A topological group observation on the Banach–Mazur separable quotient problem

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Abstract

The Banach-Mazur problem, which asks if every infinite-dimensional Banach space has an infinite-dimensional separable quotient space, has remained unsolved for 85 years, but has been answered in the affirmative for special cases such as reflexive Banach spaces. It is also known that every infinite-dimensional non-normable Fréchet space has an infinite-dimensional separable quotient space, namely $\mathbb{R}^\omega$. It is proved in this paper that every infinite-dimensional Fréchet space (including every infinite-dimensional Banach space), indeed every locally convex space which has a subspace which is an infinite-dimensional Fréchet space, has an infinite-dimensional (in the topological sense) separable metrizable quotient group, namely $\mathbb{T}^\omega$, where $\mathbb{T}$ denotes the compact unit circle group.

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1. Introduction

The famous problem of Stefan Banach and Stanisław Mazur in the 1930s asks if every (real) Banach space has a separable infinite-dimensional quotient space. This has been shown to be true for reflexive Banach spaces [13] and more generally Banach spaces which are dual spaces [2], and weakly compactly-generated Banach spaces [1], with further results in [9, 10, 11, 15, 16], but the general question remains open. However, M. Eidelheit [5] proved that every non-normable Fréchet space has a separable quotient locally convex space, namely $\mathbb{R}^\omega$. We note that $\mathbb{R}^\omega$ cannot be a quotient locally convex space (or even a quotient group) of a Banach space, but every separable Banach space is indeed homeomorphic to $\mathbb{R}^\omega$, see [4].

Noting that $\mathbb{T}^\omega$ is a quotient group of $\mathbb{R}^\omega$, where $\mathbb{T}$ is the compact unit circle group, we see that every non-normable Fréchet space has $\mathbb{T}^\omega$ as a quotient group. This leads us to ask if every Banach space also has $\mathbb{T}^\omega$ as a quotient group. In fact we prove this and more in the theorem below.

2. Results

Theorem 2.1. *Let E be a (Hausdorff) locally convex space over the field $\mathbf{F}$, where $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$. If E has a subspace which is an infinite-dimensional Fréchet space, then E has $\mathbb{T}^\omega$ as a quotient group.*

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Proof. By [17, Theorem 2], F has a basic sequence $\{e_n : n \in \omega\}$, that is, each element of the closure H of the span of $\{e_n : n \in \omega\}$ can be written uniquely as $x = \sum_{n\in\omega} a_n e_n$ with $a_n \in \mathbf{F}$ and the coefficient functionals $\chi'_n : H \to \mathbf{F}, \chi'_n(x) := a_n$, are continuous. For every $n \in \omega$, denote by χ_n some extension of χ_n onto the whole space E. For every $n \in \omega$, choose $m_n \in \omega$ such that $e_n/m_n \to 0$ in E (such numbers exist since F is metrizable) and set $f_n := \mathrm{Re}(\chi_n)$ (so f_n is a real continuous linear functional of E). Define a homomorphism $R : E \to \mathbb{T}^\omega$ by

$$R(x) := \left(e^{2\pi i\, 2^n m_n f_n(x)}\right), \quad x \in E.$$

Let us show that $R|_H$ is surjective. Indeed, fix arbitrarily $z = \left(e^{2\pi i a_n}\right) \in \mathbb{T}^\omega$, where $a_n \in [0, 1)$ for every $n \in \omega$. For every $n \in \omega$, set $b_n := a_n/(2^n m_n)$ and $x_n = \sum_{i=0}^n b_n e_n$. Then, for every $N < n < k$ and each seminorm p on the Fréchet space F, we have

$$p(x_k - x_n) \le \sum_{i=n+1}^{k} \frac{a_n}{2^n} p\big(e_n/m_n\big) \to 0 \quad \text{at } N \to \infty.$$

Therefore the sequence $\{x_n\}$ is Cauchy, and hence it converges to $x = \sum_{n\in\omega} b_n e_n \in F$. Since all the numbers b_n are real, it follows that $R(x) = z$. Thus $R|_H$ is surjective.

Since the group $\mathbb{T}^\omega$ carries the product topology and all the homomorphisms $x \mapsto e^{2\pi i\, 2^n f_n(x)} \in \mathbb{T}$ are continuous, we obtain that R is continuous. Therefore by the Open Mapping Theorem, Theorem 1.2.6 of [3], the surjective continuous homomorphism $R|_H$ from the separable Fréchet space H onto the compact metrizable group $\mathbb{T}^\omega$ is open. Thus the map R also is open. □

Corollary 2.2. *Every infinite-dimensional Fréchet space, and in particular every infinite-dimensional Banach space, has $\mathbb{T}^\omega$ as a quotient group.*

Remark 2.3. We note that by Theorem 8.4.6 of [8] and Corollary 2.2: a compact group G is a quotient group of an infinite-dimensional Fréchet space if and only if G is topologically isomorphic to $\mathbb{T}^N$ for some $N \le \aleph_0$.

Example 2.4 shows the the condition in Theorem 2.1 is not necessary, but Example 2.5 shows that the condition cannot be dropped entirely since there is a complete locally convex space which does not have $\mathbb{T}^\omega$ as a quotient group.

Example 2.4. *Let (E, τ_E) be the Banach space ℓ_∞ endowed with the topology τ_E induced from the complete metrizable space $\mathbb{R}^\omega$. Then (1) E does not contain an infinite-dimensional Fréchet subspace; and (2) E has $\mathbb{T}^\omega$ as a quotient group.*

Proof. (1) Let (F, τ_F) be a Fréchet subspace of (E, τ_E). Then it is a subspace of $\mathbb{R}^\omega$. Hence F with the weak topology denoted by F_w is also a subspace of $\mathbb{R}^\omega$. Therefore F_w is metrizable. On the other hand, since $F = \bigcup_{n\in\omega}(F \cap nB)$, where B is the closed unit ball of ℓ_∞, the Baire category theorem implies that $F \cap B$ is a neighborhood of zero in F (note that B is closed in (E, τ_E)). Therefore the topology τ_F of F is finer than the norm topology τ_∞ induced from ℓ_∞. Clearly, we also have $\tau_F \le \tau_\infty$, and hence (F, τ_F) is a subspace of the Banach space ℓ_∞. Thus (F, τ_F) is a Banach space. But a Banach space in the weak topology is metrizable if and only if it is finite-dimensional, see for example Theorem 1.5 of [6]. Thus F is finite-dimensional.

(2) To show that (E, τ_E) has $\mathbb{T}^\omega$ as a quotient group consider the map

$$T : E \to \mathbb{T}^\omega, \quad T(x_n) = \left(e^{2\pi i x_n}\right) \text{ for } (x_n) \in E.$$

It is clear that T is a surjective and open continuous homomorphism. □

Denote by φ the complete countably infinite-dimensional locally convex space which is the strong dual space of $\mathbb{R}^\omega$. We note that φ is the inductive limit of $\mathbb{R}^n$s.

Example 2.5. *There is no continuous surjective homomorphism from φ onto $\mathbb{T}^\omega$.*

Proof. Let $T : \varphi \to \mathbb{T}^\omega$ be any continuous homomorphism. We have to show that T is not onto. For every $n \geq 1$, denote by T_n the restriction of T to the closed vector subspace $\mathbb{R}^n$ of φ. Then T_n induces a continuous monomorphism S_n from $\mathbb{R}^n/\ker(T_n)$ to $\mathbb{T}^\omega$. By Theorem 9.11 of [7] this implies that the group $\mathbb{R}^n/\ker(T_n)$ is topologically isomorphic to $\mathbb{R}^{k_n} \times \mathbb{T}^{s_n}$ for some integers k_n and s_n. Hence $S_n : \mathbb{R}^{k_n} \times \mathbb{T}^{s_n} \to \mathbb{T}^\omega$ is a continuous injective homomorphism. It follows that

$$T(\varphi) = \bigcup_{n\geq 1} \left(\bigcup_{p\in\mathbb{N}} S_n\big([-p,p]^{k_n} \times \mathbb{T}^{s_n}\big) \right). \tag{2.1}$$

Since S_n is a homeomorphism on the compact finite-dimensional topological space $[-p,p]^{k_n} \times \mathbb{T}^{s_n}$, the equality (2.1) implies that $T(\varphi)$ is countable dimensional topological space. However, the metrizable group $\mathbb{T}^\omega$ is not a countable dimensional topological space by Corollary 3.13.6 of [14]. Thus T is not surjective.□

Remark 2.6. We mention that A. Leiderman, M. Tkachenko and the second author in their paper [12] have examined the question: which topological groups G have separable quotient groups? Their results include substantial information on the cases that G is (i) a compact group, (ii) a pro-Lie group, (iii) a pseudocompact group and (iv) a precompact group.

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