PROOFS OF SOME RAMANUJAN SERIES FOR $1/\pi$ USING A ZEILBERGER'S PROGRAM

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Abstract. We show with some examples how to prove some Ramanujan-type series for $1/\pi$ in an elementary way by using terminating identities.

Introduction. Up till now, we know how to prove 11 Ramanujan-type series for $1/\pi$ by using the WZ (Wilf and Zeilberger) method [6]. Here we will show how to prove some more using a related Zeilberger's algorithm.

1. The WZ algorithm as a black box

Let $G(n,k)$ be hypergeometric in n and k, that is such that $G(n+1,k)/G(n,k)$ and $G(n,k+1)/G(n,k)$ are rational functions. Then, we can use the Zeilberger's Maple package `SumTools[Hypergeometric]);`. The output of `Zeilberger(G(n,k),k,n,K)[1];` is an operator $O(K)$ of the following form

$$O(K) = P_0(k) + P_1(k)\,K + P_2(k)\,K^2 + \cdots + P_m(k)\,K^m,$$

where $P_0(k)$, $P_1(k), \ldots, P_m(k)$ are polynomials of k, and K is an operator which increases k in 1 unity, that is $KG(n,k) = G(n,k+1)$. The output of `Zeilberger(G(n,k),k,n,K)[2];` gives a function $F(n,k)$ such that

$$O(K)G(n,k) = F(n+1,k) - F(n,k).$$

If we sum for $n \geq 0$, we get

$$O(K)r_k = \lim_{n\to\infty} F(n,k) - F(0,k), \qquad r_k = \sum_{n=0}^{\infty} G(n,k).$$

If the above limit and $F(0,k)$ are equal to zero, we have

$$O(K)r_k = 0,$$

which is a recurrence of order m.

Example 1. Prove that:

$$\sum_{n=0}^{\infty} \frac{\left(\frac12\right)_n \left(\frac14\right)_n \left(\frac34\right)_n}{(1)_n^3} (-1)^n \left(\frac{16}{63}\right)^{2n} (65n+8) = \frac{9\sqrt{7}}{\pi}. \tag{1}$$

We have not found a WZ-pair which proves this Ramanujan series. However our proof is closely related to the WZ-method.

Proof. Let

$$A(n,k) = 3\left(\frac{64}{63}\right)^k \frac{(-k)_n \left(\frac12\right)_n^2}{\left(\frac12-k\right)_n^2 (1)_n} \left(\frac{1}{64}\right)^n (42n+5),$$

$$B(n,k) = \frac{(-k)_n \left(\frac{-k}{2}\right)_n \left(\frac12-\frac{k}{2}\right)_n}{\left(\frac12-k\right)_n^2 (1)_n} (-1)^n \left(\frac{16}{63}\right)^{2n} (130n-2k+15),$$

We define the sequences

$$r_k = \sum_{n=0}^{\infty} A(n,k), \qquad s_k = \sum_{n=0}^{\infty} B(n,k).$$

Then we use a Zeilberger's program which finds recurrences. Writing in a Maple session

```
with(SumTools[Hypergeometric]);
s:=subs(n=0, Zeilberger(A(n,k),k,n,K)[2]);
t:=subs(n=0, Zeilberger(B(n,k),k,n,K)[2]);
```

we see that $s = t = 0$. Then, writing

```
u:=Zeilberger(A(n,k),k,n,K)[1];
v:=Zeilberger(B(n,k),k,n,K)[1];
```

and executing it, we see that r_k and s_k satisfy a common recurrence of order 3. Then observe that the sums which define r_k and s_k are finite because the terms with $n > k$ are equal to zero due to presence of $(-k)_n$. By direct evaluation, we check that $r_0 = s_0$, $r_1 = s_1$ and $r_2 = s_2$. Hence, as the three first terms are equal, all of them are. Let

$$r(k) = 3\left(\frac{64}{63}\right)^k \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{2}\right)_n^2}{\left(\frac{1}{2}-k\right)_n^2 (1)_n} \left(\frac{1}{64}\right)^n (42n+5),$$

$$s(k) = \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{-k}{2}\right)_n \left(\frac{1}{2}-\frac{k}{2}\right)_n}{\left(\frac{1}{2}-k\right)_n^2 (1)_n} (-1)^n \left(\frac{16}{63}\right)^{2n} (130n - 2k + 15),$$

Applying Carlson's theorem [1, p. 39], we can deduce that for all complex values of k we have $r(k) = s(k)$. Finally replacing $k = -1/2$, we get

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (-1)^n \left(\frac{16}{63}\right)^{2n} (130n+16) = \frac{9\sqrt{7}}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \left(\frac{1}{64}\right)^n (42n+5)$$

But in 2002, we used the WZ-method to prove

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \left(\frac{1}{64}\right)^n (42n+5) = \frac{16}{\pi},$$

in an elementary way. Hence we are done. □

Example 2. Prove that:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left(\frac{2}{11}\right)^{3n} (126n+10) = \frac{11\sqrt{33}}{2\pi}. \tag{2}$$

Proof. It is completely similar to that in our first example: Use `Zeilberger` to prove the identity

$$11\left(\frac{32}{33}\right)^k \sum_{n=0}^{\infty} \frac{(-3k)_n \left(\frac{1}{3}-k\right)_n \left(\frac{1}{6}-2k\right)_n}{\left(\frac{2}{3}-2k\right)_n \left(\frac{1}{3}-4k\right)_n (1)_n} \left(\frac{-1}{8}\right)^n (6n+1)$$
$$= \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{3}-k\right)_n \left(\frac{2}{3}-k\right)_n}{\left(\frac{5}{6}-k\right)_n \left(\frac{2}{3}-2k\right)_n (1)_n} \left(\frac{2}{11}\right)^{3n} (126n+6k+11),$$

and take $k = -1/6$. □

Example 3. Prove that:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left(\frac{-4}{5}\right)^{3n} (63n+8) = \frac{5\sqrt{15}}{\pi}. \tag{3}$$

Proof. As in the preceeding examples, first use `Zeilberger` to show that

$$5\sum_{n=0}^{\infty} \frac{(-3k)_n \left(\frac{2}{3}+k\right)_n \left(\frac{1}{3}-k\right)_n}{\left(\frac{5}{6}-k\right)_n \left(\frac{2}{3}-2k\right)_n (1)_n} \left(\frac{1}{64}\right)^n (42n+5)$$
$$= \left(\frac{15}{16}\right)^{3k} \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{3}-k\right)_n \left(\frac{2}{3}-k\right)_n}{\left(\frac{5}{6}-k\right)_n \left(\frac{2}{3}-2k\right)_n (1)_n} \left(\frac{-64}{125}\right)^n (252n-42k+25).$$

Then take $k=-1/6$. □

Example 4. With `Zeilberger`, we can also prove the following general identity:

$$\sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{2}\right)_n^2}{\left(\frac{1}{2}-k\right)_n^2 (1)_n} z^n = (1-z)^k \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{-k}{2}\right)_n \left(\frac{1}{2}-\frac{k}{2}\right)_n}{\left(\frac{1}{2}-k\right)_n^2 (1)_n} \left(\frac{-4z}{(1-z)^2}\right)^n,$$

which is a particular case of a multi-parameter formula due to Whipple [5]. Applying to it the operator $5+42\theta$ at $z=1/64$, where $\theta = z\,d/dz$ (Zudilin's translation method [9]), we get an identity which we have proved directly in Example 1. In a similar way, If we apply the operator $1+6\theta$ at $z=-1/8$, we get an identity which we can reprove directly with `Zeilberger`. From this identity we immediatly get an elementary proof of the formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \left(\frac{32}{81}\right)^n (7n+1) = \frac{9}{2\pi}, \tag{4}$$

as there is a WZ-method proof of the series in the other side of the identity [6].

Example 5. With `Zeilberger`, we can also prove the following general identity:

$$\sum_{n=0}^{\infty} \frac{(-3k)_n \left(\frac{1}{3}-k\right)_n \left(\frac{1}{6}-2k\right)_n}{\left(\frac{2}{3}-2k\right)_n \left(\frac{1}{3}-4k\right)_n (1)_n} z^n = 2\,(4-z)^{3k} \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{3}-k\right)_n \left(\frac{2}{3}-k\right)_n}{\left(\frac{5}{6}-k\right)_n \left(\frac{2}{3}-2k\right)_n (1)_n} \left(\frac{27z^2}{(4-z)^3}\right)^n,$$

which is a particular case of a multi-parameter formula due to Bailey [5]. Applying to it the operator $1+6\theta$ at $z=-1/8$, we get an identity which we have proved directly in Example 2. In a similar way, if we apply the operators: $1+4\theta$ at $z=-1$; $1+6\theta$ at $z=1/4$ and $5+42\theta$ at $z=1/64$, we get identities which we can reprove directly with `Zeilberger`. From these identities we can derive respectively the formulas

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left(\frac{3}{5}\right)^{3n} (28n+3) = \frac{5\sqrt{5}}{\pi}, \tag{5}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left(\frac{4}{125}\right)^{n} (11n+1) = \frac{5\sqrt{15}}{6\pi}, \tag{6}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left(\frac{4}{85}\right)^{3n} (133n+8) = \frac{85\sqrt{255}}{54\pi}, \tag{7}$$

in an elementary way taking into account that we have shown that they are equal to series that we had already proved by the WZ-method [6].

Remarks.

(1) Our proofs are elementary (we do not use the modular theory).
(2) Formulas (6) and (7) are due to Ramanujan [8]. Formulas (1) and (4) are due to Berndt, Chan and Liaw [3]. Formulas (2) and (5) are due to the Borweins [4]. Formula (3) is due to Baruah and Berndt [2].
(3) For other elementary methods to prove these and other Ramanujan series see [9] and [7]. Those methods are based in the variable z, while the proofs in this paper are based in the free parameter k.

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