

ON THE LOCAL FACTORS OF IRREDUCIBLE REPRESENTATIONS OF QUATERNIONIC UNITARY GROUPS

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ABSTRACT. In this paper, we give a precise definition of the analytic γ -factor of irreducible representations of quaternionic unitary groups, which extends a work of Lapid-Rallis.

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1. INTRODUCTION

The doubling method of Piatetski-Shapiro and Rallis [7, 19] is a theory of integral representation of standard L -functions. Lapid-Rallis [17] elaborated on the doubling method, and gave a definition of the analytic γ -factor of irreducible representations of general linear groups, orthogonal groups, symplectic groups and unitary groups. This was extended by Gan [5] to metaplectic groups. Moreover, Yamana [26] established their analytic properties and verified basic properties for classical groups containing isometry groups of hermitian or skew-hermitian forms over quaternion algebras. However, his work was not enough to characterize the γ -factor. The purpose of this paper is to give a precise definition of the analytic γ -factor and to characterize it in this case.

Now, we explain our result in more detail. Let F be a local field of characteristic zero and let D be a quaternion algebra over F . We consider an ϵ -hermitian space over D (see §2.1). Then the *quaternionic unitary group* is defined as the isometry group of the ϵ -hermitian space. Let G be either a general linear group $\mathrm{GL}_n(D)$ or a quaternionic unitary group, let π be an irreducible representation of G , let ω be a character of F^\times , let ψ be a non-trivial additive character of F . In this paper, we define the γ -factor of π by

$$\gamma^{\mathcal{V}}\left(s + \frac{1}{2}, \pi \times \omega, \psi\right) = \Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi) c_{\pi}(-1) R(s, \omega, A, \psi)$$

where

- A is some element of the Lie algebra of G^{\square} (for definition, see §3.1);

- $\Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi)$ is a “normalized Γ -factor”, which is defined in §4.1. This factor is obtained from a functional equation of doubling zeta integrals;
- c_{π} is the central character of π ;
- $R(s, \omega, A, \psi)$ is a correction term, which is defined in §4.2.

We expect the γ -factor to satisfy

$$(1.1) \quad \gamma^{\mathcal{V}}(s, \pi \times \omega, \psi) = \gamma(s, \phi_{\pi} \otimes \omega, \text{std}, \psi)$$

if the L -parameter ϕ_{π} is attached to π . Here, we denote by std the standard representation of the L -group ${}^L G$ into $\text{GL}_N(\mathbb{C})$. As in [17], one can show that the γ -factor $\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi)$ satisfies the global functional equation. Yamana showed that the γ -factor $\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi)$ satisfies some required properties: the multiplicativity, the self-duality, the (local) functional equation, and the equation (1.1) for $G = \text{GL}_n(D)$. In this paper, we prove the equation (1.1) for G in the archimedean case. Moreover, we prove that the γ -factor $\gamma^{\mathcal{V}}(s, \pi, \psi)$ is characterized by some required properties. Both are stated in Theorem 5.7.

In the rest of the introduction, we explain the contents of this paper. In §2-§4, we explain the (local) framework of the doubling method. In §5, we give a definition of the γ -factor and state the main theorem. We also recall the definition of the Lapid-Rallis γ -factor. We also define the L -factor and the ϵ -factor as in [17, §10]. In §6 and §7, we prove the main theorem.

In §6.1, we prove that the properties of the γ -factor $\gamma(s, \pi \times \omega, \psi)$ stated in the main theorem determine it uniquely by using a global argument. In §6.2, we treat some properties which come from the framework of the doubling method. For example, the multiplicativity, the functional equation and the self duality. Note that there is a minor error in the proof of the multiplicativity of [17]. So we correct it at this opportunity. In §6.3, we prove that the γ -factor defined in §5 is nothing other than Lapid-Rallis γ -factor in the split case. Finally, in §6.4, we discuss the desired property (1.1). In the archimedean case, we prove it as in [17]. In the non-archimedean case, we prove it admitting the local Langlands correspondence. In §7, we discuss some explicit calculations to complete the proof of the main theorem.

In §8, we give two applications. First, we determine the local root number of irreducible representations of quaternionic unitary groups. This extends the result of Lapid-Rallis [17, Theorem 1]. Second, over a non-archimedean local field of odd residual characteristic, we give an explicit formula of the doubling zeta integrals of some spherical representations with respect to a certain subgroup. In the proof, we use the explicit formula of the γ -factor of the trivial representation obtained in §7.

Acknowledgments: The author would like to thank my supervisor A. Ichino for many advices. The author also would like to thank the referee for sincere and useful comments.

2. QUATERNIONIC UNITARY GROUPS

Let F be a field of characteristic zero, and let D be a quaternion algebra over F . Denote by $*$ the main involution of D . For a finite dimensional right vector space V over D , we denote the reduced norm of $\text{End}_D(V)$ by N_V . For simplicity, we denote the reduced norm of $M_n(D)$ by N . Note that D is possibly split since it appears as a localization of a division quaternion algebra over a number field. If D is split, then “finite dimensional right vector space over D ” means “free right module over D of finite rank”. We discuss the split case more in §2.3.

2.1. Hermitian and skew-hermitian spaces over D . Fix $\epsilon = \pm 1$. Let $\mathcal{V} = (V, h)$ be a pair consisting of an n -dimensional right vector space V over D and a map $h : V \times V \rightarrow D$ such that

- (1) $h(va, wb) = a^* h(v, w) b$
- (2) $h(v_1 + v_2, w) = h(v_1, w) + h(v_2, w)$
- (3) $h(w, v) = \epsilon h(v, w)^*$

for $a, b \in D$ and $v, w, v_1, v_2 \in V$. We call a pair $\mathcal{V} = (V, h)$ a *hermitian space* (resp. a *skew-hermitian space*) over D if h is non-degenerate and $\epsilon = 1$ (resp. h is non-degenerate and $\epsilon = -1$). We use the term “ ϵ -hermitian space” when we treat hermitian space and skew-hermitian space at the same time. In this paper, we consider the three cases: either the linear case (i.e. the case $h = 0$), the hermitian case or the skew-hermitian case.

Let \mathcal{V} be an n -dimensional ϵ -hermitian space over D . We define the *discriminant* of \mathcal{V} by

$$\mathfrak{d}(\mathcal{V}) := (-1)^n N((h(v_i, v_j))_{ij}) \in F^\times / F^{\times 2}$$

where v_1, \dots, v_n is a basis of V . Then $\mathfrak{d}(\mathcal{V})$ does not depend on the choice of basis.

2.2. Parabolic subgroups. Let

$$G = \text{Isom}(\mathcal{V}) = \{g \in \text{GL}(V) \mid h(gv, gw) = h(v, w) \text{ for } v, w \in V\}$$

be the isometry group of \mathcal{V} where $\text{GL}(V)$ is the general linear group of V . If P is a maximal parabolic subgroup of G over F , then there is a totally isotropic subspace W (in the linear case, it is just a subspace) of V such that P coincides with the stabilizer

$$P(W) = \{g \in G \mid gW = W\}$$

of W . We denote the unipotent radical of $P(W)$ by $U(W)$. We put

$$\mathcal{W}_0 := (W, 0), \quad \mathcal{W}_1 := (W^\perp/W, h)$$

where

$$W^\perp := \{v \in V \mid h(v, x) = 0 \text{ for all } x \in W\}.$$

Then there is the exact sequence

$$1 \rightarrow U(W) \rightarrow P(W) \rightarrow \text{GL}(W) \times \text{Isom}(\mathcal{W}_1) \rightarrow 1,$$

and any Levi subgroup of P is isomorphic to

$$\text{GL}(W) \times \text{Isom}(\mathcal{W}_1).$$

We denote the Lie algebra of G (resp. $P(W), U(W)$) by \mathfrak{g} (resp. $\mathfrak{p}(W), \mathfrak{u}(W)$).

2.3. Morita equivalence. In this subsection we assume that D is split. Then we may identify D with $M_2(F)$. Put

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then for an n -dimensional right vector space V over D , $V^\natural = Ve$ is a $2n$ -dimensional vector space over F . For a D -linear map $f : V \rightarrow V'$ of right vector spaces over D , the restriction $f^\natural : V^\natural \rightarrow V'^\natural$ of f is an F -linear map. For a zero or an ϵ -hermitian form h on V , we define a bilinear form h^\natural on V^\natural such that

$$h(xe, ye) = \begin{pmatrix} 0 & 0 \\ h^\natural(xe, ye) & 0 \end{pmatrix}$$

for $x, y \in V$. Then, the following properties hold:

- (1) if $h = 0$, then $h^\natural = 0$;
- (2) if h is hermitian, then h^\natural is symplectic;
- (3) if h is skew-hermitian, then h^\natural is symmetric.

Put $\mathcal{V}^\natural = (V^\natural, h^\natural)$ and

$$G^\natural = \text{Isom}(\mathcal{V}^\natural) = \{g \in \text{GL}(V^\natural; F) \mid h^\natural(gv, gw) = h^\natural(v, w) \text{ for } v, w \in V^\natural\}.$$

Then we have an isomorphism $\iota_G : G \rightarrow G^\natural : g \mapsto g^\natural$. If π is a representation of G , then we denote by π^\natural the representation of G^\natural such that $\pi = \pi^\natural \circ \iota_G$.

3. DOUBLING ZETA INTEGRALS

3.1. Doubling ϵ -hermitian spaces and unitary groups. Let $\mathcal{V}^\square = (V^\square, h^\square)$ be a pair, where $V^\square = V \times V$ and $h^\square = h \oplus (-h)$, that is, the map defined by

$$h^\square((v_1, v_2), (w_1, w_2)) = h(v_1, w_1) - h(v_2, w_2)$$

for $v_1, v_2, w_1, w_2 \in V$. Let $G^\square = \text{Isom}(\mathcal{V}^\square)$. Then $G \times G$ acts on $V \times V$ by

$$(g_1, g_2) \cdot (v_1, v_2) = (g_1 v_1, g_2 v_2),$$

so that $G \times G$ can be embedded naturally in G^\square . Consider the maximal totally isotropic subspaces

$$V^\Delta = \{(v, v) \in V^\square \mid v \in V\},$$

$$V^\nabla = \{(v, -v) \in V^\square \mid v \in V\}.$$

Note that $V = V^\Delta \oplus V^\nabla$. Then $P(V^\Delta)$ is a maximal parabolic subgroup of G^\square and its Levi part is isomorphic to

$$\begin{cases} \text{GL}(V^\Delta) \times \text{GL}(V^\square/V^\Delta) & \text{in the linear case} \\ \text{GL}(V^\Delta) & \text{in the } \epsilon\text{-hermitian case.} \end{cases}$$

3.2. Zeta integrals and intertwining operators. Assume that F is a local field of characteristic zero. Denote by Δ_{V^Δ} the character of $P(V^\Delta)$ given by

$$\Delta_{V^\Delta}(x) = \begin{cases} N_{V^\Delta}(x) \cdot N_{V^\square/V^\Delta}(x)^{-1} & \text{in the linear case,} \\ N_{V^\Delta}(x) & \text{in the } \epsilon\text{-hermitian case.} \end{cases}$$

Here, $N_{V^\Delta}(x)$ (resp. $N_{V^\square/V^\Delta}(x)$) is the reduced norm of the image of x in $\text{End}_D V^\Delta$ (resp. $\text{End}_D V^\square/V^\Delta$). Let $\omega : F^\times \rightarrow \mathbb{C}^\times$ be a character. For $s \in \mathbb{C}$, put $\omega_s = \omega \cdot |\cdot|^s$. Choose a maximal compact subgroup K of G^\square such that $G^\square = P(V^\Delta)K$. Denote by $I(s, \omega)$ the degenerate principal series representation

$$\text{Ind}_{P(V^\Delta)}^{G^\square}(\omega_s \circ \Delta_{V^\Delta})$$

consisting of the smooth right K -finite functions $f : G^\square \rightarrow \mathbb{C}$ satisfying

$$f(pg) = \delta_{P(V^\Delta)}^{\frac{1}{2}}(p) \cdot \omega_s(\Delta_{V^\Delta}(p)) \cdot f(g)$$

for $p \in P(V^\Delta)$ and $g \in G^\square$, where $\delta_{P(V^\Delta)}$ is the modular function of $P(V^\Delta)$. We may extend $|\Delta_{V^\Delta}|$ to a right K -invariant function on G^\square uniquely. For $f \in I(0, \omega)$, put $f_s = f \cdot |\Delta_{V^\Delta}|^s \in I(s, \omega)$. Define an intertwining operator $M(s, \omega) : I(s, \omega) \rightarrow I(-s, \omega^{-1})$ by

$$(M(s, \omega)f_s)(g) = \int_{U(V^\Delta)} f_s(w_1 u g) du$$

where $w_1 = (1, -1) \in G \times G \subset G^\square$. This integral defining $M(s, \omega)$ converges absolutely for $\Re s \gg 0$ and admits a meromorphic continuation to \mathbb{C} . Let π be an irreducible representation of G . For a matrix coefficient ξ of π , and for $f \in I(\omega, 0)$, define the zeta integral by

$$Z^\vee(f_s, \xi) = \int_G f_s((g, 1))\xi(g) dg.$$

Then the zeta integral satisfies the following properties, which is stated in [26, Theorem 4.1]. This gives a generalization of [17, Theorem 3].

Theorem 3.1. (1) *The integral $Z^\vee(f_s, \xi)$ converges absolutely for $\Re s \gg 0$ and extends to a meromorphic function in s . Moreover, if F is non-archimedean, the function $Z^\vee(f_s, \xi)$ is a rational function of q^{-s} . Here q denotes the cardinality of the residue field of F .*

(2) *There is a meromorphic function $\Gamma^\vee(s, \pi, \omega)$ such that*

$$Z^\vee(M(s, \omega)f_s, \xi) = \Gamma^\vee(s, \pi, \omega)Z^\vee(f_s, \xi)$$

for all matrix coefficient ξ of π and $f_s \in I(s, \omega)$.

4. INTERTWINING OPERATOR AND WHITTAKER NORMALIZATION

4.1. Degenerate Whittaker functionals. We use the notation of §2 and §3. Note that D is possibly split. We regard $\mathfrak{u}(V^\Delta)$ as a subspace of $\text{End}_D(V^\square)$ and we denote by $\mathfrak{u}(V^\Delta)_{reg}$ the set of $A \in \mathfrak{u}(V)$ of rank n . Fix a non-trivial additive character $\psi : F \rightarrow \mathbb{C}^\times$ and $A \in \mathfrak{u}(V^\Delta)_{reg}$. We define

$$\psi_A : U(V^\nabla) \rightarrow \mathbb{C}^\times : u \mapsto \psi(\text{Tr}_{V^\square}(uA))$$

where Tr_{V^\square} denotes the reduced trace of $\text{End}_D(V^\square)$. For $f \in I(\omega, 0)$ we define

$$l_{\psi_A}(f_s) = \int_{U(V^\nabla)} f_s(u)\psi_A(u) du.$$

Then this integral defining l_{ψ_A} converges for $\Re s \gg 0$ and admits a holomorphic continuation to \mathbb{C} . For the proof, see [13, §3.3] in the non-archimedean case, [23, Theorem 7.1, Theorem 7.2] in the archimedean case. The functional l_{ψ_A} is called a *degenerate Whittaker functional*, which is a $(U(V^\nabla), \overline{\psi_A})$ -equivariant functional on $I(s, \omega)$. On the other hand, the space of $(U(V^\nabla), \overline{\psi_A})$ -equivariant functionals on $I(s, \omega)$ is one dimensional for all $s \in \mathbb{C}$ (see [13, Theorem 3.2]). Hence we have the following proposition.

Proposition 4.1. *There is a meromorphic function $c(s, \omega, A, \psi)$ of s such that*

$$l_{\psi_A} \circ M(s, \omega) = c(s, \omega, A, \psi)l_{\psi_A}.$$

Then we define the normalized intertwining operator

$$M^*(s, \omega, A, \psi) = c(s, \omega, A, \psi)^{-1}M(s, \omega)$$

and put

$$\Gamma^\vee(s, \pi, \omega, A, \psi) = c(s, \omega, A, \psi)^{-1}\Gamma^\vee(s, \pi, \omega).$$

Clearly, $\Gamma^\vee(s, \pi, \omega, A, \psi)$ is a meromorphic function of s satisfying

$$(4.1) \quad Z(M^*(s, \omega, A, \psi)f_s, \xi) = \Gamma^\vee(s, \pi, \omega, A, \psi)Z(f_s, \xi)$$

for any $f \in I(\omega, 0)$ and any matrix coefficient ξ of π . Note that the function $\Gamma^\vee(s, \pi, \omega, A, \psi)$ does not depend on the choice of measure on G .

4.2. The normalization factor. In this subsection, we study the basic properties of the normalizing factor $c(s, \omega, A, \psi)$. First, we give an explicit formula of $c(s, \omega, A, \psi)$. Second, we study the dependence of $C(s, \omega, A, \psi)$ by the change of ψ .

To give an explicit formula, it is necessary to specify the Haar measure du in the definition of $M(s, \omega)$ (§3.2). Note that we do not have to pay attention to the choice of Haar measure in the later sections since the normalized intertwining operator $M(s, \omega, A, \psi)$ does not depend on that. Let v_1, \dots, v_n be a basis of V , and let e_1, \dots, e_{2n} be a basis

$$e_j = \begin{cases} (v_j, v_j) & 1 \leq j \leq n \\ (v_{j-n}, -v_{j-n}) & n+1 \leq j \leq 2n. \end{cases}$$

Then, we may regard $G^\square \subset \text{GL}_{2n}(D)$ by this basis. In the linear case, putting $\mathfrak{u} = M_n(D)$, we have a bijection

$$\iota : \mathfrak{u} \rightarrow \mathfrak{u}(V^\Delta) : X \mapsto \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

In the ϵ -hermitian case, putting

$$\mathfrak{u} = \{X \in M_n(D) \mid {}^t X^* = -\epsilon X\},$$

we have a bijection

$$\iota : \mathfrak{u} \rightarrow \mathfrak{u}(V^\Delta) : X \mapsto \begin{pmatrix} 0 & XR \\ 0 & 0 \end{pmatrix}$$

where $R = (h(v_i, v_j))_{ij} \in \mathrm{GL}_n(D)$. Let $d^\psi X$ be a self-dual Haar measure on \mathfrak{u} with respect to the pairing

$$\mathfrak{u} \times \mathfrak{u} \rightarrow \mathbb{C} : (X, Y) \mapsto \psi(T(XY)).$$

In this subsection, we choose the push-forward measure $\iota_*(d^\psi X)$ for du .

Now we state the explicit formula of $c(s, \omega, A, \psi)$. Note that we do not use it in the proof of the main theorem. We denote $e(G)$ the invariant of Kottwitz [15]. If D is split, then we have $e(G) = 1$. If D is not split, then we have

$$e(G) = \begin{cases} (-1)^n & \text{in the linear case} \\ (-1)^{\frac{1}{2}n(n-1)} & \text{in the skew-hermitian case} \\ (-1)^{\frac{1}{2}n(n+1)} & \text{in the hermitian case.} \end{cases}$$

Let $A \in \mathfrak{u}(V^\Delta)_{reg}$. We define $R(s, \omega, A, \psi)$ by

$$R(s, \omega, A, \psi) = \begin{cases} \omega_s(N_V(\frac{1}{2}A))^{-2} & \text{in the linear case,} \\ \omega_s(N_V(A))^{-1} \gamma(s + \frac{1}{2}, \omega \chi_{\mathfrak{d}(A)}, \psi) \epsilon(\frac{1}{2}, \chi_{\mathfrak{d}(A)}, \psi)^{-1} & \text{in the hermitian case,} \\ \omega_s(N_V(A))^{-1} \epsilon(\frac{1}{2}, \chi_{\mathfrak{d}(\mathfrak{v})}, \psi) & \text{in the skew-hermitian case.} \end{cases}$$

Proposition 4.2. (1) *In the linear case, we have*

$$c(s, \omega, A, \psi) = e(G) \cdot |2|^{-4ns} \omega^{-2n}(4) \cdot \prod_{i=0}^{2n-1} \gamma(2s - i, \omega^2, \psi)^{-1} \cdot R(s, \omega, A, \psi)^{-1},$$

(2) *In the ϵ -hermitian case, we have*

$$c(s, \omega, A, \psi) = e(G) \cdot |2|^{-2ns+n(n-\frac{1}{2})} \omega^{-n}(4) \cdot \prod_{i=0}^{n-1} \gamma(2s - 2i, \omega^2, \psi)^{-1} \cdot R(s, \omega, A, \psi)^{-1}.$$

Proof. In the linear case, by the analogue of [11, Lemma 3.1], we have

$$c(s, \omega, A, \psi) = e(G) \omega_{2s}^2(N_V(A)) \gamma_{\mathrm{GL}_n(D)}^{GJ}(2s - n + \frac{1}{2}, \omega^2 \circ N, \psi)^{-1}$$

where the γ -factor in the right hand side is the Godement-Jacquet γ -factor of the representation $\omega^2 \circ N : \mathrm{GL}_n(D) \rightarrow \mathbb{C}^\times$. Hence we have the claim (1). In the skew-hermitian case, the claim is proved in [27, Theorem 3.1]. In the hermitian case, the claim is proved in [10, Appendix] considering the analogue of [11, Lemma 3.1]. \square

In the later part of this subsection, we study the dependence of $c(s, \omega, A, \psi)$ by the change of ψ . For $a \in F^\times$, we denote by ψ_a the additive character $x \mapsto \psi(ax)$ of F , and we denote

$$T_N(s, \omega, a) = \begin{cases} \omega_{s-\frac{1}{2}}(a)^N & \text{in the linear case, the hermitian case,} \\ \omega_{s-\frac{1}{2}}(a)^N \chi_{\mathfrak{d}(\mathfrak{v})}(a) & \text{in the skew-hermitian case} \end{cases}$$

where

$$N = \begin{cases} 4n & \text{in the linear case,} \\ 2n + 1 & \text{in the hermitian case,} \\ 2n & \text{in the skew-hermitian case.} \end{cases}$$

Lemma 4.3. *Let $a \in F^\times$. Then we have*

$$c(s, \omega, A, \psi_a) = T_N^{-1}(s, \omega, a) \cdot c(s, \omega, A, \psi).$$

Proof. We can prove this lemma as in [17, Lemma 10]. Note that we can also prove it directly from Proposition 4.2. \square

5. STATEMENT OF THE MAIN THEOREM

5.1. Definition of the γ -factor. We use the setting of §2-§4. Note that D is possibly split. Fix $A \in \mathfrak{u}(V^\Delta)_{reg}$. Then the image of A is V^Δ . Define $\varphi_A \in \text{End}_D(V)$ so that the following diagram is commutative:

$$\begin{array}{ccc} V^\nabla & \xrightarrow{A} & V^\Delta \\ p \downarrow & & \downarrow p \\ V & \xrightarrow{\varphi_A} & V \end{array}$$

where p is the first projection of $V \times V$ to V . We define

$$(5.1) \quad N_V(A) := N_V(\varphi_A), \quad \mathfrak{d}(A) := (-1)^n N_V(A) \in F^\times / F^{\times 2}.$$

For $\mathfrak{d} \in F^\times / F^{\times 2}$, denote by $\chi_{\mathfrak{d}}$ the character $F^\times \rightarrow \mathbb{C}^\times : x \mapsto (x, \mathfrak{d})_F$ where $(\cdot, \cdot)_F$ is the Hilbert symbol of F .

Definition 5.1. *Let π be an irreducible representation of G , let ω be a character of F^\times , and let ψ be a non-trivial character of F . Then we define the γ -factor of π by*

$$\gamma^{\mathcal{V}}(s + \frac{1}{2}, \pi \times \omega, \psi) = \Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi) \cdot c_\pi(-1) \cdot R(s, \omega, A, \psi)$$

where $\Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi)$ is the meromorphic function defined by (4.1), $R(s, \omega, A, \psi)$ is a meromorphic function defined in §4.2, and c_π is the central character of π .

By Proposition 4.2, the factor $\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi)$ does not depend on the choice of A . Note that we interpret $N_V(A) = \mathfrak{d}(A) = 1$, $\mathfrak{d}(V) = 1$ and $\Gamma^{\mathcal{V}}(s, \pi, \omega, A, \pi) = 1$ when $n = 0$.

We can also define the L -factor and the ϵ -factor as in [17, §10]. Note that Yamana gave another definition of the L -factor by g.c.d property and showed that both L -factors coincide [26].

5.2. Lapid-Rallis γ -factors. If D is split, then $\dim_F V^{\mathfrak{h}} = 2n$ and $h^{\mathfrak{h}}$ is either zero, symplectic form or symmetric form (see §2.3). In this subsection, we consider the Lapid-Rallis γ -factor defined in [17, §9]. We will prove that the Lapid-Rallis γ -factor coincides with the γ -factor defined in Definition 5.1 (see Theorem 5.7 (3) below). Note that we need to treat the case where “ A is not split” (for definition, see the end of §5 in [17]), since such A may appear as a localization.

Let F be a local field of characteristic 0, and let \mathcal{V} be a pair (V, h) consisting of a $2n$ -dimensional vector space V and a bilinear map $h : V \times V \rightarrow F$. We assume that h is either zero, a non-degenerate symplectic form, or a non-degenerate symmetric form. Let G denote $\text{Isom}(\mathcal{V})$, let π be an irreducible representation of G , let ω be a character of F^\times , and let ψ be a non-trivial additive character of F . In this subsection, \mathcal{V}^\square denotes $(V \times V, h^\square)$ where $h^\square = h \oplus (-h)$, V^Δ denotes a totally isotropic subspace $\{(v, v) \mid v \in V\}$ of \mathcal{V}^\square , and $\mathfrak{u}(V^\Delta)$ denotes the Lie algebra of the unipotent radical of the parabolic subgroup of $\text{Isom}(\mathcal{V}^\square)$ corresponding V^Δ . We regard $\mathfrak{u}(V^\Delta)$ as a subspace of $\text{End}_F(V^\Delta)$ and we denote by $\mathfrak{u}(V^\Delta)_{reg}$ the set of $A \in \mathfrak{u}(V^\Delta)$ of rank $2n$. Take a basis v_1, \dots, v_{2n} of V . Then we define the *discriminant* of \mathcal{V} by

$$\mathfrak{d}(\mathcal{V}) := (-1)^n \det((h(v_i, v_j))_{ij}) \in F^\times / F^{\times 2}.$$

We define $N_V(A)$ and $\mathfrak{d}(A)$ for $A \in \mathfrak{u}(V^\Delta)_{reg}$ as in §5.1. Then we define the Lapid-Rallis γ -factor by

$$(5.2) \quad \gamma^{LR}(s + \frac{1}{2}, \pi \times \omega, \psi) = \Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi) \cdot c_\pi(-1) \cdot Q(s, \omega, A, \psi)$$

where $\Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi)$ is the Γ -factor as defined in [17, §5], c_π is the central character of π , and

$$Q(s, \omega, A, \psi) = \begin{cases} \omega_s(N_V(\frac{1}{2}A))^{-2} & \text{in the linear case,} \\ \omega_s(N_V(A))^{-1} \gamma(s + \frac{1}{2}, \omega \chi_{\mathfrak{d}(A)}, \psi) \epsilon(\frac{1}{2}, \chi_{\mathfrak{d}(A)}, \psi)^{-1} & \text{in the symplectic case,} \\ \omega_s(N_V(A))^{-1} \epsilon(\frac{1}{2}, \chi_{\mathfrak{d}(\mathcal{V})}, \psi) & \text{in the symmetric case.} \end{cases}$$

By Proposition 4.2, The right hand side of (5.2) does not depend on $A \in \mathfrak{u}(V^\Delta)_{reg}$. Lapid and Rallis moreover defined the L -factor and the ϵ -factor in [17, §10]. We denote them by $L^{LR}(s, \pi \times \omega)$ and $\epsilon^{LR}(s, \pi \times \omega, \psi)$.

5.3. Some Remarks. The definition [17, (25)] seems not to be correct, but this can be fixed as follows.

Remark 5.2. We note here the corrections in the linear case. First, the factor $\pi(-1)$ of [17, (25)] should be replaced with $\pi \otimes \omega(-1)$ (see [5, §7.2]). Second, the factor $\theta(\det_V A)^{-1}$ of [17, (25)] should be replaced with $\omega_s(\theta(\det_V(A)))^{-1}$. Consequently, the Lapid-Rallis γ -factor should be defined by

$$\gamma^{\mathcal{V}}(s + \frac{1}{2}, \pi \times \omega, \psi) = \Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi) \cdot c_{\pi \otimes \omega}(-1) \cdot Q(s, \omega, A)$$

where $\Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi)$ is the Γ -factor as defined in [17, §5], $c_{\pi \otimes \omega}$ is the central character of $\pi \otimes \omega$ and

$$Q(s, \omega, A, \psi) = \omega_s(\det_V(A))^{-1} \omega_s(\theta(\det_V(A)))^{-1}.$$

Note that ω is a character of E^\times where $E = F$ of E is a quadratic extension of F .

Remark 5.3. In the symmetric case, the symplectic case, and the hermitian case, the factor $\omega_s(\det_V(A))$ in [17, (25)] should be replaced with $\omega_s(\det_V(2A))$. See Remark 6.3 for the necessity of this modification. Since

$$\det_V(2A) = N_V(A)$$

where $\det_V(\cdot)$ and $N_V(\cdot)$ are defined in [17, p.337] and the analogue of (5.1) respectively, our definition of the Lapid-Rallis γ -factor (5.2) is consistent with this modification.

Remark 5.4. In the case of hermitian spaces over a quadratic extension E of F , the definition [17, (25)] needs to be modified more: let

$$\epsilon(\mathcal{V}) = \chi((-1)^{n(n-1)/2} \det((h(v_i, v_j))_{ij}))$$

where χ is the non-trivial quadratic character of $F^\times/N_{E/F}(E^\times)$, and (v_1, \dots, v_n) is a basis of V over E . Then as explained in [6, §10.1], the right hand side of [17, (25)] should be multiplied by $\epsilon(\mathcal{V})^{n+1}$. Note that $\epsilon(\mathcal{V})^{n+1} = 1$ when n is odd. Consequently, in the hermitian case over E , the Lapid-Rallis γ -factor should be defined by

$$\gamma^{\mathcal{V}}(s + \frac{1}{2}, \pi \times \omega, \psi) = \Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi) \cdot c_{\pi \otimes \omega}(-1) \cdot Q(s, \omega, A)$$

where $\Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi)$ is the Γ -factor as defined in [17, §5], $c_{\pi \otimes \omega}$ is the central character of $\pi \otimes \omega$ and

$$Q(s, \omega, A) = \epsilon(\mathcal{V})^{n+1} \det_V(2A).$$

Remark 5.5. Recall that in the hermitian case, the character ψ_A of [17, §5] is defined by $\psi_A(X) = \psi_F(\text{tr}_{E/F}(\text{tr}(XA)))$, but Gan-Ichino [6, p539] said that ψ_A should be given by $\psi_A(X) = \psi_F(\text{tr}(XA))$. Taking this into account, the definition of γ -factor stated in [6, §10.1] is correct as stated in this case. However, in the other cases (Case B, C', C'', D), their definition [6, §10.1] requires the same modification explained above.

Remark 5.6. As with the Lapid-Rallis γ -factor, the γ -factor for metaplectic groups defined in [5, §5] needs to be modified. The factor $\det(A)$ appeared in the expression (which defines the γ -factor) in the fourth line from the bottom of p.76 should be replaced with $\det(2A)$.

5.4. Main theorem. We use the setting of §5.1. For a non-trivial character ψ of F and an irreducible representation ρ of $\text{GL}_m(D)$, we can attach the ‘‘Godement-Jacquet γ -factor’’ as

$$\gamma^{GJ}(s, \rho, \psi) = \varepsilon^{GJ}(s, \rho, \psi) \frac{L^{GJ}(1-s, \tilde{\rho})}{L^{GJ}(s, \rho)}$$

where $\tilde{\rho}$ is the contragredient representation of ρ and $L^{GJ}(s, -)$ (resp. $\varepsilon^{GJ}(s, -, \psi)$) is the Godement-Jacquet L -factor (resp. ϵ -factor) (see [8, Theorems 3.3, 8.7]).

Theorem 5.7 (Main). *The factor $\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi)$ satisfies the following properties:*

- (1) (unramified twisting)

$$\gamma^{\mathcal{V}}(s, \pi \times \omega_{s_0}, \psi) = \gamma^{\mathcal{V}}(s + s_0, \pi \times \omega, \psi)$$

for $s_0 \in \mathbb{C}$.

- (2) (multiplicativity) Let W be a totally isotropic subspace of V , and let $\sigma = \sigma_0 \otimes \sigma_1$ be an irreducible representation of $\text{GL}(W) \times \text{Isom}(\mathcal{W}_1)$ (see §2.2). If π is a constituent of $\text{Ind}_{P(W)}^G(\sigma)$, then

$$\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi) = \gamma^{\mathcal{W}_0}(s, \sigma_0 \times \omega, \psi) \gamma^{\mathcal{W}_1}(s, \sigma_1 \times \omega, \psi).$$

- (3) (split factor) If D is split, then

$$\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi) = \gamma^{LR}(s, \pi^{\natural} \times \omega, \psi).$$

- (4) (functional equation)

$$\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi) \gamma^{\mathcal{V}}(1-s, \tilde{\pi} \times \omega^{-1}, \psi^{-1}) = 1.$$

- (5) (self duality)

$$\gamma^{\mathcal{V}}(s, \tilde{\pi} \times \omega, \psi) = \gamma^{\mathcal{V}}(s, \pi \times \omega, \psi).$$

- (6) (dependence on ψ) Denote by ψ_a the additive character $x \mapsto \psi(ax)$ of F for $a \in F^{\times}$. Then

$$\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi_a) = T_N(s, \omega, a) \cdot \gamma^{\mathcal{V}}(s, \pi \times \omega, \psi)$$

where N and $T_N(s, \omega, a)$ are defined in §4.2

- (7) (minimal cases) Suppose $F = \mathbb{R}$, \mathcal{V} is ϵ -hermitian, $n \leq 1$, $\omega = 1$ and π is trivial. Let ϕ_{π} be the L -parameter of π . Then

$$\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi) = \gamma(s, \phi_{\pi} \otimes \omega, \text{std}, \psi).$$

Here, if $n = 0$ we interpret the right hand side as $\gamma(s, \omega, \psi)$ (resp. 1) when \mathcal{V} is hermitian (resp. skew-hermitian).

(8) (GL_n -factor) In the linear case,

$$\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi) = \gamma^{GJ}(s, \pi \otimes \omega, \psi) \gamma^{GJ}(s, \tilde{\pi} \otimes \omega, \psi).$$

(9) (archimedean Langlands compatibility) If F is archimedean then

$$\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi) = \gamma(s, \phi_{\pi} \otimes \omega, \mathrm{std}, \psi)$$

where ϕ_{π} is the Langlands parameter corresponding to π and std is the standard representation of ${}^L G$ to $\mathrm{GL}_N(F)$.

(10) (global functional equation) Let \mathbb{F} be a number field, and let \mathbb{D} be a quaternion algebra over \mathbb{F} . Let $G = \mathrm{Isom}(\mathcal{V})$ where $\mathcal{V} = (V, h)$ is a pair consisting of an n -dimensional right vector space V over \mathbb{D} and either 0, hermitian form or skew-hermitian form h on V . Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_{\mathbb{F}})$. Then for any finite set S of places of \mathbb{F} containing all places where \mathbb{D} is not split, the functional equation

$$L_S^{LR}(s, \pi \times \omega) = \prod_{v \in S} \gamma_v^{\mathcal{V}}(s, \pi_v \times \omega_v, \psi_v) \cdot \epsilon_S^{LR}(s, \pi \times \omega, \psi) L_S^{LR}(1-s, \tilde{\pi} \times \omega^{-1})$$

holds, where

$$L_S^{LR}(s, \pi \times \omega) = \prod_{v \notin S} L^{LR}(s, \pi_v^{\natural} \times \omega_v)$$

and

$$\epsilon_S^{LR}(s, \pi \times \omega, \psi) = \prod_{v \notin S} \epsilon^{LR}(s, \pi_v^{\natural} \times \omega_v, \psi_v).$$

Moreover, the properties (1),(2),(3),(6),(7),(8) and (10) determine $\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi)$ uniquely.

Remark 5.8. The γ -factor appears in a functional equation of the zeta integrals. Suppose that \mathcal{V} is an ϵ -hermitian space for simplicity. Let $f_s \in I(s, \omega)$, and let ξ be a matrix coefficient of π . We can rewrite (4.1) as

$$Z(M(s, \omega)f_s, \xi) = c(s, \omega, A, \psi) \Gamma^{\mathcal{V}}(s, \omega, A, \psi) Z(f_s, \xi).$$

We choose the Haar measure du in the definition of $M(s, \omega)$ as in §4.2. Then, by Proposition 4.2, we can obtain the functional equation

$$(5.3) \quad \begin{aligned} Z^{\mathcal{V}}(M(s, \omega)f_s, \xi) &= e(G) \omega_{\pi}(-1) \gamma^{\mathcal{V}}(s + \frac{1}{2}, \pi \times \omega, \psi) \\ &\times |2|^{-2ns + n(n-\frac{1}{2})} \omega^{-n}(4) \prod_{i=0}^{n-1} \gamma(2s - 2i, \omega^2, \psi)^{-1} Z^{\mathcal{V}}(f_s, \xi) \end{aligned}$$

where $e(G)$ is the invariant of Kottwitz (see. §4.2).

6. PROOF OF THE MAIN THEOREM

In §6 and §7, we prove Theorem 5.7. Once (7) minimal cases is proved, then the other parts of the proof are not difficult or are similar to [17, Theorem 4]. However, the proof of the uniqueness is important: it explains the reason why the minimal cases contains only the cases where $F = \mathbb{R}$, $n = 0, 1$, and π, ω are trivial. In this section, we write down the proof of the uniqueness, (3) split factor, and (9) archimedean Langlands properties for the readers. The minimal cases will be proved in §7.

6.1. Uniqueness. In this subsection, we prove the uniqueness of the γ -factor, which is stated at the end of Theorem 5.7. Let γ' be another function satisfying the conditions (1),(2),(3),(6),(7),(8) and (10) of Theorem 5.7. Then we will prove the equation

$$(6.1) \quad \gamma'(s, \pi \times \omega, \psi) = \gamma^{\mathcal{V}}(s, \pi \times \omega, \psi).$$

We denote the skew-field of Hamilton's quaternions by

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

with the elements i, j, k satisfying

$$i^2 = j^2 = k^2 = -1, k = ij, ij = -ji.$$

We use the setting of §5.1. Recall that F is an arbitrary local field of characteristic zero, D is a quaternion algebra over F , and \mathcal{V} is an n -dimensional ϵ -hermitian space over D . By the condition (2) and (8), we may assume that all the coefficients of π have compact support. By the condition (3), we may assume that D is a division quaternion algebra. Then, we have the following:

Lemma 6.1. *Let \mathcal{V}' be an ϵ -hermitian space over \mathbb{H} such that $\dim \mathcal{V}' = n$. Then there is a quintuple $(\mathbb{F}, \mathbb{D}, \underline{\mathcal{V}}, \underline{\omega}, \underline{\psi})$ where*

- \mathbb{F} is a number field such that there are two (different) places v_1, v_2 with $\mathbb{F}_{v_1} = F, \mathbb{F}_{v_2} = \mathbb{R}$,
- \mathbb{D} is a division quaternion algebra over \mathbb{F} such that \mathbb{D} is not split precisely at the two places v_1, v_2 and $\mathbb{D}_{v_1} = D, \mathbb{D}_{v_2} = \mathbb{H}$,
- $\underline{\mathcal{V}}$ is an ϵ -hermitian space over \mathbb{D} such that $\underline{\mathcal{V}}_{v_1} = \mathcal{V}, \underline{\mathcal{V}}_{v_2} = \mathcal{V}'$
- $\underline{\omega}$ is a Hecke character of \mathbb{F} such that $\omega \cdot \underline{\omega}_{v_1}^{-1} = |\cdot|_{v_1}^t$ for some $t \in \mathbb{C}$ and $\omega_{v_2} = 1$,
- $\underline{\psi}$ is a non-trivial additive character of \mathbb{A}/\mathbb{F} where \mathbb{A} is the ring of adèles of \mathbb{F} .

Proof. The existence of such \mathbb{F}, \mathbb{D} , and ω is well-known. Besides, by using the weak approximation, we have an ϵ -hermitian space $\underline{\mathcal{V}}$ satisfying the condition of the lemma. \square

We apply this lemma to \mathcal{V}' having a $\lfloor n/2 \rfloor$ -dimensional totally isotropic subspace. By [9, Appendice I], there is an irreducible automorphic cuspidal representation Π of $\text{Isom}(\underline{\mathcal{V}})(\mathbb{A})$ such that $\Pi_{v_1} = \pi$. Then, by the conditions (1), (3), (6), and (10), we have

$$\gamma'_{v_1}(s, \Pi_{v_1} \times \underline{\omega}_{v_1}, \underline{\psi}_{v_1}) \gamma'_{v_2}(s, \Pi_{v_2} \times \underline{\omega}_{v_2}, \underline{\psi}_{v_2}) = \gamma_{v_1}^{\mathcal{V}}(s, \Pi_{v_1} \times \underline{\omega}_{v_1}, \underline{\psi}_{v_1}) \gamma_{v_2}^{\mathcal{V}}(s, \Pi_{v_2} \times \underline{\omega}_{v_2}, \underline{\psi}_{v_2}).$$

Thus, we can reduce the equation (6.1) to the case $F = \mathbb{R}$, $\omega = 1$ and \mathcal{V} has a $\lfloor n/2 \rfloor$ -dimensional totally isotropic subspace. By Casselman's embedding theorem ([4, Corollary 5.2]) and the condition (2), we may assume that $n = 0, 1$.

Again by the global argument as above, we may assume moreover that π is trivial. In this case, the equation (6.1) clearly holds by the condition (7).

6.2. Formal properties. The properties (1), (2), (4), (5), (6), and (10) can be deduced from the framework of the doubling method. They can be proved as in [17, §9]. However we explain (2) here to give the detail of Remark 5.3.

We use the setting of (2). We may assume that π is a constituent of $\text{Ind}_{P(W)}^{G^\square} \sigma$. We denote by \mathcal{W} the orthogonal sum $\mathcal{W}_0 \perp \mathcal{W}_1$ (§2.2). We define $I^{\mathcal{W}}(s, \omega)$ as in §3.2 and we define an intertwining map

$$\Psi(s, \omega) : I^{\mathcal{V}}(s, \omega) \rightarrow \text{Ind}_{P(W^\square)}^{G^\square}(I^{\mathcal{W}}(s, \omega) \otimes |\Delta_{\mathcal{W}:\mathcal{V}}|) : f_s \mapsto (g \mapsto [\Psi(s, \omega) f_s]_g)$$

as in [17, Proposition 1]. Fix $A \in \mathfrak{u}(V^\Delta)_{reg}$. We may assume $A(W^\square) \subset W^\square$. Then A induces the following maps

- $A_0 : W^\square \rightarrow W^\square$,

- $A_1 : (W^\perp/W)^\square \rightarrow (W^\perp/W)^\square$.

For $f'_s \in I^{\mathcal{W}}(s, \omega)$ and for

$$(B, C) \in \mathfrak{u}((W^\perp(W^\perp/W))^\Delta) = \mathfrak{u}(W^\Delta) \times \mathfrak{u}((W^\perp/W)^\Delta),$$

we define $l_{\psi_{(B,C)}}^{\mathcal{W}}(f'_s)$ as in §4.1. Then, we have the following:

Proposition 6.2. *Let $f_s \in I(s, \omega)$.*

- (1) *In the linear case, we have*

$$l_{\psi_A}(f_s) = \int_{U(V^\nabla) \cap P(W^\square) \setminus U(V^\nabla)} l_{\psi_{(A_0, A_1)}}^{\mathcal{W}}([\Psi(s)f_s]_u) \psi_A(u) du.$$

- (2) *In the ϵ -hermitian case, we have*

$$l_{\psi_A}(f_s) = \int_{U(V^\nabla) \cap P(W^\square) \setminus U(V^\nabla)} l_{\psi_{(2A_0, A_1)}}^{\mathcal{W}}([\Psi(s)f_s]_u) \psi_A(u) du.$$

Remark 6.3. *Proposition 6.2 corrects the inaccuracy in the statement of [17, Lemma 8]. The functional $l_{\psi_B}^{\mathcal{W}}$ in [17, Lemma 8] corresponds to $l_{\psi_{(A_0, A_1)}}^{\mathcal{W}}$ in our notation. However, it should be replaced with $l_{\psi_{(2A_0, A_1)}}^{\mathcal{W}}$ in the symplectic case, the symmetric case, and the hermitian case. This causes the modification explained in Remark 5.3 of §5.3.*

As in [17, §9], we have

$$\Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi) = \Gamma^{\mathcal{W}_0}(s, \sigma_0, \omega, 2A_0, \psi) \cdot \Gamma^{\mathcal{W}_1}(s, \sigma_1, \omega, A_1, \psi).$$

Observe that

$$R^{\mathcal{V}}(s, \omega, A, \psi) = R^{\mathcal{W}_0}(s, \omega, 2A_0, \psi) \cdot R^{\mathcal{W}_1}(s, \omega, A_1, \psi),$$

thus we have

$$\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi) = \gamma^{\mathcal{W}_0}(s, \sigma_0 \times \omega, \psi) \cdot \gamma^{\mathcal{W}_1}(s, \sigma_1 \times \omega, \psi),$$

as desired.

6.3. Split case. In this subsection, we prove the property (3). Suppose that D is split. Take $A \in \mathfrak{u}(V^\Delta)_{reg}$. Then one can express $I(s, \omega)$, $Z(-, \xi)$ and l_{ψ_A} in the term of V^\natural , π^\natural and A^\natural where A^\natural corresponds to A via the Morita equivalence between $\text{End}_F(V^{\square^\natural})$ and $\text{End}_D(V^\square)$ (§2.3). Note that $V^{\square^\natural} = V^{\square}$ and $A^\natural \in \mathfrak{u}(V^{\natural\Delta})_{reg}$ (§5.2). Then we have

$$\begin{aligned} \Gamma^{\mathcal{V}^\natural}(s, \omega, \pi^\natural, A^\natural, \psi) &= \Gamma^{\mathcal{V}}(s, \omega, \pi, A, \psi), \\ Q^{\mathcal{V}^\natural}(s, \omega, A^\natural, \psi) &= R^{\mathcal{V}}(s, \omega, A, \psi). \end{aligned}$$

Thus, we have the property (3).

6.4. Relation with the local Langlands Correspondence. The properties (7), (8), and (9) of Theorem 5.7 say that the equation (1.1) is true as desired in special cases. Note that the property (8) is proved by Yamana [25, Appendix]. The property (7) will be proved in §7. In this subsection, we prove the property (9) admitting the consequence of §7. Moreover we consider the non-archimedean case (§6.4.4).

6.4.1. *Notations.* Before starting the proof, we give some notations involving the quaternionic unitary groups over archimedean local fields and their representations. We use the setting of §5.1. Suppose that F is archimedean. If D is split over F , the equation (1.1) is proved by Lapid and Rallis [17]. Hence, we may assume $F = \mathbb{R}$ and $D = \mathbb{H}$. By unramified twisting, we may assume that $\omega = 1$ or the sign character sgn . Moreover, by the multiplicativity, we may assume that $\mathcal{V} = (\mathbb{H}^n, \langle I_n \rangle)$ (resp. $\mathcal{V} = (\mathbb{H}, \langle i \rangle)$) in the hermitian case (resp. the skew-hermitian case). Here,

$$\begin{aligned} \langle I_n \rangle : \mathbb{H}^n \times \mathbb{H}^n &\rightarrow \mathbb{H} : (x, y) \mapsto {}^t x^* \cdot y, \\ \langle i \rangle : \mathbb{H} \times \mathbb{H} &\rightarrow \mathbb{H} : (x, y) \mapsto x^* i y. \end{aligned}$$

Then G is a subgroup of $\text{GL}_n(\mathbb{H})$. We choose a basis e_1, \dots, e_{2n} of $(\mathbb{H}^n)^\square = \mathbb{H}^n \times \mathbb{H}^n$ defined by

$$e_j = \begin{cases} \frac{1}{\sqrt{2}}(v_j, v_j) & 1 \leq j \leq n, \\ \frac{1}{\sqrt{2}}(v_{j-n}, -v_{j-n}) & n+1 \leq j \leq 2n, \end{cases}$$

where $v_1 = {}^t(1 \ 0 \ \dots \ 0), \dots, v_n = {}^t(0 \ \dots \ 0 \ 1)$. We may regard $G^\square \subset \text{GL}_{2n}(\mathbb{H})$ by this basis. Put

$$K = \{g \in G^\square \mid g \cdot {}^t g^* = 1\}.$$

Then K is a maximal compact subgroup of G^\square , and the embedding

$$G \times G \rightarrow K : (a, b) \mapsto w_0 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} w_0^{-1}$$

is an isomorphism. Here

$$w_0 := \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}.$$

Moreover, we have the decomposition $G^\square = P(V^\Delta)K$ by the analogue of [7, Lemma 2.1].

In the later part of this subsection, we explain the finite dimensional representations of the Weil group $W_{\mathbb{R}}$ of \mathbb{R} . We regard \mathbb{C} as a subfield of \mathbb{H} by identifying $i \in \mathbb{H}$ with the imaginary unit of \mathbb{C} . The Weil group $W_{\mathbb{R}}$ of \mathbb{R} is given by $\mathbb{C}^\times \cup j\mathbb{C}^\times \subset \mathbb{H}^\times$. For any character ω' of \mathbb{R}^\times , we also denote by ω' the one-dimensional representation of $W_{\mathbb{R}}$ defined by the composition $\omega' \circ \alpha$ where $\alpha : W_{\mathbb{R}} \rightarrow \mathbb{R}^\times$ is the homomorphism defined by

$$\alpha(j) = -1, \quad \alpha(z) = z\bar{z} \text{ for } z \in \mathbb{C}^\times.$$

For $l \in \mathbb{Z}$, we denote by D_l the two-dimensional representation of $W_{\mathbb{R}}$ defined by

$$D_l(j) = \begin{pmatrix} 0 & (-1)^l \\ 1 & 0 \end{pmatrix}, \quad D_l(re^{i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ for } r \in \mathbb{R}_{>0}, \theta \in \mathbb{R}.$$

Then $D_l \cong D_{-l}$ and $D_l \otimes \text{sgn} \cong D_l$ for $l \in \mathbb{Z}$. Note that all finite dimensional representations of $W_{\mathbb{R}}$ are completely reducible, and the finite dimensional irreducible representations of $W_{\mathbb{R}}$ are $1, \text{sgn}, D_l$ for $l = 1, 2, \dots$, and their unramified twistings (cf. [14, §3]). Take the non-trivial additive character ψ given by $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. Put

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2), \quad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s} \Gamma(s).$$

Then, the γ -factors of $1, \text{sgn}$ and D_l for $l \in \mathbb{Z}$ are given by

$$\gamma\left(s + \frac{1}{2}, 1, \text{std}, \psi\right) = \frac{\Gamma_{\mathbb{R}}(-s + \frac{1}{2})}{\Gamma_{\mathbb{R}}(s + \frac{1}{2})}, \quad \gamma\left(s + \frac{1}{2}, \text{sgn}, \text{std}, \psi\right) = i \cdot \frac{\Gamma_{\mathbb{R}}(-s + \frac{3}{2})}{\Gamma_{\mathbb{R}}(s + \frac{3}{2})}$$

and

$$\gamma\left(s + \frac{1}{2}, D_l, \text{std}, \psi\right) = i^{|l|+1} \cdot \frac{\Gamma_{\mathbb{C}}(-s + \frac{|l|+1}{2})}{\Gamma_{\mathbb{C}}(s + \frac{|l|+1}{2})}.$$

In what follows, we first describe the γ -factor of the representation $\text{std} \circ \phi_\pi$ of $W_{\mathbb{R}}$.

6.4.2. *Hermitian case.* We use the setting of §6.4.1. Suppose that \mathcal{V} is hermitian. Recall that $\omega = 1$ or sgn . In this case, $G = \text{Sp}(n)$. Choose the maximal torus

$$T = \{\text{diag}(z_1, \dots, z_n) \in \text{Sp}(n) \mid z_1, \dots, z_n \in \mathbb{C}^\times, |z_1| = \dots = |z_n| = 1\}$$

of $\text{Sp}(n)$. We identify the character group $X^*(T)$ of T with \mathbb{Z}^n . We fix the “standard” positive system. We denote by $\rho = (\rho_1, \dots, \rho_n)$ the half sum of the positive roots of G . Then $\rho_j = n+1-j$. Let

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geq \dots \geq \lambda_n \geq 0$$

be the highest weight of π . Then

$$\begin{aligned} \text{std} \circ (\phi_\pi \otimes \omega) &= \left(\bigoplus_{j=1}^n D_{2(\lambda_j + \rho_j)} \right) \oplus \text{sgn}^n \otimes \omega \\ &= \left(\bigoplus_{j=1}^n D_{2(\lambda_j + \rho_j)} \right) \oplus (\text{sgn}^n \otimes \omega) \end{aligned}$$

where sgn^n is the character of $W_{\mathbb{R}}$ defined by $\text{sgn}^n(j) = (-1)^n$, $\text{sgn}^n(z) = 1$ for $z \in \mathbb{C}^\times$. Since

$$\prod_{j=1}^n i^{2(\lambda_j + \rho_j) + 1} = i^{n(n+2)} \prod_{j=1}^n (-1)^{\lambda_j} = \epsilon \left(\frac{1}{2}, \text{sgn}^n, \psi \right)^{-1} c_\pi(-1),$$

we have

$$\begin{aligned} &\gamma\left(s + \frac{1}{2}, \phi_\pi \otimes \omega, \text{std}, \psi\right) \\ &= c_\pi(-1) \gamma\left(s + \frac{1}{2}, \text{sgn}^n \otimes \omega, \psi\right) \epsilon \left(\frac{1}{2}, \text{sgn}^n, \psi \right)^{-1} \prod_{j=1}^n \frac{\Gamma_{\mathbb{C}}(-s + \frac{1}{2} + \lambda_j + \rho_j)}{\Gamma_{\mathbb{C}}(s + \frac{1}{2} + \lambda_j + \rho_j)}. \end{aligned}$$

We next compute $\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi)$. Observe that

$$I(s, \omega) = I(s, 1), \quad \Gamma^{\mathcal{V}}(s, \pi, \omega, A, \psi) = \Gamma^{\mathcal{V}}(s, \pi, 1, A, \psi), \quad \omega_s(A) = |N_V(A)|^s$$

for $A \in \mathfrak{u}(V^\Delta)_{\text{reg}}$. Hence, if we prove the equation

$$(6.2) \quad \Gamma^{\mathcal{V}}(s, \pi, 1, A, \psi) = |N_V(A)|^s \prod_{j=1}^n \frac{\Gamma_{\mathbb{C}}(-s + \frac{1}{2} + \lambda_j + \rho_j)}{\Gamma_{\mathbb{C}}(s + \frac{1}{2} + \lambda_j + \rho_j)},$$

then we can conclude (1.1). We prove (6.2) by induction on

$$|\lambda| := \sum_{i=1}^v |\lambda_i|$$

by using the “strong adjacency” (see [17, p. 347]).

We first explain the “strong adjacency”. Since $G^\square = P(V^\Delta)K$, letting \hat{G} the set of the equivalence classes of the irreducible representations of G , we have the decomposition

$$I(0, 1) = \bigoplus_{\pi \in \hat{G}} \tilde{\pi} \otimes \pi$$

as $G \times G$ -modules. We define $i_\pi^{(s)}$ by the composition

$$\tilde{\pi} \otimes \pi \hookrightarrow I(0, 1) \rightarrow I(s, 1)$$

where $I(0, 1) \rightarrow I(s, 1)$ is the map $f \mapsto f_s$ (see §3.2). Let \mathfrak{s} be the orthogonal complement of \mathfrak{k} in \mathfrak{g}^\square with respect to the Killing form κ . Then $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $St \otimes \tilde{St}$ as a representation of $K \cong G \times G$, where St is the representation of G defined by the action

$$g \cdot x = (g \otimes 1)^{\natural} x$$

for $g \in \mathrm{Sp}(n), x \in (\mathbb{H}^n \otimes_{\mathbb{R}} \mathbb{C})^{\natural}$ (for the definition of \natural , see §2.3). We identify $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$ with a subrepresentation of $I(0, 1)$ through the map $\omega^{(\nu)}$ which is defined in [2, 1.f].

Definition 6.4. Let π, π' be irreducible representations of G . We say that π and π' are strongly adjacent if the image of $(\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}) \otimes (\pi \otimes \tilde{\pi})$ of the multiplication map

$$I(0, 1) \otimes I(s, 1) \rightarrow I(s, 1) : f' \otimes f_s \mapsto f' f_s$$

contains $\pi' \otimes \tilde{\pi}'$.

If π and π' are strongly adjacent, then

$$(6.3) \quad \text{either } \mathrm{Hom}_G(St \otimes \pi, \pi') \text{ or } \mathrm{Hom}_G(\pi, St \otimes \pi') \text{ is non-zero .}$$

Conversely, one can show that the (6.3) implies the strongly adjacency. By the branching rule of $St \otimes \pi$ (see the end of [16, §2.5]), we have the following lemma.

Lemma 6.5. Let π, π' be irreducible representations of G . We denote by $\lambda = (\lambda_1, \dots, \lambda_n)$ (resp. $\lambda' = (\lambda'_1, \dots, \lambda'_n)$) the highest weight of π (resp. π'). Then the following are equivalent:

- (1) π and π' are strongly adjacent,
- (2) there exists a unique integer l with $1 \leq l \leq n$ such that

$$|\lambda_j - \lambda'_j| = \begin{cases} 1 & j = l \\ 0 & j \neq l \end{cases}$$

for $j = 1, \dots, n$.

Now we prove the property (9) admitting (6.2) for the trivial representation.

Proposition 6.6. Let π, π' be irreducible representations of G of highest weights $\lambda, \lambda' \in \mathbb{Z}^n$, respectively. Suppose π and π' are strongly adjacent, and moreover $\lambda'_l = \lambda_l + 1$ for some unique l . Then

$$(\lambda_l + \rho_l + \frac{1}{2} - s) \Gamma^{\vee}(s, \pi, 1, A, \psi) = (\lambda_l + \rho_l + \frac{1}{2} + s) \Gamma^{\vee}(s, \pi', 1, A, \psi).$$

To prove this, we first note the following lemma:

Lemma 6.7. Let $\pi \in \hat{G}$, and let $A \in \mathfrak{u}(V^{\Delta})_{reg}$. For $s \in \mathbb{C}$, the following diagram is commutative:

$$\begin{array}{ccc} I(s, 1) & \xrightarrow{M^*(s, 1, A, \psi)} & I(-s, 1) \\ \uparrow i_{\pi}^{(s)} & & \uparrow i_{\pi}^{(-s)} \\ \tilde{\pi} \otimes \pi & \xrightarrow{\Gamma^{\vee}(s, \pi, 1, A, \psi)} & \tilde{\pi} \otimes \pi \end{array}$$

Proof. The normalized intertwining operator $M^*(s, 1, A, \psi)$ acts on $\tilde{\pi} \times \pi$ as the multiplication by a scalar for almost all $s \in \mathbb{C}$. We denote the scalar by b . On the other hand, we have $Z(f_{-s}, \xi) = Z(f_s, \xi) \neq 0$ for $f \in \tilde{\pi} \otimes \pi \subset I(0, 1)$ with $f \neq 0$. Hence we have $b = \Gamma(s, \pi, \omega, A\psi)$, and then we have the lemma. \square

Then, applying [2, (2.14)] as in [17, §9], we have Proposition 6.6.

Hence, to complete the induction, it only remains to verify (6.2) in the case $\lambda = 0$, that is, π is trivial. We prove (6.2) for the trivial representation in §7 below to finish the proof of the property (9).

6.4.3. *Skew-hermitian case.* We use the setting of §6.4.1. Suppose that \mathcal{V} is skew-hermitian. Recall that $\omega = 1$ or sgn . In this case, $G = \mathbb{C}^1$ and

$$U(V^\nabla) = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in i\mathbb{R} \right\}.$$

Any irreducible representation of G is of the form

$$(6.4) \quad \pi_l : \mathbb{C}^1 \rightarrow \mathbb{C}^\times : z \mapsto z^l$$

for some $l \in \mathbb{Z}$. We know that

$$\text{std} \circ \phi_{\pi_l} \otimes \omega \cong D_{2l} \otimes \omega \cong D_{2l},$$

so we have

$$\gamma\left(s + \frac{1}{2}, \phi_{\pi_l} \otimes \omega, \psi, \text{std}\right) = i(-1)^l \frac{\Gamma_{\mathbb{C}}(-s + \frac{1}{2} + |l|)}{\Gamma_{\mathbb{C}}(s + \frac{1}{2} + |l|)}.$$

It is obvious that $\gamma(s, \phi_{\pi_l} \otimes \omega, \text{std}, \psi)$ does not depend on ω . On the other hand, $\gamma^{\mathcal{V}}(s, \pi \times \omega, \psi)$ does also not depend on ω . Thus we may assume $\omega = 1$. Moreover, we may assume $l \geq 0$ by the property (5). Then, by Proposition 7.1 (3) below, we have

$$\gamma\left(s + \frac{1}{2}, \phi_{\pi_l} \otimes \omega, \psi, \text{std}\right) = \gamma^{\mathcal{V}}\left(s + \frac{1}{2}, \pi_l \times \omega, \psi\right).$$

Remark 6.8. *Let \mathcal{V} be an anisotropic ϵ -hermitian space over \mathbb{H} . Lemma 6.7 holds even in the skew-hermitian case. By the lemma, we have*

$$(6.5) \quad l_{\psi_A}(f_s) = \Gamma^{\mathcal{V}}(s, \pi, 1, A, \psi) l_{\psi_A}(f_{-s}).$$

for an irreducible representation π of G , $f \in \tilde{\pi} \otimes \pi \subset I(s, 0)$, and $A \in \mathfrak{u}(V^\Delta)_{\text{reg}}$. This equation is useful to compute the γ -factor in the case where \mathcal{V} is anisotropic.

6.4.4. *A remark on the non-archimedean case.* In the non-archimedean case, the local Langlands correspondence is partly proved. Although it is not completed yet in general, we can conclude the equation (1.1) admitting the local Langlands correspondence.

Let F be a non-archimedean local field of characteristic zero, let ω be a character of F^\times , let D be a quaternion algebra over F (it may be split), let $\mathcal{V} = (V, h)$ be an ϵ -hermitian space over D and let G be the isometry group of \mathcal{V} . We may fix a globalization of $(F, D, \mathcal{V}, G, \omega)$ as follows:

- a number field \mathbb{F} and a place v_0 such that $\mathbb{F}_{v_0} = F$;
- a quaternion algebra \mathbb{D} over \mathbb{F} such that $\mathbb{D}_{v_0} = D$ and \mathbb{D}_v are split for all non-archimedean places $v \neq v_0$;
- an ϵ -hermitian space $\underline{\mathcal{V}}$ over \mathbb{D} and its isometry group \underline{G} such that $\underline{\mathcal{V}}_{v_0} = \mathcal{V}$, $\underline{\mathcal{V}}_v$ is unramified and $\underline{G}(F_v)$ is quasi split over F_v for all non-archimedean places $v \neq v_0$;
- a Hecke character $\underline{\omega}$ of \mathbb{A}^\times such that $\underline{\omega}_{v_0} = \omega$ is $\underline{\omega}_v$ are unramified for all non-archimedean places $v \neq v_0$, where \mathbb{A} is the ring of adèles of \mathbb{F} .

We admit the following two (expected) hypotheses;

- (1) The local Langlands correspondence for G ;
- (2) Existence of the global functorial lifting to GL_N associated to the standard representation of ${}^L\underline{G}$ into $\text{GL}_N(\mathbb{C})$ for an irreducible cuspidal automorphic representation of $\underline{G}(\mathbb{A})$.

Remark 6.9. *These hypotheses were proved by Arthur [1] and Mok [18] for quasi-split classical groups. Moreover, Kaletha, Minguez, Shin, and White extended their work to inner-forms of unitary groups [12].*

By a property of the local Langlands correspondence, it suffices to show the equation (1.1) for all irreducible tempered representations. On the other hand, an irreducible tempered representation π can be realized as a direct summand of a representation induced from an irreducible square integrable representation σ of some Levi subgroup, and the L -parameter of π factors through that of σ . Thus, it suffices to prove the equation (1.1) for all square integrable representations.

Let π be an irreducible square integrable representation of G . Then, one can apply [21, Theorem 5.8] with $S = \{v_0\}$ and $\widehat{U} = \{\pi\}$. Thus, there is an irreducible cuspidal automorphic representation Π of $\underline{G}(\mathbb{A})$ such that

- $\Pi_{v_0} \cong \pi$,
- Π_v is unramified for all non-archimedean places $v \neq v_0$.

Note that, by a property of functorial lifting, we have

$$\gamma(s, \phi_{\Pi_v} \otimes \underline{\omega}_v, \text{std}, \psi) = \gamma^{GJ}(s, L(\Pi)_v \otimes \underline{\omega}_v, \psi)$$

for all places v . Here, we denote by $L(\Pi)$ the global functorial lifting of Π . Then, by the global functional equation of $\pi \times \omega$ (Theorem 5.7 (10)) and that of $L(\Pi) \otimes \omega$, we can conclude the equation (1.1) at v_0 from those at $v \neq v_0$.

7. CALCULATIONS

In §6.4, we prove the properties (7) and (9) of Theorem 5.7 admitting a formula of the γ -factor of the trivial representation (and the characters in the skew-hermitian case) of quaternionic unitary groups. In this section, we compute them to finish the proof of Theorem 5.7.

Proposition 7.1. *Let F be a local field of characteristic 0, let ψ be a non-trivial additive character of F , let D be a quaternion algebra over F , and let \mathcal{V} be an arbitrary n -dimensional ϵ -hermitian space over D . Note that D is possibly split. We denote by 1 the trivial representation of G .*

(1) *In the hermitian case, we have*

$$\gamma^{\mathcal{V}}(s + \frac{1}{2}, 1 \times 1, \psi) = \prod_{j=-n}^n \gamma_F(s + \frac{1}{2} + j, 1, \psi).$$

(2) *In the skew-hermitian case, we have*

$$\gamma^{\mathcal{V}}(s + \frac{1}{2}, 1 \times 1, \psi) = \gamma_F(s + \frac{1}{2}, \chi_{\mathfrak{d}(\mathcal{V})}, \psi) \prod_{j=-(n-1)}^{n-1} \gamma_F(s + \frac{1}{2} + j, 1, \psi).$$

(3) *In the case $F = \mathbb{R}$, $D = \mathbb{H}$, and $\mathcal{V} = (D, \langle i \rangle)$, for $A \in \mathfrak{u}(V^{\Delta})_{reg}$, we have*

$$\gamma^{\mathcal{V}}(s + \frac{1}{2}, \pi_l \times 1, \psi) = i(-1)^l \frac{\Gamma_{\mathbb{C}}(-s + \frac{1}{2} + l)}{\Gamma_{\mathbb{C}}(s + \frac{1}{2} + l)}$$

where π_l is the character of G defined by (6.4) and we fix ψ as $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$.

Proof. When D is split, we know that this formulas (1) and (2) hold. Thus, we may assume that D is a division quaternion algebra over F . Once (1) and (2) in the case where $F = \mathbb{R}$, $D = \mathbb{H}$, $n = 0, 1$ are proved, by using multiplicativity (Theorem 5.7 (10)), we have (1) and (2) in the case where $F = \mathbb{R}$, $D = \mathbb{H}$ and \mathcal{V} has a $\lfloor n/2 \rfloor$ -dimensional totally isotropic subspace. Then, by using the global argument as in §6.1, we have (1) and (2) in the case \mathcal{V} is anisotropic. Moreover, by using multiplicativity (Theorem 5.7 (10)) again, we have (1) and (2) in the general case.

Now we prove (1). By the above discussion, it only remains to show 1 in the case $F = \mathbb{R}$, $D = \mathbb{H}$, and $n = 0, 1$. If $n = 0$, the claim is obvious. Let $n = 1$. We may fix ψ as $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. We may put $\mathcal{V} = (D, \langle 1 \rangle)$ and

$$(7.1) \quad A = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \in \mathfrak{u}(V^\Delta).$$

Take the K -invariant section $f_s \in I(s, \omega)$ with $f_s(e) = 1$. The equation

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+xx^*}} & \frac{x^*}{\sqrt{1+xx^*}} \\ 0 & \sqrt{1+xx^*} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+xx^*}} & \frac{-x^*}{\sqrt{1+xx^*}} \\ \frac{x}{\sqrt{1+xx^*}} & \sqrt{1+xx^*} \end{pmatrix}$$

gives an Iwasawa decomposition of

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in U(V^\nabla)$$

in G^\square . Hence we have

$$f_s\left(\begin{pmatrix} 1 & 0 \\ iy + jz + kw & 1 \end{pmatrix}\right) = \left(\frac{1}{1+y^2+z^2+w^2}\right)^{s+\frac{3}{2}}$$

for $y, z, w \in \mathbb{R}$ and then we have

$$\begin{aligned} & \Gamma_{\mathbb{C}}\left(s + \frac{3}{2}\right) l_{\psi_A}(f_s) \\ &= 2(2\pi)^{-s-\frac{3}{2}} \int_0^\infty \int_{\mathbb{R}^3} \left(\frac{1}{1+y^2+z^2+w^2}\right)^{s+\frac{3}{2}} e^{-t} t^{s+\frac{3}{2}} e^{4\pi i y} dy d^\times t \\ &= 2 \int_0^\infty \int_{\mathbb{R}^3} e^{-2\pi(1+y^2+z^2+w^2)t} e^{4\pi i y} dy dz dw t^{s+\frac{3}{2}} d^\times t \\ &= \frac{1}{\sqrt{2}} \int_0^\infty e^{-2\pi(t+\frac{1}{t})t^s} d^\times t. \end{aligned}$$

Note that this integral is not zero. By the change of variable $t \leftrightarrow t^{-1}$, we get

$$\Gamma_{\mathbb{C}}\left(s + \frac{3}{2}\right) l_{\psi_A}(f_s) = \Gamma_{\mathbb{C}}\left(-s + \frac{3}{2}\right) l_{\psi_A}(f_{-s}).$$

Therefore, by (6.5), we have

$$\Gamma^{\mathcal{V}}\left(s + \frac{1}{2}, \pi \times 1, \psi\right) = \frac{\Gamma_{\mathbb{C}}\left(-s + \frac{3}{2}\right)}{\Gamma_{\mathbb{C}}\left(s + \frac{3}{2}\right)}.$$

Since $\omega_s(N_V(A)) = 1$, $c_\pi(-1) = 1$ and

$$\gamma\left(s + \frac{1}{2}, \text{sgn}, \psi\right) = i \frac{\Gamma_{\mathbb{R}}\left(-s + \frac{3}{2}\right)}{\Gamma_{\mathbb{R}}\left(s + \frac{3}{2}\right)}, \quad \epsilon\left(\frac{1}{2}, \text{sgn}, \psi\right) = i,$$

we have

$$\begin{aligned} \gamma^{\mathcal{V}}\left(s + \frac{1}{2}, \pi \times 1, \psi\right) &= \frac{\Gamma_{\mathbb{C}}\left(-s + \frac{3}{2}\right) \Gamma_{\mathbb{R}}\left(-s + \frac{3}{2}\right)}{\Gamma_{\mathbb{C}}\left(s + \frac{3}{2}\right) \Gamma_{\mathbb{R}}\left(s + \frac{3}{2}\right)} \\ &= \frac{\Gamma_{\mathbb{R}}\left(-s - \frac{1}{2}\right) \Gamma_{\mathbb{R}}\left(-s + \frac{1}{2}\right) \Gamma_{\mathbb{R}}\left(-s + \frac{3}{2}\right)}{\Gamma_{\mathbb{R}}\left(s - \frac{1}{2}\right) \Gamma_{\mathbb{R}}\left(s + \frac{1}{2}\right) \Gamma_{\mathbb{R}}\left(s + \frac{3}{2}\right)}. \end{aligned}$$

Hence we have (1) with $n = 1$, and hence we complete the proof of (1).

Now we prove (3). Note that we also obtain (2) with $F = \mathbb{R}, n = 1$ by putting $l = 0$. Take $f \in \pi_{-l} \otimes \pi_l \subset I(s, 0)$ such that $f(e) = 1$, and take A as in (7.1). Then,

$$(7.2) \quad \Gamma_{\mathbb{C}}(s + \frac{1}{2} + l) l_{\psi_A}(f_s) = \int_0^{\infty} e^{-2\pi(t + \frac{1}{t})} P(t) t^s d^{\times} t$$

where

$$P(t) = t^{\frac{1}{2} + l} \int_{\mathbb{R}} e^{-2\pi t(y - \frac{1}{t})^2} (iy - 1)^{2l} dy.$$

By Cauchy's integral theorem, $P(t)$ becomes

$$\int_{\mathbb{R}} e^{-2\pi y^2} (iy - (\frac{1}{\sqrt{t}} + \sqrt{t}))^{2l} dy,$$

which is invariant under the permutation $t \leftrightarrow t^{-1}$. Note that the right hand side of (7.2) is not zero since it is the Mellin transform of the non-zero function $e^{-2\pi(t + t^{-1})} P(t)$. Thus, by the change of variable $t \leftrightarrow t^{-1}$ of the right hand side of (7.2), we get

$$\Gamma_{\mathbb{C}}(s + \frac{1}{2} + l) l_{\psi_A}(f_s) = \Gamma_{\mathbb{C}}(-s + \frac{1}{2} + l) l_{\psi_A}(f_{-s}).$$

Then, by (6.5), we have (3), and hence we have (2). □

Proposition 7.1 contains the things which are not proved in §6 (see §6.4). Hence we complete the proof of Theorem 5.7.

8. APPLICATIONS

In this section, we give two applications of the main theorem.

8.1. The local root number. In [17], they determine the root number of irreducible representations for symplectic groups and (even) orthogonal groups. We can consider the analogue of their work. In this subsection, we suppose that \mathcal{V} is an ϵ -hermitian space and $\omega^2 = 1$.

At first, we note the irreducibility of $I(0, \omega)$, which is a special case of [24, Theorem 1, 2].

Proposition 8.1. *The degenerate principal series representation $I(0, \omega)$ is irreducible as a representation of G^{\square} .*

Then, the normalized intertwining operator $M(0, \omega, A, \psi)$ acts on $I(0, \omega)$ as the multiplication by a scalar for $A \in \mathfrak{u}(V^{\Delta})_{reg}$. However, by the relation $l_{\psi_A} = M(0, \omega, A, \psi) \circ l_{\psi_A}$, the scalar is 1. Thus, we have

$$\epsilon^{\mathcal{V}}(\frac{1}{2}, \pi \times \omega, \psi) = \gamma^{\mathcal{V}}(\frac{1}{2}, \pi \times \omega, \psi) = c_{\pi}(-1) R(0, \omega, A, \psi).$$

By computing the correcting factor, we have the formula of the root number:

Proposition 8.2.

$$\epsilon^{\mathcal{V}}(\frac{1}{2}, \pi \times \omega, \psi) = c_{\pi}(-1) \omega(-1)^n \begin{cases} \epsilon(\frac{1}{2}, \omega, \psi) & \text{in the hermitian case} \\ \omega(\mathfrak{d}(\mathcal{V})) \epsilon(\frac{1}{2}, \chi_{\mathfrak{d}(\mathcal{V})}, \psi) & \text{in the skew-hermitian case} \end{cases}$$

Proof. Let \mathcal{V} be a hermitian space. Since $\omega^2 = 1$, we have

$$\gamma(\frac{1}{2}, \omega \chi_{\mathfrak{d}(A)}, \psi) \epsilon(\frac{1}{2}, \chi_{\mathfrak{d}(A)}, \psi)^{-1} = \omega(\mathfrak{d}(A)) \epsilon(\frac{1}{2}, \omega, \psi)$$

([22, §3, Corollary 2]). Hence, we have

$$\begin{aligned}\epsilon^{\mathcal{V}}\left(\frac{1}{2}, \pi \times \omega, \psi\right) &= c_{\pi}(-1)\omega(N_V(A))\gamma\left(\frac{1}{2}, \omega\chi_{\mathfrak{d}(A)}, \psi\right)\epsilon\left(\frac{1}{2}, \chi_{\mathfrak{d}(A)}, \psi\right)^{-1} \\ &= c_{\pi}(-1)\omega(-1)^n\epsilon\left(\frac{1}{2}, \omega, \psi\right).\end{aligned}$$

Let \mathcal{V} be a skew-hermitian space. Then, $\mathfrak{d}(A) = \mathfrak{d}(\mathcal{V})$ in $F^{\times}/F^{\times 2}$. Hence, we have

$$\begin{aligned}\epsilon^{\mathcal{V}}\left(\frac{1}{2}, \pi \times \omega, \psi\right) &= c_{\pi}(-1)\omega(N_V(A))\epsilon\left(\frac{1}{2}, \chi_{\mathfrak{d}(\mathcal{V})}, \psi\right) \\ &= c_{\pi}(-1)\omega(-1)^n\omega(\mathfrak{d}(\mathcal{V}))\epsilon\left(\frac{1}{2}, \chi_{\mathfrak{d}(\mathcal{V})}, \psi\right).\end{aligned}$$

□

8.2. The doubling zeta integral of representations induced from a minimal parabolic subgroup. In this subsection, over a non-archimedean local field of odd residual characteristic, we compute the zeta integral of some spherical representations with respect to a certain subgroup. Note that if G is unramified, then the spherical representations above are the unramified representations.

Let F a non-archimedean local field of characteristic 0, and let D be a division quaternion algebra of F . In this subsection, we assume that the residue characteristic of F is not 2. Let \mathcal{V} be an n -dimensional ϵ -hermitian space. We can take a basis v_1, \dots, v_n of V such that

$$(h(v_i, v_j))_{ij} = \begin{pmatrix} 0 & 0 & J_r \\ 0 & R_0 & 0 \\ \epsilon J_r & 0 & 0 \end{pmatrix}$$

with $R_0 = \text{diag}(\alpha_1, \dots, \alpha_{n_0})$, $\text{ord}_D(\alpha_i) = 0, -1$ for $i = 1, \dots, n_0$, $2r + n_0 = n$, and

$$J_r = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \in \text{GL}_r(D).$$

We choose a basis e_1, \dots, e_{2n} of $V^{\square} = V \times V$ defined by

$$e_j = \begin{cases} (v_j, v_j) & 1 \leq j \leq n \\ (v_{j-n}, -v_{j-n}) & n+1 \leq j \leq 2n. \end{cases}$$

We may regard $G^{\square} \subset \text{GL}_{2n}(D)$ by this basis, and we choose the maximal compact subgroup

$$K = \left\{ g \in G^{\square} \mid \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & R^{-1} \end{pmatrix} \in \text{GL}_{2n}(\mathcal{O}_D) \right\}.$$

Let W_i be a subspace of V spanned by v_1, \dots, v_i for $i = 1, \dots, r$, and let P_0 be the stabilizer of the flag $(0 \subsetneq W_1 \subsetneq \dots \subsetneq W_r)$ of \mathcal{V} . Then, P_0 is a minimal parabolic subgroup of G and its Levi subgroup M_0 is canonically isomorphic to $(D^{\times})^r \times G_0$ where G_0 is the unitary group of the ϵ -hermitian space $(D^{n_0}, \langle R_0 \rangle)$. Let σ_0 be a trivial representation of G_0 , let σ_i be a character of D^{\times} defined by $\sigma_i(x) = |N_D(x)|^{s_i}$ for some $s_i \in \mathbb{C}$, and let σ the representation $\otimes_{i=0}^r \sigma_i$ of M_0 . Put $C_0 := G \cap \text{GL}_n(\mathcal{O}_D)$, and put

$$C_1 := \{g \in C_0 \mid R(g-1) \in M_n(\mathcal{O}_D)\}.$$

Then C_1 is an open compact subgroup of G .

Proposition 8.3. *Let $f_s^\circ \in I(s, 1)^K$ be a non-zero K invariant section with $f_s^\circ(e) = 1$, π be a constituent of $\text{Ind}_{P_0}^G \sigma$. Suppose that π has a non-zero C_1 fixed vector. If we denote by ξ a bi- C_1 invariant matrix coefficient of π with $\xi(e) = 1$, then we have*

$$Z^\mathcal{V}(f_s^\circ, \xi) = \frac{\text{Vol}(C_1)}{d^\mathcal{V}(s)} \prod_{i=0}^r L^{\mathcal{V}_i}(s + \frac{1}{2}, \sigma_i \times 1)$$

where

$$d^\mathcal{V}(s) = \begin{cases} \zeta_F(s + m + \frac{1}{2}) \prod_{i=1}^{\lfloor n/2 \rfloor} \zeta_F(2s + 2n + 1 - 4i) & \text{in the hermitian case} \\ \prod_{i=1}^{\lfloor n/2 \rfloor} \zeta_F(2s + 2n + 3 - 4i) & \text{in the skew-hermitian case.} \end{cases}$$

Note that if $n_0 = 0$, then $L^{\mathcal{V}_0}(s, \sigma_0)$ denotes

$$\begin{cases} 1 & \text{in the hermitian case,} \\ L_F(s, \chi_{\mathfrak{d}(\mathcal{V})}) & \text{in the skew-hermitian case.} \end{cases}$$

This proposition is proved at the end of this subsection. We start with a basic lemma:

Lemma 8.4. *For $g \in G$, we have $|\Delta((g, 1))| \leq 1$. Moreover, $|\Delta((g, 1))| = 1$ if and only if $g \in C_1$.*

Proof. By considering the Iwasawa decomposition in G^\square , we can take $a \in \text{GL}_n(D)$ and $X \in \text{M}_n(D)$ with ${}^tX^* = -\epsilon X$ such that

$$(8.1) \quad \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & 0 \\ 0 & {}^t a^{*-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} (g, 1) \begin{pmatrix} 1 & 0 \\ 0 & R^{-1} \end{pmatrix} \in \text{GL}_{2n}(\mathcal{O}_D).$$

Consider two submodules

$$L_1 = R(g-1) \cdot \mathcal{O}_D^n, \quad L_2 = R(g+1)R^{-1} \cdot \mathcal{O}_D^n$$

of the vector space D^n , then (8.1) concludes that

$$L_1 + L_2 = {}^t a^{*-1} \mathcal{O}_D^n.$$

On the other hand, considering in ${}^t g^{-1} = RgR^{-1}$, we have

$$L_1 = ({}^t g^{-1} - 1)R \cdot \mathcal{O}_D^n \supset ({}^t g^{-1} - 1) \cdot \mathcal{O}_D^n, \quad L_2 = ({}^t g^{-1} + 1) \cdot \mathcal{O}_D^n,$$

and thus $L_1 + L_2 \supset \mathcal{O}_D^n$. Hence we have

$$|\Delta((g, 1))| = |N(a)| = |N({}^t a^{*-1})|^{-1} \leq 1.$$

Moreover, $|\Delta((g, 1))| = 1$ if and only if $L_1 \subset \mathcal{O}_D^n$ and $L_2 \subset \mathcal{O}_D^n$, which is equivalent to the condition of the lemma. \square

Consider the partition of the integral

$$Z^\mathcal{V}(f_s^\circ, \xi) = \int_{C_1} \xi(g) dg + \int_{G-C_1} f_s^\circ((g, 1)) \xi(g) dg.$$

If s_0 be a sufficiently large real number so that $Z^\mathcal{V}(f_{s_0}^\circ, \xi)$ converges absolutely, then, by Lemma 8.4, we have

$$\begin{aligned} \left| \int_{G-C_1} f_s^\circ((g, 1)) \xi(g) dg \right| &\leq \int_{G-C_1} |\Delta((g, 1))|^{s-s_0} |f_{s_0}^\circ((g, 1)) \xi(g)| dg \\ &\leq q^{-(\Re s - s_0)} \int_G |f_{s_0}^\circ((g, 1)) \xi(g)| dg \end{aligned}$$

for $\Re s > s_0$. Thus we have

$$(8.2) \quad \lim_{\Re s \rightarrow \infty} Z^{\mathcal{V}}(f_s^{\circ}, \xi) = \text{Vol}(C_1).$$

Now we prove Proposition 8.3. Note that $Z^{\mathcal{V}}(f_s^{\circ}, \xi) \neq 0$ by (8.2). Put $\Xi(q^{-s})$ by

$$\frac{Z^{\mathcal{V}}(f_s^{\circ}, \xi)}{\prod_{i=0}^r L^{\mathcal{V}_i}(s + \frac{1}{2}, \sigma_i \times 1)}.$$

The ‘‘g.c.d property’’ [26, Theorem 5.2] and [26, Lemma 6.1] conclude that $\Xi(q^{-s})$ is a polynomial in q^{-s} and q^s . Moreover, (8.2) implies that it is a polynomial in q^{-s} with the constant term $\text{Vol}(C_1)$. We define the polynomial $D(q^{-s})$ by $d^{\mathcal{V}}(s)^{-1}$. Since the action of the normalized intertwining operator on f_s° is given by

$$M^*(s, \omega, A, \psi) f_s^{\circ} = e(G) q^{-n's} \cdot \frac{D(q^{-s})}{D(q^s)} f_{-s}^{\circ}$$

(Proposition 4.2 and [20, Proposition 3.5] (or [3, Theorem 3.1])), and since the γ -factor of π is given by

$$\gamma^{\mathcal{V}}(s + \frac{1}{2}, \pi \times 1, \psi) = e(G) q^{-n's} \prod_{i=0}^r \frac{L^{\mathcal{V}_i}(-s + \frac{1}{2}, \sigma_i^{\vee} \times 1)}{L^{\mathcal{V}_i}(s + \frac{1}{2}, \sigma_i \times 1)}$$

where

$$n' = \begin{cases} 2\lceil \frac{n}{2} \rceil & \text{in the hermitian case,} \\ 2\lfloor \frac{n}{2} \rfloor & \text{in the skew-hermitian case.} \end{cases}$$

(Proposition 7.1), we can rewrite the functional equation (5.3) as

$$\Xi(q^{-s}) D(q^s) = \Xi(q^s) D(q^{-s}).$$

However, for sufficiently large m , $q^{-ms} D(q^s)$ is coprime to $D(q^{-s})$ as polynomials. Thus, comparing the constant term of $D(q^{-s})$ and $\Xi(q^{-s})$, we have

$$\Xi(q^{-s}) = \text{Vol}(C_1) \cdot D(q^{-s}).$$

Hence we have the proposition.

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