Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link

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ABSTRACT

There is a question asking whether a handle-irreducible summand of every stable-ribbon surface-link is a unique ribbon surface-link. This question for the case of a trivial surface-link is affirmatively answered. That is, a handle-irreducible summand of every stably trivial surface-link is only a trivial 2-link. By combining this result with an old result of F. Hosowaka and the author that every surface-knot with infinite cyclic fundamental group is a stably trivial surface-knot, it is concluded that every surface-knot with infinite cyclic fundamental group is a trivial (i.e., an unknotted) surface-knot.

Keywords: Trivial surface-link, Stably trivial surface-link, Orthogonal 2-handle pair.
Mathematics Subject Classification 2010: Primary 57Q45; Secondary 57N13

1 Introduction

A *surface-link* is a closed oriented (possibly disconnected) surface F embedded in the 4-space $\mathbf{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When a (possibly disconnected) closed surface $\mathbf{F}$ is fixed, it is also called an $\mathbf{F}$-*link*. If $\mathbf{F}$ is the disjoint union of some copies of the 2-sphere S^2, then it is also called a 2-*link*. When $\mathbf{F}$ is connected, it is also called a *surface-knot*, and a 2-*knot* for $\mathbf{F} = S^2$.

Two surface-links F and F' are *equivalent* by an *equivalence* f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism (or piecewise-linear homeomorphism) $f : \mathbf{R}^4 \to \mathbf{R}^4$. The notation $F \cong F'$ is used for equivalent surface-links F, F'. A *trivial* surface-link is a surface-link F which is the boundary of the union of mutually disjoint handlebodies smoothly embedded in $\mathbf{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, a solid torus or a boundary-disk sum of some number of solid tori. A trivial surface-knot is also called an *unknotted* surface-knot. A trivial disconnected surface-link is also called an *unknotted and unlinked* surface-link. For any given closed oriented (possibly disconnected) surface $\mathbf{F}$, a trivial $\mathbf{F}$-link exists uniquely up to equivalences (see [6]). A *ribbon* surface-link is a surface-link F which is obtained from a trivial 2-link O by the surgery along an embedded 1-handle system (see [10, 11, 12, 13], [16, II]). A *stabilization* of a surface-link F is a connected sum $F^{\#sT} = F\#_{k=1}^{s}T_k$ of F and a system of trivial torus-knots $T_k\,(k = 1, 2, \ldots, s)$. By granting $s = 0$, we understand that a surface-link F itself is a stabilization of F. The trivial torus-knot system $T_k\,(k = 1, 2, ..., s)$ is called the *stabilizer* on the stabilization $F^{\#sT}$ of F.

A *stable-ribbon* surface-link is a surface-link F such that a stabilization $F^{\#sT}$ of F is a ribbon surface-link. For every surface-link F, there is a surface-link F^* with minimal total genus such that F is equivalent to a stabilization of F^*. The surface-link F^* is called a *handle-irreducible summand* of F. The following question is a central question.

Question 1.0. A handle-irreducible summand of every stable-ribbon surface-link is a ribbon surface-link which is unique up to equivalences ?

A *stably trivial* surface-link is a surface-link F such that a stabilization of F is a trivial surface-link.

In this paper, the following theorem is shown answering affirmatively this question for the case of a stably trivial surface-link. This question in the general case will be answered affirmatively in [15].

Theorem 1.1. Any handle-irreducible summand of every stably trivial surface-link is a trivial 2-link.

The following corollary is directly obtained from Theorem 1.1:

Corollary 1.2. Every stably trivial surface-link is a trivial surface-link.

If a surface-knot F has an infinite cyclic fundamental group, then F is a TOP-trivial surface-knot, which was shown by Freedman for a 2-knot and by [3, 9] for a higher genus surface-knot. In the case of a piecewise linear surface-knot(equivalent to a smooth surface-knot), it is known by [6] that a stabilization of the surface-knot F is a trivial surface-knot, namely the surface-knot F is a stably trivial surface-knot. Hence the following corollary is directly obtained from Corollary 1.2 answering the problem [17, Problem 1.55(A)] on unknotting of a 2-knot positively (see [14] for the surface-link version):

Corollary 1.3. A surface-knot F is a trivial surface-knot if the fundamental group $\pi_1(\mathbf{R}^4 \setminus F)$ is an infinite cyclic group.

The *exterior* of a surface-knot F is the 4-manifold $E = \mathrm{cl}(\mathbf{R}^4 \setminus N(F))$ for a tubular neighborhood $N(F)$ of F in $\mathbf{R}^4$. Then the boundary ∂E is a trivial circle bundle over F. A surface-knot F is *of Dehn's type* if there is a section F' of F in the bundle ∂E such that the inclusion $F' \to E$ is homotopic to a constant map. By [3, Corollary 4.2], the fundamental group $\pi_1(\mathbf{R}^4 \setminus F)$ of a surface-knot F of Dehn's type is an infinite cyclic group. Thus, we have the following corollary(answering the problem [17, Problem 1.51)] on unknotting of a 2-knot of Dehn's type positively):

Corollary 1.4. A surface-knot of Dehn's type is a trivial surface-knot.

Unknotting Conjecture asks whether an n-knot K^n(i.e., a smooth embedding image of the n-sphere S^n in the(n+2)-sphere S^{n+2}) is unknotted (i.e., bounds a smooth $(n+1)$-ball in S^{n+2}) if and only if the complement $S^{n+2} \backslash K^n$ is homotopy equivalent to S^1 (see [8] for example). This conjecture was previously known to be true for $n > 3$ by [18], for $n = 3$ by [20] and for $n = 1$ by [5, 19]. The conjecture for $n = 2$ was known only in the TOP category by [1](see also [2]). Corollary 1.3 answers this finally remained smooth unknotting conjecture affirmatively and hence Unknotting Conjecture can be changed into the following:

Unknotting Theorem. A smooth S^n-knot K^n in S^{n+2} is unknotted if and only if the complement $S^{n+2} \backslash K^n$ is homotopy equivalent to S^1 for every $n \geq 1$.

A main idea in our argument is to use the surgery of a surface-link on an orthogonal 2-handle pair, which is much different from the surgery of a surface-link on a single 2-handle. It is known that every surface-link F in $\mathbf{R}^4$ is obtained from a higher genus trivial surface-knot F' by the surgery of F' on a system of mutually disjoint 2-handles, because a handlebody in $\mathbf{R}^4$ is obtained from a connected Seifert hypersurface of F by removing mutually disjoint 1-handles (see [6]). Thus, for example, every 2-twist spun 2-bridge knot in [21] is obtained from a trivial torus-knot T in $\mathbf{R}^4$ by the surgery of T on a single 2-handle, because it bounds a once-punctured lens space as a Seifert hypersurface.

In Section 2, it is shown that every stably trivial surface-link is a trivial surface-link if and only if the uniqueness of an orthogonal 2-handle pair on every trivial surface-link holds. In Section 3, the uniqueness of every orthogonal 2-handle pair on every surface-link is shown, by which Theorem 1.1 is obtained.

2 A triviality condition on a stably trivial surface-link

A 2-*handle* on a surface-link F in $\mathbf{R}^4$ is an embedded 2-handle $D \times I$ on F with D a core disk such that $D \times I \cap F = \partial D \times I$, where I denotes a closed interval containing 0 and $D \times 0$ is identified with D. If D is an immersed disk, then call it an *immersed 2-handle*. Two (possibly immersed) 2-handles $D \times I$ and $E \times I$ on F are *equivalent* if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f|_F : F \to F$ is the identity map and $f(D \times I) = E \times I$.

An *orthogonal 2-handle pair* (or simply, an *O2-handle pair*) on F is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I$ on F such that

$$D \times I \cap D' \times I = \partial D \times I \cap \partial D' \times I$$

and $\partial D \times I$ and $\partial D' \times I$ *meet orthogonally* on F, that is, the boundary circles ∂D and $\partial D'$ meet transversely at one point p and the intersection $\partial D \times I \cap \partial D' \times I$ is homeomorphic to the square $Q = p \times I \times I$ (see Fig. 1).

Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link F. Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from F by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union of the plumbed disk

$$\delta = \delta_{D \times I, D' \times I} = D \times \partial I \cup Q \cup D' \times partial I$$

and the surface

$$F^c_\delta = \mathrm{cl}(F \setminus (\partial D \times I \cup \partial D' \times I).$$

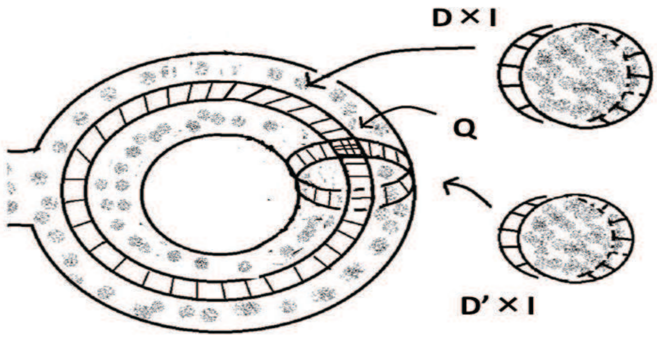

Figure 1: An orthogonal 2-handle pair(=: an O2-handle pair)

A once-punctured torus T^o in a 3-ball B is *trivial* if T^o is smoothly and properly embedded in B which splits B into two solid tori. A *bump* of a surface-link F is a 3-ball B in $\mathbf{R}^4$ with $F \cap B = T^o$ a trivial once-punctured torus in B. Let $F(B)$ be a surface-link $F^c_B \cup \delta_B$ for the surface $F^c_B = \mathrm{cl}(F \setminus T^o)$ and a disk δ_B in ∂B with $\partial\delta_B = \partial T^o$, where note that $F(B)$ is uniquely determined up to cellular moves on δ_B keeping F^c_B fixed. Here, a *cellular move* of a surface P in $\mathbf{R}^4$ is a surface $\tilde{P}$ in $\mathbf{R}^4$ such that the complements $d = \mathrm{cl}(P \setminus P_0)$ and $\tilde{d} = \mathrm{cl}(\tilde{P} \setminus P_0)$ of the intersection $P_0 = P \cap P'$ are disks in the interiors of P and $\tilde{P}$, respectively and the union $d \cup \tilde{d}$ is a 2-sphere bounding a 3-ball smoothly embedded in $\mathbf{R}^4$ and not meeting $P_0 \setminus \partial d = P_0 \setminus \partial\tilde{d}$.

For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F, let $\Delta = D \times I \cup D' \times I$ is a 3-ball in $\mathbf{R}^4$ called the *2-handle union*. Consider the 3-ball Δ as a Seifert hypersurface of the trivial S^2-knot $K = \partial\Delta$ in $\mathbf{R}^4$ to construct a 3-ball B_Δ obtained from Δ by adding an outer boundary collar. This 3-ball B_Δ is a bump of F, which we call the *associated bump* of the O2-handle pair $(D \times I, D' \times I)$. When the 3-ball Δ and a boundary collar of F^c_δ are deformed into the 3-space $\mathbf{R}^3$, this associated bump B_Δ is also considered as a regular neighborhood of Δ in $\mathbf{R}^3$ (see Fig. 2).

The following lemma shows that giving an O2-handle unordered pair on a surface-link F is the same as giving a bump of F.

Lemma 2.1. An O2-handle unordered pair $(D \times I, D' \times I)$ on a surface-link F is uniquely constructed from any given bump B of F in $\mathbf{R}^4$ with $F(D \times I, D' \times I) \cong F(B)$.

Proof of Lemma 2.1. For a bump B of F, the set of two solid tori bounded by $T^o = F \cap B$ is unique, whose meridian-longitude disk pair is an O2-handle pair. □

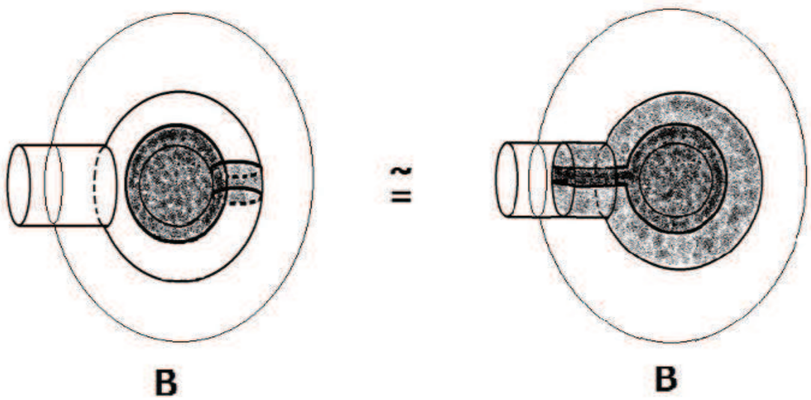

Figure 2: An associated bump B of a 2-handle union

The following lemma shows the uniqueness of the surgery of a surface-link F by an O2-handle pair.

Lemma 2.2. For any O2-handle pair $(D \times I, D' \times I)$ on any surface-link F and the associated bump B, there are equivalences

$$F(B) \cong F(D \times I, D' \times I) \cong F(D \times I) \cong F(D' \times I).$$

Further, these equivalences are attained by cellular moves keeping F^c_δ fixed.

Proof of Lemma 2.2. By definition, we have $F(B) \cong F(D \times I, D' \times I)$. The surface-link $F(D \times I, D' \times I)$ is equivalent to $F(D \times I)$ and $F(D' \times I)$ by cellular moves on the 3-balls $D' \times I$ and $D \times I$, respectively. □

Two O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on a surface-link F with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$ are *equivalent* if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f|_F : F \to F$ is the identity map and $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$.

The following characterization of equivalent O2-handle pairs is useful.

Lemma 2.3. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be O2-handle pairs on a surface-link F with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Let

$$F(D \times I, D' \times I) = F^c_\delta \cup \delta_{D\times I, D'\times I} \quad \text{and} \quad F(E \times I, E' \times I) = F^c_\delta \cup \delta_{E\times I, E'\times I}$$

for the plumbed disks $\delta_{D\times I, D'\times I}$ and $\delta_{E\times I, E'\times I}$. Then the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ are equivalent if and only if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(D \times I, D' \times I)$ to $F(E \times I, E' \times I)$ such that the restriction $f|_{F^c_\delta} : F^c_\delta \to F^c_\delta$ is the identity map and $f(\delta_{D\times I, D'\times I}) = \delta_{E\times I, E'\times I}$.

Proof of Lemma 2.3. It suffices to show the "if"part since the "only if"part is obtained from the definition of equivalent O2-handle pairs. Assume that there is an equivalence f from $F(D \times I, D' \times I)$ to $F(E \times I, E' \times I)$ such that the restriction $f|_{F^c_\delta} : F^c_\delta \to F^c_\delta$ is the identity map and $f(\delta_{D\times I, D'\times I}) = \delta_{E\times I, E'\times I}$. The map f is isotopic to a diffeomorphism $f' : \mathbf{R}^4 \to \mathbf{R}^4$ sending the associated bump $B_{\Delta(D\times I, D'\times I)}$ of $(D \times I, D' \times I)$ to the associated bump $B_{\Delta(E\times I, E'\times I)}$ of $(E \times I, E' \times I)$ by regarding $B_{\Delta(D\times I, D'\times I)}$ and $B_{\Delta(E\times I, E'\times I)}$ as collars of $\delta_{D\times I, D'\times I}$ and $\delta_{E\times I, E'\times I}$, respectively. The diffeomorphism $f' : \mathbf{R}^4 \to \mathbf{R}^4$ is modified into an equivalence $f'' : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f''|_F : F \to F$ is the identity map and $f''(D \times I) = E \times I$ and $f''(D' \times I) = E' \times I$. Thus, the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ are equivalent. □

The following corollary is a concrete application of Lemma 2.3.

Corollary 2.4. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be O2-handle pairs on a surface-link F with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. If the surface-link $F(D \times I, D' \times I)$ is obtained from the surface-link $F(E \times I, E' \times I)$ by a finite number of cellular moves on $D \times I$, $D' \times I$, $E \times I$ and $E' \times I$ keeping F^c_δ fixed, then the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ are equivalent.

Proof of Corollary 2.4. By the assumption, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(D \times I, D' \times I)$ to $F(E \times I, E' \times I)$ such that the restriction $f|_{F^c_\delta} : F^c_\delta \to F^c_\delta$ is the identity map and $f(\delta_{D\times I, D'\times I}) = \delta_{E\times I, E'\times I}$. By Lemma 2.3, the result is obtained. □

A surface-link F has *only unique O2-handle pair* if any two O2-handle pairs on F with the same attaching part are equivalent. A surface-link not admitting any O2-handle pair is understood as a surface-link with only unique O2-handle pair.

We have the following characterization on a stably trivial surface-link.

Lemma 2.5. The following (1)-(3) are mutually equivalent.

(1) If a connected sum $F\#T$ of a surface-link F and a trivial torus-knot T is a trivial surface-link, then F is a trivial surface-link.

(2) If F is a trivial surface-link and $(D \times I, D' \times I)$ is an O2-handle pair on F, then $F(D \times I, D' \times I)$ is a trivial surface-link.

(3) Any trivial surface-link has only unique O2-handle pair.

Proof of Lemma 2.5. (1) $\Rightarrow$ (2): Let B be the associated bump of the O2-handle pair $(D \times I, D' \times I)$. A 4-ball A obtained by taking a bi-collar $c(B \times [-1,1])$ of B in $\mathbf{R}^4$ with $c(B \times 0) = B$ gives a connected sum decomposition $F \cong F(D \times I, D' \times I)\#T$. By (1), $F(D \times I, D' \times I)$ is a trivial surface-link.

(2) $\Rightarrow$ (3): Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be O2-handle pairs with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Let $F(D \times I, D' \times I) = F^c_\delta \cup \delta_{D \times I, D' \times I}$ and $F(E \times I, E' \times I) = F^c_\delta \cup \delta_{E \times I, E' \times I}$ be trivial surface-links for disks $\delta_{D \times I, D' \times I}$ and $\delta_{E \times I, E' \times I}$ in the boundaries $\partial\Delta(D \times I, D' \times I)$ and $\partial\Delta(E \times I, E' \times I)$ of the 2-handle unions $\Delta(D \times I, D' \times I)$ and $\Delta(E \times I, E' \times I)$, respectively. Let $F(D \times I, D' \times I)_0$ and $F(E \times I, E' \times I)_0$ be the components of $F(D \times I, D' \times I)$ and $F(E \times I, E' \times I)$ containing the loop $\partial\delta_{D \times I, D' \times I} = \partial\delta_{E \times I, E' \times I}$, respectively, which are made split from the other components in $\mathbf{R}^4$ because all the components of every trivial surface-link are split in $\mathbf{R}^4$. Since $F(D \times I, D' \times I)_0$ and $F(E \times I, E' \times I)_0$ are trivial surface-knots of the same genus, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ sending $F(D \times I, D' \times I)_0$ to $F(E \times I, E' \times I)_0$ orientation-preservingly and the other components identically. By a cellular move of $\delta_{D \times I, D' \times I}$ in $F(D \times I, D' \times I)_0$, this map f is modified to have $f(\delta_{D \times I, D' \times I}) = \delta_{E \times I, E' \times I}$. Further, this map f is modified to send $F^c_\delta \cup \delta_{D \times I, D' \times I}$ to $F^c_\delta \cup \delta_{E \times I, E' \times I}$ by sending all the components except for $F(D \times I, D' \times I)_0$ and $F(E \times I, E' \times I)_0$ identically. Thus, we have an equivalence f with $f(F^c_\delta) = F^c_\delta$ and $f(\delta_{D \times I, D' \times I}) = \delta_{E \times I, E' \times I}$. By Lemma 2.3, the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ are equivalent.

(3) $\Rightarrow$ (1): Let $F_i\,(i = 0, 1, \dots, r)$ be the components of F, and $F\#T = F_0\#T \cup F_1 \cup \dots \cup F_r$ a trivial surface-link. Let V be the disjoint union of handlebodies $V_i\,(i = 0, 1, \dots, r)$ in $\mathbf{R}^4$ such that $\partial V_0 = F_0\#T$ and $\partial V_i = F_i\,(i = 1, 2, \dots, r)$.

A *loop basis* of $F_0\#T$ of genus $g + 1$ is a system of oriented simple loop pairs $(e_j, e'_j)\,(j = 0, 1, 2, \dots, g)$ on $F_0\#T$ representing a basis for $H_1(F_0\#T; \mathbb{Z})$ such that $e_j \cap e_{j'} = e'_j \cap e'_{j'} = e_j \cap e'_{j'} = \emptyset$ for all distinct j, j' and $e_j \cap e'_j$ is one point with the intersection number $\mathrm{Int}(e_j, e'_j) = +1$ in $F_0\#T$ for all j. A loop basis $(e_j, e'_j)\,(j = 0, 1, 2, \dots, g)$ of $F_0\#T$ is *spin* if the $\mathbb{Z}_2$-quadratic function $q : H_1(F_0\#T; \mathbb{Z}_2) \to \mathbb{Z}_2$ associated with the surface-knot $F_0\#T$ has $q(e_j) = q(e'_j) = 0$ for all j. The following result is obtained from [3, Lemma 2.2] where a non-oriented spin loop basis

$(e_j, e'_j)\,(j = 0, 1, 2, \ldots, g)$ of $F_0\#T$ is constructed.

(2.5.1) For a surface-knot $F_0\#T$ of genus $g+1$ in $\mathbf{R}^4$, there is a spin loop basis $(e_j, e'_j)\,(j = 0, 1, 2, \ldots, g)$ of $F_0\#T$. In particular, for a trivial surface-knot $F_0\#T$ bounded by a handlebody V_0 in $\mathbf{R}^4$, every loop basis $(e_j, e'_j)\,(j = 0, 1, 2, \ldots, g)$ on ∂V_0 with $e'_j\,(j = 0, 1, 2, \ldots, g)$ a meridian loop system of V_0 has $q(e'_j) = 0$ and either $q(e_j) = 0$ or $q(e_j + e'_j) = 0$ for all j, where $e_j + e'_j$ denotes a Dehn twist of e_j along e'_j.

The following result is obtained from [4]:

(2.5.2) For any two loop bases $(e_j, e'_j)\,(j = 0, 1, 2, \ldots, g)$ and $(\tilde{e}_j, \tilde{e}'_j)\,(j = 0, 1, 2, \ldots, g)$ on a trivial genus g surface-knot $F_0\#T$ with $q(e_j) = q(\tilde{e}_j)$ and $q(e'_j) = q(\tilde{e}'_j)$ for all j, there is an orientation-preserving diffeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ with $f(F_0\#T) = F_0\#T$ such that $f(e_j) = \tilde{e}_j$ and $f(e'_j) = \tilde{e}'_j$ for all j.

Let $(D \times I, D' \times I)$ be an O2-handle pair on $F\#T$ in $\mathbf{R}^4$ attached to T^o such that $(F\#T)(D \times I, D' \times I) \cong F$. By (2.5.1), there is a spin loop basis for $F_0\#T$ containing the pair $(\partial D, \partial D')$. Also, let $(e_i, e'_i)\,(i = 0, 1, 2, \ldots, g)$ be a spin loop basis for $F_0\#T$ such that e_0 bounds a disk d in $\mathbf{R}^4$ with $d \cap V = e_0$ and e'_0 bounds a meridian disk d' of V_0. Since the handlebodies $V_i\,(i = 0, 1, \ldots, r)$ are splittable in $\mathbf{R}^4$ by [6], we see from (2.5.2) that there is an orientation-preserving diffeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ with $f(F_0\#T) = F_0\#T$ and $f|_{V_i} = 1\,(i = 1, 2, \ldots, r)$ such that $f(\partial D) = e_0$ and $f(\partial D') = e'_0$. A thickening pair $(d \times I, d' \times I)$ of the disk pair (d, d') is an O2-handle pair with $(F\#T)(d \times I, d' \times I)$ is a trivial surface-knot. Since $(f(D) \times I, f(D') \times I)$ is an O2-handle pair on $F\#T$, we obtain from (3) that

$$\begin{aligned} F &\cong (F\#T)(D \times I, D' \times I) \\ &\cong (F\#T)(f(D) \times I, f(D') \times I) \\ &\cong (F\#T)(d \times I, d' \times I). \end{aligned}$$

Thus, F is a trivial surface-link. □

3 Uniqueness of an orthogonal 2-handle pair

The following theorem is our main result.

Theorem 3.1. Any (not necessarily trivial) surface-link has only unique O2-handle pair.

Theorem 1.1 is proved by Theorem 3.1 and Lemma 2.5, which is done as follows:

Proof of Theorem 1.1. Let F be a stably trivial link. That is, assume that a stabilization $F^{\#sT} = F\#_{k=1}^{s} T_k$ of F is a trivial link for some $s \geq 1$. By Theorem 3.1 and Lemma 2.5, $F\#_{k=1}^{s-1} T_k$ is a trivial surface-link. Inductively, F is a surface-link, so that any handle-irreducible summand F^* of F is a trivial S^2-link. □

The following lemma is a key lemma to Theorem 3.1.

Lemma 3.2. Let $(D \times I, D' \times I)$ and $(E' \times I, E' \times I)$ be O2-handle pairs on a surface-link F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Then there is a 2-handle $D'_* \times I$ on F with $\partial D'_* = \partial D'$ such that the pair $(E \times I, D'_* \times I)$ is an O2-handle pair on F and the 2-handle $D'_* \times I$ on F is equivalent to the 2-handle $D' \times I$.

By assuming Lemma 3.2, the proof of of Theorem 3.1 is done as follows:

Proof of Theorem 3.1. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be O2-handle pairs on a surface-link F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Then there is a 2-handle $D'_* \times I$ on F be a 2-handle on F given by Lemma 3.2 such that $(E \times I, D'_* \times I)$ is an O2-handle pair on F and there is an equivalence f from F to itself such that the restriction $f|_F$ is the identity map on F and $f(D'_* \times I) = D' \times I$. By Lemma 2.2 and Corollary 2.4, the O2-handle pair $(E \times I, E' \times I)$ on F is equivalent to the O2-handle pair $(E \times I, D'_* \times I)$ on F, which is equivalent to the O2-handle pair $(f(E) \times I, D' \times I)$ on F and hence to the O2-handle pair $(D \times I, D' \times I)$ on F. Thus, the O2-handle pair $(D \times I, D' \times I)$ on F is equivalent to an O2-handle pair $(E \times I, E' \times I)$ on F. This completes the proof of Theorem 3.1. □

Throughout the remainder of this section, the proof of Lemma 3.2 is done.

Proof of Lemma 3.2. For the core disks D, D' E and E' of $D \times I$, $D' \times I$, $E \times I$ and $E' \times I$, respectively, assume the following conditions (see Fig. 3):
(a) A neighborhood $n(\partial D)$ of ∂D in D coincides with a neighborhood $n(\partial E)$ of ∂E in E and $(\partial D') \times I \cap \partial E' = \emptyset$ by slightly sliding $\partial E'$ along F,

(b) The disk interiors $\mathrm{Int}D$, $\mathrm{Int}D'$, $\mathrm{Int}E$ and $\mathrm{Int}E'$ meet transversely except for the part $n(\partial D) = n(\partial E)$ and $D \cap D' = \partial D \cap \partial D' = \{p_{D\cap D'}\}$ and $E \cap E' = \partial E \cap \partial E' = \{p_{E\cap E'}\}$ for distinct points $p_{D\cap D'}$ and $p_{E\cap E'}$.

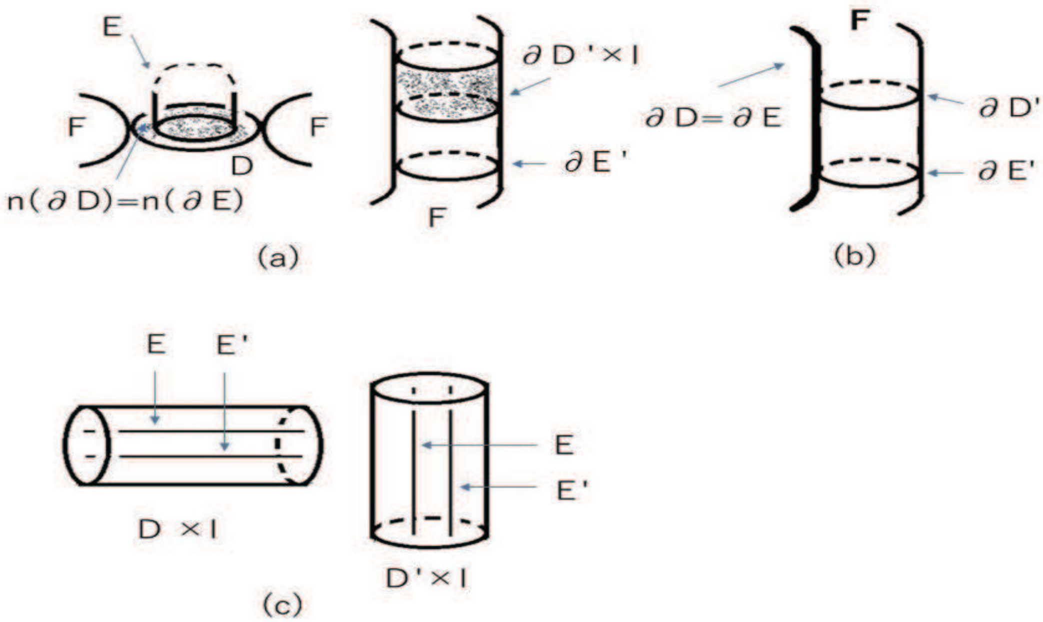

Figure 3: Positions among the core disks D, D', E and E'

(c) The disk interiors $E \backslash n(\partial E)$ and IntE' meet $D \times I$ with a finite number of mutually disjoint arcs which are parallel to a fiber I of the line bundle $D \times I$ over D. Similarly, the disk interiors Int$E$ and IntE' meet $D' \times I$ with a finite number of mutually disjoint arcs which are parallel to a fiber I of the line bundle $D' \times I$ over D'.

The following operation, called *Finger Move Canceling* eliminates an intersection point $x \in \mathrm{Int}E \cap \mathrm{Int}D'$ by creating a disk D'' with $\partial D'' = \partial D'$ from the disk D'.

Finger Move Canceling. Let S be a trivial S^2-knot in $\mathbf{R}^4$ such that the 2-sphere S^2 is disjoint from F and D' and meets the disk interior Int$E$ transversely in just one point x. Let y be a double point between the disk interiors Int$E$ and IntD', and L a simple arc in the disk E joining x and y not meeting the other double points between E' and D. Let V_L be a solid tube in $\mathbf{R}^4$ around the arc L such that $V_L \cap E = L$ and V_L joins a disk neighborhood d_x of x in the disk D' and a disk neighborhood d_y of y in the 2-sphere S. Then a disk D'' with $\partial D'' = \partial D'$ and $E \cap D'' = E \cap D' \setminus \{x\}$ is constructed so that

$$D'' = \mathrm{cl}(D' \setminus d_x) \cup \mathrm{cl}(\partial V_L \setminus (d_x \cup d_y)) \cup \mathrm{cl}(S \setminus d_y).$$

A trivial S^2-knot S used in Finger Move Canceling is constructed as follows:

Claim 3.2.1. After an isotopic deformation of F, E and E' keeping D and D', there is a trivial S^2-knot S in $\mathbf{R}^4$ such that

(1) $S \cap D = S \cap E = \{x\}$ for a point $x \in n(\partial D) = n(\partial E)$,

(2) $S \cap (F \cup D' \times I \cup E') = \emptyset$,

(3) There is a 3-ball B^S in $\mathbf{R}^4$ with $\partial B^S = S$ such that $B^S \cap (F \cup D' \times I) = D'$.

By assuming Claim 3.2.1, let D'_1 be a disk parallel to the core disk D' of the 2-handle $D' \times I$ on the surface-link F such that $D'_1 \cap F = \partial D'_1$ and $D'_1 \cap (D' \times I) = \emptyset$. Let y be a double point between the disk interiors $\mathrm{Int}D'_1$ and $\mathrm{Int}E$. Apply Finger Move Canceling to the trivial S^2-knot S in Claim 3.2.1 along an arc c in E from the point x to the point $x \in S \cap E$ which avoids the double point set $E \cap D'_1 \setminus \{y\}$ to obtain a disk D'_2 such that

(1) $\partial D'_2 = \partial D'_1$,

(2) $E \cap D'_2 = (E \cap D'_1) \setminus \{y\}$, and

(3) $D'_2 \cap F = \partial D'_2$ and $D'_2 \cap (D' \times I) = \emptyset$.

By continuing this Finger Move Canceling on a trivial S^2-knot parallel to S, a 2-handle $D'_* \times I$ on F with $\partial D'_* = \partial D'_1$ such that $(E \times I, D'_* \times I)$ is an O2-handle pair on F is obtained. The following claim shows that this 2-handle $D'_* \times I$ on the surface-link F is a desired 2-handle in Lemma 3.2.

Claim 3.2.2. The 2-handle $D_1^* \times I$ on F is equivalent to the 2-handle $D'_1 \times I$.

This completes the proof of Lemma 3.2 under the assumptions of Claims 3.2.1 and 3.2.2.

The proof of Claim 3.2.1 is done as follows:

Proof of Claim 3.2.1. Let Δ is the handle union of the O2-handle pair $(D \times I, D' \times I)$, and $B = B_\Delta$ an associated bump of Δ (see Fig. 2). Assume that the bump B is in the 3-space $\mathbf{R}^3$ by an isotopic deformation of B. Let $T^o_B = F \cap B$ be an unknotted once-punctured torus in B. Let $F^c = \mathrm{cl}(F \setminus T^o)$. For the sub-surface $T^o_\Delta = F \cap \Delta$ of T^o, the closed complement $A(T^o) = \mathrm{cl}(T^o_B \setminus T^o_\Delta)$ is an annulus bounded by the loops $o_F = \partial T^o_B = \partial \delta_B = \partial F^c_B$ and $o_\Delta = \partial T^o_\Delta = \partial \delta_{D \times I, D' \times I}$.

Assume that the disk E meets the associated bump B with the union of the loop ∂E, a set $J^E_{D\times I}$ of trivial parallel arcs and a set $J^E_{D'\times I}$ of trivial parallel arcs such that

(i) the set $J^E_{D\times I}$ of trivial proper parallel arcs in B is obtained by extending the intersection set $\mathrm{Int}E \cap (D\times I)$ of trivial parallel arcs in $D\times I$ and

(ii) the set $J^E_{D'\times I}$ of trivial proper parallel arcs in B is obtained by extending the intersection set $\mathrm{Int}E \cap (D'\times I)$ of trivial parallel arcs in $D'\times I$.

Similarly, assume that the disk E' meets the associated bump B with the union of the loop $\partial E'$, a set $J^{E'}_{D\times I}$ of trivial proper parallel arcs in B and a set $J^{E'}_{D'\times I}$ of trivial proper parallel arcs in B such that

(i)$'$ the set $J^{E'}_{D\times I}$ of trivial proper parallel arcs in B is obtained by extending the intersection set $\mathrm{Int}E' \cap (D\times I)$ of trivial parallel arcs in $D\times I$ and

(ii)$'$ the set $J^{E'}_{D'\times I}$ of trivial proper parallel arcs in B is obtained by extending the intersection $\mathrm{Int}E' \cap (D'\times I)$ of trivial parallel arcs in $D'\times I$.

Let

$$J = J^E_{D\times I} \cup J^E_{D'\times I} \cup J^{E'}_{D\times I} \cup J^{E'}_{D'\times I}.$$

Let $o_E = \partial n(\partial E)\setminus \partial E$. Let $d(D')$ be a disk in the associated bump B containing the disk D' in the interior such that the link $o_E \cup \partial d(D')$ for the boundary loop $\partial d(D')$ is a trivial link in B and $\partial d(D')$ transversely meets the disks E and D with just one point in the interior of the part $n(\partial D) = n(\partial E)$. A situation of the intersections of the disks E and E' with the associated bump B of the O2-handle pair $(D\times I, D'\times I)$ is illustrated in Fig. 4.

Notations. For a subspace A of $\mathbf{R}^3[0]$ and a subinterval K of $\mathbf{R}$ the notation

$$AK = \{(x,t)\in \mathbf{R}^4 |\, x\in A, t\in K\}$$

is used for a subspace of $\mathbf{R}^4$ as it is used in [16]. Since the associated bump $B = B_\Delta$ of the handle union Δ of the O2-handle pair $(D\times I, D'\times I)$ is assumed to be in the 3-space $\mathbf{R}^3 = \mathbf{R}^3[0]$, the 4-ball

$$B[-1,1]\subset \mathbf{R}^3[-1,1]\subset \mathbf{R}^4$$

is a bi-collar of the associated bump of B in the 4-space $\mathbf{R}^4$. To avoid a confusion, the notation AK_B is used for the subspace AK in $B[-1,1]$ defined for a subspace A of B and a subinterval K of $[-1,1]$.

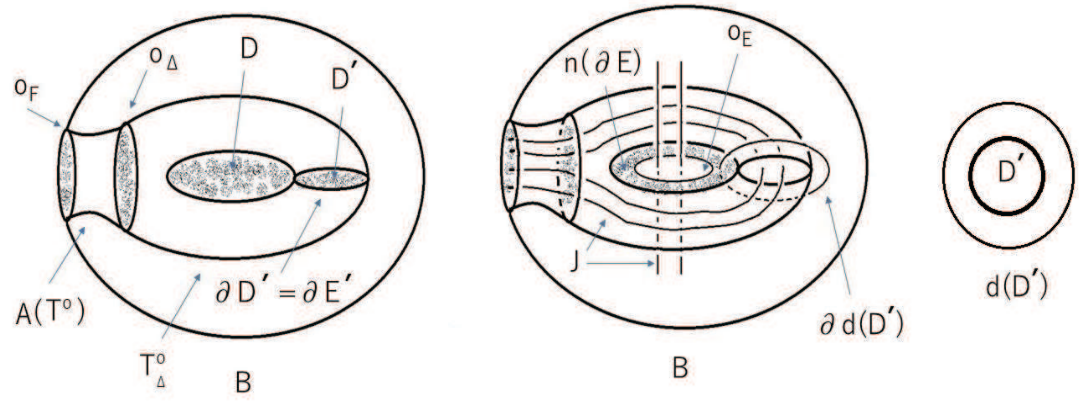

Figure 4: A situation of the intersections of the disks E and E' with the associated bump B

The following situation may be imposed on the intersection of the union $F \cup E \cup E'$ with the 4-ball $B[-1,1]$:

(3.2.1.1) The surface-knot F and the disks E and E' meet the 4-ball $B[-1,1]$ such that

$$(F \cup E \cup E') \cap B[t]_B = \begin{cases} (o_\Delta \cup J \cup o_E \cup \partial E')[t]_B, & \text{for } 0 < t \leq 1, \\ (T^o_\Delta \cup J \cup n(\partial E))[t]_B, & \text{for } t = 0, \\ J[t]_B, & \text{for } -1 \leq t < 0. \end{cases}$$

In (3.2.1.1), note that the annulus $A(T^o) \subset B$ bounded by $o_\Delta \cup o_F$ is deformed into the annulus $o_\Delta[0,1]_B \subset B[-1,1]$ identifying $o_\Delta \subset B$ with $o_\Delta[0]_B \subset B[0]_B$ and $o_F \subset B$ with $o_\Delta[1]_B \subset B[1]_B$.

Consider the 4-ball $U = \mathrm{cl}(\bar{\mathbf{R}}^4 \setminus B[-1,1])$ for the one-point-compactification $\bar{\mathbf{R}}^4$ of the 4-space $\mathbf{R}^4$ and the proper surfaces

$$\begin{aligned} R(F) &= \mathrm{cl}(F \setminus F \cap B[-1,1]), \\ R(E) &= \mathrm{cl}(E \setminus E \cap B[-1,1]), \\ R(E') &= \mathrm{cl}(E' \setminus E' \cap B[-1,1]) \end{aligned}$$

in the 4-ball U. The link

$$\mathbf{L} = \partial R(F) \cup \partial R(E) \cup \partial R(E')$$

in the 3-sphere $\partial U = B[-1]_B \cup (\partial B)[-1,1]_B \cup B[1]_B$ is illustrated in Fig. 5, where $\partial R(F)$ and $\partial R(E) \cup \partial R(E')$ are given as follows:

$$\begin{aligned} \partial R(F) &= o_\Delta[1]_B \subset \partial U, \\ \partial R(E) \cup \partial R(E') &= o_E[1]_B \cup \partial E'[1]_B \cup \mathbf{L}' \subset \partial U \\ &\qquad \text{for} \quad \mathbf{L}' = J[-1]_B \cup (\partial J)[-1,1]_B \cup J[1]_B. \end{aligned}$$

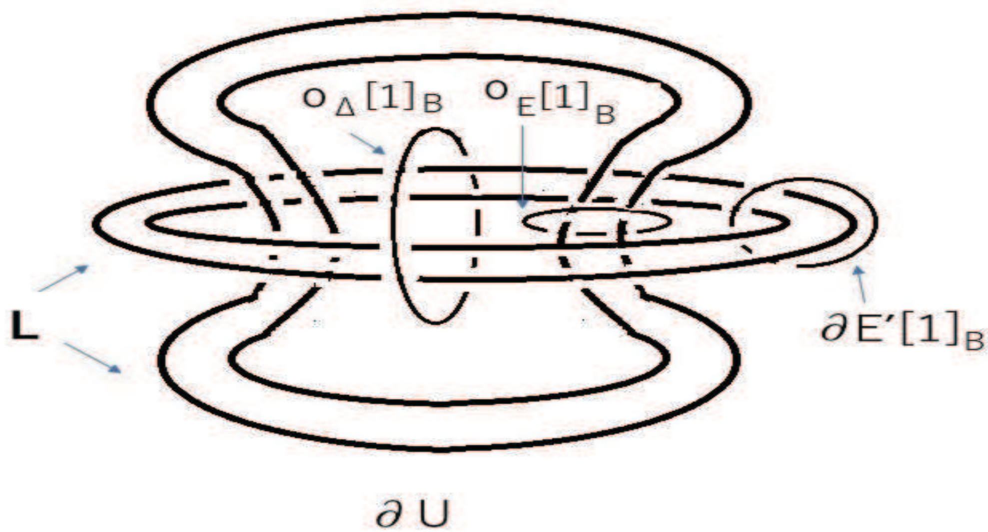

Figure 5: The link $\mathbf{L}$ in the 3-sphere ∂U

Consider the pair $(U, \partial U)$ as the one-point-compactification of the pair of the upper-half 4-space

$$\mathbf{R}^4_+ = \{(x,t) \in \mathbf{R}^3 \times \mathbf{R} |\, x \in \mathbf{R}^3, t \in \mathbf{R}\}$$

and the boundary 3-space $\partial \mathbf{R}^4_+ = \mathbf{R}^3 = \mathbf{R}^3[0]$. The same notations for the proper surface $R(F) \cup R(E) \cup R(E')$ in the 4-ball U and the link $\mathbf{L} = o_\Delta[1]_B \cup o_E[1]_B \cup \partial E'[1]_B \cup L'$ in the boundary 3-sphere ∂U are used for the corresponding proper surface in $\mathbf{R}^4_+$ and the corresponding link in the boundary 3-space $\mathbf{R}^3 = \mathbf{R}^3[0]$.

By an argument of [16], a normal form of the surface $R(F) \cup R(E) \cup R(E')$ in $\mathbf{R}^4_+$ is considered to obtain the following surface G from the surface $R(F) \cup R(E) \cup R(E')$ by an ambient isotopy of $\mathbf{R}^4_+$ keeping the boundary $\mathbf{R}^3 = \mathbf{R}^3[0]$ fixed:

(3.2.1.2) The surface G in $\mathbf{R}^4_+$ is given by

$$G \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 3 \\ \mathbf{d}(\mathbf{O})[t], & \text{for } t = 3, \\ \mathbf{O}[t], & \text{for } 2 < t < 3, \\ (\mathbf{L} \cup \mathbf{o} \cup \mathbf{b})[t], & \text{for } t = 2, \\ (\mathbf{L} \cup \mathbf{o})[t], & \text{for } 1 < t < 2, \\ (\mathbf{L} \cup \mathbf{d})[t], & \text{for } t = 1, \\ \mathbf{L}[t], & \text{for } 0 \leq t < 1, \end{cases}$$

where

- $\mathbf{d}$ is a disk system in $\mathbf{R}^3$ disjoint from the link L and $\mathbf{o} = \partial \mathbf{d}$, a trivial link,
- $\mathbf{b}$ is a band system in $\mathbf{R}^3$ spanning the link $L \cup \mathbf{o}$,
- $\mathbf{O}$ is a trivial link obtained from the link $\mathbf{L} \cup \mathbf{o}$ by the surgery along $\mathbf{b}$ and $\mathbf{d}(\mathbf{O})$ is a disk system bounding the trivial link $\mathbf{O}$.

Let $(d(D')[0], D'[0])$ be the disk pair in $\mathbf{R}^3[0]$ corresponding to the disk pair $(d(D')[1]_B, D'[1]_B)$ in the 3-ball $B[1]_B \subset \partial U$ obtained from the disk pair $(d(D'), D')$ in B. Let $(\iota \cdot d(D')[0], \iota \cdot D'[0])$ be the disk pair in $\mathbf{R}^3[0]$ corresponding to the disk pair $(d(D')[-1]_B, D'[-1]_B)$ in the 3-ball $B[-1]_B \subset \partial U$ obtained from the disk pair $(d(D'), D')$ in B, where note that the disk pair $(d(D')[-1]_B, D'[-1]_B)$ is the image of the disk pair $(d(D')[1]_B, D'[1]_B)$ by the reflection ιin $B[-1, 1]$ sending the point (x, t) to the point $(x, -t)$ for $x \in B$ and $t \in [-1, 1]$.

By a replacement to a narrow band and a band slide on the band system $\mathbf{b}[2]$ in (3.2.1.2), the following condition cab be imposed:

(3.2.1.3) The band system $\mathbf{b}[2]$ does not meet the disks $d(D')[2]$ and $\iota \cdot d(D')[2]$. Thus, for every t with $0 \leq t \leq 3$, we have:

$$\begin{aligned} d(D')[3] \cap G &= d(D')[3] \cap \mathbf{d}(\mathbf{O})[3], \\ d(D')[t] \cap G &= (D' \cap \mathbf{L})[t], \quad \text{for } 0 \leq t < 3; \\ \iota \cdot d(D')[3] \cap G &= \iota \cdot d(D')[3] \cap \mathbf{d}(\mathbf{O})[3], \\ \iota \cdot d(D')[t] \cap G &= (\iota \cdot D' \cap \mathbf{L})[t], \quad \text{for } 0 \leq t < 3. \end{aligned}$$

Let $\mathbf{p} = D' \cap \mathbf{L}$ be the point system in B, and $\mathbf{p}[0]$ the point system in $\mathbf{R}^3[0]$ representing the point system $\mathbf{p}[1]_B$ in 3-ball $B[1]_B \subset \partial U$. Similarly, let $\iota \cdot \mathbf{p}[0]$ be the point system in $\mathbf{R}^3[0]$ representing the point system $\mathbf{p}[-1]_B$ in 3-ball $B[-1]_B \subset \partial U$ which is ι-reflection image of the point system $\mathbf{p}[1]_B$.

In (3.2.1.3), the intersection $d(D')[3]\cap\mathbf{d}(\mathbf{O})[3]$ is the disjoint union of an improper arc system $\boldsymbol{\alpha}[3]$ joining the point system $\mathbf{p}[3]$ with a point system $\mathbf{p}^d[3]$ in the loop $\partial d(D')[3]$ and a proper arc system $\boldsymbol{\beta}[3]$ in the disk $d(D')[3]$.

Similarly, the intersection $\iota\cdot d(D')[3]\cap\mathbf{d}(\mathbf{O})[3]$ is the disjoint union of an improper arc system $\iota\cdot\boldsymbol{\alpha}[3]$ joining the point system $\iota\cdot\mathbf{p}[3]$ with a point system $\iota\cdot\mathbf{p}^d[3]$ in the loop $\partial\iota\cdot d(D')[3]$ and a proper arc system $\iota\cdot\boldsymbol{\beta}[3]$ in the disk $\iota\cdot d(D')[3]$.

Let $\boldsymbol{\beta}^+[3]$ and $\iota\cdot\boldsymbol{\beta}^+[3]$ be slightly extended arc systems of the arc systems $\boldsymbol{\beta}[3]$ and $\iota\cdot\boldsymbol{\beta}[3]$ in $\mathbf{d}(\mathbf{O})[3]$, respectively. Let $\boldsymbol{\gamma}$ and $\iota\cdot\boldsymbol{\gamma}$ be the arc systems in $\mathbf{R}^3[3,4]$ obtained respectively by deforming the extended arc systems $\boldsymbol{\beta}^+[3]$ and $\iota\cdot\boldsymbol{\beta}^+[3]$ as follows:

(3.2.1.4) For every t with $3\le t\le 4$, the arc systems $\boldsymbol{\gamma}$ and $\iota\cdot\boldsymbol{\gamma}$ in $\mathbf{R}^3[3,4]$ are given by

$$\boldsymbol{\gamma}\cap\mathbf{R}^3[t]=\begin{cases}\boldsymbol{\beta}\sqcap^+[t], & \text{for } t=4,\\ \partial\boldsymbol{\beta}^+[t], & \text{for } 3\le t<4,\end{cases}$$

where $\boldsymbol{\beta}\sqcap^+[4]$ is an arc system which is deformed from the arc system $\boldsymbol{\beta}^+[4]$ with $\partial\boldsymbol{\beta}\sqcap^+[4]=\partial\boldsymbol{\beta}^+[4]$ and $\boldsymbol{\beta}\sqcap^+[4]\cap d(D')[4]=\emptyset$ (see Fig. 6), and

$$\iota\cdot\boldsymbol{\gamma}\cap\mathbf{R}^3[t]=\begin{cases}\iota\cdot\boldsymbol{\beta}\sqcap^+[t], & \text{for } t=4,\\ \partial\iota\cdot\boldsymbol{\beta}^+[t], & \text{for } 3\le t<4,\end{cases}$$

where $\iota\cdot\boldsymbol{\beta}\sqcap^+[4]$ is an arc system which is deformed from the arc system $\iota\cdot\boldsymbol{\beta}^+[4]$ with $\partial\iota\cdot\boldsymbol{\beta}\sqcap^+[4]=\partial\iota\cdot\boldsymbol{\beta}^+[4]$ and $\iota\cdot\boldsymbol{\beta}\sqcap^+[4]\cap\iota\cdot d(D')[4]=\emptyset$ (see Fig. 6).

The deformation from the extended arc systems $\boldsymbol{\beta}^+[3]$ and $\iota\cdot\boldsymbol{\beta}^+[3]$ into the arc systems $\boldsymbol{\gamma}$ and $\iota\cdot\boldsymbol{\gamma}$ in (3.2.1.4) turns the disk system $\mathbf{d}(\mathbf{O})[3]$ into a disk system $\mathbf{d}'(\mathbf{O})\subset\mathbf{R}^3[3,4]$ with the intersection

$$\mathbf{d}''(\mathbf{O})[3]=\mathbf{d}'(\mathbf{O})\cap\mathbf{R}^3[3]$$

a compact multi-punctured disk system such that

$$d(D')[3,4]\cap\mathbf{d}'(\mathbf{O})=\boldsymbol{\alpha}[3]\quad\text{and}\quad\iota\cdot d(D')[3,4]\cap\mathbf{d}'(\mathbf{O})=\iota\cdot\boldsymbol{\alpha}[3].$$

Let $\mathbf{q}$ be a point system in the arc system $J^{E'}_{D\times I}\cup J^{E'}_{D'\times I}$ in B which is not in the 2-handle union Δ. Let $\mathbf{a}$ be an arc system in the link $\mathbf{L}$ in B joining the point system $\mathbf{p}$ with the point system $\mathbf{q}$. Let $\mathbf{a}[0]$ and $\iota\cdot\mathbf{a}[0]$ be the arc systems in $\mathbf{R}^3[0]$ representing the arc system $\mathbf{a}[1]_B$ in $B[1]_B$ and the arc system $\mathbf{a}[-1]_B$ in $\iota(B[1]_B)=B[-1]_B$, respectively. By a replacement to a narrow band on the band system $\mathbf{b}[2]$ and a band slide, assume that the band system $\mathbf{b}[2]$ does not attach to the arc systems $\mathbf{a}[2]$ and

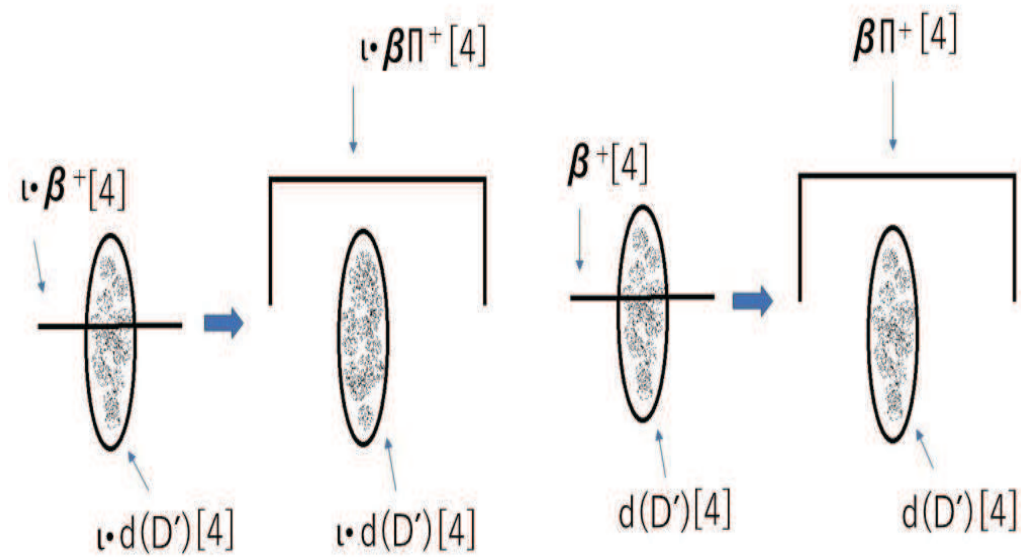

Figure 6: The arc systems $\boldsymbol{\beta}\sqcap^+[4]$ and $\iota\cdot\boldsymbol{\beta}\sqcap^+[4]$ deformed from $\boldsymbol{\beta}^+[4]$ and $\iota\cdot\boldsymbol{\beta}^+[4]$

$\iota\cdot\mathbf{a}[2]$. Then the arc systems $\mathbf{a}[3]$ and $\iota\cdot\mathbf{a}[3]$ are in the boundary of the multi-punctured disk system $\mathbf{d}''(\mathbf{O})[3]$ with $\partial\mathbf{a}[3]=\mathbf{p}[3]\cup\mathbf{q}[3]$ and $\partial\iota\cdot\mathbf{a}[3]=\iota\cdot\mathbf{p}[3]\cup\iota\cdot\mathbf{q}[3]$.

Let $\mathbf{a}^d[3]$ and $\iota\cdot\mathbf{a}^d[3]$ be arc systems in the multi-punctured disk system $\mathbf{d}''(\mathbf{O})[3]$ such that $\partial\mathbf{a}^d[3]=\mathbf{p}^d[3]\cup\mathbf{q}[3]$ and $\partial\iota\cdot\mathbf{a}^d[3]=\iota\cdot\mathbf{p}^d[3]\cup\iota\cdot\mathbf{q}[3]$. See Fig. 7 for this situation where $T^0_\Delta[3]$ and $\iota\cdot T^0_\Delta[3]$ denote the copies of $T^0_\Delta\subset B$ in $\mathbf{R}^3[3]$ via the copy in $B[1]$ and the reflection image in $\iota(B[1])=B[-1]$ for the reflection ι in $B[-1,1]$, respectively.

Let $n(\mathbf{a}^d)[3]$ and $n(\iota\cdot\mathbf{a}^d)[3]$ be regular neighborhood disk systems of the arc systems $\mathbf{a}^d[3]$ and $\iota\cdot\mathbf{a}^d[3]$ in the multi-punctured disk system $\mathbf{d}''(\mathbf{O})[3]$.

Let $\mathbf{d}^*(\mathbf{O})=\mathrm{cl}(\mathbf{d}'(\mathbf{O})\setminus(n(\mathbf{a}^d)[3]\cup n(\iota\cdot\mathbf{a}^d)[3]))$, and $\mathbf{O}^*[t]$ the trivial link obtained from the trivial link $\mathbf{O}[t]$ by the surgery along the disk systems $n(\mathbf{a}^d)[t]$ and $n(\iota\cdot\mathbf{a}^d)[t]$ for every t with $2<t<3$. Also, let $\mathbf{L}^*[t]$ be the link obtained from the link $\mathbf{L}[t]$ by surgery along the disk systems $n(\mathbf{a}^d)[t]$ and $n(\iota\cdot\mathbf{a}^d)[t]$ for every t with $1\leq t\leq 2$. Then the surface G^* in $\mathbf{R}^4_+$ which is isotopic to G by an ambient isotopy keeping

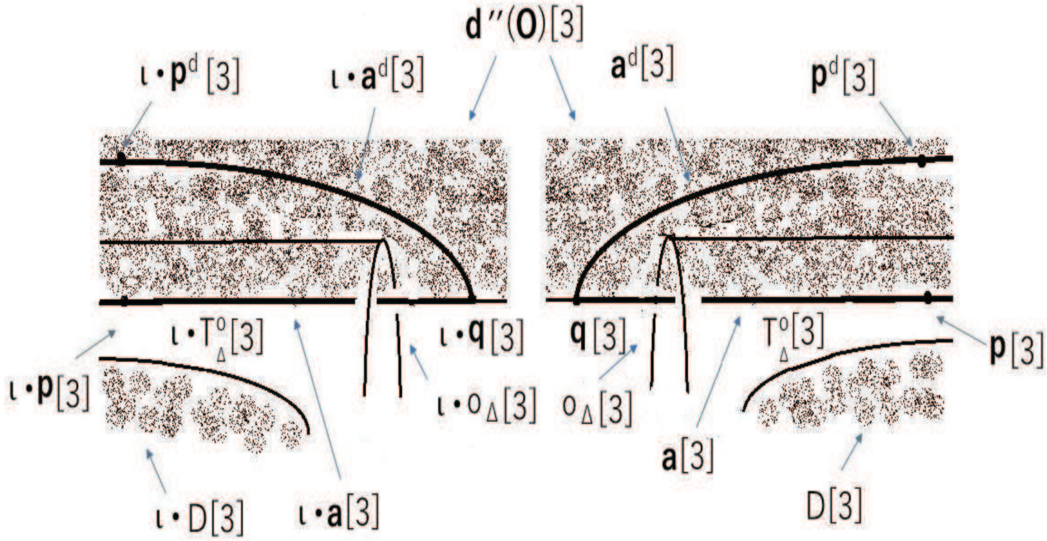

Figure 7: Arc systems $\mathbf{a}^d[3]$ and $\iota \cdot \mathbf{a}^d[3]$

$\mathbf{R}^3[0]$ fixed is given by

$$G^* \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 4 \\ \mathbf{d}'(\mathbf{O}) \cap \mathbf{R}^3[t], & \text{for } 3 < t \leq 4, \\ \mathbf{d}^*(\mathbf{O})[t], & \text{for } t = 3, \\ \mathbf{O}^*[t], & \text{for } 2 < t < 3, \\ (\mathbf{L}^* \cup \cap \mathbf{o} \cup \mathbf{b})[t], & \text{for } t = 2, \\ (\mathbf{L}^* \cup \mathbf{o})[t], & \text{for } 1 < t < 2, \\ (\mathbf{L}^* \cup \mathbf{d})[t], & \text{for } t = 1, \\ \mathbf{L}^*[t], & \text{for } 0 \leq t < 1. \end{cases}$$

Let $J^*[1]_B \cup J^*[-1]_B$ be the arc system in the 3-sphere $\partial(B[-1,1]) = \partial U$ obtained from $J[1]_B \cup J[-1]_B$ by replacing the link $\mathbf{L}[1]_B$ with the link $\mathbf{L}^*[1]_B$ in $\partial(B[-1,1]) = \partial U$.

The multi-punctured disk system $\mathbf{d}''(\mathbf{O})[3]$ is deformed in $\mathbf{R}^3[3]$ so that $T^o_\Delta[0]$ does not meet the neighborhood disk systems $n(\mathbf{a}^d)[3]$ and $n(\iota \cdot \mathbf{a}^d)[3]$. Then the arc systems $J^*[1]_B$ and $J^*[-1]_B$ extend to the disk system $J^*[-1,1]_B$ in $B[-1,1]$.

Let F^*, E^* and E'^* be the deformation results of F, E and E' using G^* and $J^*[-1,1]_B$, which are obtained by isotopic deformations on F, E and E' keeping D and D' fixed. Let D^S and $\iota \cdot D^S$ be the disks in $\mathbf{R}^3[0,4]$ defined by

$$D^S \cap \mathbf{R}^3[t] = \begin{cases} d(D')[t], & \text{for } t = 4, \\ \partial d(D')[t], & \text{for } 0 \leq t < 4, \end{cases}$$

$$\iota \cdot D^S \cap \mathbf{R}^3[t] = \begin{cases} \iota \cdot d(D')[t], & \text{for } t = 4, \\ \partial\iota \cdot d(D')[t], & \text{for } 0 \le t < 4, \end{cases}$$

Let S be the 2-sphere obtained from the disks D^S and $\iota \cdot D^S$ by connecting the tube $\partial d(D')[-1,1]_B$ in the 4-ball $B[-1,1]$ bounded by the loops ∂D^S and $\partial\iota \cdot D^S$. By construction, this 2-sphere S does not meet the surface-link F^* and the disks D', E'^* and meets the disks D and E^* with just one point in the part $n(\partial D) = n(\partial E^*)$. By construction, there is a 3-ball B^S in $\mathbf{R}^4$ with $\partial B^S = S$ such that $B^S \cap (F^* \cup D' \times I) = D'$. Thus, S is a desired 2-sphere. This completes the proof of Claim 3.2.1. □

The proof of Claim 3.2.2 is done as follows:

Proof of Claim 3.2.2. Let S be a trivial 2-knot in Claim 3.2.1. Let $D'_1 \times I$ be a 2-handle on F with core disk D'_1 which is disjoint from $D' \times I$.

Let D'_2 be the disk obtained from the disk D'_1 and the 2-sphere S by taking the surgery along a 1-handle h joining a disk d' in D' and a disk d in the S^2-knot S and not meeting the interior of the 3-ball B^3. Let $D'_2 \times I$ be the 2-handle on F with D'_2 a core disk and with $\partial D'_2 \times I = \partial D'_1 \times I$ which is obtained from the 2-handle $D'_1 \times I$ and a collaring $S \times I$ of the trivial S^2-knot S and a collaring $h \times I$ of the 1-handle h. For the bounded surface $F^c_1 = \mathrm{cl}(F \setminus \partial D'_1 \times I)$, the surface-links $F(D'_1 \times I)$ and $F(D'_2 \times I)$ are given as follows:

$$\begin{aligned} F(D'_1 \times I) &= F^c_1 \cup D'_1 \times \partial I, \\ F(D'_2 \times I) &= F^c_1 \cup D'_2 \times \partial I. \end{aligned}$$

The disk union $D'_2 \times \partial I$ is obtained from the disk union $D'_1 \times \partial I$ by the surgery along the 1-handle union $h \times \partial I$. In Fig 8, it is shown that one 1-handle of the 1-handle union $h \times \partial I$ is a self-intersecting 1-handle connecting one disk of the disk union $D'_1 \times \partial I$ and one 3-ball in the 3-ball unions $B^3 \times \partial I$ for a collaring $B^3 \times I$ of B^3. This implies that the disk union $D'_2 \times \partial I$ is deformed into the disk union $D'_1 \times \partial I$ by an ambient isotopy of $\mathbf{R}^4$ keeping the surface F^c_1 fixed. Thus, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(D'_2 \times I)$ to $F(D'_1 \times I)$ keeping the surface F^c_1 identically.

The 2-handle $D'_* \times I$ on F constructed by continuing this operation has the property that the pair $(E \times I, D'_* \times I)$ is an O2-handle pair on F and there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(D'_* \times I)$ to $F(D'_1 \times I)$ keeping the surface F^c_1 identically.

Let $a' = \partial D \cap D'_1 \times I = \partial E \cap D'_* \times I$ be the arc parallel to a fiber I of the line bundle $\partial D'_1 \times I = \partial D'_* \times I$ over the circle $\partial D'_1 = \partial D'_*$. The arc a' attaching to $F(D'_1 \times I)$ is ∂-relatively isotopic to an arc parallel to F^c_1 through the disk D. Similarly, the arc a' attaching to $F(D'_* \times I)$ is also ∂-relatively isotopic to an arc parallel to F^c_1 through the disk E. This means that the equivalence f is isotopically deformed into

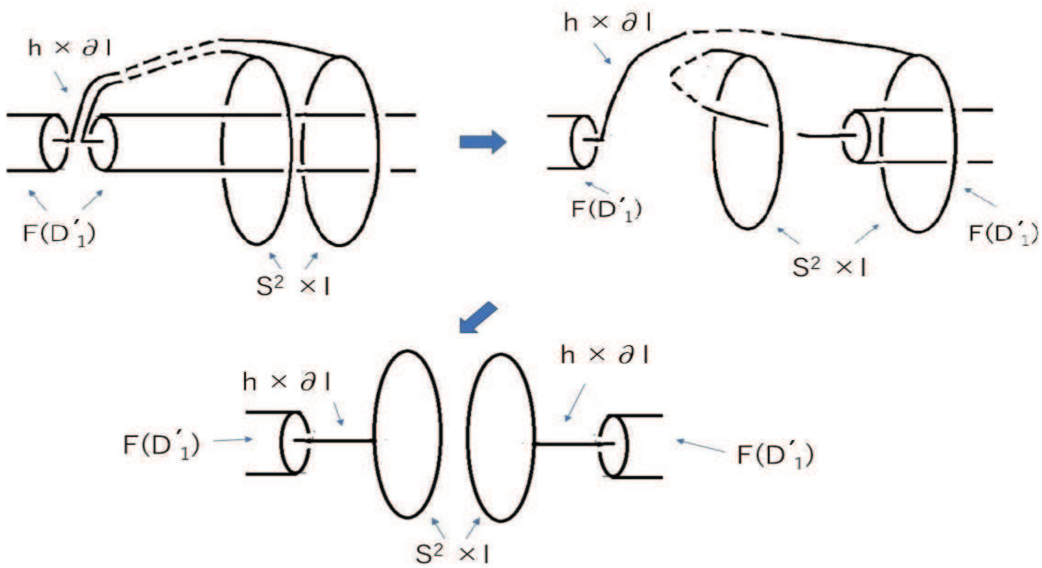

Figure 8: An equivalence from the disk D'_2 to the disk D'_1

an equivalence f' from $F(D'_* \times I)$ to $F(D'_1 \times I)$ keeping the surface F^c_1 fixed such that $f'(a') = a'$. Since the arc a' is regarded as a core of the 1-handle $D'_* \times I$ on $F(D'_* \times I)$ and a core of the 1-handle $D'_1 \times I$ on $F(D'_1 \times I)$, the equivalence f' is isotopically deformed into an equivalence f'' from F to itself such that the restriction $f'|_F$ is the identity and $f''(D'_* \times I) = D'_1 \times I$ (see [6]). This completes the proof of Claim 3.2.2. □

This completes the proof of Lemma 3.2. □

Acknowledgements. This work was partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849). An idea in this paper together with an idea in [15] was presented at the meeting "Differential Topology 19" held at Ritsumeikan Tokyo Campus on March 12, 2019, organized by Tetsuya Abe and Motoo Tange. The author would like to thank them for giving him a talk chance and the other participants for making some discussions. The first trial to this research project was done during his stay at Pusan National University in the season of cherry blossoms in full bloom of 2018 spring where the author would like to thank Sang Youl Lee and Jieon Kim for kind hospitalities during his stay. Since then several improvements of this paper were done and a nearly last improvement was done during his stay at Novosibirsk State University for "VI Russian-Chinese Conference on Knot Theory and Related Topics"held on June 17-21, 2019 where he would like to thank Nikolay Abrosimov and Andrei Vesnin for kind hospitalities during his stay. The author also would like to thank Seiichi Kamada for taking up a topic on a stabilization of a ribbon surface-knot in his lecture [7], which gave him a motivation to consider a stable surface-knot and Maggie Miller for giving sharp observations on earlier versions of this paper, Takao Matumoto for telling him where it is difficult to read, and Kengo Kawamura and Masaki Taniguchi talked at OCU Topology Seminars under a cooperation of Kouki Sato where some wrong expressions of the paper were pointed out.

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Uniqueness of an orthogonal 2-handle pair on a surface-link

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Keywords: Surface-link, Orthogonal 2-handle pair. *Mathematics Subject Classification 2010*: Primary 57Q45; Secondary 57N13

ABSTRACT

The proof of uniqueness of an orthogonal 2-handle pair on a surface-link is given from the viewpoint of a normal form of 2-handle core disks. A version to an immersed orthogonal 2-handle pair on a surface-link is also observed.

1. Introduction

A *surface-link* is a closed oriented (possibly disconnected) surface F embedded in the 4-space $\mathbf{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When $\mathbf{F}$ is connected, it is also called a *surface-knot*. Two surface-links F and F' are *equivalent* by an *equivalence* f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism (or piecewise-linear homeomorphism) $f : \mathbf{R}^4 \to \mathbf{R}^4$. A *trivial* surface-link is a surface-link F which is the boundary of disjoint handlebodies smoothly embedded in $\mathbf{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, a solid torus or a boundary-disk sum of some number of solid tori. A trivial surface-knot is also called an *unknotted* surface-knot and a trivial disconnected surface-link is also called an *unknotted and unlinked* surface-link. A trivial surface-link is unique up to equivalences (see [1]). A 2-*handle* on a surface-link F in $\mathbf{R}^4$ is an embedded 2-handle $D \times I$ on F with D a core disk such that $D \times I \cap F = \partial D \times I$, where I denotes a closed interval containing 0 and $D \times 0$ is identified with D. If D is an immersed disk, then call it an *immersed 2-handle*. Two (possibly immersed) 2-handles $D \times I$ and $E \times I$ on F are *equivalent* if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f|_F : F \to F$ is the identity map and $f(D \times I) = E \times I$. An *orthogonal 2-handle pair* (or simply, an *O2-handle pair*) on F is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I$ on F such that

$$D \times I \cap D' \times I = \partial D \times I \cap \partial D' \times I$$

and $\partial D \times I$ and $\partial D' \times I$ *meet orthogonally* on F, that is, the boundary circles ∂D and $\partial D'$ meet transversely at one point q and the intersection $\partial D \times I \cap \partial D' \times I$ is homeomorphic to the square $Q = q \times I \times I$ (see [2, Fig.1]). An important property of an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F is the following property (see [2] for the proof):

Common 2-handle property Let F be a surface-link in $\mathbf{R}^4$, and $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ O2-handle pairs on F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. If $D \times I = E \times I$ or $E' \times I = D' \times I$, then the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F are equivalent by an equivalence obtained by 3-cell moves on the unions $D \times I \cup D' \times I$ and $E \times I \cup E' \times I$ which are 3-balls.

In this paper, the following uniqueness theorem of an O2-handle pair on a surface-link is shown by using a normal form of 2-handle core disks discussed in [4] and Common 2-handle property stated above repeatedly which is announced in [2, Section 3] with incomplete proof although the tools of the present proof appear there.

Theorem 1.1. Let F be a surface-link in $\mathbf{R}^4$, and $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ O2-handle pairs on F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Then the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F are equivalent.

This theorem for a trivial surface-link is heavily used for confirming the smooth unknotting conjecture of a surface-knot in [2] and the smooth unknotting-unlinking conjecture for a surface-link in [3]. For an immersed O2-handle pair, the following proposition is provided:

Proposition 1.2. If $(D \times I, D' \times I)$ is an immersed O2-pair on a surface-link F in $\mathbf{R}^4$ with $D \times I$ immersed and $D' \times I$ embedded, then there is an embedded 2-handle $D_* \times I$ with $\partial D_* \times I = \partial D \times I$ such that $(D_* \times I, D' \times I)$ is an O2-handle pair on F.

For the proof of Proposition 1.2, Finger move canceling operation is used to cancel a double point of an immersed core disk D of the immersed 2-handle $D \times I$ on F, which is explained in Section 3. By Theorem 1.1 and Proposition 1.2, we have the following corollary.

Corollary 1.3. Let F be a surface-link in $\mathbf{R}^4$, and $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ immersed O2-handle pairs on F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$.

(1) If $D' \times I$ and $E' \times I$ are embedded, then there are embedded 2-handles $D_* \times I$ and $E_* \times I$ on F with $\partial D_* \times I = \partial D \times I$ and $\partial E_* \times I = \partial E \times I$ such that $(D_* \times I, D' \times I)$ and $(E_* \times I, E' \times I)$ are equivalent O2-handle pairs on F, so that the surface-links $F(D' \times I)$ and $F(E' \times I)$ are equivalent.

(2) If $D' \times I$ and $E \times I$ are embedded, then there are embedded 2-handles $D_* \times I$ and $E'_* \times I$ on F with $\partial D_* \times I = \partial D \times I$ and $\partial E'_* \times I = \partial E' \times I$ such that $(D_* \times I, D' \times I)$ and $(E \times I, E'_* \times I)$ are equivalent O2-handle pairs on F, so that the surface-links $F(D' \times I)$ and $F(E \times I)$ are equivalent.

The proof of Theorem 1.1 is done in Section 2 and and the proof of Proposition 1.2 is done in Section 3. Throughout the paper, the notation

$$XJ = \{(x, t) \in \mathbf{R}^4 | \, x \in X, t \in J\}$$

is used for a subspace X of $\mathbf{R}^3$ and a subinterval J of $\mathbf{R}$.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into the proof of the case of a trivial surface-knot F and the proof of the case of a general surface-link F. In the argument, the O2-handle pair $(D \times I, D' \times I)$ is fixed in the 3-space $\mathbf{R}[0]$ and consider normal forms of the core disks E, E' of the 2-handles $E \times I, E' \times I$ in $\mathbf{R}^4$. To avoid the complexity of handling the intersection point $q = E \cap E'$, a sufficiently small boundary-collar $n(\partial E')$ of E' is fixed in $\mathbf{R}^3[0]$ and consider a normal form of the disk $E'_n = \mathrm{cl}(E' \setminus n(\partial E'))$ in $\mathbf{R}^4$ together with a normal form of E.

Proof of Theorem 1.1 in the case of a trivial surface-link F. Assume that the trivial surface-knot F is embedded standardly in $\mathbf{R}^3[0]$ with a standard O2-handle pair $(D \times I, D' \times I)$ on F. By [4], the disk union $G = E \cup E'_n$ is deformed into a disk union G_1 in the following form by an isotopy of $\mathbf{R}^4$ keeping the boundary

$\partial G = \partial E \cup \partial E'_n$ (which is a trivial link in $\mathbf{R}^3[0]$), $n(\partial E')$ and F fixed:

$$G_1 \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ \mathbf{d}'[t], & \text{for } t = 2, \\ o'[t], & \text{for } 1 < t < 2, \\ (\partial G \cup \ell \cup \mathbf{b}')[t], & \text{for } t = 1, \\ (\partial G \cup \ell)[t], & \text{for } 0 \leq t < 1, \\ \ell[t], & \text{for } -1 < t < 0, \\ (o \cup \mathbf{b})[t], & \text{for } t = -1, \\ o[t], & \text{for } -2 < t < -1, \\ \mathbf{d}[t], & \text{for } t = -2, \\ \emptyset, & \text{for } t < -2, \end{cases}$$

where the notations o, o' denote trivial links in $\mathbf{R}^3$, the notations $\mathbf{d}, \mathbf{d}'$ denote disjoint disk systems in $\mathbf{R}^3$ bounded by o, o', respectively, the notations $\mathbf{b}, \mathbf{b}'$ denote disjoint band systems in $\mathbf{R}^3$ spanning o, o', respectively, and the notation ℓ denotes a link in $\mathbf{R}^3$. To obtain this disk union G_1, start the argument of [4] with the assumption that the intersection $G \cap \mathbf{R}^3[0]$ is a link $\ell[0] \cup \partial G$ in $\mathbf{R}^3[0]$ and a boundary-collar $n(\partial G)$ of ∂G in G is in $\mathbf{R}^3[0, c]$ so that

$$n(\partial G) \cap \mathbf{R}^3[t] = \partial G[t], \quad t \in [0, c]$$

for a small number $c > 0$, where ∂G is regarded to be in $\mathbf{R}^3$ under the canonical identification $\mathbf{R}^3[0] = \mathbf{R}^3$. Then pull down a minimal point of G in $\mathbf{R}^3(0, \infty)$ to $\mathbf{R}^3(-\infty, 0)$ and pull up a maximal point of G in $\mathbf{R}^3(-\infty, 0)$ to $\mathbf{R}^3(0, \infty)$. In these deformations, trivial components are increased in the intersection link $G \cap \mathbf{R}^3[0]$. After these preparations, do normalizations of $G \cap \mathbf{R}^3[0, \infty)$ and $G \cap \mathbf{R}^3(-\infty, 0]$ keeping $G \cap \mathbf{R}^3[0]$ fixed. The band systems $\mathbf{b}, \mathbf{b}'$ are made disjoint by band slide and band thinning and disjoint from ∂G by band deformation. Let $G_1 = E \cup E'_n$. The following notation is used.

Notation. The disk subsystems of the disk system $\mathbf{d}$ belonging to E or E'_n are denoted by $\mathbf{d}(E)$ or $\mathbf{d}(E'_n)$, respectively. The band subsystems of the band system $\mathbf{b}$ belonging to E or E'_n are denoted by $\mathbf{b}(E)$ or $\mathbf{b}(E'_n)$, respectively.

A next deformation of G_1 is to change the level of the band system $\mathbf{b}(E)[-1]$ into $\mathbf{b}(E)[1]$ and the level of the disk system $\mathbf{d}(E)[-2]$ into $\mathbf{d}(E)[0.5]$. To do so, it is observed that in $\mathbf{R}^3$, the boundary ∂G and the band system $\mathbf{b}(E'_n)$ meet the disk system $\mathbf{d}(E)$ in finite interior points and in finite interior simple arcs, respectively. For a point $x \in \mathbf{d}(E) \cap \partial G$, find a point $y \in \partial \mathbf{d}(E) \setminus \partial E$ and a simple arc α from x to y in $\mathbf{d}(E)$ which does not meet the band systems $\mathbf{b}, \mathbf{b}'$ by band slide and band thinning. Let $n(\alpha)$ be a disk neighborhood of α in $\mathbf{d}(E)$. Deform the disk system $\mathbf{d}'(E)$ so that $n(\alpha) \subset \mathbf{d}'(E)$. Then the intersection $e(\alpha) = n(\alpha)[-2,2] \cap G_1$ is a disk in the interior of G_1. Let $\tilde{e}(\alpha) = \mathrm{cl}((\partial(n(\alpha)[-2,2])) \setminus e(\alpha))$ be the complementary disk of the disk $e(\alpha)$ in the 2-sphere $\partial(n(\alpha)[-2,2])$. The disk union

$$\tilde{G}_1 = \mathrm{cl}(G_1 \setminus e(\alpha)) \cup \tilde{e}(\alpha)$$

induces a normal form of the union of a deformed disk $\tilde{E}$ of E and the disk E'_n with $\partial\tilde{G}_1 = \partial G_1$. Note that the disk $\tilde{E}$ may meet with the surface F and the topological position of $\tilde{E}$ in $\tilde{G}_1$ may be changed from G_1, although the disk $E' = E'_n \cup n(\partial E')$ is unchanged and the configuration of $\tilde{G}_1$ is the same as G_1. Do this deformation for all points of the finite set $\mathbf{d}(E) \cap \partial G$. Further, for an arc β in the finite arc set $\mathbf{d}(E) \cap \mathbf{b}(E'_n)$, find a simple arc α in $\mathbf{d}(E)$ extending this arc β to a point $y \in \partial\mathbf{d}(E) \setminus \partial E$ which does not meet the band systems $\mathbf{b}, \mathbf{b}'$ by band slide and band thinning. For a disk neighborhood $n(\alpha)$ in $\mathbf{d}(E)$, do the same deformation as above. Do this deformation for all arcs β in the finite arc set $\mathbf{d}(E) \cap \mathbf{b}(E'_n)$. Let $\tilde{G}_1 = \tilde{E} \cup E'_n$ be the disk union obtained from $G_1 = E \cup E'_n$ by all these deformations, which is in a normal form with the same configuration as G_1 and we have

$$\mathbf{d}(\tilde{E}) \cap (\partial E \cup n(\partial E')) = \mathbf{d}(\tilde{E}) \cap \mathbf{b}(E'_n) = \emptyset$$

although the disk $\tilde{E}$ may meet F. Now change the level of $\mathbf{b}(\tilde{E})[-1]$ into $\mathbf{b}(\tilde{E})[1]$ and the level of $\mathbf{d}(\tilde{E})[-2]$ into $\mathbf{d}(\tilde{E})[0.5]$. The resulting disk union $G_2 = \tilde{E} \cup E'_n$ is in the following form:

$$G_2 \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ \mathbf{d}'[t], & \text{for } t = 2, \\ o'[t], & \text{for } 1 < t < 2, \\ (\partial G \cup \cup o(\tilde{E}) \cup \mathbf{b}(\tilde{E}) \cup \ell(E'_n) \cup \mathbf{b}')[t], & \text{for } t = 1, \\ (\partial G \cup o(\tilde{E}) \cup \ell(E'_n))[t], & \text{for } 0.5 < t < 1, \\ (\partial G \cup \mathbf{d}(\tilde{E}) \cup \ell(E'_n))[t], & \text{for } t = 0.5, \\ (\partial G \cup \ell(E'_n))[t], & \text{for } 0 \le t < 0.5, \\ \ell(E'_n)[t], & \text{for } -1 < t < 0, \\ (o(E'_n) \cup \mathbf{b}(E'_n))[t], & \text{for } t = -1, \\ o(E'_n)[t], & \text{for } -2 < t < -1, \\ \mathbf{d}(E'_n)[t], & \text{for } t = -2, \\ \emptyset, & \text{for } t < -2. \end{cases}$$

In the configuration of the disk union G_2, the pair $(\tilde{E} \times I, E' \times I)$ is an O2-handle pair on F and hence is equivalent to the original O2-handle pair $(E \times I, E' \times I)$ on F by Common 2-handle property. Let $G_2 = E \cup E'_n$. A next deformation of G_2 is to change the level of the band system $\mathbf{b}(E'_n)[-1]$ into $\mathbf{b}(E'_n)[1]$ and the level of the disk system $\mathbf{d}(E'_n)[-2]$ into $\mathbf{d}(E'_n)[0.5]$. To do so, for a point $x \in \mathbf{d}(E'_n) \cap \partial G$, find a point $y \in \partial \mathbf{d}(E'_n) \setminus \partial E'_n$ and a simple arc α from x to y in $\mathbf{d}(E'_n)$ which does not meet the band systems $\mathbf{b}, \mathbf{b}'$ by band slide and band thinning. For a disk neighborhood $n(\alpha)$ of α in $\mathbf{d}(E'_n)$, do a similar deformation on the disk E'_n as above. Namely, deform the disk system $\mathbf{d}'(E'_n)$ so that $n(\alpha) \subset \mathbf{d}'(E'_n)$. Since the intersection $e(\alpha) = n(\alpha)[-2,2] \cap G_2$ is a disk in the interior of G_2, let $\tilde{}e(\alpha) = \mathrm{cl}((\partial(n(\alpha)[-2,2])) \setminus e(\alpha))$ be the complementary disk of the disk $e(\alpha)$ in the 2-sphere $\partial(n(\alpha)[-2,2])$. The disk union

$$\tilde{}G_2 = \mathrm{cl}(G_2 \setminus e(\alpha)) \cup \tilde{}e(\alpha)$$

induces a normal form of the union of the disk E and a deformed disk $\tilde{}E'_n$ of E'_n with $\partial \tilde{}G_2 = \partial G_2$. Note that the disk $\tilde{}E'_n$ may meet F and the topological position of $\tilde{}E'_n$ in $\tilde{}G_1$ may be changed from G_1, although the disk E is unchanged and the configuration of $\tilde{}G_2$ is the same as G_2. Do this operation for all points of the finite set $\mathbf{d}(E'_n) \cap \partial G$. Let $\tilde{G}_2 = E \cup \tilde{E}'_n$ be the disk union obtained from G_2 by all these deformations. The disk union $\tilde{G}_2 = E \cup \tilde{E}'_n$ is in a normal form with the same configuration as G_2 and has

$$\mathbf{d}(\tilde{E}'_n) \cap (\partial E \cup n(\partial E')) = \emptyset,$$

although $\tilde{E}'_n$ may meet F. Now change the level of the band system $\mathbf{b}(\tilde{E}'_n)[-1]$ into $\mathbf{b}(\tilde{E}'_n)[1]$ and the level of the disk system $\mathbf{d}(\tilde{E}'_n)[-2]$ into $\mathbf{d}(\tilde{E}'_n)[0.5]$. The resulting disk union $G_3 = E \cup \tilde{E}'_n$ is as follows:

$$G_3 \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ \mathbf{d}'[t], & \text{for } t = 2, \\ o'[t], & \text{for } 1 < t < 2, \\ (\partial G \cup o \cup \mathbf{b} \cup \mathbf{b}')[t], & \text{for } t = 1, \\ (\partial G \cup o)[t], & \text{for } 0.5 < t < 1, \\ (\partial G \cup \mathbf{d})[t], & \text{for } t = 0.5, \\ (\partial G)[t], & \text{for } 0 \le t < 0.5, \\ \emptyset, & \text{for } t < 0. \end{cases}$$

In the disk union G_3, the pair $(E \times I, \tilde{E}' \times I)$ with $\tilde{E}' = \tilde{E}'_n \cup n(\partial E')$ is an O2-handle pair on F and hence equivalent to the original O2-handle pair $(E \times I, E' \times I)$ by Common 2-handle property. Let $G_3 = E \cup E'_n$. In the configuration of G_3, the pairs $(D \times I, E' \times I)$ and $(E \times I, D' \times I)$ are O2-handle pairs on F. Thus, by Common 2-handle property, the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F are equivalent. This completes the proof of Theorem 1.1 in the case of a trivial surface-link F.

Proof of Theorem 1.1 in the case of a general surface-link F. For a general surface-link F in $\mathbf{R}^4$ and O2-handle pairs $(D\times I, D'\times I)$ and $(E\times I, E'\times I)$, let $F(D\times I, D'\times I)$ be the surface-link obtained by surgery along $(D\times I, D'\times I)$ (see [2]). Let F' be a trivial surface-knot in $\mathbf{R}^4$ obtained from the surface-link $F(D\times I, D'\times I)$ obtained by surgery along 1-handles $h_j\,(j=1,2,\dots,s)$ embedded in a connected Seifert hypersurface W for $F(D\times I, D'\times I)$ avoiding the intersection loops $E\cap W, E'\cap W$ (cf. [1]). Then there is a trivial torus-knot T in $\mathbf{R}^4$ such that the connected sum $F'\#T$ is a trivial surface-knot in $\mathbf{R}^4$ obtained from F by surgery along the 1-handles $h_j\,(j=1,2,\dots,s)$ and $(D\times I, D'\times I)$ is a standard O2-handle pair on $F'\#T$ attached to the connected summand T. By construction, the pairs $(D\times I, D'\times I)$ and $(E\times I, E'\times I)$ are O2-handles on the connected sum $F'\#T$ attached to the connected summand T whose defining 4-ball is disjoint from the "2-handles" $h_j\,(j=1,2,\dots,s)$ on $F'\#T$ attached to F'. Let $\mathbf{h}$ be the core disk system $D(h_j),(j=1,2,\dots,s)$ of the 2-handle system $h_j\,(j=1,2,\dots,s)$ on $F'\#T$ attached to F'. By the proof for the case of a trivial surface-link F, the O2-handle pair $(E\times I, E'\times I)$ is equivalent to $(D\times I, D'\times I)$ on $F'\#T$. To obtain such an equivalence without crossing the core disk system $\mathbf{h}$, the proof is revised as follows: A normal form of the disk union $\bar{G}=G\cup\mathbf{h}=E\cup E'_n\cup\mathbf{h}$ can be thought of as the following disk union $\bar{G}_1$:

$$\bar{G}_1\cap\mathbf{R}^3[t]=\begin{cases} \emptyset, & \text{for } t>2,\\ (d'(\mathbf{h})\cup\mathbf{d}')[t], & \text{for } t=2,\\ (o'(\mathbf{h})\cup o')[t], & \text{for } 1<t<2,\\ (\partial\bar{G}\cup\ell(\mathbf{h})\cup b'(\mathbf{h})\cup\ell\cup\mathbf{b}')[t] & \text{for } t=1,\\ (\partial\bar{G}\cup\ell(\mathbf{h})\cup\ell)[t], & \text{for } 0\le t<1,\\ (\ell(\mathbf{h})\cup\ell)[t], & \text{for } -1<t<0,\\ (o(\mathbf{h})\cup b(\mathbf{h})\cup o\cup\mathbf{b})[t], & \text{for } t=-1,\\ (o(\mathbf{h})\cup o)[t], & \text{for } -2<t<-1,\\ (d(\mathbf{h})\cup\mathbf{d})[t], & \text{for } t=-2,\\ \emptyset, & \text{for } t<-2, \end{cases},$$

where in addition to the notations on G_1, the following notations are also added. Namely, the notations $o(\mathbf{h}), o'(\mathbf{h})$ denote trivial links in $\mathbf{R}^3$ coming from $\mathbf{h}$, the notations $d(\mathbf{h}), d'(\mathbf{h})$ denote disjoint disk systems in $\mathbf{R}^3$ bounded by $o(\mathbf{h}), o'(\mathbf{h})$, respectively, coming from $\mathbf{h}$, the notations $b(\mathbf{h}), b'(\mathbf{h})$ denote disjoint band systems in $\mathbf{R}^3$ spanning $o(\mathbf{h}), o'(\mathbf{h})$, respectively, and the notation $\ell(\mathbf{h})$ denotes a link in $\mathbf{R}^3$ coming from $\mathbf{h}$. The band systems $\mathbf{b}, \mathbf{b}', b(\mathbf{h}), b'(\mathbf{h})$ are made disjoint by band slide and band thinning. In this normal form $\bar{G}_1$, the disk system $\mathbf{h}$ can be taken as

$$\mathbf{h}\cap D\times I=\mathbf{h}\cap D'\times I=\emptyset,$$

because the defining 4-ball of the connected summand T in the connected sum $F'\#T$

contains the union $D \times I \cup D' \times I$ and is disjoint from the 2-handles h_j $(j = 1, 2, \ldots, s)$. By a method similar to the process from G_1 to G_2, we have a deformation $\tilde{\tilde{G}}_1 = \tilde{E} \cup E'_n \cup \mathbf{h}$ of $\bar{G}_1$ with the same configuration as $\bar{G}_1$ such that

$$\mathbf{d}(\tilde{E}) \cap (\partial E \cup n(\partial E')) = \mathbf{d}(\tilde{E}) \cap \mathbf{b}(E'_n) = \mathbf{d}(\tilde{E}) \cap b(\mathbf{h}) = \emptyset,$$

although $\tilde{E}$ may meet $F' \# T$. Now change the level of $\mathbf{b}(\tilde{E})[-1]$ into $\mathbf{b}(\tilde{E})[1]$ and the level of $\mathbf{d}(\tilde{E})[-2]$ into $\mathbf{d}(\tilde{E})[0.5]$. Then the disk union $\bar{G}_2 = \tilde{E} \cup E'_n \cup \mathbf{h}$ obtained from $\tilde{\tilde{G}}_1$ is as follows:

$$\bar{G}_2 \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ (d'(\mathbf{h}) \cup \mathbf{d}')[t], & \text{for } t = 2, \\ (o'(\mathbf{h}) \cup o')[t], & \text{for } 1 < t < 2, \\ (\partial \bar{G} \cup \ell(\mathbf{h}) \cup b'(\mathbf{h}) \cup o(\tilde{E}) \cup \mathbf{b}(\tilde{E}) \cup \ell(E') \cup \mathbf{b}')[t], & \text{for } t = 1, \\ (\partial \bar{G} \cup \ell(\mathbf{h}) \cup o(\tilde{E}) \cup \ell(E'_n))[t], & \text{for } 0.5 < t < 1, \\ (\partial \bar{G} \cup \ell(\mathbf{h}) \cup \mathbf{d}(\tilde{E}) \cup \ell(E'_n))[t], & \text{for } t = 0.5, \\ (\partial \bar{G} \cup \ell(\mathbf{h}) \cup \ell(E'_n))[t], & \text{for } 0 \le t < 0.5, \\ (\ell(\mathbf{h}) \cup \ell(E'_n))[t], & \text{for } -1 < t < 0, \\ (o(\mathbf{h}) \cup b(\mathbf{h}) \cup o(E'_n) \cup \mathbf{b}(E'_n))[t], & \text{for } t = -1, \\ (o(\mathbf{h}) \cup o(E'_n))[t], & \text{for } -2 < t < -1, \\ (d(\mathbf{h}) \cup \mathbf{d}(E'_n))[t], & \text{for } t = -2, \\ \emptyset, & \text{for } t < -2. \end{cases}$$

In the configuration of $\bar{G}_2$, the pair $(\tilde{E} \times I, E' \times I)$ is an O2-handle pair on $F' \# T$ and hence equivalent to the O2-handle pair $(E \times I, E' \times I)$ on $F' \# T$ by Common 2-handle property. Let $\bar{G}_2 = E \cup E'_n \cup \mathbf{h}$. By a similar consideration from G_2 to G_3, we have a deformation $\tilde{\tilde{G}}_2 = E \cup \tilde{E}'_n \cup \mathbf{h}$ of $\bar{G}_2$ with the same configuration as $\bar{G}_2$ such that

$$\mathbf{d}(\tilde{E}'_n) \cap (\partial E \cup n(\partial E')) = \mathbf{d}(\tilde{E}'_n) \cap b(\mathbf{h}) = \emptyset,$$

although the disk $\tilde{E}'_n$ may meet $F' \# T$. Now change the level of $\mathbf{b}(\tilde{E}'_n)[-1]$ into $\mathbf{b}(\tilde{E}'_n)[1]$ and the level of $\mathbf{d}(\tilde{E}'_n)[-2]$ into $\mathbf{d}(\tilde{E}'_n)[0.5]$. Then the disk union $\bar{G}_3 =$

$E \cup \tilde{E}'_n \cup \mathbf{h}$ obtained from $\tilde{\tilde{G}}_2$ is as follows:

$$\bar{G}_3 \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ (d'(\mathbf{h}) \cup \mathbf{d}')[t], & \text{for } t = 2, \\ (o'(\mathbf{h}) \cup o')[t], & \text{for } 1 < t < 2, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup b'(\mathbf{h}) \cup o \cup \mathbf{b}' \cup \mathbf{b})[t], & \text{for } t = 1, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup o)[t], & \text{for } 0.5 < t < 1, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup \mathbf{d})[t], & \text{for } t = 0.5, \\ (\partial\bar{G} \cup \ell(\mathbf{h}))[t], & \text{for } 0 \leq t < 0.5, \\ \ell(\mathbf{h})[t], & \text{for } -1 < t < 0, \\ (o(\mathbf{h}) \cup b(\mathbf{h}))[t], & \text{for } t = -1, \\ o(\mathbf{h})[t], & \text{for } -2 < t < -1, \\ d(\mathbf{h})[t], & \text{for } t = -2, \\ \emptyset, & \text{for } t < 0. \end{cases}$$

In the configuration of G_3, the pair $(E \times I, \tilde{E}' \times I)$ with $\tilde{E}' = \tilde{E}'_n \cup n(\partial E')$ is an O2-handle pair on $F'\#T$ and hence equivalent to the O2-handle pair $(E \times I, E' \times I)$ on $F'\#T$ by Common 2-handle property. Let $G_3 = E \cup E'_n \cup \mathbf{h}$. Since $(D \times I, D' \times I)$ is in $\mathbf{R}^3[0]$, the disk system $\mathbf{h}$ is disjoint from the O2-handle pair $(D \times I, D' \times I)$, although the disk system $d'(\mathbf{h})[2]$ is isotopically deformed in $\mathbf{R}^3[2]$ in $\bar{G}_3$. Thus, in the configuration of G_3, the pairs $(D \times I, E' \times I)$ and $(E \times I, D' \times I)$ are O2-handle pairs on $F'\#T$ and disjoint from $\mathbf{h}$. This means that the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on $F'\#T$ are equivalent under 3-cell moves disjoint from the 2-handles h_j $(j = 1, 2, \ldots, s)$ by Common 2-handle property. By the back surgery from $F'\#T$ to F on the 2-handles h_j $(j = 1, 2, \ldots, s)$ on $F'\#T$, this means that the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F are equivalent under 3-cell moves disjoint from the 1-handles h_j $(j = 1, 2, \ldots, s)$ on F. This completes the proof of Theorem 1.1 in the case of a general surface-link F. □

This completes the proof of Theorem 1.1.

3. Proof of Proposition 1.2

The *Finger move canceling* is the following operation to cancel a double point of an immersed disk D in $\mathbf{R}^4$.

Finger Move Canceling. Let D be an immersed disk in $\mathbf{R}^4$ with ∂D embedded, and S a trivial S^2-knot in $\mathbf{R}^4$ meeting the immersed disk D at just one point x different from the double points of D. Let y be a double point of D, and α a simple arc in the disk D joining x and y not meeting the other double points of D. Let d_x be a disk neighborhood of x in D, and d_y a disk neighborhood d_y of y in the 2-sphere

S, regarding the disks d_x and d_y as disk fibers of a normal disk bundle over D in $\mathbf{R}^4$. Let V_α be a disk bundle over the arc α in $\mathbf{R}^4$ such that $(D \cup S) \cap V_\alpha = d_x \cup \alpha \cup d_y$. Then the immersed disk D_1 with $\partial D_1 = \partial D$ is constructed from the immersed disk D so that

$$D_1 = \mathrm{cl}(D \setminus d_x) \cup \mathrm{cl}(\partial V_\alpha \setminus (d_x \cup d_y)) \cup \mathrm{cl}(S \setminus d_y).$$

The number of the double points of D_1 is smaller than the number of the double points of D by 1.

The 2-sphere S in Finger Move Canceling is called a *canceling sphere*. If there is a canceling sphere S, then the immersed disk D is changed into an embedded disk D_* by Finger Move Canceling operations of parallel canceling spheres of S. By using Finger Move Canceling, the proof of Proposition 1.2 is done as follows:

Proof of Proposition 1.2. By assumption, the immersed O2-pair $(D \times I, D' \times I)$ on a surface-link F in $\mathbf{R}^4$ has $D \times I$ as an immersed 2-handle on F and $D' \times I$ as an embedded 2-handle on F. Let d' be a small disk neighborhood of a point $p' \in D'$ in D'. By shrinking $D' \times I$ as $d' \times I$, one finds a trivial S^2-knot S in in $\mathbf{R}^4$ such that S meets the immersed core disk D of $D \times I$ at just one point x different from the double points of D and is disjoint from F and $D' \times I$. This 2-sphere S is used for a canceling sphere for the immersed disk D. By Finger Move Canceling, the immersed disk D is changed into an embedded disk D_*, meaning that the pair $(D_* \times I, D' \times I)$ is an O2-handle pair on F. This completes the proof of Proposition 1.2. □

Acknowledgements. This work was partly supported by Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849). The author thanks Junpei Yasuda (a graduate student at Osaka University under the adviser Seiichi Kamada) for asking a question on [2, Claim 3.2.1] that motivated to write this paper.

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