

EXTENSION OF AN UNICITY CLASS FOR NAVIER-STOKES EQUATIONS

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ABSTRACT. This is a translation from French of my paper [R. May, Extension d'une classe d'unicité pour les équations de Navier-Stokes, Ann. I. H. Poincaré-AN 27 (2010) 705-718. doi:10.1016/j.anihp.2009.11.007].

Q. Chen, C. Miao, and Z. Zhang [4] have proved that weak Leray solutions of the Navier-Stokes are unique in the class $L^{\frac{2}{1+r}}([0, T].B_{\infty}^{r, \infty}(\mathbb{R}^3))$ with $r \in]-\frac{1}{2}, 1]$. In this paper, we establish that this criterion remains true for $r \in]-1, -\frac{1}{2}]$.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

We consider the Navier-Stokes equations for an incompressible fluid in the entire space $\mathbb{R}^d, d \geq 2$,

$$(NS) \quad \begin{cases} \partial_t u - \Delta u + \nabla(u \otimes u) + \vec{\nabla} p = 0, \\ \vec{\nabla} \cdot u = 0, \\ u(0, \cdot) = u_0(\cdot) \end{cases}$$

where u_0 is the initial velocity of the fluid particles, $u = u(t, x)$ designs of the particle placed in $x \in \mathbb{R}^d$ at the time $t \geq 0$, $p = p(t, x)$ is the pressure at $x \in \mathbb{R}^d$ and $t \geq 0$, $\vec{\nabla} = (\partial_{x_1}, \dots, \partial_{x_d})^t$ denotes the gradient operator, $\vec{\nabla} \cdot$ is the divergence operator, and $\nabla(u \otimes u)$ is the vector function (w_1, \dots, w_d) defined by

$$w_i = \sum_{k=1}^d \partial_{x_k} (u_k u_i) = \vec{\nabla} \cdot (u_i u).$$

Let us first recall the notion of the weak solutions for the Navier-Stokes equations that we will adopt in this paper.

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Definition 1.1. Let $T \in]0, +\infty[$ and $u_0 = (u_{01}, \dots, u_{0d}) \in (S'(\mathbb{R}^d))^d$ with divergence free. A weak solution on $]0, T[$ of the equations (NS) is a function $u : Q_T \equiv]0, T[\times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying the following properties:

- 1) $u \in L^2_{loc}(\tilde{Q}_T)$ where $\tilde{Q}_T \equiv]0, T[\times \mathbb{R}^d$.
- 2) $u \in C([0, T[, S'(\mathbb{R}^d))$.
- 3) $u(0) = u_0$.
- 4) For all $t \in [0, T[$, $\vec{\nabla} \cdot (u(t)) = 0$ in $S'(\mathbb{R}^d)$.
- 5) There exists $p \in D'(Q_T)$ such that $\partial_t u - \Delta u + \nabla(u \otimes u) + \vec{\nabla} p = 0$ in $D'(Q_T)$.

In 1934, J. Leray [16] proved that, for any initial data u_0 in $L^2(\mathbb{R}^d)$ with divergence free, the Navier-Stokes equations have at least one weak solution u on $]0, +\infty[$ which, for every $T > 0$, belongs to the Leray energy space \mathcal{L}_T defined by:

$$\mathcal{L}_T = L^\infty([0, T], L^2(\mathbb{R}^d)) \cap L^2([0, T], H^1(\mathbb{R}^d)).$$

This leads us to introduce the following notion of Leray weak solutions.

Definition 1.2. Let $T > 0$ and u_0 in $L^2(\mathbb{R}^d)$ with divergence free. We call Leray weak solutions of the equations (NS) on $]0, T[$ every weak solution on $]0, T[$ of (NS) which belongs to the Leray energy space \mathcal{L}_T .

Naturally, the question on the uniqueness of the Leray weak solutions raises. In the bi-dimension case corresponding to $d = 2$, it is well known that such solutions are unique (see for instance [22]). However, in the case $d \geq 3$, the question remains open. We only have some partial answers. In fact, the uniqueness is obtained under some variant of supplementary conditions on the regularity of the solutions. As examples, we cite the works of J. Serrin [21], W. Von Wahl [24], J. Y. Chemin [2], I. Gallagher and F. Plonchon [8], and P. Germain [10]. In This direction, Q. Chen, C. Miao, and Z. Zhang have recently proved the following uniqueness result.

Theorem 1.1 (See [4], Theorem 1.4). Let $T > 0$ and u_0 in $L^2(\mathbb{R}^d)$ with divergence free. Let u_1 and u_2 be two Leray weak solutions of the equations (NS) on $]0, T[$. Assume that

$$u_1 \in L^{\frac{2}{1-r_1}}([0, T], B_\infty^{-r_1, \infty}(\mathbb{R}^d)) \text{ and } u_2 \in L^{\frac{2}{1-r_2}}([0, T], B_\infty^{-r_2, \infty}(\mathbb{R}^d))$$

where $0 \leq r_1, r_2 < 1$ and $r_1 + r_2 < 1$. Then $u_1 = u_2$.

As a consequence, the spaces $L^{\frac{2}{1-r}}([0, T], B_\infty^{-r, \infty}(\mathbb{R}^d))$, with $r \in [0, \frac{1}{2}[$, constitute a uniqueness class of Leray weak solutions of (NS). In this paper, we extend this uniqueness criteria to $r \in [\frac{1}{2}, 1]$; which gives a positive answer to the question of Q. Chen, C. Miao, and Z. Zhang in [[4], Remark 1.7].

Before setting our results, let us introduce the following notation.

Notation 1. Let $T > 0$ and $r \in]0, 1]$. We denote by $\mathcal{P}_{r,T}$ the space $L^{\frac{2}{1-r}}([0, T], B_{\infty}^{-r, \infty}(\mathbb{R}^d))$ if $r \neq 1$ and the space $C([0, T], B_{\infty}^{-1, \infty}(\mathbb{R}^d))$ if $r = 1$.

Now we are in position to cite our main results.

Theorem 1.2. We suppose here that $d \leq 4$. Let $T > 0$ and u_0 in $L^2(\mathbb{R}^d)$ with divergence free. If u_1 and u_2 are two Leray weak solutions of the equations (NS) on $]0, T[$ such that $u_1 \in \mathcal{P}_{r_1, T}$ and $u_2 \in \mathcal{P}_{r_2, T}$ for some $r_1, r_2 \in]0, 1]$ then $u_1 = u_2$.

Thanks to the precise Sobolev inequalities proved by P.Gerard, Y. Meyer, and F. Oru [9], the proof of the former theorem will be a consequence of the following more general uniqueness result.

Theorem 1.3. Let $T > 0$, $(r_1, r_2) \in]0, 1]^2$, and $(p_i, q_i)_{i=1,2} \in \mathbb{R}^2$ such that, for each i , $q_i \geq d$ and $p_i \geq \frac{4}{1+r_i}$ if $r_i \neq 1$ and $p_i > 2$ if $r_i = 1$. If u_1 and u_2 are two Leray weak solutions on $]0, T[$ of the equations (NS) associated to the same initial data u_0 such that, for $i \in \{1, 2\}$,

$$u_i \in L^{p_i}([0, T], L^{q_i}(\mathbb{R}^d)) \cap \mathcal{P}_{r_i, T},$$

then $u_1 = u_2$.

The proof of this theorem repose essentially on the following regularity result.

Theorem 1.4. Let $T > 0$, $q \geq d, r \in]0, 1]$, and $p \geq \frac{4}{1+r}$ such that $p > 2$ if $r = 1$. If $u \in L^p([0, T], L^q(\mathbb{R}^d)) \cap \mathcal{P}_{r, T}$ is a weak solution of (NS) on $]0, T[$, then $\sqrt{t}u \in L^{\infty}([0, T], L^{\infty}(\mathbb{R}^d))$ and $\sqrt{t} \|u(t)\|_{\infty}$ tends to 0 as $t \rightarrow 0$.

Remark 1.1. This theorem implies, in particular, that every weak solution u of the equations of Navier-Stokes which belongs to the space $L^p([0, T], L^q(\mathbb{R}^d)) \cap \mathcal{P}_{r, T}$ is a classical solutions of (NS) i.e. $u \in C^{\infty}(Q_T)$ (see the proof of this result in the last section of this paper).

Remark 1.2. In the case where $r = 1$, the theorem 1.4 has been recently proved by P. G. Lemarie-Rieusset [13] when $q > d$ and by the author of this paper [18] when $q = d$. Therefore, we will prove the theorem 1.4 only in the case $r \in]0, 1[$.

Remark 1.3. P. Germain [10] proved the uniqueness of Leray weak solutions of (NS) in the class $L^{\frac{2}{1-r}}([0, T], X_r)$ with $r \in [-1, 1[$ and

$$X_r = \begin{cases} M(H^r, L^2), & \text{if } r \in]0, 1[, \\ \Lambda^r BMO, & \text{if } r \in]-1, 0[, \\ Lip, & \text{if } r = -1, \end{cases}$$

where $Lip = \{f : \mathbb{R}^d \rightarrow \mathbb{R} : \|f\|_{Lip} \equiv \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|} < \infty\}$, $\Lambda^r = (I - \Delta)^{\frac{r}{2}}$ and $M(H^r, L^2)$ is the space of functions $f \in L^2_{loc}(\mathbb{R}^d)$ such that, for every $g \in H^r(\mathbb{R}^d)$, $fg \in L^2(\mathbb{R}^d)$. The space $M(H^r, L^2)$ is endowed with the norm

$$\|f\|_{M(H^r, L^2)} = \sup_{\|g\|_{H^r(\mathbb{R}^d)} \leq 1} \|fg\|_{L^2(\mathbb{R}^d)}.$$

Since $X_r \hookrightarrow B_{\infty}^{-r, \infty}$, Theorem 1.2 of the present paper combined with Theorem 1.2 and Theorem 1.4 in [4] extend the uniqueness class of P. Germain.

Remark 1.4. H. Miura [20] proved the uniqueness of the weak solutions of the equations (NS) which belongs to the space

$$\mathcal{M}_T = L^2([0, T], L^2_{uloc}) \cap C([0, T], vmo^{-1}) \cap L^{\infty}_{loc}([0, T], L^{\infty}),$$

where vmo^{-1} is the space of $f \in S'(\mathbb{R}^d)$ satisfying

$$\forall T > 0, \|f\|_{BMO_T^{-1}} \equiv \sup_{x_0 \in \mathbb{R}^d, 0 < R^2 < T} R^{-\frac{d}{2}} \left(\int_{[0, R^2] \times B(x_0, R)} |e^{t\Delta} f|^2 dt dy \right)^{\frac{1}{2}} < +\infty$$

and

$$\lim_{T \rightarrow 0} \|f\|_{BMO_T^{-1}} = 0.$$

The space vmo^{-1} is endowed with the norm $\|\cdot\|_{BMO_T^{-1}}$ where T is a fixed non negative real number.

Theorem 1.2 extends this uniqueness result; in fact since $vmo^{-1} \hookrightarrow B_{\infty}^{-1, \infty}$ then every weak solution of the Navier-Stokes equations in the space

$$\mathcal{M}_T^{p, q} \equiv L^2([0, T], L^2_{uloc}) \cap C([0, T], vmo^{-1}) \cap L^p_{loc}([0, T], L^q),$$

with $p > 2$ and $q \geq d$, belongs to Miura's space \mathcal{M}_T . Therefore, the family of spaces $(\mathcal{M}_T^{p, q})_{p > 2, q \geq d}$ constitute a uniqueness class of weak solutions of Navier-Stokes equations.

The remaining of this paper is organized as follows: in the next section, we first recall the notion of mild solutions of (NS) introduced in [6], then we cite some useful properties of the Besov spaces and the Chemin-Lerner spaces. In the third section, we prove how the main theorem 1.4 implies Theorem 1.3 and Theorem 1.2. The last section is devoted to the proof of Theorem 1.4 in the case $r \in]0, 1[$.

2. PRELIMINAIRES

2.1. **Notations.** (1) In this paper, all the functional spaces are defined on the whole space \mathbb{R}^d . Then, in order to simplify the notations, we will design, for instance, the spaces $L^q(\mathbb{R}^d)$, $H^s(\mathbb{R}^d)$, and $B_q^{s,p}(\mathbb{R}^d)$ respectively by L^q , H^s , and $B_q^{s,p}$.

(2) If X is a vector space and $n \in \mathbb{N}$, we often write $(f_1, \dots, f_n) \in X$ in place of $(f_1, \dots, f_n) \in X^n$.

(3) If X is a Banach space, $T > 0$, and $p \in [1, +\infty]$, we denote by $L_T^p(X)$ or $L_T^p X$ the space $L^p([0, T], X)$.

(4) Let $p \geq 1$. We design by \mathbf{E}_p the space of functions $f \in L_{loc}^p(\mathbb{R}^d)$ such that

$$\|f\|_{\mathbf{E}_p} \equiv \sup_{x_0 \in \mathbb{R}^d} \|1_{B(x_0,1)} f\|_p < \infty \text{ and } \lim_{\|x_0\| \rightarrow \infty} \|1_{B(x_0,1)} f\|_p = 0.$$

(5) If A and B are two real valued functions, the notation $A \lesssim B$ means that there exists an absolute non negative real constant α such that $A \leq \alpha B$.

2.2. **Mild solutions of the Navier-Stokes equations.** We denote by \mathbb{P} the Leray projector on the space of distributions with divergence free. We recall that $(\mathbb{P}_{ij})_{1 \leq i, j \leq d}$ is defined via Riesz transformations $(\mathcal{R}_i)_{1 \leq i \leq d}$ by the relation:

$$\mathbb{P}_{ij}(f) = \delta_{ij} f - \mathcal{R}_i \mathcal{R}_j(f)$$

where δ_{ij} is the Kronecker symbol.

Let $u_0 = (u_{01}, \dots, u_{0d}) \in S'(\mathbb{R}^d)$ a tempered distribution with divergence free. By applying formally the Leray operator \mathbb{P} to the equations (NS) we obtain the following system:

$$\begin{cases} u_t - \Delta u = -\mathbb{P} \nabla(u \otimes u), \\ u(0, \cdot) = u_0(\cdot). \end{cases}$$

Next, using Duhamel formula we transform this system to the integral equations

$$(NSI) \quad u(t) = e^{t\Delta} u_0 + \mathbf{B}(u, u)(t),$$

where $(e^{t\Delta})_{t \geq 0}$ is the heat semi-group and \mathbf{B} is the bilinear application defined by:

$$\mathbf{B}(u, v) = \mathbb{L}_{Oss}(u \otimes v).$$

The operator \mathbb{L}_{Oss} , called the Oseen integral operator, is given by

$$(2.1) \quad \mathbb{L}_{Oss}(f)(t) = - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla(f) ds.$$

In [6], G. Furioli, P. G. Lemarie-Rieusset, and F. Terraneo proved that for the solutions class $L^2_{loc}([0, T[, \mathbf{E}_2)$ the equations (NS0 and (NSI) are equivalents. This leads us to introduce the following notion of mild solutions which we adopt in this paper.

Definition 2.1. *Let $T > 0$ and $u_0 \in S'(\mathbb{R}^d)$. A mild solution of the Navier-Stokes equations on $]0, T[$ is a function $u \in L^2_{loc}([0, T[, \mathbf{E}_2)$ which satisfies, for every $t \in [0, T[$, the integral equations (NSI).*

Remark 2.1. *It is well-known (see for instance [6] and [12]) that every mild solution u of the equations (NS) on the interval $]0, T[$ belongs to the space $C([0, T[, B^{-d-1, \infty})$.*

Remark 2.2. *All solutions of the Navier-Stokes considered in this paper are mild solutions; therefore, in the sequel, if $u \in L^2_{loc}([0, T[, \mathbf{E}_2)$, then the short sentence "u is a solution of the equations (NS)" means that u is a mild solution on $]0, T[$ of the equations (NS).*

Remark 2.3. *Let u be a mild solution $]0, T[$ of the equations (NS). Using the semi-group property of $(e^{t\Delta})_{t \geq 0}$, one can easily verify that for every $0 < t_0 \leq t < T$*

$$u(t) = e^{(t-t_0)\Delta}u(t_0) - \int_{t_0}^t e^{(t-s)\Delta}\mathbb{P}\nabla(u \otimes u)ds.$$

This implies that the function $u_{t_0} \equiv u(\cdot + t_0)$ is a mild solution on $]0, T - t_0[$ of the Navier-Stokes equations associated to the initial data $u(t_0)$.

Remark 2.4. *In the sequel of this work, the hypothesis of free divergence of the solutions u of the Navier-Stokes equations (NS) will play no role.*

2.3. Besov spaces and Chemin-Lerner spaces. let us first recall the Littlewood-Paley decomposition. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ which is equal to 1 on a neighbourhood of the origin. Next we define the function $\psi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ by $\psi(\xi) = \varphi(\frac{\xi}{2}) - \varphi(\xi)$. For every $j \in \mathbb{N} \cup \{0\}$, we design by S_j and Δ_j the operators defined on $S'(\mathbb{R}^d)$ and $S'(\mathbb{R} \times \mathbb{R}^d)$ by

$$S_j f = \mathcal{F}_x^{-1}(\varphi(\frac{\xi}{2^j})\mathcal{F}_x(f)),$$

$$\Delta_j f = \mathcal{F}_x^{-1}(\psi(\frac{\xi}{2^j})\mathcal{F}_x(f)),$$

where \mathcal{F}_x and \mathcal{F}_x^{-1} are respectively the Fourier transformation with respect to the space variable $x \in \mathbb{R}^d$ and its inverse transformation.

Notation 2. *In the sequel, we often denote the operator S_0 by Δ_{-1} .*

Now we can recall the definition of a class of Besov spaces.

Definition 2.2. Let $s \in \mathbb{R}$ and $q \in [1, +\infty]$. The Besov space $B_q^{s,\infty}$ is the space of $f \in S'(\mathbb{R}^d)$ such that

$$\|f\|_{B_q^{s,\infty}} \equiv \sup_{j \geq -1} 2^{sj} \|\Delta_j f\|_q < \infty.$$

We design by $\tilde{B}_q^{s,\infty}$ the closure of $S(\mathbb{R}^d)$ in $B_q^{s,\infty}$.

We introduce now the definition of a class of Chemin-Lerner spaces [[5], [2], [3]].

Definition 2.3. Let $T > 0$, $s \in \mathbb{R}$ and $p, q \in [1, +\infty]$. The Chemin-Lerner space $\tilde{L}_T^p B_q^{s,\infty}$ is the space of $v \in S'(\mathbb{R} \times \mathbb{R}^d)$ such that

$$\|v\|_{\tilde{L}_T^p B_q^{s,\infty}} \equiv \sup_{j \geq -1} 2^{sj} \|\Delta_j v\|_{L_T^p L_x^q} < \infty.$$

We design by $\tilde{\mathbf{L}}_T^p B_q^{s,\infty}$ the space of $v \in \tilde{L}_T^p B_q^{s,\infty}$ such that

$$\lim_{T \rightarrow 0} \|v\|_{\tilde{L}_T^p B_q^{s,\infty}} = 0.$$

The following proposition gathers some simple and useful properties of Besov and Chemin-Lerner spaces.

Proposition 2.1. Let $T > 0$, $s \in \mathbb{R}$, $(p, q) \in [1, +\infty]$, and $p_1 \in [1, +\infty]$. The following assertions hold true:

- 1) $L_T^p B_q^{s,\infty} \hookrightarrow \tilde{L}_T^p B_q^{s,\infty}$, $L_T^\infty B_q^{s,\infty} = \tilde{L}_T^\infty B_q^{s,\infty}$, and $L_T^{p_1} B_q^{s,\infty} \hookrightarrow \tilde{\mathbf{L}}_T^{p_1} B_q^{s,\infty}$.
- 2) The linear operators $\mathbb{P}_{ij} \frac{\partial}{\partial x_k}$ are continuous from $B_q^{s,\infty}$ (respectively $L_T^p B_q^{s,\infty}$) to $B_q^{s-1,\infty}$ (respectively $L_T^p B_q^{s-1,\infty}$).
- 3) (Bernstein's inequality) For every $m \in [q, +\infty]$, we have

$$B_q^{s,\infty} \hookrightarrow B_q^{s+d(\frac{1}{m}-\frac{1}{q}),\infty} \quad \text{and} \quad \tilde{L}_T^p B_q^{s,\infty} \hookrightarrow \tilde{L}_T^p B_q^{s+d(\frac{1}{m}-\frac{1}{q}),\infty}.$$

The proof of this proposition is classical and simple.

It is well-known (see for instance [1], [12], [23]) that Besov spaces can be characterized via the heat semi group $(e^{t\Delta})_{t \geq 0}$. The following proposition is a particular case of such characterization.

Proposition 2.2. Let $q \in [1, +\infty]$ and $s > 0$. Then for each real number $\delta > 0$, the quantity

$$\sup_{0 < \theta < \delta} \theta^{\frac{s}{2}} \|e^{s\Delta} f\|_q$$

defines a norm on the Besov space $B_q^{-s,\infty}$ equivalent to the original norm $\|\cdot\|_{B_q^{-s,\infty}}$.

In order to study the properties of the pointwise product we introduce the following modified and simplified version of the Bony para-product. For every f and g in $S'(\mathbb{R}^d)$ (or in $S'(\mathbb{R} \times \mathbb{R}^d)$), we define formally $\Pi_1(f, g)$ and $\Pi_2(f, g)$ by

$$\Pi_1(f, g) = \sum_{j=-1}^{\infty} S_{j+1} f \Delta_j g \text{ and } \Pi_2(f, g) = \sum_{j=0}^{\infty} S_j f \Delta_j g.$$

So we have, at least formally, the equality $fg = \Pi_1(f, g) + \Pi_2(g, f)$. The operators Π_1 and Π_2 will be called "the operators of the Bony para-product".

The next proposition describes some continuity properties of the Bony para-product operators on Besov spaces and Chemin-Lerner spaces.

Proposition 2.3. *Let $T > 0$, $\sigma_2 > \sigma_1 > 0$ two non negative reals numbers, and $(p_1, q_1), (p_2, q_2) \in [1, +\infty]^2$ such that $\frac{1}{p} \equiv \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\frac{1}{q} \equiv \frac{1}{q_1} + \frac{1}{q_2} \leq 1$. Then the following assertions hold true:*

- 1) *The operators Π_1 and Π_2 are continuous from $B_{q_1}^{-\sigma_1, \infty} \times B_{q_2}^{\sigma_2, \infty}$ to $B_q^{\sigma_2 - \sigma_1, \infty}$.*
- 2) *The operators Π_1 and Π_2 are continuous from $\tilde{L}_T^{p_1} B_{q_1}^{-\sigma_1, \infty} \times \tilde{L}_T^{p_2} B_{q_2}^{\sigma_2, \infty}$ to $\tilde{L}_T^p B_q^{\sigma_2 - \sigma_1, \infty}$ and from $L_T^{p_1} L_x^{q_1} \times \tilde{L}_T^{p_2} B_{q_2}^{\sigma_2, \infty}$ to $\tilde{L}_T^p B_q^{\sigma_2, \infty}$. Moreover their norms are independent of T .*

The proof of this proposition is simple, see for instance [2] and [12] where similar results are proved.

We study now the regular effect of the heat equations measured in term of Besov spaces and Chemin-Lerner spaces.

The first result concerns the regular effect of the semi-group $(e^{t\Delta})_{t \geq 0}$.

Proposition 2.4. *[Regular effect of the heat semi-group] Let $T > 0$, $(s_1, s_2, s_3) \in \mathbb{R}^3$, and $(p, q) \in [1, +\infty]^2$. Then we have the following assertions:*

- 1) *If $s_1 \leq s_2$ then the family $(t^{\frac{s_2 - s_1}{2}} e^{t\Delta})_{0 < t \leq T}$ is bounded in the space $\mathcal{L}(B_q^{s_1, \infty}, B_q^{s_2, \infty})$.*
- 2) *The operator $e^{t\Delta}$ is continuous from $B_q^{s, \infty}$ to $\tilde{L}_T^p B_q^{s + \frac{2}{p}, \infty}$. Moreover, if $p < \infty$ then $e^{t\Delta}$ is continuous from $B_q^{s, \infty}$ to $\tilde{\mathbf{L}}_T^p B_q^{s + \frac{2}{p}, \infty}$.*

The second result concerns the regular effect of the integral Oseen operator \mathbb{L}_{Oss} defined by (2.1).

Proposition 2.5. *Let $T > 0, s \in \mathbb{R}$, and $(p_1, p_2, q) \in [1, +\infty]^3$ such that $p_1 \leq p_2$ and set $s' \equiv s + 1 - 2(\frac{1}{p_1} - \frac{1}{p_2})$. Then the Oseen operator \mathbb{L}_{Oss} maps boundly the space $\tilde{L}_T^{p_1} B_q^{s, \infty}$ into $\tilde{L}_T^{p_2} B_q^{s', \infty}$ and its norm is majorized by $C(1 + T)$ where C is a non negative constant independent of T .*

For the proofs of these two propositions, we refer the reader to [2] and [5].

3. PROOF OF THEOREMS 1.2 AND 1.3

In this short section, we will see how Theorem 1.4, which will be proved in the next section, allows to prove Theorem 1.3 and how Theorem 1.3 implies Theorem 1.2.

3.1. Proof of Theorem 1.3. First, from Theorem 1.4, we have for $i = 1$ or 2

$$u_i \in L_T^p \mathbf{E}_d, \quad \sqrt{t}u_i \in L_T^\infty L_x^\infty, \quad \text{and} \quad \lim_{t \rightarrow 0} \sqrt{t} \|u_i(t)\|_\infty = 0,$$

where $p = \inf(p_1, p_2)$; (see the section 2.1 for the definition of the space \mathbf{E}_d). Set $u \equiv u_1 - u_2$; this function satisfies the equation

$$u = \mathbf{B}(u_1, u) + \mathbf{B}(u, u_2).$$

Using the continuity on the space \mathbf{E}_d of the pointwise multiplication with a function in $L^\infty(\mathbb{R}^d)$ and the convolution with a function in $L^1(\mathbb{R}^d)$ and recalling that $e^{(t-s)\Delta} \mathbb{P} \nabla$ is a convolution operator and that the $L^1(\mathbb{R}^d)$ norm of its kernel does not exceed $C(t-s)^{-\frac{1}{2}}$ for some absolute constant $C > 0$ (see for instance [1],[12], and [19]), we easily deduce that, for every $t \in [0, T]$, we have

$$\|u(t)\|_{\mathbf{E}_d} \leq C \omega(t) \int_0^t \frac{\|u(s)\|_{\mathbf{E}_d}}{\sqrt{t-s}\sqrt{s}} ds,$$

where

$$\omega(t) \equiv \sup_{0 < s \leq t} \sqrt{s} (\|u_1(s)\|_{\mathbf{E}_d} + \|u_2(s)\|_{\mathbf{E}_d}).$$

Invoking now the continuity of the linear operator

$$L(f)(t) \equiv \int_0^t \frac{f(s)}{\sqrt{t-s}\sqrt{s}} ds$$

on the space $L^p(\mathbb{R}^+)$ (see for instance Lemma 5.2 [18]), we deduce that for every $\delta \in]0, T]$ we have

$$\sup_{0 < t \leq \delta} \|u(t)\|_{\mathbf{E}_d} \leq C_p \omega(\delta) \sup_{0 < t \leq \delta} \|u(t)\|_{\mathbf{E}_d},$$

where C_p is a constant which only depends on p . Hence, by using the fact that $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ we infer that there exists $\delta \in]0, T]$ such that $u = 0$ (and by consequent $u_1 = u_2$) on $[0, \delta]$. Finally, thanks to a classical iteration argument (see for example Lemma 27.2 in [13]) we conclude that $u_1 = u_2$ on $[0, T]$.

3.2. Proof of Theorem 1.2. Let $i = 1$ or 2 . First let us notice that the classical interpolation in the Lebesgue spaces and Sobolev spaces implies that u_i belongs to the space $L_T^{\frac{2}{r_i}} H^{r_i}$. Using now the nonhomogene version of the precise Sobolev inequalities proved by P. Gerard, Y. Meyer, and F. Oru [9] (see also [13] of an other proof)

$$\|f\|_q \lesssim (\|f\|_{W^{\alpha,p}})^{1-\frac{\alpha}{\beta}} \left(\|f\|_{B_{\infty}^{\alpha-\beta,\infty}} \right)^{\frac{\alpha}{\beta}},$$

$$0 < \alpha < \beta, \quad 1 < p < \infty, \quad \frac{p}{q} = \left(1 - \frac{\alpha}{\beta}\right),$$

with $\alpha = r_i, \beta = 2r_i$ and $p = 2$, we get the inequality

$$\|u_i(t)\|_4 \lesssim (\|u(t)\|_{H^{r_i}})^{\frac{1}{2}} \left(\|u(t)\|_{B_{\infty}^{-r_i,\infty}} \right)^{\frac{1}{2}}$$

which, thanks to Holder inequality, implies that

$$u_i \in L^4([0, T], L^4(\mathbb{R}^d)).$$

Hence, applying Theorem 1.3 completes the proof of Theorem 1.2.

4. PROOF OF THE THEOREM 1.4

This section is devoted to the proof of the main theorem 1.4 in the case where $r \in]0, 1[$ (see Remark 1.2). The proof repose on some intermediate results.

The first proposition is a local uniqueness result under a supplementary regularity hypothesis on the initial data u_0 .

Proposition 4.1. *Let $T > 0, r \in]0, 1[, q \geq d$, and $p \geq \frac{4}{1+r}$. If the initial data $u_0 \in L^q(\mathbb{R}^d)$ and $u_1, u_2 \in L_T^{\frac{2}{1-r}}(B_{\infty}^{-r,\infty}) \cap L_T^p L_x^q$ are two mild solutions on $]0, T[$ of the equations (NS), then there exists $\delta \in]0, T[$ such that $u_1 = u_2$ on $[0, \delta]$.*

In the second proposition, we prove a result of regularity persistency and a criterion of eventual finite time explosion of regular solutions of Navier-Stokes equations.

Proposition 4.2. *Let $q \geq d$ a real number and $u_0 \in L^q(\mathbb{R}^d)$. Then the following assertions hold:*

- 1) *The Navier-Stokes equations (NS) havent a unique maximal solution u in the space $C([0, T^*[, L^q(\mathbb{R}^d))$. Moreover, for every $\sigma > 0$, $u \in C^\infty([0, T^*[, \tilde{B}_{\infty}^{\sigma,\infty})$.*
- 2) *If in addition $u_0 \in B_{\infty}^{-r,\infty}$ for some $r \in]0, 1[$ then the maximal solution u belongs to the space $L_{loc}^{\infty}([0, T^*[, B_{\infty}^{-r,\infty})$.*

3) If $T^* < \infty$ then for every $r \in]0, 1[$ there exists a constant $\varepsilon_{r,d} > 0$, which depends only on r and d , such that

$$(4.1) \quad \underline{\lim}_{t \rightarrow T^*} (T^* - t)^{\frac{1-r}{2}} \|u(t)\|_{B_\infty^{-r,\infty}} \geq \varepsilon_{r,d}.$$

In particular,

$$\int_{T^*/2}^{T^*} \|u(t)\|_{B_\infty^{-r,\infty}} dt = +\infty.$$

Remark 4.1. Estimation (4.1) improve a similar result of Y. Giga [11] where the Besov space $B_\infty^{-r,\infty}$ is replaced by the Lebesgue space $L_r^{\frac{d}{r}}(\mathbb{R}^d)$. (Recall that $L_r^{\frac{d}{r}}(\mathbb{R}^d) \hookrightarrow B_\infty^{-r,\infty}$).

The last preliminary result concerns the behavior as $t \rightarrow 0$ of the regular solutions $u(t)$ of the equations (NS) which belong to the class $L_T^{\frac{2}{1-r}}(B_\infty^{-r,\infty})$.

Proposition 4.3. Let $r \in]0, 1[, T > 0$, and $u \in C(]0, T], B_\infty^{1,\infty}) \cap L_T^{\frac{2}{1-r}}(B_\infty^{-r,\infty})$ a mild solution on $]0, T[$ of the equations of Navier-Stokes. Then $\sqrt{t} \|u(t)\|_\infty \rightarrow 0$ as $t \rightarrow 0$.

Let us now see how the above propositions allow together to prove the main theorem 1.4.

Proof of Theorem 1.4. Set $\Omega_{q,r} \equiv \{t_0 \in]0, T] : u(t_0) \in L^q(\mathbb{R}^d) \cap B_\infty^{-r,\infty}\}$. let t_0 be an arbitrary element of $\Omega_{q,r}$. According to Proposition 4.2, the equations (NS) with initial data $u(t_0)$ have a unique maximal solution $v \in C([0, T^*[, L^q(\mathbb{R}^d)) \cap L_{loc}^\infty([0, T^*[, B_\infty^{-r,\infty})$. Hence Remark 2.1 and Proposition 4.1 insure the existence of $\delta \in]0, \delta_0 \equiv \min(T^*, T - t_0)[$ such that $v = u(\cdot + t_0)$ on $[0, \delta]$. This allow to define

$$\delta_* \equiv \sup\{\delta \in]0, \delta_0[: v = u(\cdot + t_0) \text{ on } [0, \delta]\}.$$

Suppose that $\delta_* < \delta_0$; then the facts that v is in $C([0, \delta_0[, L^q(\mathbb{R}^d))$ and $u(\cdot + t_0)$ belongs to $C([0, \delta_0[, B_\infty^{-d-1,\infty})$ (see Remark 2.1) imply that $v(\delta_*) = u(\delta_* + t_0) \in L^q(\mathbb{R}^d)$. Hence, by applying another time the proposition 4.1 to the Navier-stokes equations with initial data $v(\delta_*)$, we deduce the existence of $\delta' > \delta_*$ such that $v = u(\cdot + t_0)$ on $[0, \delta']$. This contradicts the definition of δ_* ; we then infer that $v = u(\cdot + t_0)$ on $[0, \delta_0[$. Therefore since by assumption $u \in L_T^{\frac{2}{1-r}}(B_\infty^{-r,\infty})$, we get $v \in L^{\frac{2}{1-r}}([0, \delta_0[, B_\infty^{-r,\infty})$ which implies, thanks to the last assertion of 4.2, that $u(\cdot + t_0) = v$ on $[0, T - t_0]$. Using now the regularity property of v ensured by the first assertion of 4.2 and the fact that $\Omega_{q,r}$ is dense in $]0, T]$, we conclude that the solution u belongs to the space $\cap_{\sigma>0} C^\infty(]0, T], B_\infty^{\sigma,\infty})$. Finally, Proposition 4.3 ends the proof of Theorem 1.4. \square

4.1. Proof of Proposition 4.1. In order to prove this proposition, we will follow an approach inspired by the paper [2] of J.Y. Chemin. We will decompose our proof into two steps.

4.1.1. *The first step.* Let $u_0 \in L^q(\mathbb{R}^d)$ and $u \in L_T^{\frac{2}{1-r}}(B_\infty^{-r,\infty}) \cap L_T^p L_x^q$ a solution of the Navier-Stokes equations with initial data u_0 . We will prove that there exists $T_0 \in]0, T]$ such that $u \in \tilde{\mathbf{L}}_{T_0}^{\frac{2}{1+r}}(B_q^{1+r,\infty})$. To do this, we will need the following useful lemmas.

Lemma 4.1. *Let $\delta \in]0, T]$, $\rho \in [\frac{2}{1+r}, +\infty[$, $m \in [1, +\infty[$, and $\sigma \in]r, +\infty[$. Then the linear operator \mathbb{L}_u , defined by:*

$$(4.2) \quad \mathbb{L}_u(f) = \sum_{k=1}^2 \mathbb{L}_{oss}(\Pi_k(u, f)),$$

is bounded on the space $\tilde{\mathbf{L}}_\delta^\rho(B_m^{\sigma,\infty})$ and its norm is less than $C \|u\|_{L_\delta^{\frac{2}{1-r}}(B_\infty^{-r,\infty})}$ where C is an absolute non negative constant independent of δ .

Lemma 4.2. *Set $\omega = \mathbf{B}(u, u)$ and $\omega_0 = \mathbb{L}_u(e^{t\Delta}u_0)$ where \mathbb{L}_u is the operator defined by (4.2). Then we have*

- 1) $\omega \in \tilde{\mathbf{L}}_T^{\frac{p}{2}}(B_{\frac{q}{2}}^{1,\infty})$.
- 2) $\omega_0 \in \tilde{\mathbf{L}}_T^{\frac{2}{1+r}}(B_q^{1+r,\infty})$.
- 3) $\omega_0 \in \tilde{\mathbf{L}}_T^{\frac{p}{2}}(B_{\frac{q}{2}}^{1+\frac{2}{p},\infty})$.

Lemma 4.3. *Let X_1 and X_2 be two Banach space and let f be a function defined on X_1 and X_2 such that $f : X_1 \rightarrow X_1$ and $f : X_2 \rightarrow X_2$ are contractions. Then the fixed point of f in X_1 belongs to X_2 .*

Let us admit for a moment these lemmas and prove that there exists $T_0 \in]0, T]$ such that $u \in \tilde{\mathbf{L}}_{T_0}^{\frac{2}{1+r}}(B_q^{1+r,\infty})$.

Proof. Set $\omega = \mathbf{B}(u, u)$ and $\omega_0 = \mathbb{L}_u(e^{t\Delta}u_0)$ as in Lemma 4.2, and consider the following decomposition of ω :

$$\omega = \omega_0 + \mathbb{L}_u(\omega) \equiv F_u(\omega).$$

Lemma 4.1 and the last two assertions of Lemma 4.2 ensure that, for $T_0 \in]0, T]$ small enough such that $\|u\|_{L_{T_0}^{\frac{2}{1-r}}(B_\infty^{-r,\infty})}$ be less than an absolute constant depending only on r, p , and q , the function F_u is a contraction on the Banach spaces $\tilde{\mathbf{L}}_{T_0}^{\frac{2}{1+r}}(B_q^{1+r,\infty})$ and $\tilde{\mathbf{L}}_{T_0}^{\frac{p}{2}}(B_{\frac{q}{2}}^{1+\frac{2}{p},\infty})$. But the first assertion of Lemma 4.2 and the definition of the application

F_u imply that ω is the fixed point of F_u in the space $\tilde{\mathbf{L}}_{T_0}^{\frac{p}{2}}(B_q^{1+\frac{2}{p},\infty})$; hence lemma 4.3 yields $\omega \in \tilde{\mathbf{L}}_{T_0}^{\frac{2}{1+r}}(B_q^{1+r,\infty})$. Therefore, thanks to the first assertion of Lemma 4.2 and the fact that $u = \omega_0 + \omega$, we conclude that u belongs to the space $\tilde{\mathbf{L}}_{T_0}^{\frac{2}{1+r}}(B_q^{1+r,\infty})$. \square

Let us now prove the above three lemmas.

Proof of Lemma 4.1. It is a direct consequence of the injection $L_T^{\frac{2}{1-r}}(B_\infty^{-r,\infty}) \hookrightarrow \tilde{\mathbf{L}}_T^{\frac{2}{1-r}}(B_\infty^{-r,\infty})$, the continuity of the Bony para-product operators Π_k on the Chemin-Lerner spaces (see Proposition 2.3), and the regularizing effect of the Ossen integral operator \mathbb{L}_{Oss} (see Proposition 2.5). \square

Proof of Lemma 4.2. From Holder inequality, $u \otimes u \in L_T^{\frac{p}{2}}L_x^{\frac{q}{2}}$. Hence the injection $L_T^{\frac{p}{2}}L_x^{\frac{q}{2}}$ in $\tilde{\mathbf{L}}_T^{\frac{p}{2}}(B_q^{0,\infty})$ and Proposition 2.5 imply that $\omega \in \tilde{\mathbf{L}}_T^{\frac{p}{2}}(B_q^{1,\infty})$. To prove the second and the third assertion, we first notice in view of the injection of $L^q(\mathbb{R}^d)$ in $\tilde{B}_q^{0,\infty}$ and Proposition 2.4 we have $e^{t\Delta}u_0$ belongs to the space $\tilde{\mathbf{L}}_T^{\frac{2}{1+r}}(B_q^{1+r,\infty})$ and the space $\tilde{\mathbf{L}}_T^p(B_q^{\frac{2}{p},\infty})$. Therefore, Lemma 4.1 implies that $\omega_0 \in \tilde{\mathbf{L}}_T^{\frac{2}{1+r}}(B_q^{1+r,\infty})$. Finally, the continuity of the Bony para-product operators

$$\Pi_k : L_T^p L_x^q \times \tilde{\mathbf{L}}_T^p(B_q^{\frac{2}{p},\infty}) \rightarrow \tilde{\mathbf{L}}_T^{\frac{p}{2}}(B_q^{\frac{2}{p},\infty})$$

and the regularizing effect of the Ossen integral operator \mathbb{L}_{Oss} (see Proposition 2.5) yield that $\omega_0 \in \tilde{\mathbf{L}}_T^{\frac{p}{2}}(B_q^{1+\frac{2}{p},\infty})$. \square

Proof of Lemma 4.3. We consider the Banach space $X = X_1 \cap X_2$ endowed with the norm $\|\cdot\| = \|\cdot\|_{X_1} + \|\cdot\|_{X_2}$. It is clear that f is a contraction on X , hence in view of the Banach's fixed point theorem it has a unique fixed point z' in X which, thanks to the fact that $X \subset X_1$ and the Banach's fixed point theorem, is the unique fixed point of f in the space X_1 . \square

4.1.2. *The second step.* Let $u_1, u_2 \in L_T^{\frac{2}{1-r}}(B_\infty^{-r,\infty}) \cap L_T^p L_x^q$ be two solutions of the Navier-Stokes equations with the same initial data $u_0 \in L^q(\mathbb{R}^d)$. According to above step, $u_1, u_2 \in \mathcal{Z}_{T_0} \equiv \tilde{\mathbf{L}}_{T_0}^{\frac{2}{1+r}}(B_q^{1+r,\infty}) \cap \tilde{\mathbf{L}}_{T_0}^{\frac{2}{1-r}}(B_\infty^{1-r,\infty})$ for some $T_0 \in]0, T]$. Let $\delta \in]0, T_0]$ to be fixed later. A simple application of Proposition 2.3 and Proposition 2.5 implies that the Bilinear operator \mathbf{B} is continuous from $\mathcal{Z}_\delta \times \mathcal{Z}_\delta$ to $\tilde{\mathbf{L}}_\delta^{\frac{2}{1+r}}(B_q^{1+r,\infty}) \cap \tilde{\mathbf{L}}_\delta^{\frac{2}{1-r}}(B_q^{1-r,\infty})$ and its norm is bounded by a constant C independent of δ . Using now the Bernstein's

injection $\tilde{\mathbf{L}}_\delta^{\frac{2}{1-r}}(B_q^{1-r,\infty}) \hookrightarrow \tilde{\mathbf{L}}_\delta^{\frac{2}{1-r}}(B_\infty^{1-r-\frac{d}{q},\infty})$ and the assumption $q \geq d$, we deduce that $\mathbf{B} : \mathcal{Z}_\delta \times \mathcal{Z}_\delta \rightarrow \mathcal{Z}_\delta$ is continuous and therefore

$$\|u_1 - u_2\|_{\mathcal{Z}_\delta} \leq C(\|u_1\|_{\mathcal{Z}_\delta} + \|u_2\|_{\mathcal{Z}_\delta}) \|u_1 - u_2\|_{\mathcal{Z}_\delta},$$

with C independent of δ . Hence, up to choose δ small enough, we conclude that $u_1 = u_2$ on $[0, \delta]$.

4.2. Proof of Proposition 4.2. The proof of the first assertion of the proposition 4.2 is classical and well-known (see for instance [1], [12], [19]). The prove of the assertions (2) and (3) repose essentially on the following elementary lemma where the following notation is used.

Notation 3. Let $T, \mu > 0$ be two non negative real numbers. We denote by $L_{\mu,T}^\infty$ the space of measurable functions $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$\|f\|_{L_{\mu,T}^\infty} \equiv \sup_{0 < s < T} s^{\frac{\mu}{2}} \|f(s)\|_\infty < \infty.$$

Lemma 4.4. Let $r \in]0, 1[$ and $T > 0$. Then the bilinear operator B is continuous from $L_{1,T}^\infty \times L_{r,T}^\infty$ (respectively $L_{r,T}^\infty \times L_{r,T}^\infty$) into $L_{r,T}^\infty$ with norm less than $C_{r,d}$ (respectively $C_{r,d}T^{\frac{1-r}{2}}$) where $C_{r,d}$ is a non negative constant which depends only on r and d .

The proof of this lemma is simple, we just have to recall that $e^{(t-s)\Delta} \mathbb{P} \nabla$ is a convolution operator with an integrable function with $L^1(\mathbb{R}^d)$ norm of order $\frac{1}{\sqrt{t-s}}$.

Now we are ready to prove the last two assertions of the proposition 4.2.

Proof. It is well-known (see for example [12]) that there exists $T_0 \in]0, T]$ such that the solution u given in the first assertion is the limit in the Banach space $X_{T_0} \equiv L_{1,T_0}^\infty \cap L_{r,T_0}^\infty$ of the sequence $(u_n)_n$ defined by:

$$\begin{aligned} u_{(0)} &= e^{t\Delta} u_0, \\ \forall n \in \mathbb{N}, \quad u_{(n+1)} &= u_{(0)} + \mathbf{B}(u_n, u_n), \end{aligned}$$

and $(\sigma_n \equiv \|u_{(n+1)} - u_{(n)}\|_{X_{T_0}}) \in l^1(\mathbb{N})$. We will prove that $(u_n)_n$ is a cauchy sequence in the space L_{r,T_0}^∞ . First, notice that since $u_0 \in B_\infty^{-r,\infty}$ then, from Proposition 2.2, $u_{(0)} \in L_{r,T_0}^\infty$. Hence, by iteration, Lemma 4.4 guarantees that the sequence $(u_n)_n$ belongs to the space L_{r,T_0}^∞ and satisfies the following inequality:

$$\|u_{(n+1)} - u_{(n)}\|_{L_{r,T_0}^\infty} \leq C_{r,d} \sigma_n \left(\|u_{(n)}\|_{L_{r,T_0}^\infty} + \|u_{(n-1)}\|_{L_{r,T_0}^\infty} \right),$$

which implies (see [7]) that $(u_{(n)})_n$ is a Cauchy sequence L_{r,T_0}^∞ . Using now the fact that L_{r,T_0}^∞ is a complete space, that $L_{r,T_0}^\infty \hookrightarrow L_{1,T_0}^\infty$, and the fact that $(u_{(n)})_n$ converges to the solution u in L_{1,T_0}^∞ , we deduce that $u \in L_{r,T_0}^\infty$. Let us now show that $u \in L_{T_0}^\infty(B_\infty^{-r,\infty})$ which ends the proof of the second assertion of our proposition. First, since $u_0 \in B_\infty^{-r,\infty}$ then, from Proposition 2.4, $e^{t\Delta}u_0 \in L_{T_0}^\infty(B_\infty^{-r,\infty})$. Second, Proposition 2.2 and Young's inequalities imply that for every $t \in]0, T_0]$,

$$\begin{aligned} \|\mathbf{B}(u, u)(t)\|_{B_\infty^{-r,\infty}} &\lesssim \sup_{0 < \theta \leq 1} \theta^{\frac{r}{2}} \|e^{\theta\Delta}\mathbf{B}(u, u)(t)\| \\ &\lesssim \sup_{0 < \theta \leq 1} \theta^{\frac{r}{2}} \int_0^t \frac{1}{s^{\frac{1+r}{2}} \sqrt{t+\theta-s}} ds \|u\|_{L_{r,T_0}^\infty} \|u\|_{L_{1,T_0}^\infty} \\ &\lesssim \|u\|_{L_{r,T_0}^\infty} \|u\|_{L_{1,T_0}^\infty}. \end{aligned}$$

This completes the proof since $u = e^{t\Delta}u_0 + \mathbf{B}(u, u)$.

Let us prove the last assertion of Proposition 4.2. Suppose that $T^* < \infty$. Let $r \in]0, 1[$ and t_0 in $I_* \equiv]\max(0, T^*-), T^*[$. According to the remark 2.3, $u_{t_0} \equiv u(\cdot + t_0)$ satisfies on $[0, \delta_0 \equiv T^* - t_0[$ the following equality

$$u_{t_0}(t) = e^{t\Delta}u(t_0) + \mathbf{B}(u_{t_0}, u_{t_0})(t).$$

Hence, in view of Proposition 2.2 and Lemma 4.4, we have for every $t \in [0, \delta_0[$

$$t^{\frac{r}{2}} \|u(t)\|_\infty \leq C \left(\|u(t_0)\|_{B_\infty^{-r,\infty}} + t^{\frac{1-r}{2}} \left(\sup_{0 < s \leq t} t^{\frac{r}{2}} \|u(s)\|_\infty \right)^2 \right).$$

Define $f(t) \equiv \sup_{0 < s \leq t} t^{s\frac{r}{2}} \|u(s)\|_\infty$. It yields that for any $t \in [0, \delta_0[$,

$$f(t) \leq C \left(\|u(t_0)\|_{B_\infty^{-r,\infty}} + (T^* - t_0)^{\frac{1-r}{2}} f^2(t) \right).$$

Recalling now that $\|u(t)\|_\infty \rightarrow +\infty$ as $t \rightarrow T^*$ (see [11], [12], [17]), which is equivalent to $f(t) \rightarrow +\infty$ as $t \rightarrow T^*$; we deduce from the elementary lemma below that

$$\|u(t_0)\|_{B_\infty^{-r,\infty}} (T^* - t_0)^{\frac{1-r}{2}} \geq \varepsilon_{r,d} \equiv \frac{1}{4C^2},$$

which ends the proof of our proposition. \square

Lemma 4.5. *Let $a < b$ two real numbers and $f : [a, b[\rightarrow \mathbb{R}$ a continuous function. Assume that there exist two reals numbers $A, B > 0$ such that $4AB < 1$, $f(0) \leq 2A$, and, for every $t \in [a, b[$, $f(t) \leq A + Bf^2(t)$. Then, for every $t \in [a, b[$, $f(t) \leq 2A$.*

The proof of this lemma is simple, it suffices to apply the intermediate value theorem after noticing that if $4AB < 1$ then $f(t) \neq 2A$ for every $t \in [a, b[$.

4.3. The proof of Proposition 4.3. The proof of this paper is inspired by the paper [13] of P.G. Lemarie'-Rieusset. Let $(t_n)_n \in]0, \frac{T}{2}[$ such that $t_n \rightarrow 0$ as $n \rightarrow +\infty$. We consider the sequence of functions $(u_n)_n$ definite on $[0, \frac{T}{2}]$ by $u_n(t) = u(t_n + t)$. We have to prove that $\sup_{0 < t < \delta} \sqrt{t} \|u_n(t)\|_\infty$ converges to 0 as δ goes to 0 uniformly on n . Firstly, in order to simplify the writing, we introduce the following notations:

$$h_n(\mu, \delta) \equiv \sup_{0 < t < \delta} t^{\frac{\mu+1}{2}} \|u_n(t)\|_{B_\infty^{\mu, \infty}},$$

$$\Theta(\delta) = \sup_{0 < t < \frac{T}{2}} \|u\|_{L^{\frac{2}{1-r}}([t_0, t_0 + \delta], B_\infty^{-r, \infty})}.$$

Let $\sigma \in]r, 1[$ a fixed real number and $\delta_0 \in]0, \frac{T}{2}[$ to be chosen later. Let $n \in \mathbb{N}$, $\delta \in]0, \delta_0[$, and $t \in]0, \delta]$. Let $a = a(n, t)$ be an element of the interval $[\frac{t}{4}, \frac{t}{2}]$ such that

$$\|u_n(a)\|_{B_\infty^{-r, \infty}} = \inf_{s \in [\frac{t}{4}, \frac{t}{2}]} \|u_n(s)\|_{B_\infty^{-r, \infty}}.$$

Since u_n is a mild solution of the equations of Navier-Stokes then, according to Remark 2.3,

$$(4.3) \quad u_n(t) = e^{(t-a)\Delta} u_n(a) - \int_a^t e^{(t-s)\Delta} \mathbb{P} \nabla (u_n \otimes_n) ds$$

$$(4.4) \quad \equiv I_n(t) + J_n(t).$$

Now we will estimate the norm of the terms $I_n(t)$ and $J_n(t)$ in the Besov space $B_\infty^{\sigma, \infty}$. According to the first assertion of Proposition 2.4 and the definition of $a = a(n, t)$ we have

$$(4.5) \quad \begin{aligned} \|I_n(t)\|_{B_\infty^{\sigma, \infty}} &\lesssim t^{-\frac{\sigma+r}{2}} \|u_n(a)\|_{B_\infty^{-r, \infty}} \\ &\lesssim t^{-\frac{1+\sigma}{2}} \|u_n\|_{L^{\frac{2}{1-r}}([\frac{t}{4}, \frac{t}{2}], B_\infty^{-r, \infty})} \\ &\lesssim t^{-\frac{1+\sigma}{2}} \Theta(\delta). \end{aligned}$$

On the other hand since $J_n(t) = \mathbb{L}_{Oss}(1_{[a, t]} u_n \otimes 1_{[a, t]} u_n)$, then by using the continuity of the Bony para-product operators Π_k from $\tilde{L}_T^{\frac{2}{1-r}}(B_\infty^{-r, \infty}) \times L_T^\infty(B_\infty^{\sigma, \infty})$ to $\tilde{L}_T^{\frac{2}{1-r}}(B_\infty^{\sigma-r, \infty})$ (see Proposition 2.3) and the continuity of the operator \mathbb{L}_{Oss} from $\tilde{L}_T^{\frac{2}{1-r}}(B_\infty^{\sigma-r, \infty})$ to $L_T^\infty(B_\infty^{\sigma, \infty})$

(see Proposition 2.5) we easily deduce that

$$\begin{aligned}
\|J_n(t)\|_{B_\infty^{\sigma,\infty}} &\lesssim \|u_n\|_{\bar{L}^{\frac{2}{1-r}}([a,t],B_\infty^{-r,\infty})} \|u_n\|_{L^\infty([a,t],B_\infty^{\sigma,\infty})} \\
&\lesssim \|u_n\|_{\bar{L}^{\frac{2}{1-r}}([a,t],B_\infty^{-r,\infty})} \sup_{a \leq s \leq t} \|u_n(s)\|_{B_\infty^{\sigma,\infty}} \\
&\lesssim t^{-\frac{1+\sigma}{2}} \Theta(\delta) h_n(\sigma, \delta) \\
(4.6) \quad &\lesssim t^{-\frac{1+\sigma}{2}} \Theta(\delta_0) h_n(\sigma, \delta).
\end{aligned}$$

estimates (4.5) and (4.6) imply that there exists a constant $C_1 > 0$ independent of t, δ , and n such that

$$h_n(\sigma, \delta) \leq C_1 \Theta(\delta) + C_1 \Theta(\delta_0) h_n(\sigma, \delta).$$

Hence, by choosing δ_0 small enough such that $\Theta(\delta_0) \leq \frac{1}{2C_1}$ (which is always possible since $\Theta(\delta_0) \rightarrow 0$ as $\delta_0 \rightarrow 0$), we get

$$(4.7) \quad h_n(\sigma, \delta) \leq C_1 \Theta(\delta).$$

Now let us go back to (4.3) and (4.4) in order to estimate the norms of $I_n(t)$ and $J_n(t)$ in the Besov space $B_\infty^{-r,\infty}$. Firstly, in view of Proposition 2.4 and the definition of $a = a(n, t)$ we have

$$\begin{aligned}
\|I_n(t)\|_{B_\infty^{-r,\infty}} &\lesssim \|u_n(a)\|_{B_\infty^{-r,\infty}} \\
(4.8) \quad &\lesssim t^{-\frac{1-r}{2}} \Theta(\delta).
\end{aligned}$$

Secondly, by using the continuity of the operators Π_k from $B_\infty^{-r,\infty} \times B_\infty^{\sigma,\infty}$ to $B_\infty^{\sigma-r,\infty}$, the action of $\mathbb{P}\nabla$ on Besov spaces (Proposition 2.1), and the first assertion of Proposition 2.4, we get

$$\begin{aligned}
\|J_n(t)\|_{B_\infty^{-r,\infty}} &\lesssim \int_a^t \frac{1}{(t-s)^{\frac{1-\sigma}{2}}} \|\mathbb{P}\nabla(u_n \otimes u_n)\|_{B_\infty^{\sigma-r-1,\infty}} ds \\
&\lesssim t^{\frac{1+\sigma}{2}} \sup_{\frac{t}{4} < s < t} \|u_n(s)\|_{B_\infty^{-r,\infty}} \sup_{\frac{t}{4} < s < t} \|u_n(s)\|_{B_\infty^{\sigma,\infty}} \\
&\lesssim t^{-\frac{1-r}{2}} h_n(-r, \delta) h_n(\sigma, \delta) \\
(4.9) \quad &\lesssim t^{-\frac{1-r}{2}} h_n(-r, \delta) \Theta(\delta_0),
\end{aligned}$$

where we have used (4.7) in the last inequality.

Combining (4.8) and (4.9), we deduce the existence of an absolute constant $C_2 > 0$ independent of t, δ , and n such that

$$h_n(-r, \delta) \leq C_2 \Theta(\delta) + C_2 \Theta(\delta_0) h_n(-r, \delta).$$

Hence, for δ_0 small enough, we have

$$(4.10) \quad h_n(-r, \delta) \leq 2C_2\Theta(\delta).$$

Using now the following interpolation inequality [15]

$$\|f\|_\infty \leq (\|f\|_{B_\infty^{-r,\infty}})^{\frac{\sigma}{r+\sigma}} (\|f\|_{B_\infty^{\sigma,\infty}})^{\frac{r}{r+\sigma}},$$

we deduce from (4.7) and (4.10) that there exist two constants $C > 0$ and $\delta_0 \in]0, \frac{T}{2}]$ independent of n such that for every $\delta \in]0, \delta_0]$ we have

$$\sup_{0 < t < \delta} \sqrt{t} \|u_n(t)\|_\infty \leq C\Theta(\delta),$$

which completes the proof of Proposition 4.3.

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