# A derivation of the sharp Moser-Trudinger-Onofri inequalities from the fractional Sobolev inequalities

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#### **Abstract**

We derive the sharp Moser-Trudinger-Onofri inequalities on the standard n-sphere and CR (2n+1)- sphere as the limit of the sharp fractional Sobolev inequalities for all  $n \geq 1$ . On the 2-sphere and 4-sphere, this was established recently by S.-Y. Chang and F. Wang. Our proof uses an alternative and elementary argument.

#### 1 Introduction

In [18], E. Onofri proved the sharp Moser-Trudinger inequality on the unit 2-sphere

$$\ln \int_{\mathbb{S}^2} e^{2w} \, d\mu_{g_0} \le \int_{\mathbb{S}^2} |\nabla w|^2 \, d\mu_{g_0} + 2 \int_{\mathbb{S}^2} w \, d\mu_{g_0} \quad \text{for } w \in W^{1,2}(\mathbb{S}^2),$$

where  $g_0$  is the standard metric and  $\int_{\mathbb{S}^2} \mathrm{d}\mu_{g_0} = \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} \mathrm{d}\mu_{g_0}$ . Onofri's proof is based on a version of the Moser-Trudinger inequality due to T. Aubin [1] which holds under the additional constraint  $\int_{\mathbb{S}^2} e^{2w} x \, \mathrm{d}\mu_{g_0} = 0$ ,  $x \in \mathbb{R}^3$ ; see C. Gui and A. Moradifam [15] for the proof of sharp form of Aubin's inequality which was conjectured by S.-Y. Chang and P. Yang [11]. Until now, there have been many different proofs of the Moser-Trudinger-Onofri inequality. A collection of them can be found in the survey J. Dolbeault, M. J. Esteban, and G. Jankowiak [13]. In [19], Y. Rubinstein gave a Kähler geometric proof of the sharp inequality and obtained an optimal extension of it to the higher dimensional Kähler-Einstein manifolds. Rubinstein's proof is based on the earlier results of W. Ding and G. Tian [12] and G. Tian [20]. On the standard n- sphere  $\mathbb{S}^n$ , the Moser-Trudinger-Onofri inequality was established by T. Branson, S.-Y. Chang and P. Yang [4] and W. Beckner [2] for n=4, and by [2] for all  $n\geq 1$ .

Recently, S.-Y. Chang and F. Wang [10] derived the sharp Moser-Trudinger-Onofri inequality on the 2- and 4- spheres as the limit case of the fractional power Sobolev inequalities, which was motivated by a dimensional continuation argument of T. Branson. The proof of [10] exploits the definition of the fractional order operators as generalized Dirichlet-to-Neumann operators from scattering theory, and uses the extension formula of the fractional order operators, which was first introduced by L. Caffarelli and L. Silvestre [6] on the Euclidean spaces, and later generalized to

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operators defined on the boundaries of asymptotically hyperbolic manifolds by S.-Y. Chang and M. González [9], and J. Case and S.-Y. Chang [8]. In the final remark of [10], they commented that it is plausible that their arguments can be applied to other dimensions, but the arguments would become increasingly delicate when n is large.

In this paper, we derive the sharp Moser-Trudinger-Onofri inequality as the limit case of the fractional power Sobolev inequalities on  $\mathbb{S}^n$  for all  $n \geq 1$ . Instead of using Chang-Wang's argument from scattering theory, our proof uses the explicit formulas of the fractional order operators on the spheres. Chang-Wang's method should have broader applications in related problems on manifolds. On the dual side, E. Carlen and M. Loss [7] derived the sharp logarithmic Hardy-Littlewood-Sobolev inequality on  $\mathbb{S}^n$  from the sharp HLS inequalities via endpoint differentiation, which in turn implies the sharp Moser-Trudinger-Onofri inequality.

Our argument works in the CR setting, too. In this situation, a sharp Moser-Trudinger-Onofri inequality on CR sphere  $\mathbb{S}^{2n+1}$  was discovered by T. Branson, L. Fontana and C. Morpurgo [5] after introducing the  $\mathcal{A}'_Q$  operator of order Q=2n+2. On the other hand, R. Frank and E. Lieb [14] proved the sharp fractional Sobolev inequalities as a corollary of their sharp HLS inequalities. [14] also proved the limiting cases of HLS by differentiating HLS at the endpoints; see Corollary 2.4 and Corollary 2.5. We derive the sharp Moser-Trudinger-Onofri inequality of [5] as the limit of the sharp fractional Sobolev inequalities of [14] in a similar way.

In the next section, we extend [10] to all dimensions  $n \ge 1$  by a different approach. In section 3, we prove the analogue in the CR spheres setting.

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# 2 The standard spheres setting

Let  $n \geq 1$ ,  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$  be the unit *n*-dimensional sphere. For  $\gamma > 0$ , let

$$P_{\gamma} = \frac{\Gamma(B + \frac{1}{2} + \gamma)}{\Gamma(B + \frac{1}{2} - \gamma)}, \quad B = \sqrt{-\Delta_{g_0} + \left(\frac{n-1}{2}\right)^2},$$

where  $\Delta_{g_0}$  is the Laplace-Beltrami operator on  $\mathbb{S}^n$  with respect to the standard induced metric  $g_0$  from  $\mathbb{R}^{n+1}$ . More precisely, B and  $P_{\gamma}$  are determined by the formulas

$$B\left(Y^{(k)}\right) = \left(k + \frac{n-1}{2}\right)Y^{(k)} \quad \text{and} \quad P_{\gamma}\left(Y^{(k)}\right) = \frac{\Gamma\left(k + \frac{n}{2} + \gamma\right)}{\Gamma\left(k + \frac{n}{2} - \gamma\right)}Y^{(k)} \tag{1}$$

for every spherical harmonic  $Y^{(k)}$  of degree  $k \geq 0$ , where  $\Gamma(\cdot)$  is the Gamma function.

Let  $\gamma \in (0, n/2)$ . The sharp Sobolev inequality on  $\mathbb{S}^n$  asserts that

$$Y(n,\gamma) \left( \int_{\mathbb{S}^n} |v|^{\frac{2n}{n-2\gamma}} d\mu_{g_0} \right)^{\frac{n-2\gamma}{n}} \le \int_{\mathbb{S}^n} v P_{\gamma}(v) d\mu_{g_0} \quad \text{for } v \in C^{\infty}(\mathbb{S}^n),$$
 (2)

where  $Y(n,\gamma):=\frac{\Gamma(\frac{n}{2}+\gamma)}{\Gamma(\frac{n}{2}-\gamma)}$  and  $\int_{\mathbb{S}^n} d\mu_{g_0}=\frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} d\mu_{g_0}$ . The sharp Moser-Trudinger-Onofri inequality asserts that

$$\frac{2(n-1)!}{n} \ln \int_{\mathbb{S}^n} e^{nw} \, \mathrm{d}\mu_{g_0} \le \int_{\mathbb{S}^n} \left( w P_{n/2} w + 2(n-1)! w \right) \, \mathrm{d}\mu_{g_0} \quad \text{for } w \in C^{\infty}(\mathbb{S}^n). \tag{3}$$

See W. Beckner [2] for the proofs of the both inequalities. In particular, (2) is a consequence of the sharp HLS inequality due to E. Lieb [17] in the Euclidean spaces.

Recently, S.-Y. Chang and F. Wang [10] studied the limit of (2) when n=2 and n=4. Generalizing the cases n=2 and n=4 from [10], we have

**Proposition 1.** For  $\gamma \in (0, n/2)$  and any  $w \in C^{\infty}(\mathbb{S}^n)$ , let  $v = e^{(\frac{n}{2} - \gamma)w}$ . Denote

$$LHS_{\gamma} := \frac{4}{(n-2\gamma)^2} Y(n,\gamma) \left[ \left( \oint_{\mathbb{S}^n} |v|^{\frac{2n}{n-2\gamma}} d\mu_{g_0} \right)^{\frac{n-2\gamma}{n}} - \oint_{\mathbb{S}^n} |v|^2 d\mu_{g_0} \right]$$

and

$$RHS_{\gamma} := \frac{4}{(n-2\gamma)^2} \left[ \oint_{\mathbb{S}^n} v P_{\gamma}(v) \, d\mu_{g_0} - Y(n,\gamma) \oint_{\mathbb{S}^n} |v|^2 \, d\mu_{g_0} \right].$$

Then

$$\lim_{\gamma \to n/2} LHS_{\gamma} = \frac{2(n-1)!}{n} \ln \int_{\mathbb{S}^n} e^{n(w-\bar{w})} d\mu_{g_0}$$

$$\tag{4}$$

and

$$\lim_{\gamma \to n/2} RHS_{\gamma} = \int_{\mathbb{S}^n} w P_{n/2} w \, \mathrm{d}\mu_{g_0},\tag{5}$$

where  $\bar{w}$  is the average of w over  $\mathbb{S}^n$ .

Consequently, we immediately have

**Theorem 2.** We can derive the sharp Moser-Trudinger-Onofri inequality (3) from the sharp Sobolev inequality (2) by sending  $\gamma \to \frac{n}{2}$ .

*Proof of Proposition 1.* The proof of (4) essentially follows from the proof of Lemma 3.1 of [10]. Note that

$$\left( \int_{\mathbb{S}^n} e^{nw} \, \mathrm{d}\mu_{g_0} \right)^{\frac{n-2\gamma}{n}} - \int_{\mathbb{S}^n} e^{(n-2\gamma)w} \, \mathrm{d}\mu_{g_0}$$

$$= \left( \int_{\mathbb{S}^n} e^{nw} \, \mathrm{d}\mu_{g_0} \right)^{\frac{n-2\gamma}{n}} - 1 - \int_{\mathbb{S}^n} (e^{(n-2\gamma)w} - 1) \, \mathrm{d}\mu_{g_0}.$$

Then by L'Hôpital's rule

$$\begin{split} \lim_{\gamma \to n/2} LHS_{\gamma} &= 2\Gamma(n) \left( \frac{1}{n} \ln \int_{\mathbb{S}^n} e^{nw} \, \mathrm{d}\mu_{g_0} - \int_{\mathbb{S}^n} w \, \mathrm{d}\mu_{g_0} \right) \\ &= \frac{2(n-1)!}{n} \ln \int_{\mathbb{S}^n} e^{n(w-\bar{w})} \, \mathrm{d}\mu_{g_0}. \end{split}$$

Therefore, (4) is proved.

To prove (5), using the Taylor expansion of the exponential function, we write

$$v = e^{\frac{n-2\gamma}{2}w} = 1 + (\frac{n}{2} - \gamma)w + (n-2\gamma)^2 f,$$

where  $f = \frac{1}{8}w^2 \int_0^1 (1-s)e^{\frac{n-2\gamma}{2}ws} \, \mathrm{d}s \in C^\infty(\mathbb{S}^n)$  is uniformly bounded in  $C^{2n}$  norm as  $\gamma \to n/2$ . Then we see that

$$\int_{\mathbb{S}^{n}} v P_{\gamma}(v) d\mu_{g_{0}} 
= \int_{\mathbb{S}^{n}} \left( 1 + (\frac{n}{2} - \gamma)w + (n - 2\gamma)^{2} f \right) \left( P_{\gamma}(1) + (\frac{n}{2} - \gamma)P_{\gamma}(w) + (n - 2\gamma)^{2} P_{\gamma}(f) \right) d\mu_{g_{0}} 
= \int_{\mathbb{S}^{n}} \left( Y(n, \gamma) + (n - 2\gamma)Y(n, \gamma)w + 2(n - 2\gamma)^{2} Y(n, \gamma)f + (\frac{n}{2} - \gamma)^{2} w P_{\gamma}w \right) d\mu_{g_{0}} 
+ O((n - 2\gamma)^{3}),$$

where we have used the self-adjointness of  $P_{\gamma}$  and  $P_{\gamma}(1) = Y(n, \gamma)$ . We also see that

$$Y(n,\gamma) \oint_{\mathbb{S}^n} |v|^2 d\mu_{g_0}$$

$$= Y(n,\gamma) \oint_{\mathbb{S}^n} \left( 1 + (n-2\gamma)w + 2(n-2\gamma)^2 f + (\frac{n}{2} - \gamma)^2 w^2 + O((n-2\gamma)^3) \right) d\mu_{g_0}.$$

It follows that

$$\int_{\mathbb{S}^n} v P_{\gamma}(v) d\mu_{g_0} - Y(n, \gamma) \int_{\mathbb{S}^n} |v|^2 d\mu_{g_0}$$

$$= \left(\frac{n}{2} - \gamma\right)^2 \int_{\mathbb{S}^n} w P_{\gamma} w d\mu_{g_0} + O((n - 2\gamma)^3).$$

Let  $w = \sum_{k=0}^{\infty} Y^{(k)}$ , where  $Y^{(k)}$  are spherical harmonics of degree k. Hence,

$$\int_{\mathbb{S}^{n}} w P_{\gamma} w \, d\mu_{g_{0}} = \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{n}{2} + \gamma)}{\Gamma(k + \frac{n}{2} - \gamma)} \int_{\mathbb{S}^{n}} |Y^{(k)}|^{2} \, d\mu_{g_{0}}$$

$$\rightarrow \sum_{k=1}^{\infty} \frac{\Gamma(k + n)}{\Gamma(k)} \int_{\mathbb{S}^{n}} |Y^{(k)}|^{2} \, d\mu_{g_{0}} = \int_{\mathbb{S}^{n}} w P_{n/2} w \, d\mu_{g_{0}}$$

as  $\gamma \to \frac{n}{2}$ , where we have used (1) in the first identity, the definition of  $P_{n/2}$  in the second one and have used the smoothness of w to ensure the convergence. Therefore, (5) follows.

Proposition 1 is proved.

## 3 The CR spheres setting

Following T. Branson, L. Fontana and C. Morpurgo [5], we let  $\mathcal{H}_{j,k}$  be the space of harmonic polynomials of bidegree (j,k) on CR sphere  $\mathbb{S}^{2n+1}$ ,  $j,k=0,1,\ldots$ ; such spaces make up for the

standard decomposition of  $L^2$  into U(n+1)-invariant and irreducible subspaces, where  $n \geq 1$ . For 0 < d < Q := 2n+2, let  $\mathcal{A}_d$  be the intertwining operator of order d on CR sphere  $\mathbb{S}^{2n+1}$ , characterized by

$$\mathcal{A}_d Y^{(j,k)} = \lambda_j(d) \lambda_k(d) Y^{(j,k)}, \quad \lambda_j(d) = \frac{\Gamma(j + \frac{Q+d}{4})}{\Gamma(j + \frac{Q-d}{4})}$$
 (6)

for every  $Y^{(j,k)} \in \mathcal{H}_{j,k}$ . When d=2, it gives the CR invariant sub-Laplacian, see D. Jerison and J.M. Lee [16].

One can define the operator  $\mathcal{A}_Q := \lim_{d \to Q} \mathcal{A}_d$ . The kernel of this operator is the space of CR-pluriharmonic functions on  $\mathbb{S}^{2n+1}$  given by

$$\mathcal{P} := \bigoplus_{j>0} (\mathcal{H}_{j,0} \bigoplus \mathcal{H}_{0,j}) \bigoplus \mathcal{H}_{0,0}.$$

It was discussed in [5] that  $\mathcal{A}_Q$  is not a suitable operator which could be used to conclude a conformally invariant Moser-Trudinger-Onofri inequality. In [5], the authors defined the operator  $\mathcal{A}'_Q$  acting on the CR-pluriharmonic functions with

$$\mathcal{A}'_{Q}F = \prod_{\ell=0}^{n} \left(\frac{2}{n}\mathcal{L} + \ell\right)F = \lim_{d \to Q} \frac{1}{\lambda_{0}(d)} \mathcal{A}_{d}F, \quad \forall F \in C^{\infty}(\mathbb{S}^{2n+1}) \cap \mathcal{P}, \tag{7}$$

where  $\mathcal{L} = \mathcal{A}_2 - \frac{n^2}{4}$  is the sub-Laplacian operator satisfying

$$\mathcal{L}Y^{(j,k)} = (jk + \frac{n}{2}j + \frac{n}{2}k)Y^{(j,k)}$$
 for all  $j, k \ge 0$ .

The limit in the second equality of (7) is uniform, see Proposition 1.2 of [5]. The sharp Moser-Trudinger-Onofri inequality on CR  $\mathbb{S}^{2n+1}$  proved by [5] asserts that

$$\frac{n!}{Q} \ln \int_{\mathbb{S}^{2n+1}} e^{QF} \le \int_{\mathbb{S}^{2n+1}} F \mathcal{A}'_Q F + n! \int_{\mathbb{S}^{2n+1}} F \quad \text{for } F \in C^{\infty}(\mathbb{S}^{2n+1}) \cap \mathcal{P}. \tag{8}$$

(It is called Beckner-Onofri inequality in [5].) By duality, the sharp Hardy-Littlewood-Sobolev inequality on CR  $\mathbb{S}^{2n+1}$  due to R. Frank and E. Lieb [14] yields that

$$\lambda_0(d)^2 \left( \oint_{\mathbb{S}^{2n+1}} |v|^{\frac{2Q}{Q-d}} \right)^{\frac{Q-d}{Q}} \le \oint_{\mathbb{S}^{2n+1}} v \mathcal{A}_d(v) \quad \text{for } v \in C^{\infty}(\mathbb{S}^{2n+1}).$$
 (9)

**Proposition 3.** For any  $F \in C^{\infty}(\mathbb{S}^{2n+1}) \cap \mathcal{P}$ , let  $v = e^{\frac{Q-d}{2}F}$ . Denote

$$LHS_d := \frac{4}{(Q-d)^2} \lambda_0(d) \left[ \left( \oint_{\mathbb{S}^{2n+1}} |v|^{\frac{2Q}{Q-d}} \right)^{\frac{Q-d}{Q}} - \oint_{\mathbb{S}^{2n+1}} |v|^2 \right]$$

and

$$RHS_d := \frac{4}{(Q-d)^2} \lambda_0(d)^{-1} \left[ \oint_{\mathbb{S}^{2n+1}} v \mathcal{A}_d(v) - \lambda_0(d)^2 \oint_{\mathbb{S}^{2n+1}} |v|^2 \right].$$

Then

$$\lim_{d \to Q} LHS_d = \frac{n!}{Q} \ln \int_{\mathbb{S}^{2n+1}} e^{Q(F-\bar{F})}$$
(10)

and

$$\lim_{d \to Q} RHS_d = \int_{\mathbb{S}^{2n+1}} F \mathcal{A}'_Q F,\tag{11}$$

where  $\bar{F}$  is the average of F over  $\mathbb{S}^{2n+1}$ .

Proof. Note that

$$\left( \int_{\mathbb{S}^{2n+1}} |v|^{\frac{2Q}{Q-d}} \right)^{\frac{Q-d}{Q}} - \int_{\mathbb{S}^{2n+1}} |v|^2$$

$$= \left( \int_{\mathbb{S}^{2n+1}} e^{QF} \right)^{\frac{Q-d}{Q}} - 1 - \int_{\mathbb{S}^{2n+1}} (e^{(Q-d)F} - 1).$$

Then by L'Hôpital's rule

$$\lim_{d \to n/2} LHS_d = \Gamma(n+1) \left( \frac{1}{Q} \ln \int_{\mathbb{S}^{2n+1}} e^{QF} - \int_{\mathbb{S}^{2n+1}} F \right)$$
$$= \frac{n!}{Q} \ln \int_{\mathbb{S}^{2n+1}} e^{Q(F-\bar{F})}.$$

Therefore, (10) is proved.

To prove (11), using the Taylor expansion of the exponential function, we write

$$v = e^{\frac{Q-d}{2}F} = 1 + \frac{1}{2}(Q-d)F + (Q-d)^2f,$$

where  $f = \frac{1}{8}F^2 \int_0^1 (1-s)e^{\frac{Q-d}{2}Fs} \, \mathrm{d}s \in C^\infty(\mathbb{S}^{2n+1})$  is uniformly bounded in  $C^{4n}$  norm as  $d \to Q$ . Then we see that

$$\begin{split} & \oint_{\mathbb{S}^{2n+1}} v \mathcal{A}_d(v) \\ & = \oint_{\mathbb{S}^{2n+1}} (1 + \frac{1}{2}(Q - d)F + (Q - d)^2 f)(\mathcal{A}_d(1) + \frac{1}{2}(Q - d)\mathcal{A}_d(F) + (Q - d)^2 \mathcal{A}_d(f)) \\ & = \oint_{\mathbb{S}^{2n+1}} \left( \lambda_0(d)^2 + (Q - d)\lambda_0(d)^2 F + 2(Q - d)^2 \lambda_0(d)^2 f \right. \\ & \quad + \frac{1}{4}(Q - d)^2 F \mathcal{A}_d F + (Q - d)^3 f \mathcal{A}_d F \right) + O((Q - d)^4), \end{split}$$

where we have used the self-adjointness of  $A_d$  and  $A_d(1) = \lambda_0(d)^2$ . We also see that

$$\lambda_0(d)^2 \oint_{\mathbb{S}^{2n+1}} |v|^2$$

$$= \lambda_0(d)^2 \oint_{\mathbb{S}^{2n+1}} \left( 1 + (Q-d)F + 2(Q-d)^2 f + \frac{1}{2}(Q-d)^2 F^2 + O((Q-d)^3) \right).$$

It follows that

$$\int_{\mathbb{S}^{2n+1}} v \mathcal{A}_d(v) - \lambda_0(d)^2 \int_{\mathbb{S}^{2n+1}} |v|^2 
= \frac{1}{4} (Q - d)^2 \int_{\mathbb{S}^{2n+1}} (F \mathcal{A}_d F + 4(Q - d) f \mathcal{A}_d F) + O((Q - d)^4).$$

By the second equality of (7), (11) follows immediately.

Therefore, Proposition 3 is proved.

Similarly, we immediately obtain

**Theorem 4.** We can derive the sharp Moser-Trudinger-Onofri inequality (8) from the sharp Sobolev inequality (9) by sending  $d \to Q$ .

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