DISTRIBUTIONAL CHAOS IN MULTIFRACTAL ANALYSIS, RECURRENCE AND TRANSITIVITY

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Abstract. There are lots of results to study dynamical complexity on irregular sets and level sets of ergodic average from the perspective of density in base space, Hausdorff dimension, Lebesgue positive measure, positive or full topological entropy (and topological pressure) etc.. However, it is unknown from the viewpoint of chaos. There are lots of results on the relationship of positive topological entropy and various chaos but it is known that positive topological entropy does not imply a strong version of chaos called DC1 so that it is non-trivial to study DC1 on irregular sets and level sets. In this paper we will show that for dynamical system with specification property, there exist uncountable DC1-scrambled subsets in irregular sets and level sets. On the other hand, we also prove that several recurrent levels of points with different recurrent frequency all have uncountable DC1-scrambled subsets. The main technique established to prove above results is that there exists uncountable DC1-scrambled subset in saturated sets.

1. Introduction

Throughout this paper, let (X, d) be a nondegenerate(i.e, with at least two points) compact metric space, and $f : X \to X$ is a continuous map. (X, f) is called a dynamical system.

1.1. **Multifractal Analysis.** The theory of multifractal analysis is a subfield of the dimension theory of dynamical systems. Briefly, multifractal analysis studies the dynamical complexity of the level sets of the invariant local quantities obtained from a dynamical system. There are lots of results to study dynamical complexity on irregular sets and level sets of ergodic average from the perspective of density in base space, Hausdorff dimension, Lebesgue positive measure, positive or full topological entropy (and topological pressure) etc., for example, see [52, 9, 51, 16, 67, 23, 5, 66, 28] (for topological entropy or Hausdorff dimension), [68, 69] (for topological pressure), [64, 38] (for Lebesgue positive measure) and references therein. However, it is unknown from the viewpoint of chaos. From chaos theory, we know that Li-Yorke chaotic and distributional chaotic are also good ways to describe the dynamical complexity. In this paper, we firstly study dynamical complexity of irregular set and level sets in the viewpoint of a strong chaotic property called DC1. Pikula showed in [55] that positive topological entropy does not imply DC1 so that it is not expected to show DC1 of irregular sets and level sets by using the results in [52, 9, 8, 66] that irregular set and level sets carry positive (and full) topological entropy.

The notion of chaos was first introduced in mathematic language by Li and Yorke in [44] in 1975. For a dynamical system (X, f), they defined that (X, f) is Li-Yorke chaotic if there is an uncountable scrambled set $S \subseteq X$, where S is called a scrambled set if for any pair of distinct two points x, y of S,

$$\liminf_{n\to+\infty} d(f^n x, f^n y) = 0,\ \limsup_{n\to+\infty} d(f^n x, f^n y) > 0.$$

Since then, several refinements of chaos have been introduced and extensively studied. One of the most important extensions of the concept of chaos in sense of Li and Yorke is distributional chaos as introduced in [63]. The stronger form of chaos has three variants: DC1(distributional chaotic of type 1), DC2 and DC3 (ordered from strongest to weakest). In this paper, we focus on DC1. Readers can refer to [26, 61, 62] for the definition of DC2 and DC3 and see [1, 49, 14, 22, 10, 37, 47, 11] and references therein for related

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topics on chaos theory if necessary. A pair $x, y \in X$ is DC1-scrambled if the following two conditions hold:

$$\forall t>0,\ \limsup_{n\to\infty}\frac{1}{n}|\{i\in[0,n-1]:\ d(f^i(x),f^i(y))<t\}|=1,$$

$$\exists t_0>0,\ \liminf_{n\to\infty}\frac{1}{n}|\{i\in[0,n-1]:\ d(f^i(x),f^i(y))<t_0\}|=0.$$

In other words, the orbits of x and y are arbitrarily close with upper density one, but for some distance, with lower density zero.

Definition 1.1. A set S is called a DC1-scrambled set if any pair of distinct points in S is DC1-scrambled.

1.1.1. *DC1 in Irregular set.* For a continuous function φ on X, define the $\varphi-$*irregular set* as

$$I_\varphi(f) \quad := \quad \left\{x\in X:\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^ix)\ \text{ diverges}\right\}.$$

φ-regular set and the irregular set, the union of $I_\varphi(f)$ over all continuous functions of φ (denoted by $IR(f)$), arise in the context of multifractal analysis and have been studied a lot, for example, see [52, 9, 51, 16, 69, 23]. The irregular points are also called points with historic behavior, see [57, 64]. From Birkhoff's ergodic theorem, the irregular set is not detectable from the point of view of any invariant measure. However, the irregular set may have strong dynamical complexity in sense of Hausdorff dimension, Lebesgue positive measure, topological entropy and topological pressure etc.. Pesin and Pitskel [52] are the first to notice the phenomenon of the irregular set carrying full topological entropy in the case of the full shift on two symbols. There are lots of advanced results to show that the irregular points can carry full entropy in symbolic systems, hyperbolic systems, non-uniformly expanding or hyperbolic systems, and systems with specification-like or shadowing-like properties, for example, see [9, 51, 16, 69, 23, 46, 71]. For topological pressure case see [69] and for Lebesgue positive measure see [64, 38]. Now let us state our first main theorem to study dynamical complexity of irregular set from the perspective of DC1.

Theorem A. *Suppose that (X, f) has specification property, φ is a continuous function on X and $I_\varphi(f)\neq\emptyset$. Then there is an uncountable DC1-scrambled subset in $I_\varphi(f)$.*

1.1.2. *DC1 in Level sets.* Level sets is a natural concept to slice points with convergent Birkhoffs average operated by some continuous function, regarded as the multifractal decomposition [18, 29]. Let $\varphi: X\to\mathbb{R}$ be a continuous function. For any $a\in L_\varphi$, consider the level set

$$R_\varphi(a) \quad := \quad \left\{x\in X:\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^ix)=a\right\}.$$

Denote $R_\varphi=\bigcup_{a\in L_\varphi}R_\varphi(a)$, then R_φ represents the regular points for φ. Many authors have considered the entropy of the $R_\varphi(a)$. For example, Barreira and Saussol proved in [8] that the following properties for a dynamical system (X, f) whose function of metric entropy is upper semi-continuous. Consider a Hölder continuous function ϕ (see [5, 6] for almost additive functions with tempered variation) which has a unique equilibrium measure, then for any constant $a\in\operatorname{int}(L_\phi)$

$$h_{top}(R_\varphi(a))=t_a, \tag{1.1}$$

where

$$t_a=\sup_{\mu\in\mathcal{M}_f(X)}\left\{h_\mu:\int\varphi d\mu=a\right\},$$

$h_{top}(R_\varphi(a))$ denotes the entropy of $R_\varphi(a)$, h_μ denotes the measure entropy of μ. For ϕ being an arbitrary continuous function (hence there may exist more than one equilibrium measures), (1.1) was established by Takens and Verbitski [66] under the assumption that f has the specification property. This result was further generalized by Pfister and Sullivan [54] to dynamical systems with g-product property(see [68, 70] for more related discussions). The method used in [6, 8] mainly depends on thermodynamic formalism such as differentiability of pressure function while the method in [66, 54] is a direct approach by constructing fractal sets. Here we consider the distributional chaotic of $R_\varphi(a)$ and R_φ. Let $\mathcal{M}(X)$,

$\mathcal{M}_f(X)$, $\mathcal{M}_f^e(X)$ denote the space of probability measures, f-invariant, f-ergodic probability measures respectively. For a continuous function φ on X, denote

$$L_\varphi = \left[\inf_{\mu\in\mathcal{M}_f(X)}\int\varphi d\mu,\ \sup_{\mu\in\mathcal{M}_f(X)}\int\varphi d\mu\right] \text{ and } Int(L_\varphi) = \left(\inf_{\mu\in\mathcal{M}_f(X)}\int\varphi d\mu,\ \sup_{\mu\in\mathcal{M}_f(X)}\int\varphi d\mu\right).$$

Note that if $I_\varphi(f)\neq\emptyset$, then $Int(L_\varphi)\neq\emptyset$. The inverse is also true if the system has specification property, see [69] (see [67] for the case of almost specification), and it is easy to check the continuous functions with $Int(L_\varphi)\neq\emptyset$ form an open and dense subset in the space of continuous functions so that so do the functions with $I_\varphi(f)\neq\emptyset$ if the system has specification property or almost specification.

Theorem B. *Suppose that (X,f) has specification property, φ is a continuous function on X and $Int(L_\varphi)\neq\emptyset$. Then for any $a\in Int(L_\varphi)$, there is an uncountable DC1-scrambled subset in $R_\varphi(a)$.*

As a corollary, there are uncountable number of disjoint uncountable DC1-scrambled subsets.

Corollary A. *Suppose that (X,f) has specification property but is not uniquely ergodic. Then there exist a collection of subsets of X, $\{S_\alpha\}_{\alpha\in(0,1)}$ such that*
(1). For any $0<\alpha_1<\alpha_2<1$, $S_{\alpha_1}\cap S_{\alpha_2}=\emptyset$, and
(2). For any $\alpha\in(0,1)$, S_α is an uncountable DC1-scrambled set.

Let us explain why this result holds. By assumption there are two different invariant measures μ,ν so that by weak* topology there exists a continuous function ϕ such that $\int\phi d\mu\neq\int\phi d\nu$. Thus $Int(L_\phi)\neq\emptyset$. Let $\varphi := \frac{1}{L}(\phi-\inf_{\mu\in\mathcal{M}_f(X)}\int\phi d\mu)$ where L denotes the length of interval L_ϕ. Then $Int(L_\varphi)=(0,1)$ and Theorem B implies this corollary since $R_\varphi(a)\cap R_\varphi(b)=\emptyset$ if $a\neq b$.

Theorem 1.2. *Suppose that (X,f) has specification property, φ is a continuous function on X. Then there is an uncountable DC1-scrambled subset in R_φ.*

Let us explain why Theorem 1.2 holds. If $Int(L_\varphi)\neq\emptyset$, then one can get this from Theorem B by taking one $a\in Int(L_\varphi)$ since $R_\varphi(a)\subseteq R_\varphi$. On the other hand, $Int(L_\varphi)=\emptyset$, then $R_\varphi=X$ and one can get this result by [49] (or see [48]).

1.2. **DC1 in recurrence.** In classical study of dynamical systems, an important concept is recurrence. Recurrent points such as periodic points, minimal points are typical objects to be studied. It is known that whole recurrent points set has full measure for any invariant measure under f and minimal points set is not empty[31]. A fundamental question in dynamical systems is to search the existence of periodic points. For systems with Bowen' specification(such as topological mixing subshifts of finite type and topological mixing uniformly hyperbolic systems), the set of periodic points is dense in the whole space [21]. Further, many people pay attention to more refinements of recurrent points according to the 'recurrent frequency' such as weakly almost periodic points and quasi-weakly almost periodic points and measure them[33, 75]. In [35, 70] the authors considered various recurrence and showed many different recurrent levels carry strong dynamical complexity from the perspective of topological entropy. In present paper, one of our aim is to consider these different recurrent levels from the perspective of chaos.

For any $x\in X$, the orbit of x is $\{f^n x\}_{n=0}^\infty$, denoted by $orb(x,f)$. The ω-limit set of x is the set of all limit points of $orb(x,f)$, denoted by $\omega_f(x)$.

Definition 1.3. A point $x\in X$ is **recurrent**, if $x\in\omega_f(x)$. If $\omega_f(x)=X$, we say x is a transitive point of f. A point $x\in X$ is **almost periodic**, if for any open neighborhood U of x, there exists $N\in\mathbb{N}$ such that $f^k(x)\in U$ for some $k\in[n,n+N]$ for every $n\in\mathbb{N}$. A point x is called periodic, if there exists natural number n such that $f^n(x)=x$.

We denote the sets of all recurrent points, transitive points, almost periodic points and periodic points by Rec, $Trans$, AP and Per respectively. Now we recall some notions of recurrence by using density. Let $S\subseteq\mathbb{N}$, we denote

$$\overline{d}(S) := \limsup_{n\to\infty}\frac{|S\cap\{0,1,\cdots,n-1\}|}{n},\quad \underline{d}(S) := \liminf_{n\to\infty}\frac{|S\cap\{0,1,\cdots,n-1\}|}{n},$$

$$B^*(S) := \limsup_{|I|\to\infty} \frac{|S\cap I|}{|I|}, \quad B_*(S) := \liminf_{|I|\to\infty} \frac{|S\cap I|}{|I|},$$

where $|A|$ denotes the cardinality of the set A. They are called the upper density of S and the lower density, Banach upper density and Banach lower density of S respectively. Let $U, V \subseteq X$ be two nonempty open sets and $x \in X$. Define sets of visiting time

$$N(U,V) := \{n \geq 1 | U \cap f^{-n}(V) \neq \emptyset\} \text{ and } N(x,U) := \{n \geq 1 | f^n(x) \in U\}.$$

Definition 1.4. A point $x \in X$ is called Banach upper recurrent, if $N(x, B_\varepsilon(x))$ has positive Banach upper density where $B_\varepsilon(x)$ denotes the ball centered at x with radius ε. Similarly, one can define the Banach lower recurrent, upper recurrent, and lower recurrent.(see [35])

Let BR denote the set of all Banach upper recurrent points and let QW, W denote the set of upper recurrent points and lower recurrent points respectively. Note that AP coincides with the set of all Banach lower recurrent points. From [33, 74, 75, 72] W, QW, BR, Rec all have full measure for any invariant measure but AP maybe not. Note that

$$AP \subseteq W \subseteq QW \subseteq BR \subseteq Rec.$$

So the recurrent set can be decomposed into several disjoint 'periodic-like' recurrent level sets which reflect different recurrent frequency:

$$Rec = AP \sqcup (W \setminus AP) \sqcup (QW \setminus W) \sqcup (BR \setminus QW) \sqcup (Rec \setminus BR).$$

A question appeared in [70] is that

How much dierence are there between these 'periodic-like' recurrences?

One main basic idea firstly considered in [70] is to search which recurrent level set carries the same dynamical complexity as the whole system, for example, by using topological entropy since W, QW, BR, Rec all carry full topological entropy as the whole space. It was showed that these recurrent level sets except $Rec \setminus BR$ all have full topological entropy studied in [70] for $QW \setminus W$ and $W \setminus AP$, [35] for $BR \setminus QW$, [25] for AP. From [48] Oprocha proved that there exists an uncountable DC1-scrambled subset in $Rec \setminus AP$. Recall that Pikula showed in [55] that positive topological entropy does not imply DC1. Thus, motivated by these results we can also ask the similar question from the perspective of chaos. That is, whether there is an uncountable DC1-scrambled set in every recurrent level set of $Rec \setminus BR$, $BR \setminus QW$, $QW \setminus W$, $W \setminus AP$ and AP. We will mainly show there are uncountable $DC1$-scrambled subsets in $BR \setminus QW$ and $QW \setminus W$ if the system has specification property (and we also discuss uncountable $DC1$-scrambled subset in $W \setminus AP$ under more assumptions and uncountable $DC2$-scrambled subset in AP in the last section).

Theorem C. *Suppose that (X, f) has specification property but is not uniquely ergodic. Then there exist uncountable DC1-scrambled subsets in $QW \setminus W$ and $BR \setminus QW$. Moreover, the points in these subsets can be chosen transitive.*

We will prove Theorem C in Section 4.

Corollary B. *Suppose that (X, f) has specification property. Then there exist uncountable DC1-scrambled subset in $Trans$.*

Let us explain why this result holds. By assumption if the system is not uniquely ergodic, then it can be deduced from Theorem C. Otherwise, the system is uniquely ergodic. By [20] (or see [35]) minimal points are dense in the whole space so that the system must also be minimal. In this case $Tran = AP = X$ so that one only needs to show uncountable DC1-scrambled in X which is the result of [49] (or see [48]).

1.3. **Combination of Multifractal Analysis and Recurrence.** We give a DC1 result in combined sets of multifractal analysis and recurrence.

Theorem D. *Suppose that (X, f) has specification property, φ is a continuous function on X and $Int(L_\varphi) \neq \emptyset$. Then*
(1) there exsit uncountable DC1-scrambled subsets in $I_\varphi \cap (QW \setminus W)$ and $I_\varphi \cap (BR \setminus QW)$ respectively.
(2) for any $a \in Int(L_\varphi)$, there exsit uncountable DC1-scrambled subsets in $R_\varphi(a) \cap (QW \setminus W)$ and

$R_\varphi(a) \cap (BR \setminus QW)$ *respectively.*
Moreover, the points in these subsets can be chosen transitive.

Theorem D imply Theorems A and B so that we only need to prove Theorem D in Section 4. As a corollary of Theorem D, we state a following result.

Corollary C. *Suppose that* (X, f) *has specification property,* φ *is a continuous function on* X *and* $Int(L_\varphi) \neq \emptyset$. *Then there exist uncountable DC1-scrambled subset in* $Trans \cap I_\varphi$. *And for any* $a \in Int(L_\varphi)$, *there exsit uncountable DC1-scrambled subset in* $R_\varphi(a) \cap Trans$.

1.4. **DC1 in Recurrent Level Sets Characterized by Statistical $\omega-$limit Sets.** Recently several concepts of statistical $\omega-$limit sets were introduced in [24] (some notions also see [2, 3]). They also can describe different levels of recurrence and some cases coincide with above classifications of Banach recurrence.

Definition 1.5. For $x \in X$ and $\xi = \overline{d}$, $\underline{d}$, B^*, B_*, a point $y \in X$ is called $x - \xi-$accessible, if for any $\varepsilon > 0$, $N(x, V_\varepsilon(y))$ has positive density w. r. t. ξ, where $V_\varepsilon(x)$ denotes the ball centered at x with radius ε. Let

$$\omega_\xi(x) := \{y \in X \mid y \text{ is } x - \xi - \text{accessible}\}.$$

For convenience, it is called $\xi - \omega$*-limit set of* x. $\omega_{B_*}(x)$ is also called *syndetic center* of x.

With these definitions, one can immediately note that

$$\omega_{B_*}(x) \subseteq \omega_{\underline{d}}(x) \subseteq \omega_{\overline{d}}(x) \subseteq \omega_{B^*}(x) \subseteq \omega_f(x). \tag{1.2}$$

For any $x \in X$, if $\omega_{B_*}(x) = \emptyset$, then from [24] we know that x satisfies one and only one of following twelve cases:

Case (1) : $\emptyset = \omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) = \omega_{\overline{d}}(x) = \omega_{B^*}(x) = \omega_f(x)$;
Case (1') : $\emptyset = \omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) = \omega_{\overline{d}}(x) = \omega_{B^*}(x) \subsetneq \omega_f(x)$;
Case (2) : $\emptyset = \omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) = \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) = \omega_f(x)$;
Case (2') : $\emptyset = \omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) = \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) \subsetneq \omega_f(x)$;
Case (3) : $\emptyset = \omega_{B_*}(x) = \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) = \omega_{B^*}(x) = \omega_f(x)$;
Case (3') : $\emptyset = \omega_{B_*}(x) = \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) = \omega_{B^*}(x) \subsetneq \omega_f(x)$;
Case (4) : $\emptyset = \omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) = \omega_{B^*}(x) = \omega_f(x)$;
Case (4') : $\emptyset = \omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) = \omega_{B^*}(x) \subsetneq \omega_f(x)$;
Case (5) : $\emptyset = \omega_{B_*}(x) = \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) = \omega_f(x)$;
Case (5') : $\emptyset = \omega_{B_*}(x) = \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) \subsetneq \omega_f(x)$;
Case (6) : $\emptyset = \omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) = \omega_f(x)$;
Case (6') : $\emptyset = \omega_{B_*}(x) \subsetneq \omega_{\underline{d}}(x) \subsetneq \omega_{\overline{d}}(x) \subsetneq \omega_{B^*}(x) \subsetneq \omega_f(x)$.

Theorem 1.6. *Suppose that* (X, f) *has specification property but is not uniquely ergodic, then* $\{x \in Rec \mid x \; satisfies \; Case \; (i)\}$, $i = 2, 3, 4, 5, 6$ *contains an uncountable DC1-scrambled subset in* $Trans$. *Further, if* φ *is a continuous function on* X *and* $I_\varphi(f) \neq \emptyset$, *then for any* $a \in Int(L_\varphi)$, *the recurrent level set of* $\{x \in Rec \mid x \; satisfies \; Case \; (i)\}$ *contains an uncountable DC1-scrambled subset in* $Trans \cap I_\varphi(f)$, $Trans \cap R_\varphi(a)$ *and* $Trans \cap R_\varphi$, *respectively,* $i = 2, 3, 4, 5, 6$.

We will prove this theorem in in Section 4. Case (1) is also known if the system has more assumptions, see the last section, but Cases (1')-(6') restricted on recurrent points all are still unknown to have DC1 or weaker ones such as Li-Yorke chaos. Chaotic behavior in non-recurrent points and various non-recurrent levels by using above statistical ω-limit sets will be discussed in another forthcoming paper.

1.5. **DC1 in Saturated sets.** To show above results on irregular set, level sets and different recurrence, one main proof idea is motivated by Oprocha and Štefánková's result in [49] (or see [47]) that there is an uncountable DC1-scrambled subset in X when (X, f) has specification. One can construct corresponding uncountable DC1-scrambled subset one by one but everyone needs a long construction proof so that it is not a good choice to do these constructions directly. Recall that in the case of entropy estimate on recurrent levels, one main technique chosen in [70, 35] is using (transitively) saturated property which can avoid to do a long construction proof for every considered object. So here we follow the way of [70, 35] to give a DC1 result in saturated sets.

Given $x \in X$, denote $V_f(x) \subseteq \mathcal{M}_f(X)$ the set of all accumulation points of the empirical measures

$$\mathcal{E}_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)},$$

where δ_x is the Dirac measure concentrate on x. The system (X, f) is called to have *saturated* property, if for any compact connected nonempty set $K \subseteq \mathcal{M}_f(X)$,

$$G_K \neq \emptyset \quad \text{and} \quad h_{top}(T, G_K) = \inf\{h_\mu(T) \mid \mu \in K\}, \tag{1.3}$$

where $G_K = \{x \in X \mid V_f(x) = K\}$ (called saturated set), $h_{top}(A)$ denotes the topological entropy of A defined by Bowen in [13] and $h_\mu(T)$ denotes its metric entropy of μ. The existence of saturated sets is proved by Sigmund [58] for systems with uniform hyperbolicity or specification and generalized to non-uniformly hyperbolic systems in [45]. The property on entropy estimate was firstly established by Pfister and Sullivan in [54] and then was generalized to transitively-saturated version in [35], provided that the system has g-product property (which is weaker than specification) and uniform separation property (which is weaker than expansiveness). In this subsection we aim to establish DC1 in saturated sets. A point $x \in X$ is generic for some invariant measure μ means that $V_f(x) = \mu$(or equivalently, Birkhoff averages of all continuous map converge to the integral of μ.) Let G_μ denote the set of all generic points for μ.

Theorem E. *Suppose that (X, f) has specification and K be a connected non-empty compact subset of $\mathcal{M}_f(X)$. If there is a $\mu \in K$ such that $\mu = \theta\mu_1 + (1-\theta)\mu_2$* ($\mu_1 = \mu_2$ could happens) *where $\theta \in [0, 1]$, and G_{μ_1}, G_{μ_2} both have distal pair. Then for any non-empty open set $U \subseteq X$, there exists an uncountable DC1-scrambled set $S_K \subseteq G_K \cap U \cap Trans$.*

We will prove this theorem in Section 3. Since an ergodic measure with nondegenerate minimal support has two generic points as a distal pair, see Proposition 4.4 below, one has a following result as a corollary of Theorem E.

Corollary D. *Suppose that (X, f) has specification. For any ergodic measure μ, if its support is non-degenerate and minimal, then there exists an uncountable DC1-scrambled set $S \subseteq Trans$ such that any point in S is generic for μ.*

Here μ admits to have zero metric entropy. If the system is not minimal, then above set S has zero measure for μ, since $S \subseteq Trans$, $S_\mu \neq X$ and by Birkhoff ergodic theorem $\mu(S_\mu \cap G_\mu) = 1$.

2. Preliminaries

2.1. **Specification Properties.**

Specification was first introduced by Bowen in [12]. Before giving the definition, we make a notion that for (X, f) and $x, y \in X$, $a, b \in \mathbb{N}$, we say x *ε-traces* y on $[a, b]$ if $d(f^i x, f^{i-a} y) < \varepsilon$ $\forall i \in [a, b]$. The following definition mainly refers to [21, 49].

Definition 2.1. We say (X, f) has **strong specfication property**, if for any $\varepsilon > 0$, there is a positive integer K_ε such that for any integer s ≥ 2, any set $\{y_1, y_2, \cdots, y_s\}$ of s points of X, and any sequence

$$0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$$

of $2s$ integers with

$$a_{m+1} - b_m \geq K_\varepsilon$$

for $m = 1, 2, \cdots, s-1$, there is a point x in X such that the following two conditions hold:

(a) x *ε-traces* y_m on $[a_m, b_m]$ for all positive integers m $\leq$ s;

(b) $f^n(x) = x$, where $n = b_s + K_\varepsilon$.

If the periodicity condition (b) is omitted, we say that f has **specification property**.

Proposition 2.2. [27] *Suppose that (X, f) has specification property, then $\mathcal{M}_f^e(X)$ is dense in $\mathcal{M}_f(X)$.*

Proposition 2.3. [21] *A dynamical system (X, f) with specification property has measure with full support. Moreover, the set of such measure is dense in $\mathcal{M}_f(X)$.*

2.2. **Levels of Recurrence.** Let us recall some equivalent statements of recurrence referring to [33, 74, 75, 35]. For a measure μ, define the support of μ by $S_\mu := supp(\mu) = \{x \in X | \ \mu(U) > 0$ for any neighborhood U of $x\}$.

Proposition 2.4. [33] *For (X, f), let $x \in Rec$. Then the following conditions are equivalent.*
(a) $x \in W$;
(b) $x \in C_x = S_\mu$ *for any* $\mu \in V_f(x)$;
(c) $S_\mu = \omega_f(x)$ *for any* $\mu \in V_f(x)$.

Proposition 2.5. [33] *For (X, f), let $x \in Rec$. Then the following conditions are equivalent.*
(a) $x \in QW$;
(b) $x \in C_x$;
(c) $C_x = \omega_f(x)$.

Proposition 2.6. *For (X, f) with specification property, $x \in Trans$ implies $x \in BR$.*

Proposition 2.6 is direct consequence by combining Proposition 2.3 and [35, Lemma 4.3].

3. Proof of Theorem E

One main proof idea is motivated by Oprocha and Štefánková's result in [49] that there is uncountable DC1-scrambled subset in X when (X, f) has specification. Before proof we introduce some basic facts and lemmas.

3.1. **Ergodic Average.** We write $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{N}^+ = \{1, 2, \cdots\}$. If $r, s \in \mathbb{N}, r \leq s$, we set $[r, s] := \{j \in \mathbb{N} | \ r \leq j \leq s\}$, and the cardinality of a finite set Λ is denoted by $|\Lambda|$. We set

$$\langle f, \mu \rangle \ := \ \int_X f d\mu.$$

There exists a countable and separating set of continuous functions $\{f_1, f_2, \cdots\}$ with $0 \leq f_k(x) \leq 1$, and such that

$$d(\mu, \nu) \ := \| \ \mu - \nu \ \| := \ \sum_{k \geq 1} 2^{-k} \mid \langle f_k, \mu - \nu \rangle \mid$$

defines a metric for the weak*-topology on $\mathcal{M}_f(X)$. We refer to [54] and use the metric on X as following defined by Pfister and Sullivan.

$$d(x, y) := d(\delta_x, \delta_y),$$

which is equivalent to the original metric on X. Readers will find the benefits of using this metric in our proof later.

Lemma 3.1. *For any $\varepsilon > 0, \delta > 0$ and two sequences $\{x_i\}_{i=0}^{n-1}, \{y_i\}_{i=0}^{n-1}$ of X such that $d(x_i, y_i) < \varepsilon$ holds for any $i \in [0, n-1]$, then for any $J \subseteq \{0, 1, \cdots, n-1\}$, $\frac{n-|J|}{n} < \delta$, one has:*
(a) $d(\frac{1}{n}\sum_{i=0}^{n-1}\delta_{x_i}, \frac{1}{n}\sum_{i=0}^{n-1}\delta_{y_i}) < \varepsilon$.
(b) $d(\frac{1}{n}\sum_{i=0}^{n-1}\delta_{x_i}, \frac{1}{|J|}\sum_{i\in J}\delta_{y_i}) < \varepsilon + 2\delta$.

Lemma 3.1 is easy to be verified and shows us that if any two orbit of x and y in finite steps are close in the most of time, then the two empirical measures induced by x, y are also close.

Lemma 3.2. *Suppose that (X, f) has specification. Let K be a connected non-empty compact subset of $\mathcal{M}_f(X)$ and $\mu \in K$. Then for any $\varepsilon > 0$ there exists a $N_\varepsilon^\mu \in \mathbb{N}$ such that for any $\alpha \in K$, any $N > N_\varepsilon^\mu$ and any $M > N$, there is an $x \in X$ and $N^* > M$ such that*

- **(a):** $\mathcal{E}_n(x) \in B(\mu, \varepsilon), \ \forall n \in [N_\varepsilon^\mu, N]$;
- **(b):** $\mathcal{E}_n(x) \in B(K, \varepsilon), \ \forall n \in [N, N^*]$;
- **(c):** $\mathcal{E}_{N^*}(x) \in B(\alpha, \varepsilon)$.

Proof. For any fixed $\varepsilon > 0$, by Proposition 2.2, there exists $p^\mu \in X$ and $n^\mu \in \mathbb{N}$ such that $\mathcal{E}_n(p^\mu) \in B(\mu, \varepsilon/6)$ holds for any $n \geq n^\mu$. Set $N^\mu_\varepsilon := n^\mu$, we will prove that such N^μ_ε makes this lemma true. Note that K is connected, so for any $\alpha \in K$, we can find a sequence $\{\beta_1, \beta_2, \cdots, \beta_{m_\varepsilon}\} \subseteq K$ such that $d(\beta_{i+1}, \beta_i) < \varepsilon$, $\forall i \in \{1, 2, \cdots, m_\varepsilon - 1\}$ and $\beta_1 = \mu, \beta_{m_\varepsilon} = \alpha$. By Proposition 2.2, for any $i \in \{2, \cdots, m_\varepsilon\}$, there exists $p^{\beta_i} \in X$ and $n^{\beta_i} \in \mathbb{N}$ such that $\mathcal{E}_n(p^{\beta_i}) \in B(\beta_i, \varepsilon/6)$ holds for any $n \geq n^{\beta_i}$. For any $N > N^\mu_\varepsilon$ and $M > N$, we choose $\{T_i\}_{i=1}^{2m_\varepsilon}$ with $T_i \in \mathbb{N}$ such that for $i \in \{1, \cdots, m_\varepsilon - 1\}$

$$T_1 = 0,\ T_2 = N. \tag{3.1}$$

$$T_{2i+1} = T_{2i} + K_{\varepsilon/6}, \text{where } K_{\varepsilon/6} \text{ defined in the } Definiton \text{ 2.1.} \tag{3.2}$$

$$\frac{\varepsilon}{12}(T_{2i} - T_{2i-1}) > n^{\beta_{i+1}}. \tag{3.3}$$

$$\frac{K_{\varepsilon/6} + T_{2i-1}}{T_{2i} - T_{2i-1}} < \frac{\varepsilon}{12}. \tag{3.4}$$

So far, we have fixed $\{T_i\}_{i=1}^{2m_\varepsilon - 1}$. We choose T_{2m_ε} large enough such that

$$T_{2m_\varepsilon} \geq \max\{M, T_{2m_\varepsilon - 1} + n^{\beta_{m_\varepsilon}}\}. \tag{3.5}$$

$$\frac{T_{2m_\varepsilon - 1}}{T_{2m_\varepsilon}} < \frac{\varepsilon}{12}. \tag{3.6}$$

By (3.2), we can use specification property. So there is an $x \in X$ that x $\varepsilon/6$-*traces* x^* on $[T_1, T_2]$ and $\varepsilon/6$-*traces* p^{β_i} on $[T_{2i-1}, T_{2i}]$, $\forall i \in \{2, \cdots, m_\varepsilon\}$. Now we claim that such x and $N^* = T_{2m_\varepsilon}$ satisfy the items (**a**)(**b**)(**c**). (**a**)(**c**) is easy to check by (3.1)(3.5)(3.6) and Lemma 3.1. Here we check the (**b**). If $n \in (T_{2i}, T_{2i+1})$ for some $i \in \{1, \cdots, m_\varepsilon - 1\}$, we have

$$\frac{n - T_{2i} + T_{2i-1}}{T_{2i} - T_{2i-1}} < \frac{\varepsilon}{12},$$

by (3.2)(3.4). So, by Lemma 3.1, we have

$$\begin{aligned} d(\mathcal{E}_n(x), \beta_i) &< d(\mathcal{E}_n(x), \mathcal{E}_{T_{2i}-T_{2i-1}}(p^{\beta_i})) + d(\mathcal{E}_{T_{2i}-T_{2i-1}}(p^{\beta_i}), \beta_i) \\ &< \frac{\varepsilon}{6} + 2 \cdot \frac{\varepsilon}{12} + \frac{\varepsilon}{6} \\ &= \frac{\varepsilon}{2}. \end{aligned} \tag{3.7}$$

If $n \in [T_{2i-1}, T_{2i}]$ for some $i \in \{2, 3, \cdots, m_\varepsilon\}$, we split this situation into the following two cases.

Case 1: $\frac{n - T_{2i-1}}{T_{2i-2} - T_{2i-3}} < \frac{\varepsilon}{12}$. Then

$$d(\mathcal{E}_n(x), \beta_{i-1}) < \frac{\varepsilon}{6} + 2 \cdot (\frac{\varepsilon}{12} + \frac{\varepsilon}{12}) + \frac{\varepsilon}{6} = \frac{2\varepsilon}{3}, \tag{3.8}$$

by Lemma 3.1 and (3.4).

Case 2: $\frac{n - T_{2i-1}}{T_{2i-2} - T_{2i-3}} \geq \frac{\varepsilon}{12}$. If so, we have $n - T_{2i-1} > n^{\beta_i}$ by (3.3), which implies $\mathcal{E}_{n-T_{2i-1}}(p^{\beta_i}) \in B(\beta_i, \varepsilon/6)$. We consider $d(\mathcal{E}_n(x), \beta_i)$ and $d(\mathcal{E}_n(x), \beta_{i-1})$.

$$\begin{aligned} d(\mathcal{E}_n(x), \beta_i) &= d(\frac{T_{2i-1}}{n}\mathcal{E}_{T_{2i-1}}(x) + \frac{n - T_{2i-1}}{n}\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_i) \\ &\leq \frac{T_{2i-1}}{n} d(\mathcal{E}_{T_{2i-1}}(x), \beta_i) + \frac{n - T_{2i-1}}{n} d(\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_i) \\ &\leq \frac{T_{2i-1}}{n} d(\mathcal{E}_{T_{2i-1}}(x), \beta_{i-1}) + \frac{T_{2i-1}}{n} d(\beta_{i-1}, \beta_i) + \frac{n - T_{2i-1}}{n} d(\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x), \beta_i) \\ &< \frac{T_{2i-1}}{n}(\frac{\varepsilon}{6} + 2 \cdot \frac{\varepsilon}{12} + \frac{\varepsilon}{6}) + \frac{T_{2i-1}}{n}\varepsilon + \frac{n - T_{2i-1}}{n}(\frac{\varepsilon}{6} + \frac{\varepsilon}{6}) \\ &< \frac{\varepsilon}{2} + \frac{T_{2i-1}}{n}\varepsilon, \end{aligned}$$

$$
\begin{aligned}
d(\mathcal{E}_n(x),\beta_{i-1}) &= d(\frac{T_{2i-1}}{n}\mathcal{E}_{T_{2i-1}}(x)+\frac{n-T_{2i-1}}{n}\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x),\beta_{i-1})\\
&\le \frac{T_{2i-1}}{n}d(\mathcal{E}_{T_{2i-1}}(x),\beta_{i-1})+\frac{n-T_{2i-1}}{n}d(\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x),\beta_{i-1})\\
&\le \frac{T_{2i-1}}{n}d(\mathcal{E}_{T_{2i-1}}(x),\beta_{i-1})+\frac{n-T_{2i-1}}{n}d(\mathcal{E}_{n-T_{2i-1}}(f^{T_{2i-1}}x),\beta_{i})+\frac{n-T_{2i-1}}{n}d(\beta_i,\beta_{i-1})\\
&< \frac{T_{2i-1}}{n}(\frac{\varepsilon}{6}+2\cdot\frac{\varepsilon}{12}+\frac{\varepsilon}{6})+\frac{n-T_{2i-1}}{n}(\frac{\varepsilon}{6}+\frac{\varepsilon}{6})+\frac{n-T_{2i-1}}{n}\varepsilon\\
&< \frac{\varepsilon}{2}+\frac{n-T_{2i-1}}{n}\varepsilon.
\end{aligned}
$$

So,

$$\min\{d(\mathcal{E}_n(x),\beta_i),d(\mathcal{E}_n(x),\beta_{i-1})\}<\varepsilon. \tag{3.9}$$

With the combination of (3.7) (3.8) (3.9), one has (**b**). □

Lemma 3.3. *Suppose that (X,f) has specification. Let K be a connected non-empty compact subset of $\mathcal{M}_f(X)$ and $\mu\in K$. Then for any $\varepsilon>0$, there exists a $M^\mu_\varepsilon\in\mathbb{N}$ such that for any $\alpha\in K$ and any $M>M^\mu_\varepsilon$, there exist $t_2>t_1>M$ and $x\in X$ such that*

- **(a):** $\mathcal{E}_n(x)\in B(\mu,\varepsilon),\ \forall n\in[M^\mu_\varepsilon,M]$;
- **(b):** $\mathcal{E}_n(x)\in B(K,\varepsilon),\ \forall n\in[M,t_1]$;
- **(c):** $\mathcal{E}_{t_1}(x)\in B(\alpha,\varepsilon)$;
- **(d):** $\mathcal{E}_n(x)\in B(K,\varepsilon),\ \forall n\in[t_1,t_2]$;
- **(e):** $\mathcal{E}_{t_2}(x)\in B(\mu,\varepsilon)$.

Proof. By Lemma 3.2, for $\varepsilon/3$, we obtain $N^\mu_{\varepsilon/3}$ and $N^\alpha_{\varepsilon/3}$ such that for any $N_1>N^\mu_{\varepsilon/3}$, there is an x_1 and N^* such that

$$N^*>\max\{N_1,\frac{K_{\varepsilon/3}+N^\alpha_{\varepsilon/3}}{\varepsilon/6}\}, \tag{3.10}$$

$$
\begin{aligned}
&\mathcal{E}_n(x_1)\in B(\mu,\varepsilon/3),\ \forall n\in[N^\mu_{\varepsilon/3},N_1];\\
&\mathcal{E}_n(x_1)\in B(K,\varepsilon/3),\ \forall n\in[N_1,N^*];\\
&\mathcal{E}_{N^*}(x_1)\in B(\alpha,\varepsilon/3),
\end{aligned}
$$

and for

$$N_2>\max\{N^\alpha_{\varepsilon/3},\frac{N^*+K_{\varepsilon/3}}{\varepsilon/6}\}, \tag{3.11}$$

there exist $N^{**}>N_2$ and x_2 such that

$$
\begin{aligned}
&\mathcal{E}_n(x_2)\in B(\alpha,\varepsilon/3),\ \forall n\in[N^\alpha_{\varepsilon/3},N_2];\\
&\mathcal{E}_n(x_2)\in B(K,\varepsilon/3),\ \forall n\in[N_2,N^{**}];\\
&\mathcal{E}_{N^{**}}(x_2)\in B(\mu,\varepsilon/3).
\end{aligned} \tag{3.12}
$$

By specification property, we can obtain an $x\in X$ such that x $\varepsilon/3$-*traces* x_1 on $[0,N^*]$ and $\varepsilon/3$-*traces* x_2 on $[N^*+K_{\varepsilon/3},N^*+K_{\varepsilon/3}+N^{**}]$. Now we consider $\mathcal{E}_n(x),\ n\in[N^\mu_{\varepsilon/3},N^*+K_{\varepsilon/3}+N^{**}]$ and split into the following cases

Case 1: When $n\in[N^\mu_{\varepsilon/3},N^*]$, we have $d(\mathcal{E}_n(x),\mathcal{E}_n(x_1))<\varepsilon/3$. So

$$
\begin{aligned}
&\mathcal{E}_n(x)\in B(\mu,\varepsilon),\ \forall n\in[N^\mu_{\varepsilon/3},N_1];\\
&\mathcal{E}_n(x)\in B(K,\varepsilon),\ \forall n\in[N_1,N^*];\\
&\mathcal{E}_{N^*}(x)\in B(\alpha,\varepsilon).
\end{aligned}
$$

Case 2: When $n\in[N^*,N^*+K_{\varepsilon/3}+N^\alpha_{\varepsilon/3}]$, we have $d(\mathcal{E}_n(x),\mathcal{E}_{N^*}(x_1))<2\varepsilon/3$ by (3.10) and Lemma 3.1. So $d(\mathcal{E}_n(x),\alpha)<\varepsilon$.

Case 3: When $n \in [N^* + K_{\varepsilon/3} + N^{\alpha}_{\varepsilon/3}, N_2]$,

$$\begin{aligned} d(\mathcal{E}_n(x), \alpha) &= d(\frac{N^* + K_{\varepsilon/3}}{n}\mathcal{E}_{N^*+K_{\varepsilon/3}}(x) + \frac{n - N^* - K_{\varepsilon/3}}{n}\mathcal{E}_{n-N^*-K_{\varepsilon/3}}(f^{N^*+K_{\varepsilon/3}}x), \alpha) \\ &\le \frac{N^* + K_{\varepsilon/3}}{n} d(\mathcal{E}_{N^*+K_{\varepsilon/3}}(x), \alpha) + \frac{n - N^* - K_{\varepsilon/3}}{n} d(\mathcal{E}_{n-N^*-K_{\varepsilon/3}}(f^{N^*+K_{\varepsilon/3}}x), \alpha). \end{aligned}$$

Note that $n - N^* - K_{\varepsilon/3} \ge N^{\alpha}_{\varepsilon/3}$ and $n \le N_2$, then we have $d(\mathcal{E}_{n-N^*-K_{\varepsilon/3}}(f^{N^*+K_{\varepsilon/3}}x), \alpha) < \varepsilon$ by (3.12). So

$$d(\mathcal{E}_n(x), \alpha) < \frac{N^* + K_{\varepsilon/3}}{n}\varepsilon + \frac{n - N^* - K_{\varepsilon/3}}{n}\varepsilon = \varepsilon.$$

Case 4: When $n \in [N_2, N^{**}]$. Note that $N^{**} > N_2 > \frac{N^*+K_{\varepsilon/3}}{\varepsilon/6}$, so by Lemma 3.1, we have

$$d(\mathcal{E}_n(x), \mathcal{E}_{n-N^*-K_{\varepsilon/3}}(x_2)) < 2\varepsilon/3.$$

Thus

$$\begin{aligned} &\mathcal{E}_n(x) \in B(K, \varepsilon),\ \forall n \in [N_2, N^{**}]; \\ &\mathcal{E}_{N^{**}}(x_2) \in B(\mu, \varepsilon). \end{aligned}$$

Set $M^{\mu}_{\varepsilon} = N^{\mu}_{\varepsilon/3}$, $M = N_1$ $t_1 = N^*$ $t_2 = N^{**}$, we finish the proof. □

For a dynamical system (X, f), we say a pair $p, q \in X$ is distal if $\liminf_{i\to\infty} d(f^i p, f^i q) > 0$. Obviously, $\inf\{d(f^i p, f^i q)|\ i \in \mathbb{N}\} > 0$ if p, q is distal. We say a subset $M \subseteq X$ has distal pair if there are distinct $p, q \in M$ such that p, q is distal.

Lemma 3.4. *Suppose that (X, f) has specification. Suppose there are $\mu_1, \mu_2 \in \mathcal{M}_f(X)$ such that G_{μ_1}, G_{μ_2} have distal pair (p_1, q_1), (p_2, q_2) respectively. Let $\zeta = \min\{\inf\{d(f^i p_1,$ $f^i q_1)|\ i \in \mathbb{N}\}, \inf\{d(f^i p_2, f^i q_2)|\ i \in \mathbb{N}\}\}$, then for any $\delta > 0$, any $0 < \varepsilon < \zeta$ and any $\theta \in [0, 1]$, there exists $x_1, x_2 \in X$ and $N \in \mathbb{N}$ such that for any $n > N$,*

- **(a):** $\mathcal{E}_n(x_1) \in B(\theta\mu_1 + (1-\theta)\mu_2, \varepsilon + \delta)$ and $\mathcal{E}_n(x_2) \in B(\theta\mu_1 + (1-\theta)\mu_2, \varepsilon + \delta)$;
- **(b):** $\frac{|\{0 \le i \le n-1 | d(f^i x_1, f^i x_2) < \zeta - \varepsilon\}|}{n} < \delta$.

Proof. We just proof this lemma for θ is rational. Then, the lemma naturally holds for any $\theta \in [0, 1]$ by the denseness of rational numbers. For any fixed $\delta > 0$, $0 < \varepsilon < \zeta$ and $\frac{\theta}{1-\theta} = \frac{s}{t}$, where $s, t \in \mathbb{N}^+$, we can obtain an M_1 such that $\mathcal{E}_n(p_i) \in B(\mu_i, \varepsilon/2)$ and $\mathcal{E}_n(q_i) \in B(\mu_i, \varepsilon/2), i = \{1, 2\}$ hold for any $n \ge M_1$. We choose $M, r \in \mathbb{N}^+$ such that

$$M > \max\{M_1, \frac{4K_{\varepsilon/2}}{\delta}\}, \tag{3.13}$$

$$r > \frac{4}{\delta}. \tag{3.14}$$

For any $k \ge 1$, by specification property, we can obtain an x_1^k such that for any $j \in [0, k-1],\ i \in [0, s-1]$, x_1^k $\varepsilon/2$-*traces* p_1 on $[j(s+t)(M+K_{\varepsilon/2})+i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2})+(i+1)M+iK_{\varepsilon/2}]$ and for any $j \in [0, k-1],\ i \in [s, s+t-1]$, x_1^k $\varepsilon/2$-*traces* p_2 on $[j(s+t)(M+K_{\varepsilon/2})+i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2})+(i+1)M+iK_{\varepsilon/2}]$. Also we can obtain an x_2^k such that for any $j \in [0, k-1],\ i \in [0, s-1]$, x_2^k $\varepsilon/2$-*traces* q_1 on $[j(s+t)(M+K_{\varepsilon/2})+i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2})+(i+1)M+iK_{\varepsilon/2}]$ and for any $j \in [0, k-1],\ i \in [s, s+t-1]$, x_2^k $\varepsilon/2$-*traces* q_2 on $[j(s+t)(M+K_{\varepsilon/2})+i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2})+(i+1)M+iK_{\varepsilon/2}]$. We can assume that(take subsequence if necessary) $x_1 = \lim_{k\to\infty} x_1^k,\ x_2 = \lim_{k\to\infty} x_2^k$. By the continuity of f, we have for any $j \in \mathbb{N},\ i \in [0, s-1]$, x_1 $\varepsilon/2$-*traces* p_1 on $[j(s+t)(M+K_{\varepsilon/2})+i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2})+(i+1)M+iK_{\varepsilon/2}]$ and for any $j \in \mathbb{N},\ i \in [s, s+t-1]$, x_1 $\varepsilon/2$-*traces* p_2 on $[j(s+t)(M+K_{\varepsilon/2})+i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2})+(i+1)M+iK_{\varepsilon/2}]$. Similarly, for any $j \in \mathbb{N},\ i \in [0, s-1]$, x_2 $\varepsilon/2$-*traces* q_1 on $[j(s+t)(M+K_{\varepsilon/2})+i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2})+(i+1)M+iK_{\varepsilon/2}]$ and for any $j \in \mathbb{N},\ i \in [s, s+t-1]$, x_2 $\varepsilon/2$-*traces* q_2 on $[j(s+t)(M+K_{\varepsilon/2})+i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2})+(i+1)M+iK_{\varepsilon/2}]$. Set

$N := r(s+t)(M+K_{\varepsilon/2})$, we will show that such N and x_1, x_2 satisfy (**a**) and (**b**). For any $n > N$, n lies in $[k(s+t)(M+K_{\varepsilon/2}), (k+1)(s+t)(M+K_{\varepsilon/2})]$ for some $k \geq r$. By (3.14) and Lemma 3.1, we have

$$d(\mathcal{E}_n(x_1), \mathcal{E}_{k(s+t)(M+K_{\varepsilon/2})}(x_1)) < \frac{\delta}{2}; \quad d(\mathcal{E}_n(x_2), \mathcal{E}_{k(s+t)(M+K_{\varepsilon/2})}(x_2)) < \frac{\delta}{2} \tag{3.15}$$

Note that for any $j \in \mathbb{N}$, $i \in [0, s-1]$, x_1 $\varepsilon/2$-*traces* p_1 on $[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}]$ and for any $j \in \mathbb{N}$, $i \in [s, s+t-1]$, x_1 $\varepsilon/2$-*traces* p_2 on $[j(s+t)(M+K_{\varepsilon/2}) + i(M+K_{\varepsilon/2}), j(s+t)(M+K_{\varepsilon/2}) + (i+1)M + iK_{\varepsilon/2}]$. We have

$$\begin{aligned}
&d(\mathcal{E}_{k(s+t)(M+K_{\varepsilon/2})}(x_1), \theta\mathcal{E}_M(p_1) + (1-\theta)\mathcal{E}_M(p_2)) \\
\leq & d(\sum_{i=1}^{k} \frac{1}{k}\mathcal{E}_{(s+t)(M+K_{\varepsilon/2})}(f^{(i-1)(s+t)(M+K_{\varepsilon/2})}x_1), \theta\mathcal{E}_M(p_1) + (1-\theta)\mathcal{E}_M(p_2)) \\
\leq & \frac{1}{k}\sum_{i=1}^{k} d(\mathcal{E}_{(s+t)(M+K_{\varepsilon/2})}(f^{(i-1)(s+t)(M+K_{\varepsilon/2})}x_1), \theta\mathcal{E}_M(p_1) + (1-\theta)\mathcal{E}_M(p_2)) \\
\leq & \frac{1}{k}\sum_{i=1}^{k} [d(\frac{s}{s+t}\mathcal{E}_{s(M+K_{\varepsilon/2})}(f^{(i-1)(s+t)(M+K_{\varepsilon/2})}x_1), \theta\mathcal{E}_M(p_1)) \\
& + d(\frac{t}{s+t}\mathcal{E}_{t(M+K_{\varepsilon/2})}(f^{[(i-1)(s+t)+s](M+K_{\varepsilon/2})}x_1), (1-\theta)\mathcal{E}_M(p_2))] \\
< & \frac{1}{k}\sum_{i=1}^{k} [\theta(\varepsilon/2 + \delta/2) + (1-\theta)(\varepsilon/2 + \delta/2)] \\
= & \varepsilon/2 + \delta/2.
\end{aligned}$$

Combining with (3.15) and $\mathcal{E}_M(p_i) \in B(\mu_i, \varepsilon/2)$, we have $d(\mathcal{E}_n(x_1), \theta\mu_1 + (1-\theta\mu_2)) < \varepsilon + \delta$. Similarly, we can prove $d(\mathcal{E}_n(x_2), \theta\mu_1 + (1-\theta\mu_2)) < \varepsilon + \delta$. Hence (**a**) holds. Note that $\zeta = \min\{\inf\{d(f^ip_1, f^iq_1)| \ i \in \mathbb{N}\}, \inf\{d(f^ip_2, f^iq_2)| \ i \in \mathbb{N}\}\}$, then we have

$$\frac{|\{i|d(f^ix_1, f^ix_2) < \zeta - \varepsilon\}|}{n} < \frac{1}{k} + \frac{K_{\varepsilon/2}}{M} < \delta.$$

Hence (**b**) holds. □

3.2. **Proof of Theorem E.** We assume that (p_1, q_1), (p_2, q_2) is the distal pair of G_{μ_1}, G_{μ_2} respectively and $\min\{\inf\{d(f^ip_1, f^iq_1)| \ i \in \mathbb{N}\}, \inf\{d(f^ip_2, f^iq_2)| \ i \in \mathbb{N}\}\} = \zeta > 0$. For any non-empty open set U, we can fix an $\varepsilon > 0$ and a transitive point $z \in U$ such that $\overline{B(z,\varepsilon)} \subseteq U$ since transitive points are dense for system with specification property. Let $\varepsilon_i = \varepsilon/2^i$, $K_i = K_{\varepsilon_i}$(cf.definition of specification property). Let $\delta_1 < 1$, $\delta_i = \delta_{i-1}/2$. By [54, Page 944], there exists a sequence $\{\alpha_1, \alpha_2, \cdots\} \subseteq K$ such that

$$\overline{\{\alpha_j : j \in \mathbb{N}^+, j > n\}} = K, \ \forall n \in \mathbb{N}.$$

By Lemma 3.4, for any $s \in \mathbb{N}^+$, we can obtain $x_1^{\varepsilon_s,\delta_s}$, $x_2^{\varepsilon_s,\delta_s}$ and $N^{\varepsilon_s,\delta_s}$ such that for any $n \geq N^{\varepsilon_s,\delta_s}$

$$\mathcal{E}_n(x_1^{\varepsilon_s,\delta_s}) \in B(\mu, \varepsilon_s + \delta_s), \ \mathcal{E}_n(x_2^{\varepsilon_s,\delta_s}) \in B(\mu, \varepsilon_s + \delta_s), \tag{3.16}$$

$$\frac{|\{i \in [0, n-1]|d(f^ix_1^{\varepsilon_s,\delta_s}, f^ix_2^{\varepsilon_s,\delta_s}) < \zeta - \varepsilon\}|}{n} < \delta_s. \tag{3.17}$$

Also, for any $s \in \mathbb{N}^+$, we can obtain an $M_{\varepsilon_s}^{\mu}$ such that the result of Lemma 3.3 holds. Now, giving an $\xi = (\xi_1, \xi_2, \cdots) \in \{1,2\}^\infty$, we construct the x_ξ inductively.

Step 1: *construct* x_{ξ_1}**.** We fix $T_1 = 2K_1$. By Lemma 3.3, for a large enough $M_1 > M_{\varepsilon_1}^{\mu}$ satisfying

$$\delta_1 M_1 > \max\{T_1 + 2K_1, \ N^{\varepsilon_1,\delta_1}\}. \tag{3.18}$$

we can obtain an $x_{\varepsilon_1}^{\alpha_1}$ and $t_2^{\varepsilon_1,\alpha_1} > t_1^{\varepsilon_1,\alpha_1} > M_1$ such that

$$\begin{cases} \mathcal{E}_n(x_{\varepsilon_1}^{\alpha_1}) \in B(\mu,\varepsilon_1),\ \forall n \in [M_{\varepsilon_1}^{\mu}, M_1]; \\ \mathcal{E}_n(x_{\varepsilon_1}^{\alpha_1}) \in B(K,\varepsilon_1),\ \forall n \in [M_1, t_1^{\varepsilon_1,\alpha_1}]; \\ \mathcal{E}_{t_1^{\varepsilon_1,\alpha_1}}(x_{\varepsilon_1}^{\alpha_1}) \in B(\alpha_1,\varepsilon_1); \\ \mathcal{E}_n(x_{\varepsilon_1}^{\alpha_1}) \in B(K,\varepsilon_1),\ \forall n \in [t_1^{\varepsilon_1,\alpha_1}, t_2^{\varepsilon_1,\alpha_1}]; \\ \mathcal{E}_{t_2^{\varepsilon_1,\alpha_1}}(x_{\varepsilon_1}^{\alpha_1}) \in B(\mu,\varepsilon_1). \end{cases} \tag{3.19}$$

Set $T_{1\to 2} = T_1 + t_1^{\varepsilon_1,\alpha_1}$, $T_2 = T_1 + t_2^{\varepsilon_1,\alpha_1}$, $T_3 = T_2 + 2K_1$, T_4 large enough such that

$$\delta_1 T_4 > \max\{T_3 + 2K_2,\ M_{\varepsilon_2}^{\mu}\},\ T_4 - T_3 > N^{\varepsilon_1,\delta_1}. \tag{3.20}$$

By specification property, we can obtain an x_{ξ_1} ε_1-*traces* $z, x_{\varepsilon_1}^{\alpha_1}, x_{\xi_1}^{\varepsilon_1,\delta_1}$ on $[0,0], [T_1,T_2], [T_3,T_4]$ respectively.

Step k: *construct* $x_{\xi_1\cdots\xi_k}$. If $x_{\xi_1\cdots\xi_{k-1}}$, $\{T_i\}_{i=1}^{2k(k-1)}$ and $\{T_{4i-3\to 4i-2}\}_{i=1}^{\frac{k(k-1)}{2}}$ have been defined, we construct $x_{\xi_1\cdots\xi_k}$ in the following way. For any $i \in \{1,2,\cdots,k\}$, let $T_{2k(k-1)+4i-2}$ and $T_{2k(k-1)+4i}$ be indefinite; $T_{2k(k-1)+4i-3} = T_{2k(k-1)+4i-4} + 2K_k$ and $T_{2k(k-1)+4i-1} = T_{2k(k-1)+4i-2} + 2K_k$. By Lemma 3.3, for a large enough $M_{\frac{k(k-1)}{2}+i} > M_{\varepsilon_k}^{\mu}$ satisfying

$$\delta_k M_{\frac{k(k-1)}{2}+i} > \max\{T_{2k(k-1)+4i-3} + 2K_k,\ N^{\varepsilon_k,\delta_k}\}. \tag{3.21}$$

we can obtain an $x_{\varepsilon_k}^{\alpha_i}$ and $t_2^{\varepsilon_k,\alpha_i} > t_1^{\varepsilon_k,\alpha_i} > M_{\frac{k(k-1)}{2}+i}$ such that

$$\begin{cases} \mathcal{E}_n(x_{\varepsilon_k}^{\alpha_i}) \in B(\mu,\varepsilon_k),\ \forall n \in [M_{\varepsilon_k}^{\mu}, M_{\frac{k(k-1)}{2}+i}]; \\ \mathcal{E}_n(x_{\varepsilon_k}^{\alpha_i}) \in B(K,\varepsilon_k),\ \forall n \in [M_{\frac{k(k-1)}{2}+i}, t_1^{\varepsilon_k,\alpha_i}]; \\ \mathcal{E}_{t_1^{\varepsilon_k,\alpha_i}}(x_{\varepsilon_k}^{\alpha_i}) \in B(\alpha_i,\varepsilon_k); \\ \mathcal{E}_n(x_{\varepsilon_k}^{\alpha_i}) \in B(K,\varepsilon_k),\ \forall n \in [t_1^{\varepsilon_k,\alpha_i}, t_2^{\varepsilon_k,\alpha_i}]; \\ \mathcal{E}_{t_2^{\varepsilon_k,\alpha_i}}(x_{\varepsilon_k}^{\alpha_i}) \in B(\mu,\varepsilon_k). \end{cases} \tag{3.22}$$

Set $T_{2k(k-1)+4i-3\to 2k(k-1)+4i-2} = T_{2k(k-1)+4i-3} + t_1^{\varepsilon_k,\alpha_i}$, $T_{2k(k-1)+4i-2} = T_{2k(k-1)+4i-3} + t_2^{\varepsilon_k,\alpha_i}$. If $i < k$, we select $T_{2k(k-1)+4i}$ is large enough such that

$$\delta_k T_{2k(k-1)+4i} > \max\{T_{2k(k-1)+4i-1} + 2K_k,\ M_{\varepsilon_k}^{\mu}\}, \tag{3.23}$$

$$T_{2k(k-1)+4i} - T_{2k(k-1)+4i-1} > N^{\varepsilon_k,\delta_k}. \tag{3.24}$$

If $i = k$, $T_{2k(k-1)+4i}$ is large enough such that

$$\delta_k T_{2k(k-1)+4i} > \max\{T_{2k(k-1)+4i-1} + 2K_{k+1},\ M_{\varepsilon_{k+1}}^{\mu}\}, \tag{3.25}$$

$$T_{2k(k-1)+4i} - T_{2k(k-1)+4i-1} > N^{\varepsilon_k,\delta_k}. \tag{3.26}$$

Hence we have defined the $T_{2(k-1)k+1}, \cdots, T_{2k(k+1)}$ and $T_{2k(k-1)+4i-3\to 2k(k-1)+4i-2}$ $\forall i \in [1,k]$. By specification property, we can obtain an $x_{\xi_1\cdots\xi_k}$ ε_k-*traces* $x_{\xi_1\cdots\xi_{k-1}}, f^{k-1}z, x_{\varepsilon_k}^{\alpha_1}$, $x_{\xi_1}^{\varepsilon_k,\delta_k}, x_{\varepsilon_k}^{\alpha_2}, x_{\xi_2}^{\varepsilon_k,\delta_k}, \cdots, x_{\varepsilon_k}^{\alpha_k}, x_{\xi_k}^{\varepsilon_k,\delta_k}$ on $[0, T_{2k(k-1)}], [T_{2k(k-1)} + K_k, T_{2k(k-1)} + K_k], [T_{2k(k-1)+1}, T_{2k(k-1)+2}], \cdots, [T_{2k(k-1)+4k-1}, T_{2k(k-1)+4k}]$ respectively. Obviously, $d(x_{\xi_1\cdots\xi_{k-1}}, x_{\xi_1\cdots\xi_k}) < \varepsilon_k$, so $\{x_{\xi_1\cdots\xi_k}\}_{k=1}^{\infty}$ is a cauchy sequence in $\overline{B(z,\varepsilon)}$ since $\sum_{i=k}^{+\infty}\varepsilon_i \le 2\varepsilon_k$. Denote the accumulation point of $\{x_{\xi_1\cdots\xi_k}\}_{k=1}^{\infty}$ by x_ξ, and it is easy to verify that x_ξ $2\varepsilon_k$-*traces* $f^{k-1}z, x_{\varepsilon_k}^{\alpha_1}$, $x_{\xi_1}^{\varepsilon_k,\delta_k}, x_{\varepsilon_k}^{\alpha_2}, x_{\xi_2}^{\varepsilon_k,\delta_k}, \cdots, x_{\varepsilon_k}^{\alpha_k}, x_{\xi_k}^{\varepsilon_k,\delta_k}$ on $[T_{2k(k-1)} + K_k, T_{2k(k-1)} + K_k], [T_{2k(k-1)+1}, T_{2k(k-1)+2}], \cdots$, $[T_{2k(k-1)+4k-1}, T_{2k(k-1)+4k}]$ respectively since $\sum_{i=k}^{+\infty}\varepsilon_i \le 2\varepsilon_k$. Note that $orb(x_\xi, f)$ has a subsequence which shadows the orbit of the transitive point z closer and closer, so we can conclude that x_ξ is also a transitive point. Fix $\xi, \eta \in \{1,2\}^\infty$, we claim that $x_\xi \ne x_\eta$ and x_ξ, x_η is a DC1-scrambled pair if $\xi \ne \eta$. Suppose $\xi_s \ne \eta_s$(implied by $\xi \ne \eta$), then for any $k \ge s$ x_ξ $2\varepsilon_k$-*traces* $x_{\xi_s}^{\varepsilon_k,\delta_k}$ on $[T_{2(k-1)k+4s-1}, T_{2(k-1)k+4s}]$ and x_η $2\varepsilon_k$-*traces* $x_{\eta_s}^{\varepsilon_k,\delta_k}$ on $[T_{2(k-1)k+4s-1}, T_{2(k-1)k+4s}]$. For any fixed $\kappa < \zeta$, we can get an $I_\kappa > s$ such that $\zeta - \kappa > 5\varepsilon_{I_\kappa}$. Note that (3.17), we have

$$\frac{|\{i \in [T_{2k(k-1)+4s-1}, T_{2k(k-1)+4s}] | d(f^i x_{\xi_s}^{\varepsilon_k,\delta_k}, f^i x_{\eta_s}^{\varepsilon_k,\delta_k}) < \zeta - \varepsilon_k\}|}{T_{2k(k-1)+4s} - T_{2k(k-1)+4s-1} + 1} < \delta_k < 1$$

holds for any $k \geq I_\kappa$. So we have

$$\frac{|\{i \in [T_{2k(k-1)+4s-1}, T_{2k(k-1)+4s}] | d(f^i x_\xi, f^i x_\eta) < \zeta - 5\varepsilon_k\}|}{T_{2k(k-1)+4s} - T_{2k(k-1)+4s-1} + 1} < \delta_k < 1$$

holds for any $k \geq I_\kappa$, which implies for any $k \geq I_\kappa$, $\exists t \in [T_{2(k-1)k+4s-1}, T_{2(k-1)k+4s}]$ such that $d(f^t x_\xi, f^t x_\eta) \geq \zeta - 5\varepsilon_k > \kappa$. So $x_\xi \neq x_\eta$ and $\{x_\xi\}_{\xi \in \{1,2\}^\infty}$(denote by S) is an uncountable set. Meanwhile,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} |\{j \in [0, n-1] : \ d(f^j x_\xi, f^j x_\eta) < \kappa)\}| \\
\leq &\liminf_{k \geq I_\kappa,\ k\to\infty} \frac{1}{T_{2(k-1)k+4s}} |\{j \in [0, T_{2(k-1)k+4s} - 1] : \ d(f^j x_\xi, f^j x_\eta) < \kappa)\}| \\
\leq &\liminf_{k \geq I_\kappa,\ k\to\infty} \frac{T_{2(k-1)k+4s-1}}{T_{2(k-1)k+4s}} + \delta_k \\
\leq &\liminf_{k \geq I_\kappa,\ k\to\infty} 2\delta_k = 0.
\end{aligned}$$

On the other hand, For any fixed $t > 0$, we can choose $k_t \in \mathbb{N}$ large enough such that $4\varepsilon_k < t$ holds for any $k \geq k_t$. Note that x_ξ and x_η are both $2\varepsilon_k$-*tracea* $x_{\varepsilon_k}^{\alpha_1}$ on $[T_{2(k-1)k+1}, T_{2(k-1)k+2}]$. So

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} |\{j \in [0, n-1] : \ d(f^i x_\xi, f^i x_\eta) < t)\}| \\
\geq &\limsup_{n\to\infty} \frac{1}{n} |\{j \in [0, n-1] : \ d(f^j x_\xi, f^j x_\eta) < 4\varepsilon_{k_t})\}| \\
\geq &\limsup_{k \geq k_t,\ k\to\infty} \frac{1}{T_{2(k-1)k+2}} |\{j \in [0, T_{2(k-1)k+2} - 1] : \ d(f^j x_\xi, f^j x_\eta)) < 4\varepsilon_k\}| \\
\geq &\limsup_{k \geq k_t,\ k\to\infty} (1 - \frac{T_{2(k-1)k+1}}{T_{2(k-1)k+2}}) \\
\geq &\limsup_{k \geq k_t,\ k\to\infty} (1 - \delta_k) \\
= &1.
\end{aligned}$$

So far, we have proved that $S = \{x_\xi\}_{\xi \in \{1,2\}^\infty} \subseteq \overline{B(z, \varepsilon)} \subseteq U$ is an uncountable DC1-scrambled set. To finish this proof, we need to check that $V_f(x_\xi) = K$ for any $\xi \in \{1,2\}^\infty$. On one hand, for any fixed $s \in \mathbb{N}^+$, when $k \geq s$, note (3.21), $T_{2(k-1)k+4s-3 \to 2(k-1)k+4s-2} - T_{2(k-1)k+4s-3} > M_{\frac{k(k-1)}{2}+s}$, and x_ξ $2\varepsilon_k$-*traces* $x_{\varepsilon_k}^{\alpha_s}$ on $[T_{2(k-1)k+4s-3}, T_{2(k-1)k+4s-3 \to 2(k-1)k+4s-2}]$, so we have

$$\begin{aligned}
&d(\mathcal{E}_{T_{2(k-1)k+4s-3 \to 2(k-1)k+4s-2}}(x_\xi),\ \alpha_s) \\
\leq &d(\mathcal{E}_{T_{2(k-1)k+4s-3 \to 2(k-1)k+4s-2} - T_{2(k-1)k+4s-3}}(f^{T_{2(k-1)k+4s-3}} x_\xi),\ \alpha_s) + 2\delta_k \\
\leq &d(\mathcal{E}_{T_{2(k-1)k+4s-2} - T_{2(k-1)k+4s-3}}(x_{\varepsilon_k}^{\alpha_s}),\ \alpha_s) + 2\varepsilon_k + 2\delta_k \\
\leq &\varepsilon_k + 2\varepsilon_k + 2\delta_k \\
= &3\varepsilon_k + 2\delta_k
\end{aligned}$$

by Lemma 3.1. Let $k \to \infty$, we have $\alpha_s \in V_f(x_\xi)$ for any $s \in \mathbb{N}^+$, which implies $K \subseteq V_f(x_\xi)$.

On the other hand, for any fixed $n \in \mathbb{N}^*$, we consider $\mathcal{E}_n(x_\xi)$. Obviously, there is a $k \in \mathbb{N}$ such that $n \in [T_{2(k-1)k+1}, T_{2k(k+1)} + 2K_{k+1}]$. If n lies in $[T_{2(k-1)k+4s-3}, T_{2(k-1)k+4s-2} + 2K_k]$ for certain $s \in \{2, 3, \cdots, k\}$,

$$\begin{aligned}
\mathcal{E}_n(x_\xi) = \quad & \frac{T_{2(k-1)k+4s-3}}{n} \mathcal{E}_{T_{2(k-1)k+4s-3}}(x_\xi) \\
& + \frac{n - T_{2(k-1)k+4s-3}}{n} \mathcal{E}_{n - T_{2(k-1)k+4s-3}}(f^{T_{2(k-1)k+4s-3}} x_\xi).
\end{aligned}$$

Notice that $T_{2(k-1)k+4s-3}=T_{2(k-1)k+4(s-1)}+2K_k$, x_ξ $2\varepsilon_k$*-traces* $x_{\xi_s}^{\varepsilon_k,\delta_k}$ on $[T_{2(k-1)k+4(s-1)-1}, T_{2(k-1)k+4(s-1)}]$ and (3.16),(3.23), so by Lemma 3.1, we have

$$\begin{aligned}d(\mathcal{E}_{T_{2(k-1)k+4s-3}}(x_\xi),\mu) &< d(\mathcal{E}_{T_{2(k-1)k+4(s-1)}-T_{2(k-1)k+4(s-1)-1}}(f^{T_{2(k-1)k+4(s-1)-1}}x_\xi),\ \mu)+2\delta_k\\ &< d(\mathcal{E}_{T_{2(k-1)k+4(s-1)}-T_{2(k-1)k+4(s-1)-1}}(x_{\xi_s}^{\varepsilon_k,\delta_k}),\ \mu)+2\varepsilon_k+2\delta_k\\ &< \varepsilon_k+\delta_k+2\varepsilon_k+2\delta_k,\end{aligned}$$

i.e.,

$$d(\mathcal{E}_{T_{2(k-1)k+4s-3}}(x_\xi),\mu)<3\varepsilon_k+3\delta_k. \tag{3.27}$$

If $n\in[T_{2(k-1)k+4s-3}, T_{2(k-1)k+4s-3}+M_{\varepsilon_k}^\mu]$, note that (3.21) and $M_{\frac{2k(k-1)}{2}+s}>M_{\varepsilon_k}^\mu$, then we have $d(\mathcal{E}_n(x_\xi),\mathcal{E}_{T_{2(k-1)k+4s-3}}(x_\xi))<2\delta_k$ by Lemma 3.1. So,

$$d(\mathcal{E}_n(x_\xi),\mu)<2\delta_k+3\varepsilon_k+3\delta_k=3\varepsilon_k+5\delta_k. \tag{3.28}$$

If $n\in[T_{2(k-1)k+4s-3}+M_{\varepsilon_k}^\mu, T_{2(k-1)k+4s-3}+M_{\frac{2k(k-1)}{2}+s}]$, by (3.22), one has

$$\begin{aligned}d(\mathcal{E}_{n-T_{2(k-1)k+4s-3}}(f^{T_{2(k-1)k+4s-3}}x_\xi),\mu) &< d(\mathcal{E}_{n-T_{2(k-1)k+4s-3}}(x_{\varepsilon_k}^{\alpha_s}),\mu)+2\varepsilon_k\\ &<\varepsilon_k+2\varepsilon_k\\ &=3\varepsilon_k.\end{aligned}$$

Combine with (3.27), one has

$$d(\mathcal{E}_n(x_\xi),\mu)<3\varepsilon_k+3\delta_k. \tag{3.29}$$

If $n\in[T_{2(k-1)k+4s-3}+M_{\frac{2k(k-1)}{2}+s}, T_{2(k-1)k+4s-2}+2K_k]$, by (3.21) and Lemma 3.1, we have

$$d(\mathcal{E}_n(x_\xi),\mathcal{E}_{n-T_{2(k-1)k+4s-3}}(f^{T_{2(k-1)k+4s-3}}x_\xi))<2\delta_k. \tag{3.30}$$

Then $\mathcal{E}_n(x_\xi)\in B(K,\varepsilon_k+2\delta_k)$ by (3.22). So $\mathcal{E}_n(x_\xi)\subseteq B(K,3\varepsilon_k+5\delta_k)$ when $n\in[T_{2(k-1)k+4s-3}, T_{2(k-1)k+4s-2}+2K_k]$. In other situations of the interval where n lies, we can also prove $\mathcal{E}_n(x_\xi)\subseteq B(K,3\varepsilon_k+5\delta_k)$ with a little modification of the method above. When $n\to\infty$, forcing $k\to\infty$, $B(K,3\varepsilon_k+5\delta_k)\to K$, hence we have $\mathcal{E}_n(x_\xi)=K$. □

Remark 3.5. Theorem E just states the situation where K contains a measure μ which is the convex combination of two measures. Actually, with little modification, Theorem E also holds for any $K\subseteq\mathcal{M}_f(X)$ if K contains a measure μ which is the convex combination of finite measures. Here we omit it.

4. Proof of Theorems D, C and 1.6

Similar as [70], [35] and [25], we also deal with many refined recurrent levels which will used not only to prove Theorems D and C but also to show Theorem 1.6. Now let us recall their definitions. Given $x\in X$, let $C_x=\overline{\bigcup_{m\in V_f(x)}S_m}$. Let $BR^\#:=BR\setminus QW$,

$$\begin{aligned}W^\# &:= \{x\in BR^\#\mid S_\mu=C_x \text{ for every } \mu\in V_f(x)\},\\ V^\# &:= \{x\in BR^\#\mid \exists\mu\in V_f(x) \text{ such that } S_\mu=C_x\},\\ S^\# &:= \{x\in X\mid \cap_{\mu\in V_f(x)}S_\mu\neq\emptyset\}.\end{aligned}$$

Then we can divide $BR^\#$ into following several levels with different asymptotic behavior:

$$\begin{aligned}BR_1 &:= W^\#,\\ BR_2 &:= V^\#\cap S^\#,\quad BR_3 := V^\#,\\ BR_4 &:= V^\#\cup(BR^\#\cap S^\#),\quad BR_5 := BR^\#.\end{aligned}$$

Immediately, $BR_1\subseteq BR_2\subseteq BR_3\subseteq BR_4\subseteq BR_5$. Denote

$$\begin{aligned}W^* &:= \{x\in QW\mid S_\mu=C_x \text{ for every } \mu\in V_f(x)\},\\ V &:= \{x\in QW\mid \exists\mu\in V_f(x) \text{ such that } S_\mu=C_x\},\\ S &:= \{x\in X\mid \cap_{\mu\in V_f(x)}S_\mu\neq\emptyset\}.\end{aligned}$$

Later, we will see that $W^* = W$. Now we can divide QW into following several levels with different asymptotic behavior:

$$\begin{aligned} QW_1 &:= W^*, \\ QW_2 &:= V \cap S, \quad QW_3 := V, \\ QW_4 &:= V \cup (QW \cap S), \quad QW_5 := QW. \end{aligned}$$

These levels are related the different statistical ω-limit sets, see Section 1.4.

For a collection of subsets $Z_1, Z_2, \cdots, Z_k \subseteq X (k \geq 2)$, we denote $\mathrm{GS}\{Z_1, Z_2, \cdots, Z_k\} = \{Z_2 \setminus Z_1, Z_3 \setminus Z_2, \cdots, Z_k \setminus Z_{k-1}\}$ the gap sets of the sequence.

Definition 4.1. We say $\{Z_i\}_{i=1}^k$ has uncountable DC1-scrambled gap with respect to $Y (\subseteq X)$ if $S \cap Y$ contains an uncountable DC1-scrambled subset for any $S \in \mathrm{GS}\{Z_1, Z_2, \cdots, Z_k\}$.

Theorem 4.2. *Suppose that (X, f) has specification property, φ is a continuous function on X. Then*

- **(a):** *If (X, f) is not uniquely ergodic, $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Trans$;*
- **(b):** *If $I_\varphi \neq \emptyset$, then $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Trans \cap I_\varphi$;*
- **(c):** *If $Int(L_\varphi) \neq \emptyset$, then for any $a \in Int(L_\varphi)$, $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Trans \cap R_\varphi(a)$.*
- **(d):** *If (X, f) is not uniquely ergodic, $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Trans \cap R_\varphi$.*

Remark 4.3. If $a \in L_\varphi \setminus Int(L_\varphi)$, Theorem 4.2 may be not true even for Li-Yorke chaotic. For example, if (X, f) is full shift of two symbols (which satisfies specification), taking $orb(p, f), orb(q, f)$ to be two different periodic orbits with period ≥ 2 and letting φ be a continuous function such that $\varphi|_{orb(p,f)} = 0$, $\varphi|_{orb(q,f)} = 1$ and for any $x \in X \setminus (orb(p, f) \cup orb(q, f))$, $0 < \varphi(x) < 1$. In this case $L_\phi = [0, 1]$. Let μ_p, μ_q denote the periodic measures supported on the orbit of p, q. It is not difficult to check that $G_{\mu_p} \cap Trans \subseteq R_\phi(0) \cap Trans \subseteq BR_1$ and $G_{\mu_q} \cap Trans \subseteq R_\phi(1) \cap Trans \subseteq BR_1$ so that $\{QW_1, QW_2, QW_3, QW_4, QW_5\}$ and $\{BR_1, BR_2, BR_3, BR_4, BR_5\}$ have empty gap with respect to $R_\phi(0) \cap Trans$ and $R_\phi(1) \cap Trans$. So most cases can not have any kind of chaotic behavior with respect to $R_\phi(0) \cap Trans$ and $R_\phi(1) \cap Trans$. By Theorem E G_{μ_p}, G_{μ_q} all contain uncountable DC1-scrambled subsets and so do $R_\phi(0) \cap Trans \cap BR_1, R_\phi(1) \cap Trans \cap BR_1$. However, $R_\phi(0)$ and $R_\phi(1)$ has zero topological entropy by (1.1). In particular, this implies that there exists an uncountable DC1-scrambled set with zero topological entropy.

Theorem 4.2 implies Theorems D and C so that we only need to prove Theorem 4.2 .

4.1. Distal Pair in Minimal Sets.

Proposition 4.4. *Suppose that (X, f) has specification property, then*

$$\{\mu \in \mathcal{M}_f(X) | \mu \text{ is ergodic, } S_\mu \text{ is nondegenerate and minimal}\}$$

is dense in $\mathcal{M}_f(X)$ and for any μ in such set, G_μ has distal pair.

To prove Proposition 4.4, we need some preliminaries. An infinite set $A = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}$ is syndetic if there is an $N \in \mathbb{N}$ such that $a_{i+1} - a_i \leq N$ holds for any $i \in \mathbb{N}$. Denote $\mathcal{D}(A) = \min\{N \in \mathbb{N} \mid a_{i+1} - a_i \leq N$ holds for any $i \in \mathbb{N}\}$ and $\mathcal{F}_s = \{A \subseteq \mathbb{N} | A$ is syndetic$\}$.

Lemma 4.5. *Given (X, f), for any $p, q \in X$, if there is an $\varepsilon > 0$ such that $\{i \mid d(f^i p, f^i q) > \varepsilon\} \in \mathcal{F}_s$. Then p, q is distal.*

Proof. Suppose p, q, ε is fixed, $\mathcal{D}(\{i \mid d(f^i p, f^i q) > \varepsilon\}) = M$. Obviously f is uniform continuous since f is continuous and X is compact. So we can get η_1 such that for any $x, y \in X$, if $d(x, y) < \eta_1$, then $d(fx, fy) < \varepsilon$. By induction, we get η_k such that for any $x, y \in X$, if $d(x, y) < \eta_k$, then $d(fx, fy) < \eta_{k-1}$, until $k = M$. Set $\eta = \min\{\varepsilon, \eta_1, \eta_2, \cdots, \eta_M\}$, we claim that $\liminf_{n\to\infty} d(f^n p, f^n q) \geq \eta$. If not, there is an $n_0 \in \mathbb{N}$ such that $d(f^{n_0} p, f^{n_0} q) < \eta$. By the discussion above, we have $d(f^{n_0+k} p, f^{n_0+k} q) < \varepsilon$ for any $k \in \{0, 1, \cdots, M\}$, which conflicts with $\mathcal{D}(\{i \mid d(f^i p, f^i q) > \varepsilon\}) = M$. □

Lemma 4.6. *Given (X, f). Suppose that $\mu \in \mathcal{M}_f^e(X)$, S_μ is nondegenerate and minimal. Then, there are two distinct points $p, q \in G_\mu$ such that p, q is distal.*

Proof. By the hypothesis, we can choose two distinct points $u, v \in S_\mu$. Denote B_u, B_v the open neighborhood of u, v respectively. Here we can assume $B_u \cap B_v = \emptyset$ and $d(B_u, B_v) = \zeta > 0$ since X is a metric space. Obviously $\mu(S_\mu) = 1, \mu(B_u) > 0, \mu(B_v) > 0$. Notice that μ is ergodic, so $\mu(G_\mu) = 1$ and there exists an $M \in \mathbb{N}$ such that $\mu(B_u \cap f^{-M}B_v) > 0$. So $\mu(B_u \cap f^{-M}B_v \cap G_\mu \cap S_\mu) > 0$. Fix a $p \in B_u \cap f^{-M}B_v \cap G_\mu \cap S_\mu$, then $N(p, B_u \cap f^{-M}B_v) = \{a_1 < a_2 < \cdots\} \in \mathcal{F}_s$ since $p \in S_\mu$ is a minimal point and $B_u \cap f^{-M}B_v$ is an open neighborhood of p. Set $q = f^M p$, for any $k \in \mathbb{N}$, we have $f^{a_k}q \in B_v$ since $f_{a_k}p \in B_u \cap f^{-M}B_v$. So $d(f^{a_k}p, f^{a_k}q) \geq \zeta > 0$. Notice that $\{a_1, a_2, \cdots\} \in \mathcal{F}_s$, so p, q is distal by lemma 4.5. $p \in G_\mu \Rightarrow q \in G_\mu$. □

Proof of Proposition 4.4 For system (X, f) with specification, we have

$$\{\mu \in \mathcal{M}_f(X) | \mu \textit{ is ergodic, } S_\mu \textit{ is minimal}\}$$

is dense in $\mathcal{M}_f(X)$, which is a direct corollary of [35, Theorem A]. Here we claim that

$$\{\mu \in \mathcal{M}_f(X) | \mu \textit{ is ergodic, } S_\mu \textit{ is nondegenerate and minimal}\}$$

is also dense in $\mathcal{M}_f(X)$. If not, there will be a open set $U \subseteq \mathcal{M}_f(X)$ such that

$$\{\mu \in \mathcal{M}_f(X) | \mu \textit{ is ergodic, } S_\mu \textit{ is degenerate and minimal}\}$$

is dense in U, which implies that any measure in U can be approximated by the Dirac measure concentrate on a fix point. i.e. for any $\mu \in U$, there is a sequence $\{x_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} \delta_{x_i} = \mu$. Without loss of generality, we can assume that $\lim_{i\to\infty} x_i = x$. Then for any continuous function f on X,

$$\int f d\mu = \lim_{i\to\infty} \int f d\delta_{x_i} = \lim_{i\to\infty} f(x_i) = f(x) = \int f d\delta_x.$$

So we have $\mu = \delta_x$, which means measures in U are all Dirac measures, which conflict with Proposition 2.3. So the conflict and Lemma 4.6 end this proof. □

4.2. **Proof of Theorem 4.2.** For any $\mu_1, \mu_2 \in \mathcal{M}_f(X)$, we define

$$\mathrm{cov}\{\mu_1, \mu_2\} = \{\theta\mu_1 + (1-\theta)\mu_2 |\ \theta \in [0, 1]\}.$$

Proof of Item (a). By [35, Lemma 3.4], we can take $\mu_1, \mu_2, \cdots$ satisfying Proposition 4.4 and $\overline{\bigcup_{i=1}^\infty S_{\mu_i}} = X$. Then their support are naturally mutually disjoint and for any finite set $\Lambda \subseteq \mathbb{N}^+$, $\bigcup_{i\in\Lambda} S_{\mu_i} \neq X$ since S_{μ_i} is minimal. Let μ be a measure with full support and take $\nu_i = \frac{i-1}{i}\mu_1 + \frac{1}{i}\mu_i,\ i \in \{1, 2, \cdots\}$. then we have

$$\begin{aligned}
\lim_{i\to\infty} d(\nu_i, \mu_1) &= \lim_{i\to\infty} d(\frac{i-1}{i}\mu_1 + \frac{1}{i}\mu_i, \frac{i-1}{i}\mu_1 + \frac{1}{i}\mu_1), \\
&\leq \lim_{i\to\infty} \frac{1}{i} d(\mu_i, \mu_1), \\
&\leq \lim_{i\to\infty} \frac{1}{i}, \\
&= 0.
\end{aligned} \tag{4.1}$$

Here we consider $\bigcup_{i=1}^\infty \mathrm{cov}\{\nu_i, \nu_{i+1}\}$. By (4.1), it is easy to check that $\bigcup_{i=1}^\infty \mathrm{cov}\{\nu_i, \nu_{i+1}\}$ is connected and compact. One can observe that $S_\kappa \neq X$ for any $\kappa \in \bigcup_{i=1}^\infty \mathrm{cov}\{\nu_i, \nu_{i+1}\}$. Moreover, $\bigcap_{\kappa \in \bigcup_{i=1}^\infty \mathrm{cov}\{\nu_i,\nu_{i+1}\}} S_\kappa =$

S_{μ_1} and $\overline{\bigcup_{\kappa\in\bigcup_{i=1}^{\infty}\mathrm{cov}\{\nu_i,\nu_{i+1}\}} S_\kappa}=X$. Let

$$\begin{aligned}
K_1 &:= \mathrm{cov}\{\mu_1,\mu\},\\
K_2 &:= \mathrm{cov}\{\mu_1,\mu\}\cup\mathrm{cov}\{\mu_1,\mu_2\},\\
K_3 &:= \bigcup_{i=1}^{\infty}\mathrm{cov}\{\nu_i,\nu_{i+1}\},\\
K_4 &:= \bigcup_{i=1}^{\infty}\mathrm{cov}\{\nu_i,\nu_{i+1}\}\cup\mathrm{cov}\{\mu_1,\mu_2\},\\
K_5 &:= \{\mu_1\},\\
K_6 &:= \mathrm{cov}\{\mu_1,\nu_2\},\\
K_7 &:= \mathrm{cov}\{\mu_1,\mu_2\},\\
K_8 &:= \mathrm{cov}\{\mu_1,\nu_2\}\cup\mathrm{cov}\{\mu_1,\nu_3\},\\
K_9 &:= \mathrm{cov}\{\mu_1,\mu_2\}\cup\mathrm{cov}\{\mu_1,\mu_3\}.
\end{aligned}$$

Using Theorem E on $K_i, i\in\{1,2,3,4,5,6,7,8,9\}$, for any open set U, there is an uncountable scramble set $S_i\subseteq G_{K_i}\cap U\cap Trans$. By Proposition 2.6 and Proposition 2.5(c), we have $S_i\subseteq Trans$ implies $S_i\subseteq QW$ for $i\in\{1,2,3,4\}$ and $S_i\subseteq BR^{\#}$ for $i\in\{5,6,7,8,9\}$. One can observe that $G_{K_i}\subseteq QW_{i+1}\setminus QW_i$ for any $i\in\{1,2,3,4\}$, $G_{K_5}\subseteq BR_1\setminus QW_5$, $G_{K_i}\subseteq BR_{i-4}\setminus BR_{i-5}$ for any $i\in\{6,7,8,9\}$. Then the proof is completed. □

Remark 4.7. Let $QR=\bigcup_{\mu\in\mathcal{M}_f(X)}G_\mu$. The points in QR are called quasiregular points of f in [21]. Note that K_5 in the proof above is a single measure, so we can replace BR_1 by $BR_1\cap QR$ and the theorem still holds. The same situation will happen in the proof of item (c).

Proof of Item (b). If $I_\varphi(f)\neq\emptyset$, then there exist $\lambda_1,\lambda_2\in\mathcal{M}_f(X)$ such that $\int\varphi d\lambda_1\neq\int\varphi d\lambda_2$. Note that the measure satisfying Proposition 4.4 and measures with full support are both dense in $\mathcal{M}_f(X)$. Then we can choose $\mu_1,\mu_2,\cdots$ satisfying Proposition 4.4 and $\overline{\bigcup_{i=1}^{\infty}S_{\mu_i}}=X$ such that $\int\varphi d\mu_1\neq\int\varphi d\mu_2\neq\int\varphi d\mu_3\neq\int\varphi d\mu$. Take $\nu_i=\frac{i-1}{i}\mu_1+\frac{1}{i}\mu_i,\ i\in\{1,2,\cdots\}$. Let

$$\begin{aligned}
K_1 &:= \mathrm{cov}\{\mu_1,\mu\},\\
K_2 &:= \mathrm{cov}\{\mu_1,\mu\}\cup\mathrm{cov}\{\mu_1,\mu_2\},\\
K_3 &:= \bigcup_{i=1}^{\infty}\mathrm{cov}\{\nu_i,\nu_{i+1}\},\\
K_4 &:= \bigcup_{i=1}^{\infty}\mathrm{cov}\{\nu_i,\nu_{i+1}\}\cup\mathrm{cov}\{\mu_1,\mu_2\},\\
K_5 &:= \mathrm{cov}\{\nu_2,\frac{1}{3}\mu_1+\frac{2}{3}\mu_2\},\\
K_6 &:= \mathrm{cov}\{\mu_1,\nu_2\},\\
K_7 &:= \mathrm{cov}\{\mu_1,\mu_2\},\\
K_8 &:= \mathrm{cov}\{\mu_1,\nu_2\}\cup\mathrm{cov}\{\mu_1,\nu_3\},\\
K_9 &:= \mathrm{cov}\{\mu_1,\mu_2\}\cup\mathrm{cov}\{\mu_1,\mu_3\}.
\end{aligned}$$

One can observe that $G_{K_i}\subseteq I_\varphi(f)$ for any $i\in\{1,2,3,4,5,6,7,8,9\}$. Based on the discussion in the proof of item (a), we complete the proof. □

Proof of Item (c). If $Int(L_\varphi)\neq\emptyset$, then for any $a\in Int(L_\varphi)$, there exist $\lambda_1,\lambda_2\in\mathcal{M}_f(X)$ such that $\int\varphi d\lambda_1<a<\int\varphi d\lambda_2$. By [35, Lemma 3.4], we can take $\mu_1,\mu_2,\cdots$ satisfying Proposition 4.4 and $\overline{\bigcup_{i=1}^{\infty}S_{\mu_i}}=X$. We can assume that $\{i\in[1,+\infty)|\ \int\varphi d\mu_i>a\}$ and $\{i\in[1,+\infty)|\ \int\varphi d\mu_i<a\}$ are both infinite set since measures satisfying Proposition 4.4 are dense in $\mathcal{M}_f(x)$. Set $\{i\in[1,+\infty)|\ \int\varphi d\mu_i>$

$a\} = \{m_i\}_{i=1}^{\infty}$ and $\{i \in [1,+\infty) | \int \varphi d\mu_i < a\} = \{n_i\}_{i=1}^{\infty}$. In order to simplify the proof, we assume $\{i \in [1,+\infty) | \int \varphi d\mu_i = a\} = \emptyset$. Now, we can choose proper $\{\theta_i\}_{i=1}^{\infty} \subseteq (0,1)$ such that $\nu_i = \theta_i \mu_{m_i} + (1-\theta)\mu_{n_i}$ and $\int \varphi d\nu_i = a$ for any $i \in \{1,2,\cdots\}$. We can also choose proper $\kappa_1, \kappa_2, \in (0,1)$ such that $\rho_1 = \kappa_1 \mu_{m_1} + (1-\kappa_1)\mu_{n_2}, \rho_2 = \kappa_2 \mu_{m_1} + (1-\kappa_2)\mu_{n_3}$ and $\int \varphi d\rho_1 = \int \varphi d\rho_2 = a$. By proposition 2.3, there are μ^*, μ^{**} with full support such that $\int \varphi d\mu^* < a < \int \varphi d\mu^{**}$. Choosing proper $\iota \in (0,1)$ such that $\mu = \iota\mu^* + (1-\iota)\mu^{**}$ and $\int \varphi d\mu = a$. Take $\omega_i = \frac{i-1}{i}\nu_1 + \frac{1}{i}\nu_i,\ i \in \{1,2,\cdots\}$. Let

$$\begin{aligned}
K_1 &:= \operatorname{cov}\{\nu_1, \mu\},\\
K_2 &:= \operatorname{cov}\{\nu_1, \mu\} \cup \operatorname{cov}\{\nu_1, \nu_2\},\\
K_3 &:= \bigcup_{i=1}^{\infty} \operatorname{cov}\{\omega_i, \omega_{i+1}\},\\
K_4 &:= \bigcup_{i=1}^{\infty} \operatorname{cov}\{\omega_i, \omega_{i+1}\} \cup \operatorname{cov}\{\omega_1, \nu_2\},\\
K_5 &:= \{\nu_1\},\\
K_6 &:= \operatorname{cov}\{\nu_1, \rho_1\},\\
K_7 &:= \operatorname{cov}\{\nu_1, \nu_2\},\\
K_8 &:= \operatorname{cov}\{\nu_1, \rho_1\} \cup \operatorname{cov}\{\nu_1, \rho_2\},\\
K_9 &:= \operatorname{cov}\{\nu_1, \nu_2\} \cup \operatorname{cov}\{\nu_2, \nu_3\}.
\end{aligned}$$

One can observe that $G_{K_i} \subseteq R_\varphi(a)$, for any $i \in \{1,2,3,4,5,6,7,8,9\}$. Based on the discussion in the proof of item (a), we complete the proof. □

Proof of Item (d). If $Int(L_\varphi) \neq \emptyset$, then one can get this from item (c) by taking one $a \in Int(L_\varphi)$ since $R_\varphi(a) \subseteq R_\varphi$. On the other hand, $Int(L_\varphi) = \emptyset$, then $R_\varphi = X$ and one can get this from item(a). □

Remark 4.8. If (X, f) is not uniquely ergodic, there are two different invariant measures μ, ν so that by weak* topology there exists a continuous function ϕ such that $\int \phi d\mu \neq \int \phi d\nu$. Thus $Int(L_\phi) \neq \emptyset$. Note that $I_\phi(f) \neq \emptyset$ is equivalent to $Int(L_\phi) \neq \emptyset$ if the system has specification property, see [69]. Thus item (a) can also be deduced from item (b).

4.3. **Proof of Theorem 1.6.** The proof is based on [35, Theorem H]. From the proof of [35, Theorem H], we know that

$$x \in BR \Leftrightarrow x \in \omega_{B^*}(x) \text{ and } x \in QW \Leftrightarrow x \in \omega_{\overline{d}}(x).$$

The construction of x in the proof of Theorem 4.2 always satisfies that $x \in Trans \cap BR$, which implies $\omega_{B^*}(x) = \omega_f(x) = X$ by [35, Lemma 4.6]. Since the dynamical systems with specification are not minimal but minimal points are dense, so for any $x \in Trans,\ \omega_{B_*}(x) = \emptyset$. Thus one can check that the uncountable DC1-scrambled sets constructed by K_1, K_2, K_5, K_6, K_7 in the proof of Theorem 4.2 satisfy the five cases, which ends the proof. □

5. Applications

5.1. **Examples with Specification.** It is known from [15] that any topologically mixing interval map satisfies specification. For example, [36] showed that there exists a set of parameter values $\Lambda \subseteq [0,4]$ of positive Lebesgue measure such that if $\lambda \in \Lambda$, then the logistic map $f_\lambda(x) = \lambda x(1-x)$ is topological mixing.

Moreover, maps satisfying the specification property includes the mixing subshift of finite type, mixing sofic subshift, topological mixing uniformly hyperbolic systems and the time-1 map of the geodesic flow of compact connected negative curvature manifolds, for example, see [58, 69].

So, all the results of Theorems A - E are all suitable for such systems.

5.2. **Examples Without Specification.** Now, we use our theorem on a type of subshift which may not have specification property. Before the statement, we need some preparations.

For any finite alphabet A, the *full symbolic space* is the set $A^{\mathbb{Z}} = \{\cdots x_{-1}x_0x_1\cdots : x_i \in A\}$, which is viewed as a compact topological space with the discrete product topology. The set $A^{\mathbb{N}_+} = \{x_1x_2\cdots : x_i \in A\}$ is called *one-side full symbolic space*. The *shift action* on *one-side full symbolic space* is defined by

$$\sigma: \ A^{\mathbb{N}_+} \to A^{\mathbb{N}_+}, \quad x_1x_2\cdots \mapsto x_2x_3\cdots.$$

$(A^{\mathbb{N}_+}, \sigma)$ forms a dynamical system under the discrete product topology which we called a shift. A closed subset $X \subseteq A^{\mathbb{N}_+}$ is called *subshift* if it is invariant under the shift action σ. $\mathbf{w} \in A^n \triangleq \{x_1x_2\cdots x_n : x_i \in A\}$ is a *word* of *subshift* X if there is an $x \in X$ and $k \in \mathbb{N}$ such that $\mathbf{w} = x_kx_{k+1}\cdots x_{k+n-1}$. Here we call n the length of $\mathbf{w}$ denoted by $|\mathbf{w}|$. The *language* of a subshift X, denoted by $\mathcal{L}(X)$, is the set of all words of X. Denote $\mathcal{L}_n(X) \triangleq \mathcal{L}(X)\bigcap A^n$ all the words of X with length n.

Now we introduce the typical subshift of one-side full shift space β-shift. Basic references are [56, 59, 53]. It is worth mentioning that from [15] the set of parameters of β for which specification holds, is dense in $(1, +\infty)$ but has Lebesgue zero measure.

Let $\beta > 1$ be a real number. We denote by $[x]$ and $\{x\}$ the integer and fractional part of the real number x. Considering the β-*transformation* $f_\beta : [0, 1) \to [0, 1)$ given by

$$f_\beta(x) = \beta x \pmod 1$$

For $\beta \notin \mathbb{N}$, let $b = [\beta]$ and for $\beta \in \mathbb{N}$, let $b = \beta - 1$. Then we split the interval $[0, 1)$ into $b + 1$ partition as below

$$J_0 = \left[0, \frac{1}{\beta}\right), \ J_1 = \left[\frac{1}{\beta}, \frac{2}{\beta}\right), \cdots, \ J_1 = \left[\frac{b}{\beta}, 1\right).$$

For $x \in [0, 1)$, let $i(x, \beta) = (i_n(x, \beta))_1^\infty$ be the sequence given by $i_n(x, \beta) = j$ when $f^{n-1}x \in J_j$. We call $i(x, \beta)$ the greedy β-*expansion* of x and we have

$$x = \sum_{n=1}^{\infty} i_n(x, \beta)\beta^{-n}.$$

We call (Σ_β, σ) β-shift, where σ is the shift map, Σ_β is the closure of $\{i(x, \beta)\}_{x\in[0,1)}$ in $\prod_{i=1}^{\infty}\{0, 1, \cdots, b\}$.

From the discussion above, we can also define the greedy β-*expansion* of 1, denoted by $i(1, \beta)$. Parry showed that the set of sequence with belong to Σ_β can be characterised as

$$\omega \in \Sigma_\beta \Leftrightarrow f^k(\omega) \le i(1, \beta) \text{ for all } k \ge 1,$$

where $\le$ is taken in the lexicographic ordering [50]. By the definition of Σ_β above, $\Sigma_{\beta_1} \subsetneq \Sigma_{\beta_2}$ for $\beta_1 < \beta_2$([50]). Now we introduce some lemmas about β-shift, which indicate that β-shift has a certain degree of transitive property.

Lemma 5.1. *For any $\mathbf{w} \in \mathcal{L}_n(\Sigma_\beta)$, if there is a $j \in [1, n]$ such that $\mathbf{w}_j \neq 0$, then for any $\eta \in \Sigma_\beta$, $\mathbf{w}_1\cdots(\mathbf{w}_j - 1)\cdots\mathbf{w}_n\eta \in \Sigma_\beta$.*

The proof is a easy part of [53, Proposition 5.1].

Lemma 5.2. *For any $\omega \in \Sigma_\beta$ and any open set $U \subseteq \Sigma_\beta$, we can find an $\eta \in U$ and a $k \in \mathbb{N}$ such that $\sigma^k\eta = \omega$.*

Proof. U is open, so we can find a point $\xi = \xi_1\xi_2\cdots \in U$ such that $\xi < i(1, \beta)$. So we can find a $k \in \mathbb{N}$ large enough, such that $\xi' \triangleq \xi_1\xi_2\cdots(\xi_k + 1)\xi_{k+1}\xi_{k+2}\cdots < i(1, \beta)$ and $\xi' \in U$. Then by Lemma 5.1, we conclude that $\eta \triangleq \xi_1\xi_2\cdots\xi_k\omega \in U$ and $\sigma^k\eta = \omega$. □

Lemma 5.3. *For β-shift, there exists an increasing sequence $\{\Sigma_\beta^n\}$ of compact σ-invariant subsets of Σ_β with the following properties:*

(a) *Each $\{\Sigma_\beta^n\}$ is a sofic shift and has specification property*

(b) *For any $\mu \in \mathcal{M}_f(\Sigma_\beta)$, and any neighborhood U of μ in $\mathcal{M}_f(\Sigma_\beta)$, there exist $n \ge 1$ and $\mu' \in \mathcal{M}_f^e(\Sigma_\beta^n)\bigcap U$.*

Lemma 5.3 is a main application in [17]. Reader can refer to [17] for the details of the proof. The lemma above shows us that to figure out the irregular set for the whole space(Σ_β), it is sufficient to study the irregular set for certain asymptotic 'horseshoe-like'(Σ_β^n) of the whole space.

Theorem 5.4. *For any $\beta > 1$ and (Σ_β, σ), suppose φ is a continuous function on Σ_β, then*

- **(a):** $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ *has uncountable DC1-scrambled gap with respect to $Rec(\sigma)$;*
- **(b):** *If $I_\varphi(\sigma) \neq \emptyset$, then $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Rec(\sigma) \cap I_\varphi(\sigma)$;*
- **(c):** *If $Int(L_\varphi) \neq \emptyset$, then for any $a \in Int(L_\varphi)$, $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Rec(\sigma) \cap R_\varphi(a)$.*
- **(d):** $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ *has uncountable DC1-scrambled gap with respect to $Rec(\sigma) \cap R_\varphi$.*

Proof. **(a)**: Refer to [59], we have $\{\beta \in (1, +\infty) \mid (\Sigma_\beta, \sigma)$ has specification property$\}$ is dense in $(1, +\infty)$. Then for any $\beta > 1$, we can find an $\alpha < \beta$ such that (Σ_α, σ) has specification property. By Theorem 4.2, for (Σ_α, σ), we have $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1,$
$BR_2, BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Trans_{\sigma|_{\Sigma_\alpha}}$. Note that $\Sigma_\alpha \subsetneq \Sigma_\beta$, so the transitive points of Σ_α must be the recurrent points of Σ_β. Moreover, it is easy to see that for any $S_a \in \mathrm{GS}\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2,$
$BR_3, BR_4, BR_5\}$ for system (Σ_α, σ) is a subset of some $S_b \in \mathrm{GS}\{QW_1, QW_2, QW_3, QW_4,$
$QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ for system (Σ_β, σ). Then item**(a)** has been proved.

(b): If $I_\varphi(\sigma) \neq \emptyset$, there exist $\lambda_1, \lambda_2 \in \mathcal{M}_\sigma(\Sigma_\beta)$ such that $\int \varphi d\lambda_1 \neq \int \varphi d\lambda_2$. By Lemma 5.3, we have (Σ_β^n, σ) which has specification property and $\mu_1, \mu_2 \in \mathcal{M}_\sigma(\Sigma_\beta^n)$ such that $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$. By the proof of Theorem 4.2, for (Σ_β^n, σ), we have $\{QW_1, QW_2, QW_3, QW_4,$
$QW_5, BR_1, BR_2, BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Trans_{\sigma|_{\Sigma_\beta^n}} \cap I_\varphi(\sigma)$. Similarly, item**(b)** has been proved.

(c): If $Int(L_\varphi) \neq \emptyset$, then for any $a \in Int(L_\varphi)$, there exist λ_1, λ_2 such that $\int \varphi d\lambda_1 < a < \int \varphi d\mu_2$. By Lemma 5.3, we have (Σ_β^n, σ) which has specification property and $\mu_1, \mu_2 \in \mathcal{M}_f(\Sigma_\beta^n)$ such that $\int \varphi d\mu_1 < a < \int \varphi d\mu_2$. By the proof of Theorem 4.2, for (Σ_β^n, σ), we have $\{QW_1, QW_2, QW_3, QW_4, QW_5, BR_1, BR_2,$
$BR_3, BR_4, BR_5\}$ has uncountable DC1-scrambled gap with respect to $Trans_{\sigma|_{\Sigma_\beta^n}} \cap R_\varphi(a)$. Similarly, item**(c)** has been proved.

(d): If $Int(L_\varphi) \neq \emptyset$, item**(d)** is from item**(c)**. Otherwise, $R_\varphi = X$ so that item**(d)** is from item**(a)**. □

6. Comments and Questions

6.1. **Weakly almost periodic points.** The reason why we can't analyse whether there is an uncountable DC1-scrambled set in W by our method is that we didn't find a measure μ with full support and G_μ has distal pair. For a point $x \in W \cap Trans$, we can observe that x must be a element of the generic point of a measure with full support. But Theorem E don't cover this situation.

Theorem 6.1. *Suppose that (X, f) has specification property. If for any invariant measure μ with full support, G_μ has distal pair, then*
(1) there is an uncountable DC1-scrambled set $S \subseteq W \cap Trans$.
(2) If φ is a continuous function on X and $I_\varphi(f) \neq \emptyset$. Then there is an uncountable DC1-scrambled set $S \subseteq W \cap Trans \cap I_\varphi(f)$.
(3) If φ is a continuous function on X and $Int(L_\varphi) \neq \emptyset$. Then for any $a \in L_\varphi$, there is an uncountable DC1-scrambled set $S \subseteq W \cap Trans \cap R_\varphi(a)$.
(4) For any continuous function φ on X, there is an uncountable DC1-scrambled set $S \subseteq W \cap Trans \cap R_\varphi$.

Remark 6.2. The set of points with Case (1) restricted on recurrent set coincides with the set of $W \setminus AP$. For systems with specification, note that $W \cap Trans \subseteq W \setminus AP$ so that above result can be also stated for the set of points with Case (1) restricted on recurrent set or $W \setminus AP$.

Remark 6.3. For a transitive system (X, f) without periodic points with period m, it is easy to check for any $x \in Trans$, $(x, f^m x)$ must be a distal pair. This implies that for any invariant measure μ (not necessarily with full support), $G_\mu \cap Trans$ has distal pair. So Theorem 6.1 are suitable for systems with specification but without periodic points with period m for some m. In particular, it apllies in mixing subshifts of finite type without periodic points with period m for some m. For example it can be a subshift of finite type defined by a graph with two distinct cycles of length $m+1$ and $m+2$ starting from the same vertex. For such dynamical systems, Theorem E holds for any nonempty compact connected set K, since G_μ has distal pair for any μ in K.

Proof. Let μ be an invariant measure with full support.

(1) Take $K = \{\mu\}$. Then one can use Proposition 2.4 and Theorem E to give the proof.

(2) By Proposition 2.3, one can choose an invariant measure μ' with full support such that $\int \varphi d\mu \neq \int \varphi d\mu'$. Take $K = \text{cov}\{\mu, \mu'\}$. Then one can use Proposition 2.4 and Theorem E to give the proof.

(3) If $\int \varphi d\mu = a$, take $\omega = \mu$. Otherwise, by Proposition 2.3, one can choose an invariant measure μ' with full support such that $\int \varphi d\mu' < a < \int \varphi d\mu$ or $\int \varphi d\mu < a < \int \varphi d\mu'$. Take suitable $\theta \in (0,1)$ such that $\omega = \theta\mu + (1-\theta)\mu'$ such that $\int \varphi d\omega = a$. In this case take $K = \{\omega\}$. Then one can use Proposition 2.4 and Theorem E to give the proof.

(4) If $Int(L_\varphi) \neq \emptyset$, item (4) is from item (3). Otherwise, $R_\varphi = X$ so that item (4) is from item (1). □

6.2. **Minimal points.** For minimal points, it is still unknown whether DC1 appear but we remark that DC-2 appear.

Theorem 6.4. *Suppose that (X, f) has specification property (or almost specification, or shadowing property with positive entropy). Then there is an uncountable DC-2 scrambled set $S \subseteq AP(f)$.*

Proof. From [26] a dynamical system with positive entropy has DC-2 scrambled set so that if a minimal subsystem has positive entropy, then the proof is completed. In fact, from [25], we know there exist minimal subsystems arbitrarily close to full entropy (and thus $AP(f)$ carries full topological entropy). □

From [15] the set of parameters of β for which specification holds, is dense in $(1, +\infty)$ but has Lebesgue zero measure. However, every β shift has almost specification by [54] so that Theorem 6.4 applies in all β shifts.

Let $C(M)$be the set of continuous maps on a compact manifold M and $H(M)$ the set of homeomorphisms on M. Recall that C^0 generic $f \in H(M)$ (or $f \in C(M)$) has the shadowing property and infinite topological entropy (see [41] and [39, 40], respectively). Thus Theorem 6.4 applies in C^0 generic dynamical systems.

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