n^2 - 4kn - 9n + 4k^2 + 4k + 6$. Then if $G \subseteq L_n^k$, we have $WW(\overline{G}) \leqslant WW(\overline{L_n^k}) < \frac{1}{4}n^4 - (\frac{1}{2}k+1)n^3 + (\frac{3}{4}k^2 + \frac{9}{4}k + \frac{3}{2})n^2 - (\frac{3}{4}k^2 + \frac{3}{4}k + \frac{1}{4})n - (\frac{3}{2}k^2 + \frac{7}{2}k + 2)$, a contradiction.

If $G \subseteq N_n^k$. Note that $WW(\overline{N_n^k}) = 3n^2 - 10kn - 3n + 9k^2 + 5k$. Then if $G \subseteq N_n^k$, we have $WW(\overline{G}) \leqslant WW(\overline{N_n^k}) < \frac{1}{4}n^4 - (\frac{1}{2}k+1)n^3 + (\frac{3}{4}k^2 + \frac{9}{4}k + \frac{3}{2})n^2 - (\frac{3}{4}k^2 + \frac{3}{4}k + \frac{1}{4})n - (\frac{3}{2}k^2 + \frac{7}{2}k + 2)$, a contradiction.

This completes the proof. ■

THEOREM 7.4 *Let k be an integer and $\overline{G}$ be a connected graph of order $n \geqslant 6k+5$. If $\delta(G) \geqslant k$ and $H(\overline{G}) < \frac{(2k+2)n^2-(3k^2+11k+8)n+6k^2+14k+8}{2n-2}$, then G is hamiltonian.*

Proof. Since $H(\overline{G})\frac{(2k+2)n^2-(3k^2+11k+8)n+6k^2+14k+8}{2n-2}$, by lemma 2.6, we get $e(G) > \binom{n-k-1}{2} + (k+1)^2$. By lemma 7.1, we obtain that G is hamiltonian unless $G \subseteq L_n^k$ or N_n^k.

If $G \subseteq L_n^k$. Note that $H(\overline{L_n^k}) = \frac{1}{4}(n^2 - 3n + 2kn - 2k^2 - 2k + 2)$. Then if $G \subseteq L_n^k$, we have $H(\overline{G}) \geqslant H(\overline{L_n^k}) > \frac{(2k+2)n^2-(3k^2+11k+8)n+6k^2+14k+8}{2n-2}$, a contradiction.

If $G \subseteq N_n^k$. Note that $H(\overline{N_n^k}) = \frac{1}{4}(n^2 - n - 3k^2 + k)$, Then if $G \subseteq N_n^k$, we have $H(\overline{G}) \geqslant H(\overline{N_n^k}) > \frac{(2k+2)n^2-(3k^2+11k+8)n+6k^2+14k+8}{2n-2}$, a contradiction.

This completes the proof. ■

8 Hamilton-connected of Graphs

LEMMA 8.1 *[20] Let G be a k-connected graph of order n, where $k \geqslant 2$. If $e(G) > \frac{n(n-1)-k(n-k-1)}{2}$, then G is Hamilton-connected.*

By lemmas 2.4, 2.5, 2.6 and 8.1, and by direct computations, we get theorems 8.2, 8.3, and 8.4, respectively.

THEOREM 8.2 *Let G be a k-connected graph of order n, $\overline{G}$ be a connected graph, where $k \geqslant 2$. If*

$$W(\overline{G}) > \frac{1}{2}n^3 - (\frac{1}{2}k+1)n^2 + \frac{1}{2}(k^2+3k+1)n - k^2 - k,$$

then G is Hamilton-connected.

THEOREM 8.3 *Let G be a k-connected graph of order n, $\overline{G}$ be a connected graph, where $k \geqslant 2$. If*

$$WW(\overline{G}) > \frac{1}{4}n^4 - (\frac{1}{4}k+\frac{1}{2})n^3 + (\frac{1}{4}k^2+\frac{1}{2}k+\frac{1}{4})n^2 - (\frac{1}{4}k^2-\frac{1}{4}k)n - \frac{1}{2}k^2 - \frac{1}{2}k,$$

then G is Hamilton-connected.

THEOREM 8.4 *Let G be a k-connected graph of order n, $\overline{G}$ be a connected graph, where $k \geqslant 2$. If*

$$H(\overline{G}) < \frac{(k+1)n^2 - (k^2+3k+1)n + 2k^2 + 2k}{2(n-1)},$$

then G is Hamilton-connected.

9 Traceable from every vertex of Graphs

Lemma 9.1 *[20] Let G be a k-connected graph of order n, where $k \geqslant 2$. If $e(G) > \frac{n(n-1)-k(n-k)}{2}$, then G is traceable from every vertex.*

By lemmas 2.4, 2.5, 2.6 and 9.1, and by a direct computation, we get theorems 9.2, theorem 9.3, theorem 9.4, respectively.

Theorem 9.2 *Let G be a k-connected graph of order n, $\overline{G}$ be a connected graph, where $k \geqslant 2$. If*

$$W(\overline{G}) > \frac{1}{2}n^3 - (\frac{1}{2}k+1)n^2 + \frac{1}{2}(k^2+2k+1)n - k^2,$$

then G is traceable from every vertex.

Theorem 9.3 *Let G be a k-connected graph of order n, $\overline{G}$ be a connected graph, where $k \geqslant 2$. If*

$$WW(\overline{G}) > \frac{1}{4}n^4 - (\frac{1}{4}k+\frac{1}{2})n^3 + (\frac{1}{4}k^2+\frac{1}{4}k+\frac{1}{4})n^2 - (\frac{1}{4}k^2-\frac{1}{2}k)n - \frac{1}{2}k^2,$$

then G is traceable from every vertex.

Theorem 9.4 *Let G be a k-connected graph of order n, $\overline{G}$ be a connected graph, where $k \geqslant 2$. If*

$$H(\overline{G}) < \frac{(k+1)n^2 + (-k^2-2k-1)n + 2k^2}{2n-2},$$

then G is traceable from every vertex.

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