

# INSTABILITY OF THE SOLITARY WAVE SOLUTIONS FOR THE GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION IN THE ENDPOINT CASE

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ABSTRACT. We consider the stability theory of solitary wave solutions for the generalized derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma} \partial_x u = 0,$$

where  $1 < \sigma < 2$ . The equation has a two-parameter family of solitary wave solutions of the form

$$u_{\omega,c}(t, x) = e^{i\omega t + i\frac{c}{2}(x-ct) - \frac{i}{2\sigma+2} \int_{-\infty}^{x-ct} \varphi_{\omega,c}^{2\sigma}(y) dy} \varphi_{\omega,c}(x-ct).$$

The stability theory in the frequency region of  $|c| < 2\sqrt{\omega}$  was studied previously. In this paper, we prove the instability of the solitary wave solutions in the endpoint case  $c = 2\sqrt{\omega}$ .

## 1. INTRODUCTION

**1.1. Setting of the Problem.** In this paper, we consider the stability theory of solitary wave solutions for the generalized derivative nonlinear Schrödinger (gDNLS) equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u + i|u|^{2\sigma} \partial_x u = 0, & (t, x) \in [0, T] \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where  $\sigma > 0$ .

When  $\sigma = 1$ , by a suitable gauge transformation, (1.1) is transformed to the standard derivative nonlinear Schrödinger (DNLS) equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u + i\partial_x(|u|^2 u) = 0, & (t, x) \in [0, T] \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (1.2)$$

It describes an Alfvén wave and appears in plasma physics, nonlinear optics, and so on (see [25, 26]). The Cauchy problem of (1.2) is local well-posedness in the energy space  $H^1(\mathbb{R})$  by Hayashi and Ozawa [15, 16]. That is, given  $u_0 \in H^1(\mathbb{R})$ , there exists a unique maximal solution  $u(t, x)$  of (1.2) in  $C([0, T], H^1(\mathbb{R}))$ , moreover,  $\lim_{t \rightarrow T} \|u\|_{L^2} = \infty$  if  $T < +\infty$ . See also [34, 35, 8, 30, 32, 1, 33] for some of the previous or extended results. Meanwhile, the global well-posedness was widely studied. In [15], the authors showed that  $H^1(\mathbb{R})$  initial data with  $\|u_0\|_{L^2} < \sqrt{2\pi}$  gives global and  $H^1(\mathbb{R})$  bounded solutions. Recently, the global well-posedness with  $H^1(\mathbb{R})$  initial data satisfying  $\|u_0\|_{L^2} < 2\sqrt{\pi}$ , has been established in the works by Wu [36, 37].

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*Key words and phrases.* generalized DNLS, orbital instability, solitary wave solutions, endpoint case.

In [13], a two-page's proof was presented by simplifying the argument in [37]. More recently, Jenkins, Liu, Perry and Sulem [7] proved that the Cauchy problem (1.2) is global well-posedness for any  $H^{2,2}$ -initial datum, without mass restriction. See also [16, 30, 3, 4, 24, 11, 36, 13] for the related results.

It is known (see for examples [10, 2, 37]) that (1.2) has a two-parameter family of solitary wave solutions:

$$\tilde{u}_{\omega,c}(t, x) = e^{i\omega t + i\frac{c}{2}(x-ct) - \frac{3}{4}i \int_{-\infty}^{x-ct} |\tilde{\varphi}_{\omega,c}(\eta)|^2 d\eta} \tilde{\varphi}_{\omega,c}(x-ct),$$

where  $\omega = (\omega, c) \in \{(\omega, c) \in \mathbb{R}^+ \times \mathbb{R} : c^2 < 4\omega \text{ or } c = 2\sqrt{\omega}\}$ , and  $\tilde{\varphi}_{\omega,c}$  is the solution of

$$-\partial_x^2 \tilde{\varphi} + \left(\omega - \frac{c^2}{4}\right) \tilde{\varphi} + \frac{c}{2} |\tilde{\varphi}|^2 \tilde{\varphi} - \frac{3}{16} |\tilde{\varphi}|^4 \tilde{\varphi} = 0.$$

For general  $\sigma > 0$ , (1.1) is regarded as an extension of (DNLS) equation. In energy space  $H^1(\mathbb{R})$ , Hayashi and Ozawa [17] proved that the Cauchy problem (1.1) is local well-posedness for  $\sigma > 1$ . See also [14, 31, 17, 20] for the related results. Moreover, the  $H^1$ -solution  $u(t)$  of (1.1) satisfies three conservation laws:

$$E(u(t)) = E(u_0), \quad P(u(t)) = P(u_0), \quad M(u(t)) = M(u_0),$$

for all  $t \in [0, T)$ , where

$$\begin{aligned} E(u) &= \frac{1}{2} \|\partial_x u\|_{L^2}^2 - \frac{1}{2(\sigma+1)} \operatorname{Im} \int_{\mathbb{R}} |u|^{2\sigma} u \overline{\partial_x u} dx, \\ P(u) &= \frac{1}{2} (i\partial_x u, u)_{L^2} = \frac{1}{2} \operatorname{Im} \int_{\mathbb{R}} u \overline{\partial_x u} dx, \\ M(u) &= \frac{1}{2} \|u\|_{L^2}^2. \end{aligned}$$

By using the conservation laws, Fukaya, Hayashi and Inui [5] investigated the global well-posedness of (1.1) in energy space  $H^1(\mathbb{R})$  for  $\sigma > 1$  with some suitable size restriction on the initial datum. Hayashi and Ozawa [17] proved the global existence (without uniqueness) in  $H^1(\mathbb{R})$  for  $0 < \sigma < 1$ .

Equation (1.1) also admits a two-parameter family of solitary wave solutions:

$$u_{\omega,c}(t, x) = e^{i\omega t} \phi_{\omega,c}(x-ct),$$

where  $\phi_{\omega,c}(x) = \varphi_{\omega,c}(x) e^{i\frac{c}{2}x - \frac{i}{2(\sigma+1)} \int_{-\infty}^x \varphi_{\omega,c}^{2\sigma}(y) dy}$ . Here  $\varphi_{\omega,c}$  is the solution of

$$-\partial_x^2 \varphi + \left(\omega - \frac{c^2}{4}\right) \varphi + \frac{c}{2} \varphi^{2\sigma+1} - \frac{2\sigma+1}{(2\sigma+2)^2} \varphi^{4\sigma+1} = 0.$$

**1.2. Stability theory of gDNLS.** In the continuation of these works, there are many results about the stability theory of solitary wave solutions for the generalized derivative nonlinear Schrödinger equation.

When  $\sigma = 1$ , Guo and Wu [10] proved that  $\tilde{u}_{\omega,c}(t, x)$  is stable for  $c < 0$  and  $c^2 < 4\omega$ . Further, Colin and Ohta [2] proved that  $\tilde{u}_{\omega,c}(t, x)$  is stable for  $c^2 < 4\omega$ . The endpoint case  $c = 2\sqrt{\omega}$  was studied by Kwon and Wu [18]. Recently, the stability of the multi-solitons is studied by Le Coz, Wu [19] (see also Miao, Tang, Xu [23] in the two-solitons case).

In an effort to understand the stability theory of (DNLS) equation, one may add a term " $b|u|^4u$ " with  $b > 0$  to (DNLS) equation, which brings some destabilizing effect. In this case, Ohta [29] showed that there exists  $\kappa \in (0, 1)$  such that  $\tilde{u}_{\omega,c}(t, x)$  is stable when  $-2\sqrt{\omega} < c < 2\kappa\sqrt{\omega}$ , and unstable when  $2\kappa\sqrt{\omega} < c < 2\sqrt{\omega}$ . Moreover, Ning, Ohta and Wu [27, 28] proved  $\tilde{u}_{\omega,c}(t, x)$  was unstable both in the borderline case  $c = 2\kappa\sqrt{\omega}$  and in the endpoint case  $c = 2\sqrt{\omega}$ .

When  $0 < \sigma < 1$ , Liu, Simpson and Sulem [21] proved that the solitary wave solution  $u_{\omega,c}(t, x)$  is stable for any  $-2\sqrt{\omega} < c < 2\sqrt{\omega}$ ; when  $\sigma \geq 2$ , the solitary wave solution  $u_{\omega,c}(t, x)$  is unstable for any  $-2\sqrt{\omega} < c < 2\sqrt{\omega}$ . Recently, Guo [9] proved the stability of the solitary wave solutions in the endpoint case  $c = 2\sqrt{\omega}$ ,  $\sigma \in (0, 1)$ .

When  $1 < \sigma < 2$ , Liu, Simpson and Sulem [21] proved that there exists  $z_0(\sigma) \in (0, 1)$  such that the solitary wave solution  $u_{\omega,c}(t, x)$  is stable when  $-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}$ , and unstable when  $2z_0\sqrt{\omega} < c < 2\sqrt{\omega}$ . Further, Fukaya [6] proved that the solitary waves solution  $u_{\omega,c}(t, x)$  is unstable when  $\frac{7}{6} < \sigma < 2$ ,  $c = 2z_0\sqrt{\omega}$ . Moreover, Guo, Ning and Wu [12] and Miao, Tang and Xu [22] independently proved that the solitary waves solution  $u_{\omega,c}(t, x)$  is unstable for any  $1 < \sigma < 2$  in borderline case  $c = 2z_0\sqrt{\omega}$ . After these works, the stability theory when  $c = 2\sqrt{\omega}$ ,  $\sigma \in (1, 2)$  is unsolved.

**1.3. Statement of the results.** In this paper, we aim to the unsolved case

$$c = 2\sqrt{\omega}, \quad \sigma \in (1, 2).$$

More precisely, let us define

$$u_c(t, x) = e^{i\frac{c^2}{4}t} \phi_{\frac{c^2}{4},c}(x - ct),$$

where

$$\phi_{\frac{c^2}{4},c}(x) = \varphi_{\frac{c^2}{4},c}(x) e^{i\frac{c}{2}x - \frac{i}{2(\sigma+1)} \int_{-\infty}^x \varphi_{\frac{c^2}{4},c}^{2\sigma}(y) dy}, \quad (1.3)$$

and

$$\varphi_{\frac{c^2}{4},c}(x) = \left( \frac{2c(\sigma+1)}{\sigma^2(cx)^2 + 1} \right)^{\frac{1}{2\sigma}}. \quad (1.4)$$

For simplicity, we denote  $\varphi_c$  to be  $\varphi_{\frac{c^2}{4},c}$  and  $\phi_c$  to be  $\phi_{\frac{c^2}{4},c}$  for short. Note that  $\phi_c$  is the solution of

$$-\partial_x^2 \phi + \frac{c^2}{4} \phi + ci \partial_x \phi - i|\phi|^{2\sigma} \partial_x \phi = 0, \quad (1.5)$$

and  $\varphi_c$  is the solution of

$$-\partial_x^2 \varphi + \frac{c}{2} \varphi^{2\sigma+1} - \frac{2\sigma+1}{(2\sigma+2)^2} \varphi^{4\sigma+1} = 0. \quad (1.6)$$

**Remark 1.** Compared with the case of  $-2\sqrt{\omega} < c < 2\sqrt{\omega}$ , the solution of the elliptic equation (1.6)  $\varphi_c$  is "zero mass" in the endpoint case  $c = 2\sqrt{\omega}$ . For the (DNLS) equation, there also appears "zero mass" in the endpoint case  $c = 2\sqrt{\omega}$ , see [27, 37].

From (1.4), we know that  $\phi_c, \varphi_c \notin L^2(\mathbb{R})$ , when  $\sigma \geq 2$ . Hence, compared to the definitions of stability/instability in the following, the analogous definitions should be given in a different way in the case of  $\sigma \geq 2$ .

For  $\varepsilon > 0$ , we define

$$U_\varepsilon(\phi_c) = \{u \in H^1(\mathbb{R}) : \inf_{(\theta, y) \in \mathbb{R}^2} \|u - e^{i\theta} \phi_c(\cdot - y)\|_{H^1} < \varepsilon\}.$$

**Definition 1.** We say that the solitary wave solution  $e^{i\frac{c^2}{4}t} \phi_c(x - ct)$  of (1.1) is stable if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $u_0(x) \in U_\delta(\phi_c)$ , then the solution  $u(t, x)$  of (1.1) with  $u(0, x) = u_0(x)$  exists for all  $t \in \mathbb{R}$ , and  $u(t, x) \in U_\varepsilon(\phi_c)$  for all  $t \in \mathbb{R}$ . Otherwise,  $e^{i\frac{c^2}{4}t} \phi_c(x - ct)$  is said to be unstable.

**Theorem 1.** Let  $\sigma \in (1, 2)$ , then the solitary wave solution  $e^{i\frac{c^2}{4}t} \phi_c(x - ct)$  of (1.1) is unstable.

As described in Remark 1, the new feature in the endpoint case is the “zero mass” properties, which are related to both  $\phi_{\omega, c}$  and the functional  $S_{\omega, c}$  when  $c = 2\sqrt{\omega}$ . This new feature brings obstacles in the study of the stability theory. It is worth to noting that the direction of neither  $\partial_\omega \phi_{\omega, c}$  nor  $\partial_c \phi_{\omega, c}$  makes sense when  $c \rightarrow 2\sqrt{\omega}$ . Especially,  $\partial_\omega P(\phi_{\omega, c})$  and  $\partial_c P(\phi_{\omega, c})$  go to infinity when  $c \rightarrow 2\sqrt{\omega}$ . This makes it impossible to handle this problem in the same way as the non-endpoint case. Furthermore, in the endpoint case, a two-parameter family of solitary wave solutions ( $\omega$  and  $c$ ) degenerates into only one parameter family of solitary wave solutions. This causes the absence of a nature definition to the negative direction which is orthogonal with both  $M'(\phi_c)$  and  $P'(\phi_c)$ .

Our argument is based on [27], in which the authors constructed an auxiliary function and used the cut-off trick to define the negative direction. However, the argument in [27] does not work for all  $\sigma \in (1, 2)$ , but only applies when  $\sigma$  is close to 1. To overcome the difficulty, we construct a new auxiliary function to solve the problem for any  $\sigma \in (1, 2)$ .

This paper is organized as follows. In Section 2, we give the definitions of some important functionals and some useful lemmas. In Section 3, we construct the negative direction. In Section 4, we prove the Theorem 1.

## 2. PRELIMINARIES

**2.1. Notations.** We use  $A \lesssim B$  to denote an estimate of the form  $A \leq CB$  for some constant  $C > 0$ . Also, we write  $O(A)$  to indicate any quantity  $A$  such that  $|A| \lesssim B$ . And we denote  $\langle x \rangle = \sqrt{1 + x^2}$ .

For a function  $f(x)$ , its  $L^q$ -norm  $\|f\|_{L^q} = \left( \int_{\mathbb{R}} |f(x)|^q dx \right)^{\frac{1}{q}}$  and its  $H^1$ -norm  $\|f\|_{H^1} = (\|f\|_{L^2}^2 + \|\partial_x f\|_{L^2}^2)^{\frac{1}{2}}$ . For  $u, v \in L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$ , we define

$$(u, v) = \operatorname{Re} \int_{\mathbb{R}} u(x) \overline{v(x)} dx$$

and regard  $L^2(\mathbb{R})$  as a real Hilbert space.

From the definitions of  $E$ ,  $P$  and  $M$ , we have

$$\begin{aligned} E'(u) &= -\partial_x^2 u - i|u|^{2\sigma} \partial_x u, \\ P'(u) &= i\partial_x u, \end{aligned} \tag{2.1}$$

$$M'(u) = u. \tag{2.2}$$

Let

$$\begin{aligned} S_c(u) &= E(u) + cP(u) + \frac{c^2}{4}M(u), \\ K_c(u) &= \langle S'_c(u), u \rangle. \end{aligned}$$

Then, we have

$$\begin{aligned} S'_c(u) &= E'(u) + cP'(u) + \frac{c^2}{4}M'(u) \\ &= -\partial_x^2 u - i|u|^{2\sigma} \partial_x u + ci\partial_x u + \frac{c^2}{4}u, \end{aligned} \tag{2.3}$$

and

$$K_c(u) = \|\partial_x u\|_{L^2}^2 - (i|u|^{2\sigma} \partial_x u, u) + c(i\partial_x u, u) + \frac{c^2}{4}\|u\|_{L^2}^2.$$

For the solution  $\phi_c$  to (1.5), we have

$$S'_c(\phi_c) = 0,$$

and thus  $K_c(\phi_c) = 0$ . Moreover, by (2.3), we obtain

$$\begin{aligned} S''_c(\phi_c)f &= -\partial_x^2 f + ci\partial_x f + \frac{c^2}{4}f - i\sigma|\phi_c|^{2\sigma-2}\overline{\phi_c}\partial_x \phi_c f \\ &\quad - i\sigma|\phi_c|^{2\sigma-2}\phi_c\partial_x \overline{\phi_c}f - i|\phi_c|^{2\sigma}\partial_x f. \end{aligned} \tag{2.4}$$

**2.2. Useful Lemmas.** In this section, we prove some useful lemmas.

**Lemma 1.**  $S''_c(\phi_c)$  is self-adjoint, that is, for any  $f, g \in H^1(\mathbb{R})$ ,

$$\langle S''_c(\phi_c)f, g \rangle = \langle S''_c(\phi_c)g, f \rangle.$$

Moreover,

$$S''_c(\phi_c)\partial_c \phi_c = -\frac{c}{2}M'(\phi_c) - P'(\phi_c). \tag{2.5}$$

*Proof.* Note that

$$\partial_t \partial_s S_c(\phi_c + sg + tf) = \partial_s \partial_t S_c(\phi_c + sg + tf).$$

Then, taking  $t = s = 0$ , we get the first formula in the lemma .

From  $S'_c(\phi_c) = 0$ , and differentiating it with respect to  $c$ , we know that

$$S''_c(\phi_c)\partial_c \phi_c = -\frac{c}{2}M'(\phi_c) - P'(\phi_c).$$

This finishes the proof. □

**Lemma 2.** *Let  $\sigma \in (1, 2)$ , then we have*

$$P(\phi_c) = \frac{c}{2}(1 - \sigma)M(\phi_c), \quad (2.6)$$

and

$$\|\phi_c\|_{L^{2\sigma+2}}^{2\sigma+2} = 2c(\sigma + 1)(2 - \sigma)M(\phi_c). \quad (2.7)$$

Moreover,

$$\partial_c P(\phi_c) = \frac{c}{2} \partial_c M(\phi_c). \quad (2.8)$$

*Proof.* First, we use the definition of  $M$  and the explicit formula (1.4) to derive

$$\begin{aligned} M(\phi_c) &= \frac{1}{2} \|\phi_c\|_{L^2}^2 = \frac{1}{2} \|\varphi_c\|_{L^2}^2 \\ &= \frac{1}{2} \int_{\mathbb{R}} \left( \frac{2c(\sigma + 1)}{\sigma^2(cx)^2 + 1} \right)^{\frac{1}{\sigma}} dx \\ &= \frac{1}{2} \sigma^{-1} (2\sigma + 2)^{\frac{1}{\sigma}} A_{\sigma} c^{\frac{1}{\sigma}-1}, \end{aligned} \quad (2.9)$$

where  $A_{\sigma} = \int_{\mathbb{R}} (x^2 + 1)^{-\frac{1}{\sigma}} dx > 0$ .

Next, by (1.3), we have

$$\partial_x \phi_c(x) = e^{i\frac{c}{2}x - \frac{i}{2(\sigma+1)} \int_{-\infty}^x \varphi_c^{2\sigma}(y) dy} \left[ \left( i\frac{c}{2} - \frac{i}{2\sigma+2} \varphi_c^{2\sigma} \right) \varphi_c + \partial_x \varphi_c \right]. \quad (2.10)$$

Then, combining with the definition of  $P$  yields

$$\begin{aligned} P(\phi_c) &= -\frac{c}{2} M(\phi_c) + \frac{1}{2(2\sigma + 2)} \|\varphi_c\|_{L^{2\sigma+2}}^{2\sigma+2} \\ &= -\frac{c}{2} M(\phi_c) + \frac{1}{2(2\sigma + 2)} (2\sigma + 2)^{\frac{1}{\sigma}+1} c^{\frac{1}{\sigma}+1} \int_{\mathbb{R}} \left( \frac{1}{\sigma^2(cx)^2 + 1} \right)^{\frac{1}{\sigma}+1} dx \\ &= -\frac{c}{2} M(\phi_c) + \frac{1}{2} \sigma^{-1} (2\sigma + 2)^{\frac{1}{\sigma}} B_{\sigma} c^{\frac{1}{\sigma}}, \end{aligned} \quad (2.11)$$

where  $B_{\sigma} = \int_{\mathbb{R}} (x^2 + 1)^{-\frac{1}{\sigma}-1} dx$ .

The fundamental observation is that

$$\frac{d[x(x^2 + 1)^{-\frac{1}{\sigma}}]}{dx} = \left(1 - \frac{2}{\sigma}\right) (x^2 + 1)^{-\frac{1}{\sigma}} + \frac{2}{\sigma} (x^2 + 1)^{-\frac{1}{\sigma}-1}. \quad (2.12)$$

Integration of (2.12) with  $x$  for  $\sigma \in (1, 2)$  yields

$$\frac{2}{\sigma} \int_{\mathbb{R}} (x^2 + 1)^{-\frac{1}{\sigma}-1} dx = \left(\frac{2}{\sigma} - 1\right) \int_{\mathbb{R}} (x^2 + 1)^{-\frac{1}{\sigma}} dx.$$

That is

$$B_{\sigma} = \left(1 - \frac{\sigma}{2}\right) A_{\sigma}. \quad (2.13)$$

Together with (2.9), (2.11) and (2.13), we have

$$P(\phi_c) = -\frac{c}{2} M(\phi_c) + c \left(1 - \frac{\sigma}{2}\right) M(\phi_c) = \frac{c}{2} (1 - \sigma) M(\phi_c).$$

Moreover, from (2.11), we have

$$\begin{aligned}\|\phi_c\|_{L^{2\sigma+2}}^{2\sigma+2} &= \|\varphi_c\|_{L^{2\sigma+2}}^{2\sigma+2} = 4(\sigma+1) \left[ \frac{c}{2} M(\phi_c) + P(\phi_c) \right] \\ &= 2c(\sigma+1)(2-\sigma)M(\phi_c).\end{aligned}$$

On the other hand, differentiating (2.9) with respect to  $c$ , we have

$$\begin{aligned}\partial_c M(\phi_c) &= \frac{1}{2}\sigma^{-1}(2\sigma+2)^{\frac{1}{\sigma}} A_\sigma c^{\frac{1}{\sigma}-2} \left( \frac{1}{\sigma} - 1 \right) \\ &= c^{-1} \left( \frac{1}{\sigma} - 1 \right) M(\phi_c).\end{aligned}$$

That is

$$M(\phi_c) = c\sigma(1-\sigma)^{-1}\partial_c M(\phi_c). \quad (2.14)$$

Finally, differentiating (2.6) with respect to  $c$  and together with (2.14) yields

$$\begin{aligned}\partial_c P(\phi_c) &= \frac{1}{2}(1-\sigma)M(\phi_c) + \frac{c}{2}(1-\sigma)\partial_c M(\phi_c) \\ &= \frac{1}{2}(1-\sigma)c\sigma(1-\sigma)^{-1}\partial_c M(\phi_c) + \frac{c}{2}(1-\sigma)\partial_c M(\phi_c) \\ &= \frac{c}{2}\partial_c M(\phi_c).\end{aligned}$$

This completes the proof.  $\square$

**Lemma 3.** *Let  $\sigma \in (1, 2)$ , then*

$$\langle S_c''(\phi_c)\partial_c\phi_c, \partial_c\phi_c \rangle > 0.$$

*Proof.* Using (2.5), we get

$$\langle S_c''(\phi_c)\partial_c\phi_c, \partial_c\phi_c \rangle = -\frac{c}{2}\partial_c M(\phi_c) - \partial_c P(\phi_c). \quad (2.15)$$

From (2.9) and (2.11), we have

$$\begin{aligned}-\frac{c}{2}\partial_c M(\phi_c) - \partial_c P(\phi_c) &= -\frac{c}{2}\partial_c M(\phi_c) + \frac{c}{2}\partial_c M(\phi_c) + \frac{1}{2}M(\phi_c) - \frac{1}{2}\sigma^{-2}(2\sigma+2)^{\frac{1}{\sigma}} B_\sigma c^{\frac{1}{\sigma}-1} \\ &= \frac{1}{4}\sigma^{-1}(2\sigma+2)^{\frac{1}{\sigma}} A_\sigma c^{\frac{1}{\sigma}-1} - \frac{1}{2}\sigma^{-2}(2\sigma+2)^{\frac{1}{\sigma}} B_\sigma c^{\frac{1}{\sigma}-1} \\ &= \frac{1}{4}\sigma^{-2}(2\sigma+2)^{\frac{1}{\sigma}} (\sigma A_\sigma - 2B_\sigma) c^{\frac{1}{\sigma}-1}.\end{aligned}$$

Combining with (2.15), (2.13) and (2.9), we have

$$\begin{aligned}\langle S_c''(\phi_c)\partial_c\phi_c, \partial_c\phi_c \rangle &= \frac{1}{4}\sigma^{-2}(2\sigma+2)^{\frac{1}{\sigma}} \left[ \sigma A_\sigma - 2\left(1 - \frac{\sigma}{2}\right) B_\sigma \right] \\ &= \frac{1}{4}\sigma^{-2}(2\sigma+2)^{\frac{1}{\sigma}} 2(\sigma-1)A_\sigma > 0.\end{aligned}$$

This completes the proof.  $\square$

**Lemma 4.** *Let  $\sigma \in (1, 2)$ , then*

$$\langle S_c''(\phi_c)\phi_c, \phi_c \rangle < 0.$$

*Proof.* From (2.4) and (1.5), we have

$$\begin{aligned} S_c''(\phi_c)\phi_c &= -\partial_x^2\phi_c + \frac{c^2}{4}\phi_c + ic\partial_x\phi_c - i\sigma|\phi_c|^{2\sigma-2}|\phi_c|^2\partial_x\phi_c - i\sigma|\phi_c|^{2\sigma-2}|\phi_c|^2\partial_x\phi_c - i|\phi_c|^{2\sigma}\partial_x\phi_c \\ &= -\partial_x^2\phi_c - (2\sigma+1)i|\phi_c|^{2\sigma}\partial_x\phi_c + \omega\phi_c + ic\partial_x\phi_c \\ &= -2\sigma i|\phi_c|^{2\sigma}\partial_x\phi_c. \end{aligned}$$

Hence, we obtain

$$\langle S_c''(\phi_c)\phi_c, \phi_c \rangle = (-2\sigma i|\phi_c|^{2\sigma}\partial_x\phi_c, \phi_c) = -2\sigma \operatorname{Im} \int_{\mathbb{R}} |\phi_c|^{2\sigma} \phi_c \overline{\partial_x\phi_c} dx.$$

Taking product with  $\overline{x\partial_x\phi_c}$  and  $\overline{\phi_c}$  in (1.5) respectively, and integrating, we obtain

$$\|\partial_x\phi_c\|_{L^2}^2 = \frac{c^2}{4}\|\phi_c\|_{L^2}^2, \quad (2.16)$$

and

$$\|\partial_x\phi_c\|_{L^2}^2 + \frac{c^2}{4}\|\phi_c\|_{L^2}^2 + c \operatorname{Im} \int_{\mathbb{R}} \phi_c \overline{\partial_x\phi_c} dx - \operatorname{Im} \int_{\mathbb{R}} |\phi_c|^{2\sigma} \phi_c \overline{\partial_x\phi_c} dx = 0.$$

We collect the above computations and obtain

$$\operatorname{Im} \int_{\mathbb{R}} |\phi_c|^{2\sigma} \phi_c \overline{\partial_x\phi_c} dx = c^2 M(\phi_c) + 2cP(\phi_c). \quad (2.17)$$

Thus, by (2.6), (2.17) and  $\sigma \in (1, 2)$ , we have

$$\begin{aligned} \langle S_c''(\phi_c)\phi_c, \phi_c \rangle &= -2\sigma [c^2 M(\phi_c) + c^2(1-\sigma)M(\phi_c)] \\ &= -2\sigma(2-\sigma)c^2 M(\phi_c) < 0. \end{aligned}$$

This completes the proof.  $\square$

**2.3. Variational characterization.** Next, we consider the following standard minimization problem:

$$\mu(c) = \inf\{S_c(u) : u \in H^1(\mathbb{R}) \setminus \{0\}, K_c(u) = 0\}. \quad (2.18)$$

Let  $\mathcal{M}_c$  be the set of all minimizations for (2.18), i.e.

$$\mathcal{M}_c = \{\phi \in H^1(\mathbb{R}) \setminus \{0\} : S_c(\phi) = \mu(c), K_c(\phi) = 0\}.$$

Let  $\mathcal{G}_c$  be the set of all critical points of  $S_c$ , then

$$\mathcal{G}_c = \{\phi \in H^1(\mathbb{R}) \setminus \{0\} : S_c'(\phi) = 0\}.$$

The main result of this subsection is following. Since it can be proved by the standard variational argument (see for examples [2, 18, 21], in particular, see [18] for the “zero mass” case), we omit the details of the proof here.

**Lemma 5.**  $\mathcal{G}_c = \{e^{i\theta}\phi_c(\cdot-y) : (\theta, y) \in \mathbb{R}^2\}$ , and  $\mathcal{M}_c = \mathcal{G}_c$ . In particular, if  $v \in H^1(\mathbb{R})$  satisfies  $K_c(v) = 0$  and  $v \neq 0$ , then  $S_c(\phi_c) \leq S_c(v)$ .



## 3. NEGATIVE DIRECTION AND MODULATION

For  $R > 0$ , let  $\chi_R(x) = \chi(\frac{x}{R})$ , where  $\chi \in C^\infty(\mathbb{R})$ , such that  $\chi(x) = 1$  when  $|x| \leq 1$ ;  $\chi(x) = 0$  when  $|x| \geq 2$ . Because  $\partial_c \phi_c$  does not belong to  $L^2(\mathbb{R})$ , the localization technique is employed here, as will be seen in the proof of the following lemma.

**Proposition 1.** *There exist  $\mu, \nu$  and  $R$  such that for the function  $\psi = \phi_c + \mu \chi_R \partial_c \phi_c + \nu i \partial_x \phi_c$ , the following properties hold:*

- (1)  $\psi \in H^1(\mathbb{R})$ ;
- (2)  $\langle P'(\phi_c), \psi \rangle = \langle M'(\phi_c), \psi \rangle = 0$ ;
- (3)  $\langle S_c''(\phi_c) \psi, \psi \rangle < 0$ .

*Proof.* (1) Since  $\phi_c \in H^1(\mathbb{R})$  and  $\partial_x \phi_c \in H^1(\mathbb{R})$ , we just need to verify that  $\chi_R \partial_c \phi_c \in H^1(\mathbb{R})$ . From (1.3), we have

$$\partial_c \phi_c = e^{i\frac{\sigma}{2}x - \frac{i}{2(\sigma+1)} \int_{-\infty}^x \varphi_c(y)^{2\sigma} dy} \left( \frac{i}{2} x \varphi_c - \frac{i\sigma}{\sigma+1} \varphi_c \int_{-\infty}^x \partial_c \varphi_c \varphi_c^{2\sigma-1} dy + \partial_c \varphi_c \right). \quad (3.1)$$

By (A.5)(see Appendix), we know that

$$|\partial_c \phi_c| \lesssim \langle x \rangle^{1-\frac{1}{\sigma}}, \quad \text{and} \quad |\partial_x \partial_c \phi_c| \lesssim \langle x \rangle^{1-\frac{1}{\sigma}}.$$

Since  $\chi_R(x)$  is smooth cutoff function, we have  $\chi_R \partial_c \phi_c \in H^1(\mathbb{R})$ .

(2) It is sufficient to find  $\mu, \nu$  such that

$$\begin{cases} \langle P'(\phi_c), \phi_c + \mu \chi_R \partial_c \phi_c + \nu i \partial_x \phi_c \rangle = 0, \\ \langle M'(\phi_c), \phi_c + \mu \chi_R \partial_c \phi_c + \nu i \partial_x \phi_c \rangle = 0. \end{cases}$$

Together with (2.1) and (2.2), we obtain

$$\mu = -\frac{4P(\phi_c)^2 - 2M(\phi_c) \|\partial_x \phi_c\|_{L^2}^2}{2P(\phi_c) \cdot \frac{1}{2} \partial_c \text{Im} \int_{\mathbb{R}} \chi_R \phi_c \overline{\partial_x \phi_c} dx - \|\partial_x \phi_c\|_{L^2}^2 \cdot \frac{1}{2} \partial_c \int_{\mathbb{R}} \chi_R |\phi_c|^2 dx}, \quad (3.2)$$

and

$$\nu = \frac{2P(\phi_c) \cdot \frac{1}{2} \partial_c \int_{\mathbb{R}} \chi_R |\phi_c|^2 dx - 2M(\phi_c) \cdot \frac{1}{2} \partial_c \text{Im} \int_{\mathbb{R}} \chi_R \phi_c \overline{\partial_x \phi_c} dx}{2P(\phi_c) \cdot \frac{1}{2} \partial_c \text{Im} \int_{\mathbb{R}} \chi_R \phi_c \overline{\partial_x \phi_c} dx - \|\partial_x \phi_c\|_{L^2}^2 \cdot \frac{1}{2} \partial_c \int_{\mathbb{R}} \chi_R |\phi_c|^2 dx}. \quad (3.3)$$

Inserting (2.6), (2.8), (2.16) into (3.2) and (3.3) and using Lemmas A.1 and A.2 yields

$$\mu = \frac{-2(2-\sigma)M(\phi_c)}{\partial_c M(\phi_c) + O(R^{-\frac{2}{\sigma}+1})},$$

and

$$\nu = \frac{2}{c} + O(R^{-\frac{2}{\sigma}+1}). \quad (3.4)$$

(3) According to Lemma 1 and the selection of  $\psi$ , we have

$$\langle S_c''(\phi_c) \phi_c, \phi_c \rangle = \langle S_c''(\phi_c) (\psi - \mu \chi_R \partial_c \phi_c - \nu i \partial_x \phi_c), \psi - \mu \chi_R \partial_c \phi_c - \nu i \partial_x \phi_c \rangle.$$

By the self-adjoint of  $S_c''(\phi_c)$  and a direct expansion, we obtain

$$\begin{aligned} \langle S_c''(\phi_c)\phi_c, \phi_c \rangle &= \langle S_c''(\phi_c)\psi, \psi \rangle - 2\mu \langle S_c''(\phi_c)\chi_R \partial_c \phi_c, \psi \rangle - 2\nu \langle S_c''(\phi_c)\psi, i\partial_x \phi_c \rangle \\ &\quad + \mu^2 \langle S_c''(\phi_c)\chi_R \partial_c \phi_c, \chi_R \partial_c \phi_c \rangle + 2\mu\nu \langle S_c''(\phi_c)\chi_R \partial_c \phi_c, i\partial_x \phi_c \rangle \\ &\quad + \nu^2 \langle S_c''(\phi_c)i\partial_x \phi_c, i\partial_x \phi_c \rangle. \end{aligned} \quad (3.5)$$

Using  $\psi = \phi_c + \mu\chi_R \partial_c \phi_c + \nu i\partial_x \phi_c$ , we have

$$\begin{aligned} \langle S_c''(\phi_c)\psi, i\partial_x \phi_c \rangle &= \langle S_c''(\phi_c)(\phi_c + \mu\chi_R \partial_c \phi_c + \nu i\partial_x \phi_c), i\partial_x \phi_c \rangle \\ &= \langle S_c''(\phi_c)\phi_c, i\partial_x \phi_c \rangle + \mu \langle S_c''(\phi_c)\chi_R \partial_c \phi_c, i\partial_x \phi_c \rangle \\ &\quad + \nu \langle S_c''(\phi_c)i\partial_x \phi_c, i\partial_x \phi_c \rangle. \end{aligned} \quad (3.6)$$

Together with (3.5) and (3.6), we get

$$\begin{aligned} \langle S_c''(\phi_c)\phi_c, \phi_c \rangle &= \langle S_c''(\phi_c)\psi, \psi \rangle - 2\mu \langle S_c''(\phi_c)\chi_R \partial_c \phi_c, \psi \rangle - 2\nu \langle S_c''(\phi_c)\phi_c, i\partial_x \phi_c \rangle \\ &\quad + \mu^2 \langle S_c''(\phi_c)\chi_R \partial_c \phi_c, \chi_R \partial_c \phi_c \rangle - \nu^2 \langle S_c''(\phi_c)i\partial_x \phi_c, i\partial_x \phi_c \rangle. \end{aligned} \quad (3.7)$$

Combining with (2.5) and the conclusion (2), we have

$$\langle S_c''(\phi_c)\partial_c \phi_c, \psi \rangle = 0.$$

Then, we know

$$\langle S_c''(\phi_c)\chi_R \partial_c \phi_c, \psi \rangle = - \langle S_c''(\phi_c)(1 - \chi_R)\partial_c \phi_c, \psi \rangle. \quad (3.8)$$

Inserting (3.8) into (3.7) yields

$$\begin{aligned} \langle S_c''(\phi_c)\phi_c, \phi_c \rangle &= \langle S_c''(\phi_c)\psi, \psi \rangle + 2\mu \langle S_c''(\phi_c)(1 - \chi_R)\partial_c \phi_c, \psi \rangle - 2\nu \langle S_c''(\phi_c)\phi_c, i\partial_x \phi_c \rangle \\ &\quad - \nu^2 \langle S_c''(\phi_c)i\partial_x \phi_c, i\partial_x \phi_c \rangle + \mu^2 \langle S_c''(\phi_c)(1 - \chi_R)\partial_c \phi_c, (1 - \chi_R)\partial_c \phi_c \rangle \\ &\quad - 2\mu^2 \langle S_c''(\phi_c)(1 - \chi_R)\partial_c \phi_c, \partial_c \phi_c \rangle + \mu^2 \langle S_c''(\phi_c)\partial_c \phi_c, \partial_c \phi_c \rangle. \end{aligned} \quad (3.9)$$

From Lemma A.3, we have

$$\begin{aligned} |\langle S_c''(\phi_c)(1 - \chi_R)\partial_c \phi_c, \partial_c \phi_c \rangle| &\lesssim \int \left(1 - \chi_{\frac{R}{2}}\right) \langle x \rangle^{-1-\frac{1}{\sigma}} \langle x \rangle^{-1-\frac{1}{\sigma}} dx \\ &= O(R^{-\frac{2}{\sigma}+1}), \end{aligned} \quad (3.10)$$

and

$$|\langle S_c''(\phi_c)(1 - \chi_R)\partial_c \phi_c, (1 - \chi_R)\partial_c \phi_c \rangle| = O(R^{-\frac{2}{\sigma}+1}). \quad (3.11)$$

Note that  $|\psi| \lesssim \langle x \rangle^{1-\frac{1}{\sigma}}$ , we get

$$\begin{aligned} |\langle S_c''(\phi_c)(1 - \chi_R)\partial_c \phi_c, \psi \rangle| &\lesssim \int \left(1 - \chi_{\frac{R}{2}}\right) \langle x \rangle^{-1-\frac{1}{\sigma}} |\psi| dx \\ &= O(R^{-\frac{2}{\sigma}+1}). \end{aligned} \quad (3.12)$$

Hence, inserting the estimates in (3.10)–(3.12) into (3.9), and using (3.4), we get

$$\begin{aligned} \langle S_c''(\phi_c)\phi_c, \phi_c \rangle &= \langle S_c''(\phi_c)\psi, \psi \rangle - \frac{4}{c} \langle S_c''(\phi_c)\phi_c, i\partial_x \phi_c \rangle - \nu^2 \langle S_c''(\phi_c)i\partial_x \phi_c, i\partial_x \phi_c \rangle \\ &\quad + \mu^2 \langle S_c''(\phi_c)\partial_c \phi_c, \partial_c \phi_c \rangle + O(R^{-\frac{2}{\sigma}+1}). \end{aligned} \quad (3.13)$$

Now we need the following lemma.

**Lemma 6.** *It holds that*

$$\langle S_c''(\phi_c)i\partial_x\phi_c, i\partial_x\phi_c \rangle < 0, \quad \text{and} \quad \langle S_c''(\phi_c)i\partial_x\phi_c, \phi_c \rangle < 0.$$

*Proof.* From (1.5) and (2.4), we have

$$S_c''(\phi_c)i\partial_x\phi_c = -\frac{c^2}{2}\sigma|\phi_c|^{2\sigma}\phi_c.$$

Therefore,

$$\begin{aligned} \langle S_c''(\phi_c)i\partial_x\phi_c, i\partial_x\phi_c \rangle &= -\frac{c^2}{2}\sigma(|\phi_c|^{2\sigma}\phi_c, i\partial_x\phi_c) \\ &= -\frac{c^2}{2}\sigma \operatorname{Im} \int_{\mathbb{R}} |\phi_c|^{2\sigma}\phi_c \overline{\partial_x\phi_c} dx. \end{aligned}$$

From (2.17) and (2.6), we get

$$\begin{aligned} \langle S_c''(\phi_c)i\partial_x\phi_c, i\partial_x\phi_c \rangle &= -\frac{c^2}{2}\sigma [c^2M(\phi_c) + 2cP(\phi_c)] \\ &= -\frac{c^4}{2}(2-\sigma)\sigma M(\phi_c) < 0. \end{aligned}$$

Similarly, we have

$$\langle S_c''(\phi_c)i\partial_x\phi_c, \phi_c \rangle = -\frac{c^2}{2}\sigma \|\phi_c\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

From (2.7), we obtain

$$\langle S_c''(\phi_c)i\partial_x\phi_c, \phi_c \rangle = -(\sigma+1)(2-\sigma)\sigma c^3 M(\phi_c) < 0.$$

This proves the lemma. □

Combining with (3.13) and Lemma 6, we have

$$\begin{aligned} \langle S_c''(\phi_c)\psi, \psi \rangle &= \langle S_c''(\phi_c)\phi_c, \phi_c \rangle + \frac{4}{c}\langle S_c''(\phi_c)\phi_c, i\partial_x\phi_c \rangle + \nu^2\langle S_c''(\phi_c)i\partial_x\phi_c, i\partial_x\phi_c \rangle \\ &\quad - \mu^2\langle S_c''(\phi_c)\partial_c\phi_c, \partial_c\phi_c \rangle + O(R^{-\frac{2}{\sigma}+1}) \\ &< \langle S_c''(\phi_c)\phi_c, \phi_c \rangle - \mu^2\langle S_c''(\phi_c)\partial_c\phi_c, \partial_c\phi_c \rangle + O(R^{-\frac{2}{\sigma}+1}). \end{aligned}$$

From Lemma 3 and Lemma 4, we note that the first and the second terms in the right-hand side are negative. Hence, choosing  $R$  large enough, we obtain

$$\langle S_c''(\phi_c)\psi, \psi \rangle < \langle S_c''(\phi_c)\phi_c, \phi_c \rangle < 0.$$

This concludes the proof of Proposition 1. □

**Lemma 7.** *There exists a constant  $\beta_0 > 0$  such that*

$$S_c(\phi_c + \beta\psi) < S_c(\phi_c),$$

for all  $\beta \in (-\beta_0, 0) \cup (0, \beta_0)$ .

*Proof.* By Taylor's expansion, for  $\beta \in \mathbb{R}$ , we have

$$\begin{aligned} S_c(\phi_c + \beta\psi) &= S_c(\phi_c) + \beta \langle S'_c(\phi_c), \psi \rangle + \frac{1}{2} \beta^2 \langle S''_c(\phi_c) \psi, \psi \rangle + o(\beta^2) \\ &= S_c(\phi_c) + \frac{1}{2} \beta^2 \langle S''_c(\phi_c) \psi, \psi \rangle + o(\beta^2). \end{aligned}$$

Since  $\langle S''_c(\phi_c) \psi, \psi \rangle < 0$ , there exists a constant  $\beta_0 > 0$ , such that for any  $\beta \in (-\beta_0, 0) \cup (0, \beta_0)$ , we have

$$S_c(\phi_c + \beta\psi) < S_c(\phi_c).$$

This finishes the proof.  $\square$

We denote  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ . Then we can get the following proposition.

**Proposition 2.** *Suppose  $u \in U_{\varepsilon_0}(\phi_c)$ , then exist  $\theta = \theta(u)$ ,  $y = y(u)$ , such that*

- (1)  $\langle u, ie^{i\theta} \phi_c(\cdot - y) \rangle = 0$ ,  $\langle u, e^{i\theta} \partial_x \phi_c(\cdot - y) \rangle = 0$ ;
- (2)  $\|\partial_u \theta\|_{H^1(\mathbb{R})} \leq C$  and  $\|\partial_u y\|_{H^1(\mathbb{R})} \leq C$  for any  $u \in U_{\varepsilon_0}(\phi_c)$ ;
- (3)  $\theta(e^{i\theta_0} u(\cdot - y_0)) = \theta + \theta_0$ ,  $y(e^{i\theta_0} u(\cdot - y_0)) = y + y_0$  for any  $u \in U_{\varepsilon_0}(\phi_c)$  and  $\theta_0 \in \mathbb{T}$ ,  $y_0 \in \mathbb{R}$ .

*Proof.* Denote

$$F_1(\theta, y; u) = \langle u, ie^{i\theta} \phi_c(\cdot - y) \rangle, \quad F_2(\theta, y; u) = \langle u, e^{i\theta} \partial_x \phi_c(\cdot - y) \rangle.$$

Then  $F_1(0, 0; \phi_c) = F_2(0, 0; \phi_c) = 0$ .

According to the definitions of  $F_1$  and  $F_2$ , we have

$$\partial_\theta F(u, \theta) = \begin{pmatrix} \partial_\theta F_1(\theta, y; u) & \partial_y F_1(\theta, y; u) \\ \partial_\theta F_2(\theta, y; u) & \partial_y F_2(\theta, y; u) \end{pmatrix}.$$

Moreover, we have

$$\begin{aligned} \partial_\theta F_1|_{(0,0;\phi_c)} &= -\|\phi_c\|_{L^2}^2, & \partial_y F_1|_{(0,0;\phi_c)} &= -2P(\phi_c), \\ \partial_\theta F_2|_{(0,0;\phi_c)} &= 2P(\phi_c), & \partial_y F_2|_{(0,0;\phi_c)} &= \|\partial_x \phi_c\|_{L^2}^2. \end{aligned}$$

Then, from (2.16) and (2.6), the Jacobian

$$\begin{aligned} |\partial_\theta F(u, \theta)|_{(0,0;\phi_c)} &= -\|\phi_c\|_{L^2}^2 \|\partial_x \phi_c\|_{L^2}^2 + 4P^2(\phi_c) \\ &= -\sigma(2 - \sigma)c^2 M^2(\phi_c) \neq 0. \end{aligned}$$

Therefore by implicit function theorem, there exist a  $\varepsilon_0 > 0$  and a unique  $\mathbb{C}^1$ -function  $\theta = \theta(u)$ ,  $y = y(u)$  such that for any  $u \in U_{\varepsilon_0}(\phi_c)$ ,

$$\langle u, ie^{i\theta} \phi_c(\cdot - y) \rangle = 0, \quad \langle u, e^{i\theta} \partial_x \phi_c(\cdot - y) \rangle = 0.$$

Moreover, (2) follows from the implicit function differentiability theorem, and (3) follows from the uniqueness of the implicit functions.

This concludes the proof of Proposition 2.  $\square$

## 4. PROOF OF THEOREM 1

We argue for contradiction and suppose that  $u \in U_{\varepsilon_0}(\phi_c)$ . Moreover, we define

$$A(u) = \langle iu, e^{i\theta}\psi(\cdot - y) \rangle,$$

and

$$q(u) = iA'(u).$$

Then, we have

$$q(u) = e^{i\theta}\psi(\cdot - y) + i\theta_u \langle u, e^{i\theta}\psi(\cdot - y) \rangle + iy_u \langle iu, -e^{i\theta}\partial_x\psi(\cdot - y) \rangle. \quad (4.1)$$

**Lemma 8.** For  $u \in U_{\varepsilon_0}(\phi_c)$ ,  $q(u)$  is continuous from  $U_{\varepsilon_0}(\phi_c)$  to  $H^1(\mathbb{R})$  and  $q(\phi_c) = \psi$ .

*Proof.* By Proposition 1 (2),

$$\begin{aligned} q(\phi_c) &= \psi + (\phi_c, \psi)i\theta_u(\phi_c) + (i\phi_c, -\partial_x\psi)y_u(\phi_c) \\ &= \psi + (\phi_c, \psi)i\theta_u(\phi_c) + (i\partial_x\phi_c, \psi)y_u(\phi_c) \\ &= \psi. \end{aligned}$$

Moreover, from the definition (4.1) and Proposition 2 (2), we know that  $q(u)$  is continuous from  $U_{\varepsilon_0}(\phi_c)$  to  $H^1(\mathbb{R})$ . This proves the lemma.  $\square$

Now, we prove Theorem 1.

*Proof.* From (1.1), we know  $i\partial_t u = E'(u)$ , thus

$$\partial_t A(u) = \langle A'(u), \partial_t u \rangle = \langle iA'(u), E'(u) \rangle.$$

Since  $A(e^{i\theta_0}u(\cdot - y_0)) = A(u)$ , for any  $(\theta_0, y_0) \in \mathbb{R}^2$ . Differentiating with  $\theta_0$  and  $y_0$ , we have

$$\langle iA'(u), M'(u) \rangle = \langle iA'(u), P'(u) \rangle = 0.$$

Note that  $q(u) = iA'(u)$ , then using the identities above, we have

$$\begin{aligned} \partial_t A(u(t)) &= \langle iA'(u), S'_c(u) \rangle = \langle q(u), S'_c(u) \rangle \\ &= \frac{1}{\lambda} \left[ S_c(u + \lambda q(u)) - S_c(u) - \lambda^2 \int_0^1 (1-s) \langle S''_c(\phi_c + s\lambda q(u)) q(u), q(u) \rangle ds \right]. \end{aligned}$$

Next, we denote  $\tilde{u} = e^{-i\theta}u(\cdot + y)$ . Combinig with  $u \in U_{\varepsilon_0}(\phi_c)$  and Proposition 2, we have

$$\|\tilde{u} - \phi_c\|_{H^1} \leq \varepsilon_0.$$

Then, choosing  $\lambda, \varepsilon_0$  small enough, and by Proposition 1, we get

$$\begin{aligned} \int_0^1 (1-s) \langle S''_c(\phi_c + s\lambda q(u)) q(u), q(u) \rangle ds &= \int_0^1 (1-s) \langle S''_c(\phi_c + s\lambda q(\tilde{u})) q(\tilde{u}), q(\tilde{u}) \rangle ds \\ &= \langle S''(\phi_c)\psi, \psi \rangle + O(\lambda + \|q(\tilde{u}) - q(\phi_c)\|_{H^1}) \\ &= \langle S''(\phi_c)\psi, \psi \rangle + O(\lambda + \varepsilon_0) \\ &< \frac{1}{2} \langle S''(\phi_c)\psi, \psi \rangle \\ &< 0. \end{aligned}$$

Hence, we get

$$\partial_t A(u(t)) > \frac{1}{\lambda} [S_c(u + \lambda q(u)) - S_c(u)]. \quad (4.2)$$

Now we claim that

$$\langle K'_c(\phi_c), \psi \rangle \neq 0. \quad (4.3)$$

To show this, we need the following lemma.

**Lemma 9.** *If  $v \in H^1(\mathbb{R})$  satisfies  $\langle K'_c(\phi_c), v \rangle = 0$ , then  $\langle S''_c(\phi_c)v, v \rangle \geq 0$ .*

*Proof.* See Lemma 4 in [29] for the proof.  $\square$

By Proposition 1 (3) and Lemma 9, we have (4.3). Then applying the implicit functional theorem, we can find a  $\lambda(u) \in (-\lambda_0, \lambda_0) \setminus \{0\}$ , such that for any  $u \in U_{\varepsilon_0}(\phi_c)$ ,

$$K_c(u + \lambda(u)q(u)) = 0.$$

Hence, by Lemma 5, we have

$$S_c(u + \lambda(u)q(u)) \geq S_c(\phi_c).$$

Without loss of generality, we assume  $\lambda(u) > 0$ . We choose

$$u_0 = \phi_c + \beta\psi.$$

Then, by the conservation laws, we have  $S_c(u) = S_c(\phi_c + \beta\psi)$ . Hence,

$$S_c(u + \lambda(u)q(u)) - S_c(u) \geq S_c(\phi_c) - S_c(\phi_c + \beta\psi).$$

From Lemma 7, we have  $S_c(\phi_c) - S_c(\phi_c + \beta\psi) > 0$ . Thus, by (4.2),

$$\partial_t A(u(t)) \geq \frac{1}{\lambda_0} (S_c(\phi_c) - S_c(u_0)) > 0.$$

Therefore, we get that  $A(u(t)) \rightarrow +\infty$  as  $t \rightarrow \infty$ . However,

$$|A(u(t))| \leq \|u\|_{L^2} \|\psi\|_{L^2} \leq C \text{ for any } t > 0.$$

This is a contradiction. This finishes the proof of Theorem 1.  $\square$

## APPENDIX

In this appendix, we prove the following element lemmas used in Section 3.

**Lemma A. 1.** *Let  $R > 0$ , then*

$$\frac{1}{2} \partial_c \int_{\mathbb{R}} \chi_R |\phi_c|^2 dx = \partial_c M(\phi_c) + O(R^{-\frac{2}{\sigma}+1}).$$

*Proof.* From the definition of  $M$ , we have

$$\frac{1}{2} \int_{\mathbb{R}} \chi_R |\phi_c|^2 dx = M(\phi_c) + \frac{1}{2} \int_{\mathbb{R}} (\chi_R - 1) |\phi_c|^2 dx.$$

Together with (1.3) and (1.4), we get

$$\begin{aligned}
\int_{\mathbb{R}} (\chi_R - 1) |\phi_c|^2 dx &= \int_{\mathbb{R}} (\chi_R - 1) \varphi_c^2 dx \\
&= [2c(\sigma + 1)]^{\frac{1}{\sigma}} \sigma^{-1} c^{-1} \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx \\
&= c_1(\sigma) c^{\frac{1}{\sigma}-1} \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx, \tag{A.1}
\end{aligned}$$

where  $c_1(\sigma) = \sigma^{-1} [2(\sigma + 1)]^{\frac{1}{\sigma}}$ . Now, differentiating (A.1) with respect to  $c$ , we have

$$\begin{aligned}
\partial_c \int_{\mathbb{R}} (\chi_R - 1) |\phi_c|^2 dx &= \left( \frac{1}{\sigma} - 1 \right) c_1(\sigma) c^{\frac{1}{\sigma}-2} \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx \\
&\quad - c_1(\sigma) c^{\frac{1}{\sigma}-2} \int_{\mathbb{R}} \chi' \left( \frac{x}{\sigma c R} \right) \frac{x}{\sigma c R} \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx.
\end{aligned}$$

Note that

$$\int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx, \quad \int_{\mathbb{R}} \chi' \left( \frac{x}{\sigma c R} \right) \frac{x}{\sigma c R} \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx = O(R^{-\frac{2}{\sigma}+1}),$$

we obtain

$$\partial_c \int_{\mathbb{R}} (\chi_R - 1) |\phi_c|^2 dx = O(R^{-\frac{2}{\sigma}+1}).$$

This finishes the proof.  $\square$

**Lemma A. 2.** *Let  $R > 0$ , then*

$$\frac{1}{2} \partial_c \text{Im} \int_{\mathbb{R}} \chi_R \phi_c \overline{\partial_x \phi_c} dx = \partial_c P(\phi_c) + O(R^{-\frac{2}{\sigma}+1}).$$

*Proof.* By the definition of  $P$ , we have

$$\frac{1}{2} \partial_c \text{Im} \int_{\mathbb{R}} \chi_R \phi_c \overline{\partial_x \phi_c} dx = \partial_c P(\phi_c) + \frac{1}{2} \partial_c \text{Im} \int_{\mathbb{R}} (\chi_R - 1) \phi_c \overline{\partial_x \phi_c} dx.$$

From (1.3) and (2.10), we obtain

$$\begin{aligned}
\text{Im} \int_{\mathbb{R}} (\chi_R - 1) \phi_c \overline{\partial_x \phi_c} dx &= \text{Im} \int_{\mathbb{R}} (\chi_R - 1) \varphi_c \left( -\frac{c}{2} i \varphi_c + \frac{i}{2(\sigma + 1)} \varphi_c^{2\sigma+1} \right) dx \\
&= -\frac{c}{2} \int_{\mathbb{R}} (\chi_R - 1) \varphi_c^2 dx + \frac{1}{2(\sigma + 1)} \int_{\mathbb{R}} (\chi_R - 1) \varphi_c^{2\sigma+2} dx.
\end{aligned}$$

Using (1.4), we further write  $\text{Im} \int_{\mathbb{R}} (\chi_R - 1) \phi_c \overline{\partial_x \phi_c} dx$  as

$$\begin{aligned}
& -\frac{c}{2} [2c(\sigma+1)]^{\frac{1}{\sigma}} \sigma^{-1} c^{-1} \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx \\
& + \frac{1}{2(\sigma+1)} [2c(\sigma+1)]^{\frac{1}{\sigma}+1} \sigma^{-1} c^{-1} \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}+1} dx \\
& = c_2(\sigma) c^{\frac{1}{\sigma}} \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx \\
& + c_3(\sigma) c^{\frac{1}{\sigma}} \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}+1} dx, \tag{A.2}
\end{aligned}$$

where  $c_2(\sigma) = -\frac{1}{2} \sigma^{-1} [2(\sigma+1)]^{\frac{1}{\sigma}}$  and  $c_3(\sigma) = \sigma^{-1} [2(\sigma+1)]^{\frac{1}{\sigma}}$ . Differentiating (A.2) with respect to  $c$ , and treating similarly as the proof in the previous lemma, we obtain

$$\begin{aligned}
\partial_c \text{Im} \int_{\mathbb{R}} (\chi_R - 1) \phi_c \overline{\partial_x \phi_c} dx &= \frac{1}{\sigma} c^{\frac{1}{\sigma}-1} \left[ c_2(\sigma) \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx \right. \\
& \quad \left. + c_3(\sigma) \int_{\mathbb{R}} \left[ \chi \left( \frac{x}{\sigma c R} \right) - 1 \right] \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}+1} dx \right] \\
& \quad - c_2(\sigma) c^{\frac{1}{\sigma}-1} \int_{\mathbb{R}} \chi' \left( \frac{x}{\sigma c R} \right) \frac{x}{\sigma c R} \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}} dx \\
& \quad - c_3(\sigma) c^{\frac{1}{\sigma}-1} \int_{\mathbb{R}} \chi' \left( \frac{x}{\sigma c R} \right) \frac{x}{\sigma c R} \left( \frac{1}{1+x^2} \right)^{\frac{1}{\sigma}+1} dx \\
& = O(R^{-\frac{2}{\sigma}+1}) + O(R^{-\frac{2}{\sigma}-1}) \\
& = O(R^{-\frac{2}{\sigma}+1}).
\end{aligned}$$

This proves the lemma.  $\square$

**Lemma A. 3.** *Let  $R > 0$ . Then*

$$|S_c''(\phi_c)(1 - \chi_R) \partial_c \phi_c(x)| \lesssim (1 - \chi_{\frac{R}{2}}(x)) \langle x \rangle^{-1-\frac{1}{\sigma}}.$$

*Proof.* According to (1.4), we get

$$|\varphi_c| \lesssim \langle x \rangle^{-\frac{1}{\sigma}}, \quad |\partial_x \varphi_c| \lesssim \langle x \rangle^{-1-\frac{1}{\sigma}}, \quad \text{and} \quad |\partial_{xx} \varphi_c| \lesssim \langle x \rangle^{-2-\frac{1}{\sigma}}. \tag{A.3}$$

Moreover,

$$|\partial_c \varphi_c| \lesssim \langle x \rangle^{-\frac{1}{\sigma}}, \quad \text{and} \quad |\partial_x \partial_c \varphi_c| \lesssim \langle x \rangle^{-1-\frac{1}{\sigma}}. \tag{A.4}$$

Combining with (2.10), (3.1), (A.3) and (A.4), we have

$$|\partial_x \phi_c| \lesssim \langle x \rangle^{-\frac{1}{\sigma}}, \quad |\partial_c \phi_c| \lesssim \langle x \rangle^{1-\frac{1}{\sigma}}, \quad \text{and} \quad |\partial_x \partial_c \phi_c| \lesssim \langle x \rangle^{1-\frac{1}{\sigma}}. \tag{A.5}$$

For suitable function  $f$ , we note that

$$\partial_x^2 f - ci \partial_x f - \frac{c^2}{4} f = e^{\frac{c}{2}ix} \partial_x^2 (e^{-\frac{c}{2}ix} f).$$



Together with (2.4), we can write  $S_c''(\phi_c)(1 - \chi_R)\partial_c\phi_c$  as

$$\begin{aligned} & - e^{\frac{c}{2}ix} \partial_{xx} \left[ e^{-\frac{i}{2(\sigma+1)} \int_{-\infty}^x \varphi_c(y)^{2\sigma} dy} (1 - \chi_R) \left( \partial_c \varphi_c + \frac{i}{2} x \varphi_c - \frac{i\sigma}{\sigma+1} \varphi_c \int_{-\infty}^x \partial_c \varphi_c \varphi_c^{2\sigma-1} dy \right) \right] \\ & - (1 - \chi_R) \left[ i\sigma |\phi_c|^{2\sigma-2} \overline{\phi_c} \partial_x \phi_c \partial_c \phi_c + i\sigma |\phi_c|^{2\sigma-2} \phi_c \partial_x \phi_c \overline{\partial_c \phi_c} + i |\phi_c|^{2\sigma} \partial_x \partial_c \phi_c \right]. \quad (\text{A.6}) \end{aligned}$$

We collect the computations (A.3)–(A.5) and obtain that every term can be controlled by  $\langle x \rangle^{-1-\frac{1}{\sigma}}$  in (A.6). Thus, we have

$$|S_c''(\phi_c)(1 - \chi_R)\partial_c\phi_c(x)| \lesssim \langle x \rangle^{-1-\frac{1}{\sigma}}.$$

Finally, we observe that the support of  $S_c''(\phi_c)(1 - \chi_R)\partial_c\phi_c$  is included in  $[R, +\infty)$ .

This concludes the proof of Lemma 3.  $\square$

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