

ON THE CONTINUITY OF THE HAUSDORFF DIMENSION OF THE UNIVOQUE SET

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ABSTRACT. In a recent paper [*Adv. Math.*, 305:165–196, 2017], Komornik et al. proved a long-conjectured formula for the Hausdorff dimension of the set \mathcal{U}_q of numbers having a unique expansion in the (non-integer) base q , and showed that this Hausdorff dimension is continuous in q . Unfortunately, their proof contained a gap which appears difficult to fix. This article gives a completely different proof of these results, using a more direct combinatorial approach.

1. INTRODUCTION

Fix an integer $M \geq 1$ and a real number $q \in (1, M + 1]$, and let $I_q := [0, M/(q - 1)]$. It is well known that every number $x \in I_q$ can be written in the form

$$(1.1) \quad x = \sum_{j=1}^{\infty} \frac{x_j}{q^j} =: \pi_q(x_1 x_2 \dots), \quad x_j \in \{0, 1, \dots, M\} \forall j.$$

We call such a representation a q -*expansion* of x . Such expansions were introduced by Rényi [24] and studied further by Parry [23]. They were then largely forgotten for about 30 years until Erdős et al. [11, 12] uncovered their fascinating mathematical structure. Since then, q -expansions have been the subject of a large number of research articles, many of which focus on the *univoque set*

$$\mathcal{U}_q := \{x \in I_q : x \text{ has a unique } q\text{-expansion of the form (1.1)}\}.$$

This set was shown to be of Lebesgue measure zero in [12], but its more detailed structure was first exposed in the remarkable paper by Glendinning and Sidorov [14]. For the case $M = 1$, they found that phase transitions occur at two critical values $q_G := (1 + \sqrt{5})/2$ and $q_{KL} \approx 1.78723$, as follows: \mathcal{U}_q is (i) the two-point set $\{0, M/(1 - q)\}$ for $1 < q \leq q_G$; (ii) countably infinite for $q_G \leq q < q_{KL}$; (iii) uncountable but of zero Hausdorff dimension for $q = q_{KL}$; and (iv) of positive Hausdorff dimension for $q_{KL} < q \leq M + 1$. The number q_{KL} is called the *Komornik-Loreti constant*; see Section 2 below for a precise definition. The above result was generalized to arbitrary $M \geq 1$ by Baker [7] and Kong et al. [22].

Let $\Omega := \{0, 1, \dots, M\}^{\mathbb{N}}$. The set \mathcal{U}_q is most easily understood by studying the symbolic univoque set

$$\mathbf{U}_q := \{(x_i) \in \Omega : \pi_q((x_i)) \in \mathcal{U}_q\} = \pi_q^{-1}(\mathcal{U}_q).$$

Date: 10th April 2018.

2010 Mathematics Subject Classification. Primary:11A63, Secondary: 37B10, 28A78.

Key words and phrases. Univoque set, Hausdorff dimension, topological entropy.

For any subset $X \subseteq \Omega$, we define the *topological entropy* of X by

$$h(X) := \lim_{k \rightarrow \infty} \frac{\log \#B_k(X)}{k} = \inf_{k \in \mathbb{N}} \frac{\log \#B_k(X)}{k},$$

where $B_k(X)$ is the set of all subwords of length k which occur in some sequence in X , and $\#B$ denotes the cardinality of a set B . The above limit always exists, and is equal to the infimum, since it is easily seen that $\log \#B_k(X)$ is subadditive as a function of k . When X is a subshift of the full shift Ω , $h(X)$ coincides with the dynamical notion of topological entropy.

In their paper, Glendinning and Sidorov suggested the formula

$$(1.2) \quad \dim_H \mathcal{U}_q = \frac{h(\mathbf{U}_q)}{\log q}$$

for the Hausdorff dimension of \mathcal{U}_q , and stated without proof that $h(\mathbf{U}_q)$ is continuous in q . Detailed proofs of these statements were recently given by Komornik et al. [18]. Unfortunately, as we explain in Remark 2.7 below, their proof contains a serious error, which is not easily fixed. Since the publication of [18], a number of papers (e.g. [2, 3, 4, 5]) have used the continuity of $\dim_H \mathcal{U}_q$ in a fundamental way, and it is therefore of crucial importance to have a complete proof of this result on record. Giving such a proof is the principal objective of this paper. We state our main results as follows:

Theorem 1.1. *The function $q \mapsto h(\mathbf{U}_q)$ is continuous on $(1, M + 1]$.*

Theorem 1.2. *For each $q \in (1, M + 1]$, the formula (1.2) holds.*

Our initial approach is the same as in [18]: we sandwich the set \mathbf{U}_q between two sets $\mathbf{U}_{q,n}$ and $\mathbf{V}_{q,n}$ and show that $h(\mathbf{U}_{q,n}) - h(\mathbf{V}_{q,n}) \rightarrow 0$ as $n \rightarrow \infty$. But, whereas the authors of [18] attempted to use the Perron-Frobenius theorem to compare the entropies of $\mathbf{U}_{q,n}$ and $\mathbf{V}_{q,n}$, we give instead a more direct combinatorial argument by constructing for each $k \in \mathbb{N}$ a map $f_{n,k}$ from $B_k(\mathbf{V}_{q,n})$ into $B_k(\mathbf{U}_{q,n})$ that is “not too many”-to-one; see Section 3 for the details. We observe that we use some results from [1], a paper which supercedes [18]. However, we emphasize that the continuity of $h(\mathbf{U}_q)$ is not used in [1].

Theorem 1.2 is a fairly direct consequence of Theorem 1.1, and is proved in Section 4.

2. SYMBOLIC UNIVOQUE SETS

In this section we will describe the symbolic univoque set \mathbf{U}_q and calculate its Hausdorff dimension. Let σ be the *left shift* on Ω defined by $\sigma((c_i)) = (c_{i+1})$. Then (Ω, σ) is a *full shift*. By a *word* \mathbf{c} we mean a finite string of digits $\mathbf{c} = c_1 \dots c_n$ with each digit $c_i \in \{0, 1, \dots, M\}$. For two words $\mathbf{c} = c_1 \dots c_m$ and $\mathbf{d} = d_1 \dots d_n$ we denote by $\mathbf{cd} = c_1 \dots c_m d_1 \dots d_n$ their concatenation. For a positive integer n we write $\mathbf{c}^n = \mathbf{c} \dots \mathbf{c}$ for the n -fold concatenation of \mathbf{c} with itself. Furthermore, we write $\mathbf{c}^\infty = \mathbf{c} \mathbf{c} \dots$ for the infinite periodic sequence with period block \mathbf{c} . For a word $\mathbf{c} = c_1 \dots c_m$ we set $\mathbf{c}^+ := c_1 \dots c_{m-1}(c_m + 1)$ if $c_m < M$, and set $\mathbf{c}^- := c_1 \dots c_{m-1}(c_m - 1)$ if $c_m > 0$. Furthermore, we define the *reflection* of the word \mathbf{c} by $\bar{\mathbf{c}} := (M - c_1)(M - c_2) \dots (M - c_m)$. Clearly, $\mathbf{c}^+, \mathbf{c}^-$ and $\bar{\mathbf{c}}$ are all words with digits from $\{0, 1, \dots, M\}$. For a sequence $(c_i) \in \Omega$ its reflection is also a sequence in Ω defined by $\overline{(c_i)} = (M - c_1)(M - c_2) \dots$.

For a subset $X \subseteq \Omega$, the *language* of X , denoted $\mathcal{L}(X)$, is the set of all finite words that occur in some sequence in X . So, $\mathcal{L}(X) = \bigcup_{k=1}^{\infty} B_k(X)$.

Throughout the paper we will use the lexicographical ordering \prec, \preceq, \succ and \succeq between sequences and words. More precisely, for two sequences $(c_i), (d_i) \in \Omega$ we say $(c_i) \prec (d_i)$ or $(d_i) \succ (c_i)$ if there exists an integer $n \geq 1$ such that $c_1 \dots c_{n-1} = d_1 \dots d_{n-1}$ and $c_n < d_n$. Furthermore, we write $(c_i) \preceq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$. Similarly, for two words \mathbf{c} and \mathbf{d} we say $\mathbf{c} \prec \mathbf{d}$ or $\mathbf{d} \succ \mathbf{c}$ if $\mathbf{c}0^\infty \prec \mathbf{d}0^\infty$.

Let $q \in (1, M + 1]$. The description of \mathbf{U}_q is based on the *quasi-greedy* q -expansion of 1, denoted by $\alpha(q) = \alpha_1(q)\alpha_2(q)\dots$, which is the lexicographically largest q -expansion of 1 not ending with 0^∞ (cf. [8]). The following characterization of $\alpha(q)$ was given in [10, Proposition 2.3].

Lemma 2.1. *The map $q \mapsto \alpha(q)$ is a strictly increasing bijection from $(1, M + 1]$ onto the set of all sequences $(a_i) \in \Omega$ not ending with 0^∞ and satisfying*

$$a_{n+1}a_{n+2}\dots \preceq a_1a_2\dots \quad \text{for all } n \geq 0.$$

Furthermore, the map $q \mapsto \alpha(q)$ is left-continuous.

The following lexicographic characterization of the symbolic univoque set \mathbf{U}_q was essentially established by Parry [23] (see also [18]).

Lemma 2.2. *Let $q \in (1, M + 1]$. Then $(x_i) \in \mathbf{U}_q$ if and only if*

$$\begin{cases} x_{n+1}x_{n+2}\dots \prec \overline{\alpha(q)} & \text{whenever } x_n < M, \\ x_{n+1}x_{n+2}\dots \succ \overline{\alpha(q)} & \text{whenever } x_n > 0. \end{cases}$$

By Lemmas 2.1 and 2.2 it follows that the set-valued map $q \mapsto \mathbf{U}_q$ is increasing, i.e., $\mathbf{U}_p \subseteq \mathbf{U}_q$ when $p < q$.

Next, we recall from [19] the definition of the Komornik-Loreti constant $q_{KL} = q_{KL}(M)$. Let $(\tau_i)_{i=0}^\infty = 0110100110010110\dots$ be the classical Thue-Morse sequence (cf. [6]). Then q_{KL} is given implicitly by

$$\alpha(q_{KL}) = \lambda_1\lambda_2\dots,$$

where for each $i \geq 1$,

$$\lambda_i = \lambda_i(M) := \begin{cases} k + \tau_i - \tau_{i-1} & \text{if } M = 2k, \\ k + \tau_i & \text{if } M = 2k + 1. \end{cases}$$

For example, $q_{KL}(1) \approx 1.78723$, $q_{KL}(2) \approx 2.53595$, $q_{KL}(3) \approx 2.91002$, etc.

We shall also need the set

$$\mathcal{U} := \{q \in (1, M + 1] : 1 \in \mathcal{U}_q\}.$$

This set is of Lebesgue measure zero, and $\min \mathcal{U} = q_{KL}$ (see [19]). The following characterizations of \mathcal{U} and its topological closure $\overline{\mathcal{U}}$ were established in [20] (see also [10]).

Lemma 2.3.

- (i) $q \in \mathcal{U}$ if and only if $\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \prec \alpha(q)$ for all $n \geq 1$.
- (ii) $q \in \overline{\mathcal{U}}$ if and only if $\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \preceq \alpha(q)$ for all $n \geq 1$.

The following definition was taken from [1, Definition 3.10].

Definition 2.4. Say a word $a_1 \dots a_m$ in $\mathcal{L}(\Omega)$ is *primitive* if

$$\overline{a_1 \dots a_{m-i}} \prec a_{i+1} \dots a_m \preceq a_1 \dots a_{m-i} \quad \text{for all } 0 \leq i < m.$$

For a proof of the next Lemma, see [20, Lemma 4.1].

Lemma 2.5. *Let $q \in \overline{\mathcal{U}}$. Then there are infinitely many positive integers n such that $\alpha_1(q) \dots \alpha_n(q)$ is primitive.*

Note that \mathbf{U}_q is in general not a subshift of Ω . Following [9] and [18] we introduce the set

$$\mathbf{V}_q := \left\{ (x_i) \in \Omega : \overline{\alpha(q)} \preceq x_{n+1}x_{n+2} \dots \preceq \alpha(q) \text{ for all } n \geq 0 \right\}.$$

Then \mathbf{V}_q is a subshift of Ω . Comparison of this definition with the characterization of \mathbf{U}_q in Lemma 2.2 suggests that \mathbf{U}_q and \mathbf{V}_q should have the same entropy. This is indeed the case:

Proposition 2.6. *For all $q \in (q_{KL}, M+1)$, $h(\mathbf{U}_q) = h(\mathbf{V}_q)$.*

Proof. We first show that $h(\mathbf{U}_q) \leq h(\mathbf{V}_q)$. By Lemma 2.2 it follows that for each $q \in (1, M+1]$ the set \mathbf{U}_q is contained in a countable union of affine copies of \mathbf{V}_q (see also [16, Lemma 3.2]), i.e., there exists a sequence of affine maps $\{g_i\}_{i=1}^\infty$ on Ω of the form

$$x_1x_2 \dots \mapsto ax_1x_2 \dots, \quad x_1x_2 \dots \mapsto M^m b x_1x_2 \dots \quad \text{or} \quad x_1x_2 \dots \mapsto 0^m c x_1x_2 \dots,$$

where $a \in \{1, 2, \dots, M-1\}$, $b \in \{0, 1, \dots, M-1\}$, $c \in \{1, 2, \dots, M\}$ and $m = 1, 2, \dots$, such that

$$(2.1) \quad \mathbf{U}_q \subseteq \bigcup_{i=1}^{\infty} g_i(\mathbf{V}_q).$$

Hence any word in $\mathcal{L}(\mathbf{U}_q)$ of length k is either itself in $\mathcal{L}(\mathbf{V}_q)$, or else has a prefix a , $M^m b$ or $0^m c$ followed by a word in $\mathcal{L}(\mathbf{V}_q)$, with a, b and c as above. Thus,

$$\begin{aligned} \#B_k(\mathbf{U}_q) &\leq \#B_k(\mathbf{V}_q) + (M-1)\#B_{k-1}(\mathbf{V}_q) + 2M \sum_{m=0}^{k-1} \#B_{k-m-1}(\mathbf{V}_q) \\ &\leq (2k+1)M\#B_k(\mathbf{V}_q) \end{aligned}$$

for all k , and hence $h(\mathbf{U}_q) \leq h(\mathbf{V}_q)$.

For the reverse inequality, note that $\mathbf{V}_q \setminus \mathbf{U}_q$ contains only sequences ending in $\alpha(q)$ or $\overline{\alpha(q)}$. Hence $\mathbf{V}_q \setminus \mathbf{U}_q$ is countable. This does not immediately imply that $h(\mathbf{V}_q) \leq h(\mathbf{U}_q)$, since the topological entropy of a countable set can be positive. To verify the inequality rigorously, it suffices in view of Lemma 2.3 to consider the following three cases:

Case 1: $q \in \overline{\mathcal{U}}$. We show that in this case,

$$(2.2) \quad B_k(\mathbf{V}_q) \subseteq B_k(\mathbf{U}_q) \quad \text{for all } k \in \mathbb{N}.$$

Let $x_1 \dots x_k \in B_k(\mathbf{V}_q)$. Then there is a sequence $(y_i) \in \mathbf{V}_q$ and an index $j \in \mathbb{N}$ such that $y_{j+1} \dots y_{j+k} = x_1 \dots x_k$. Since $(y_i) \in \mathbf{V}_q$, we have

$$\overline{\alpha(q)} \preceq y_{i+1}y_{i+2} \dots \preceq \alpha(q) \quad \text{for all } i \geq 0.$$

Suppose without loss of generality that $y_{i+1}y_{i+2} \dots = \alpha(q)$ for some i . Write $\alpha(q) = \alpha_1\alpha_2 \dots$. Since $q \in \overline{\mathcal{U}}$, there is by Lemma 2.5 an index $n > \max\{i, j+k\}$ such that $\alpha_1 \dots \alpha_n$ is primitive. Consider the sequence

$$\begin{aligned} z_1 z_2 \dots &= y_1 \dots y_j x_1 \dots x_k y_{j+k+1} \dots y_{n+i}^- (\alpha_1 \dots \alpha_n^-)^\infty \\ &= y_1 \dots y_i (\alpha_1 \dots \alpha_n^-)^\infty. \end{aligned}$$

Then $z_1 z_2 \cdots \in \mathbf{U}_q$. Hence $x_1 \dots x_k \in B_k(\mathbf{U}_q)$, proving (2.2).

Case 2: $\sigma^n(\alpha(q)) \prec \overline{\alpha(q)}$ for some $n \geq 1$. Then it is not possible for a sequence in \mathbf{V}_q to end in $\alpha(q)$ or $\overline{\alpha(q)}$ in view of the definition of \mathbf{V}_q . Hence, $\mathbf{V}_q \subseteq \mathbf{U}_q$.

Case 3: $\sigma^n(\alpha(q)) = \overline{\alpha(q)}$ for some $n \geq 1$. This means that, with $v := \alpha_1 \dots \alpha_n$, $\alpha(q) = (v\bar{v})^\infty$. So, if $(x_i) \in \mathbf{V}_q$ and $x_{j+1} \dots x_{j+n} = v$, it must be the case that $\sigma^j((x_i)) = (v\bar{v})^\infty$; and likewise, if $x_{j+1} \dots x_{j+n} = \bar{v}$ then $\sigma^j((x_i)) = (\bar{v}v)^\infty$. Thus, any word in $\mathcal{L}(\mathbf{V}_q)$ consists of a word in $\mathcal{L}(\mathbf{U}_q)$ followed by v or \bar{v} , followed in turn by a forced suffix. As a result,

$$\#B_k(\mathbf{V}_q) \leq 2 \sum_{j=0}^k \#B_j(\mathbf{U}_q) \leq 2(k+1)\#B_k(\mathbf{U}_q)$$

for all $k \in \mathbb{N}$, and therefore, $h(\mathbf{V}_q) \leq h(\mathbf{U}_q)$. \square

As a final preparation for the proof of Theorem 1.1, we define for each $n \in \mathbb{N}$ the sets

$$\mathbf{U}_{q,n} := \{(x_i) \in \Omega : \overline{a_1 \dots a_n} \prec x_{j+1} \dots x_{j+n} \prec a_1 \dots a_n \text{ for all } j \geq 0\}$$

and

$$\mathbf{V}_{q,n} := \{(x_i) \in \Omega : \overline{a_1 \dots a_n} \preceq x_{j+1} \dots x_{j+n} \preceq a_1 \dots a_n \text{ for all } j \geq 0\},$$

where we write $\alpha(q) = a_1 a_2 \dots$. Then $(\mathbf{U}_{q,n}, \sigma)$ and $(\mathbf{V}_{q,n}, \sigma)$ are both subshifts of finite type for any $n \geq 1$. Observe from [18, Lemma 2.7] that

$$\mathbf{U}_{q,n} \subseteq \mathbf{V}_q \subseteq \mathbf{V}_{q,n} \quad \text{for all } n \geq 1.$$

Furthermore, the set sequence $(\mathbf{U}_{q,n})$ is nondecreasing and the set sequence $(\mathbf{V}_{q,n})$ is nonincreasing.

Remark 2.7 (The error in the proof of Komornik, Kong and Li). The authors of [18] applied the Perron-Frobenius theorem to the edge graph representation $G(n)$ of $\mathbf{U}_{q,n}$ to obtain constants c_1 and c_2 such that

$$c_1 \lambda_n^k \leq \#B_k(\mathbf{U}_{q,n}) \leq c_2 k^s \lambda_n^k,$$

where λ_n is the spectral radius, and s the number of strongly connected components, of $G(n)$. However, later in their proof they treat c_1 and c_2 as absolute constants, whereas in fact they depend on n . The method of proof in [18] could be saved by finding good bounds on the growth rate of $c_2(n)$, but this turns out to be very difficult to do. Despite our best efforts, we have not been able to accomplish this; hence our resort to the combinatorial method of Section 3 below.

Let $H : (1, M+1] \rightarrow [0, \infty)$ be the map

$$H(q) := h(\mathbf{U}_q).$$

The right continuity of H is easy to prove:

Proposition 2.8. *The function H is right continuous on $(1, M+1]$.*

Proof. By [21, Theorem 2.6] it follows that H is constant on each connected component of $(1, M+1] \setminus \mathcal{U} = (1, q_{KL}) \cup \bigcup [q_0, q_0^*)$. So, we only need to prove the right continuity of H on

\mathcal{U} . Take $q \in \mathcal{U}$, then by Lemma 2.3 (i) $\alpha(q) = \beta(q)$, where $\beta(q)$ is the greedy q -expansion of 1. We first show that

$$(2.3) \quad \lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}) \leq h(\mathbf{V}_q).$$

By [18, Lemma 2.10], $B_n(\mathbf{V}_{q,n}) = B_n(\mathbf{V}_q)$ for each n . Hence

$$h(\mathbf{V}_{q,n}) = \inf_{k \in \mathbb{N}} \frac{\log \#B_k(\mathbf{V}_{q,n})}{k} \leq \frac{\log \#B_n(\mathbf{V}_{q,n})}{n} = \frac{\log \#B_n(\mathbf{V}_q)}{n},$$

and letting $n \rightarrow \infty$ gives (2.3).

Next, we can choose for each $n \in \mathbb{N}$ a base $p_n \in (q, M+1)$ sufficiently close to q such that $\alpha_i(p_n) = \alpha_i(q)$ for $i = 1, \dots, n$. Since $\alpha(q) = \beta(q)$, we have

$$\mathbf{V}_q \subseteq \mathbf{V}_p \subseteq \mathbf{V}_{q,n} \quad \text{for all } p \in (q, p_n).$$

It follows by (2.3) that $\lim_{p \searrow q} h(\mathbf{V}_p) = h(\mathbf{V}_q)$, and then also $\lim_{p \searrow q} h(\mathbf{U}_p) = h(\mathbf{U}_q)$ in view of Proposition 2.6. \square

The proof of left continuity of H is much more involved, and the next section is entirely devoted to this task.

3. LEFT CONTINUITY OF H

Let \mathcal{B} be the *bifurcation set* of the entropy function H , defined by

$$\mathcal{B} = \mathcal{B}(M) := \{q \in (1, M+1] : H(p) \neq H(q) \text{ for any } p \neq q\}.$$

Alcaraz Barrera et al. [1] proved that $\mathcal{B} \subseteq \mathcal{U}$, and hence \mathcal{B} is of zero Lebesgue measure. They also showed that \mathcal{B} has full Hausdorff dimension. Furthermore, \mathcal{B} has no isolated points and its complement can be written as

$$(1, M+1] \setminus \mathcal{B} = (1, q_{KL}] \cup \bigcup [p_L, p_R],$$

where the union on the right hand side is countable and pairwise disjoint. From the definition of \mathcal{B} it follows that each connected component $[p_L, p_R]$ is a maximal interval on which H is constant. Each closed interval $[p_L, p_R]$ is therefore called an *entropy plateau*. Observe that H is trivially left continuous on each half open connected component $(p_L, p_R]$ (including the first connected component $(1, q_{KL}]$). Hence it suffices to prove the left-continuity of H at any

$$q \in (q_{KL}, M+1] \setminus \bigcup (p_L, p_R] =: \mathcal{B}^L.$$

(The ‘‘left bifurcation set’’ \mathcal{B}^L was introduced and studied in [5].)

Theorem 3.1. *For any $q \in \mathcal{B}^L$ we have*

$$(3.1) \quad \lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}) = \lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n}).$$

Corollary 3.2. *The function H is left continuous on \mathcal{B}^L .*

Before proving Theorem 3.1, we show how to derive the corollary.

Proof of Corollary 3.2. Fix $q \in \mathcal{B}^L$. For each n we can choose a base $p_n \in (1, q)$ sufficiently close to q so that $\alpha_i(p_n) = \alpha_i(q)$ for $i = 1, \dots, n$. Then

$$\mathbf{U}_{q,n} \subseteq \mathbf{V}_p \subseteq \mathbf{V}_q \subseteq \mathbf{V}_{q,n} \quad \text{for all } p \in (p_n, q).$$

By (3.1) and the above inclusions, $\lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n}) = h(\mathbf{V}_q)$. Hence, $\lim_{p \nearrow q} h(\mathbf{V}_p) = h(\mathbf{V}_q)$, and then also $\lim_{p \nearrow q} h(\mathbf{U}_p) = h(\mathbf{U}_q)$ in view of Proposition 2.6. \square

Our approach to proving Theorem 3.1 is to construct, for arbitrarily large numbers n and for all $k \in \mathbb{N}$, a map $f_{n,k} : B_k(\mathbf{V}_{q,n}) \rightarrow B_k(\mathbf{U}_{q,n})$ that is “not too many”-to-one. (We will specify later what “not too many” means.) This will show that the set $B_k(\mathbf{V}_{q,n})$ is not too much larger than $B_k(\mathbf{U}_{q,n})$, and as a consequence, $h(\mathbf{V}_{q,n})$ is not too much larger than $h(\mathbf{U}_{q,n})$.

Recall the definition of a primitive word from Definition 2.4.

Lemma 3.3. *Let $q \in \mathcal{B}^L$ with $\alpha(q) = (a_i)$.*

(i) *There exist infinitely many integers n such that*

$$(3.2) \quad a_1 \dots a_n \text{ is primitive and } (a_1 \dots a_n^-)^\infty \succ \alpha(q_{KL}).$$

(ii) *If (3.2) holds, then $a_1 \dots a_n(\overline{a_1 \dots a_n^+})^\infty \prec (a_i)$.*

Proof. Since $\mathcal{B}^L \subseteq \overline{\mathcal{U}}$, by Lemma 2.5 there are infinitely many $n \in \mathbb{N}$ such that $a_1 \dots a_n$ is primitive. Furthermore, for all large enough n , $(a_1 \dots a_n^-)^\infty \succ \alpha(q_{KL})$ since $q > q_{KL}$. So, it suffices to show that for such a large integer n we have $a_1 \dots a_n(\overline{a_1 \dots a_n^+})^\infty \prec \alpha(q)$.

Take such a large integer n , and let $[q_L, q_R]$ be the interval determined by

$$\alpha(q_L) = (a_1 \dots a_n^-)^\infty \quad \text{and} \quad \alpha(q_R) = a_1 \dots a_n(\overline{a_1 \dots a_n^+})^\infty.$$

By a similar argument as in the proofs of [1, Lemmas 5.1 and 5.5] one can show that H is constant on $[q_L, q_R]$. Since $q > q_L$, by the definition of \mathcal{B}^L it follows that $q > q_R$, and hence $a_1 \dots a_n(\overline{a_1 \dots a_n^+})^\infty \prec \alpha(q)$ by Lemma 2.1. \square

Definition 3.4. For $q \in \mathcal{B}^L$ with $\alpha(q) = (a_i)$, let

$$\mathcal{N}(q) := \{n \in \mathbb{N} : a_1 \dots a_n \text{ is primitive and } (a_1 \dots a_n^-)^\infty \succ \alpha(q_{KL})\}.$$

Take $q \in \mathcal{B}^L$ and fix $m \in \mathcal{N}(q)$. By Lemma 3.3 (ii) it follows that $a_1 \dots a_m(\overline{a_1 \dots a_m^+})^\infty \prec \alpha(q)$. So, there exist integers $l = l(m) \geq 0$ and $r = r(m) \in \{1, \dots, m\}$ such that for $n = n(m) := m(l+1) + r$,

$$(3.3) \quad \begin{aligned} a_1 \dots a_{n-1} &= a_1 \dots a_m(\overline{a_1 \dots a_m^+})^l \overline{a_1 \dots a_{r-1}}, \\ a_n &> \overline{a_r} \quad \text{if } r < m, \\ a_n &> \overline{a_m^+} \quad \text{if } r = m. \end{aligned}$$

We point out that the integers l, r and n all depend on m (and, of course, on q). However, most of the time the base $q \in \mathcal{B}^L$ is fixed, and if the value of m is implicitly understood we will write l, r and n instead of $l(m), r(m)$ and $n(m)$.

Lemma 3.5. *Let $q \in \mathcal{B}^L$ with $\alpha(q) = (a_i)$. Suppose $m \in \mathcal{N}(q)$ and $n = m(l+1) + r$ as in (3.3). The following statements hold.*

(i) $a_{n-m+1} \dots a_n^- \succ \overline{a_1 \dots a_m}$.

- (ii) $n \in \mathcal{N}(q)$. Thus, $a_1 \dots a_n$ is primitive and $a_1 \dots a_n (\overline{a_1 \dots a_n})^+ \prec \alpha(q)$.
 (iii) $a_1 \dots a_r$ is primitive.

Proof. First we prove (i). From (3.3) we see that if $r = m$, then $a_{n-m+1} \dots a_n^- \succ \overline{a_1 \dots a_m}^+ \succ \overline{a_1 \dots a_m}$. If $r < m$ with $l > 0$, then $n - m = ml + r$, and it follows by primitivity of $a_1 \dots a_m$ that

$$a_{n-m+1} \dots a_{n-r} = \overline{a_{r+1} \dots a_m}^+ \succ \overline{a_1 \dots a_{m-r}},$$

which implies $a_{n-m+1} \dots a_n^- \succ \overline{a_1 \dots a_m}$. Furthermore, if $r < m$ with $l = 0$, then we deduce from the primitivity of $a_1 \dots a_m$ that

$$a_{n-m+1} \dots a_{n-r} = a_{r+1} \dots a_m \succ \overline{a_1 \dots a_{m-r}}.$$

Again this yields $a_{n-m+1} \dots a_n^- \succ \overline{a_1 \dots a_m}$. So (i) holds.

For (ii), since $(a_1 \dots a_n^-)^\infty \succ (a_1 \dots a_m^-)^\infty \succ \alpha(q_{KL})$, by Lemma 3.3 (ii) it suffices to prove that $a_1 \dots a_n$ is primitive. By Lemma 2.1,

$$a_{j+1} \dots a_n^- \prec a_{j+1} \dots a_n \preceq a_1 \dots a_{n-j} \quad \text{for all } 0 \leq j < n,$$

so by Definition 2.4 it suffices to prove that

$$(3.4) \quad a_{j+1} \dots a_n^- \succ \overline{a_1 \dots a_{n-j}} \quad \text{for all } 0 \leq j < n.$$

Note by (3.3) that $n > m$, and $a_1 \dots a_n^-$ begins with $a_1 \dots a_m$. The primitivity of $a_1 \dots a_m$ gives

$$a_{j+1} \dots a_m \succ \overline{a_1 \dots a_{m-j}} \quad \text{and} \quad \overline{a_{j+1} \dots a_m}^+ \succ \overline{a_1 \dots a_{m-j}}$$

for all $0 \leq j < m$. The first inequality gives (3.4) for $0 \leq j < m$; the second inequality implies (3.4) for $m \leq j < n$, using (3.3). This establishes (ii).

Now we turn to prove (iii). If $r = m$, then (iii) follows trivially from the primitivity of $a_1 \dots a_m$. Suppose $r < m$. It suffices to prove

$$a_{i+1} \dots a_r^- \succ \overline{a_1 \dots a_{r-i}} \quad \text{for all } 0 \leq i < r.$$

This follows from (ii) and (3.3), since

$$\overline{a_{i+1} \dots a_r^-} \preceq a_{n-r+i+1} \dots a_n \preceq a_1 \dots a_{r-i}$$

for all $0 \leq i < r$. □

From now on, we will fix a base $q \in \mathcal{B}^L$.

Fix a number $k > n$. We will construct a map $f_{n,k} : B_k(\mathbf{V}_{q,n}) \rightarrow B_k(\mathbf{U}_{q,n})$ and show that this map is “not too many”-to-one. The map $f_{n,k}$ will be defined as the k th iterate of an auxiliary function $F_{\mathbf{u},\mathbf{v}}$; see Definition 3.7 below. We also use the following notation:

Definition 3.6. For a primitive word $\mathbf{u} = a_1 \dots a_n$ and a word $\mathbf{x} = x_1 \dots x_k \in B_k(\mathbf{V}_{q,n})$, let

$$i_{\mathbf{u}}(\mathbf{x}) := \min\{i \leq k - n : x_{i+1} \dots x_{i+n} = \mathbf{u} \text{ or } x_{i+1} \dots x_{i+n} = \overline{\mathbf{u}}\},$$

or $i_{\mathbf{u}}(\mathbf{x}) := \infty$ if no such i exists.

Thus, $i_{\mathbf{u}}(\mathbf{x})$ indicates where the word \mathbf{u} or $\overline{\mathbf{u}}$ occurs for the first time in the word \mathbf{x} . Note that $i_{\mathbf{u}}(\mathbf{x}) = \infty$ if and only if $\mathbf{x} \in B_k(\mathbf{U}_{q,n})$.

Definition 3.7. Let $\mathbf{u} = a_1 \dots a_n$ be primitive, and let $\mathbf{v} = a_1 \dots a_m$ be a primitive prefix of \mathbf{u} . Write $\mathbf{u} = \mathbf{vz}$. Then we define the map $F_{\mathbf{u},\mathbf{v}} : B_k(\mathbf{V}_{q,n}) \rightarrow \{0, 1, \dots, M\}^k$ as follows:

- (1) If $\mathbf{x} = x_1 \dots x_k \in B_k(\mathbf{V}_{q,n})$ does not contain the word \mathbf{u} or $\bar{\mathbf{u}}$, then set $F_{\mathbf{u},\mathbf{v}}(\mathbf{x}) := \mathbf{x}$.
(2) Otherwise, let $i := i_{\mathbf{u}}(\mathbf{x})$.

- If $x_{i+1} \dots x_{i+n} = \mathbf{u}$, then we put

$$F_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = F_{\mathbf{u},\mathbf{v}}(x_1 \dots x_i \mathbf{v} \mathbf{z} x_{i+n+1} \dots x_k) := x_1 \dots x_i \mathbf{v}^- \overline{\mathbf{z} x_{i+n+1} \dots x_k}.$$

- If $x_{i+1} \dots x_{i+n} = \bar{\mathbf{u}}$, then we put

$$F_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = F_{\mathbf{u},\mathbf{v}}(x_1 \dots x_i \bar{\mathbf{v}} \mathbf{z} x_{i+n+1} \dots x_k) := x_1 \dots x_i \bar{\mathbf{v}}^+ \overline{\mathbf{z} x_{i+n+1} \dots x_k}.$$

From Definition 3.7 it follows that

$$F_{\mathbf{u},\mathbf{v}}(\bar{\mathbf{x}}) = \overline{F_{\mathbf{u},\mathbf{v}}(\mathbf{x})} \quad \text{for any } \mathbf{x} \in B_k(\mathbf{V}_{q,n}).$$

In each of the cases worked out below, the key is to choose \mathbf{u} and \mathbf{v} carefully and show that $F_{\mathbf{u},\mathbf{v}}$ maps $B_k(\mathbf{V}_{q,n})$ into itself, so that the k th iterate $F_{\mathbf{u},\mathbf{v}}^k$ is well defined and maps $B_k(\mathbf{V}_{q,n})$ into $B_k(\mathbf{U}_{q,n})$.

3.1. Construction of $f_{n,k}$: the first case. Assume first that $l(m) > 0$ for infinitely many $m \in \mathcal{N}(q)$. Take $q \in \mathcal{B}^L$ and fix $m \in \mathcal{N}(q)$ with $l = l(m) > 0$. Let $n = m(l+1) + r$ as in (3.3). Write

$$\mathbf{u} := a_1 \dots a_n = \mathbf{v}(\bar{\mathbf{v}}^+)^l \mathbf{w}, \quad \text{where } \mathbf{v} := a_1 \dots a_m, \quad \mathbf{w} := a_{n-r+1} \dots a_n.$$

With \mathbf{u} and \mathbf{v} as above, we set $F := F_{\mathbf{u},\mathbf{v}}$. The following lemma shows that F is a map from $B_k(\mathbf{V}_{q,n})$ to $B_k(\mathbf{V}_{q,n})$.

Lemma 3.8. *For any $\mathbf{x} \in B_k(\mathbf{V}_{q,n})$ we have $F(\mathbf{x}) \in B_k(\mathbf{V}_{q,n})$, and*

$$i_{\mathbf{u}}(F(\mathbf{x})) \geq i_{\mathbf{u}}(\mathbf{x}) + \frac{n}{2}.$$

Proof. Let $i := i_{\mathbf{u}}(\mathbf{x})$. By symmetry we may assume $x_{i+1} \dots x_{i+n} = \mathbf{u}$, so

$$\mathbf{x} = x_1 \dots x_i \mathbf{v}(\bar{\mathbf{v}}^+)^l \mathbf{w} x_{i+n+1} \dots x_k.$$

Let $F(\mathbf{x}) = y_1 \dots y_k$. By Definition 3.7 it follows that

$$(3.5) \quad y_1 \dots y_k = x_1 \dots x_i (\mathbf{v}^-)^{l+1} \overline{\mathbf{w} x_{i+n+1} \dots x_k}.$$

Since the entire word $x_{i+m+1} \dots x_k$ is being reflected by F , no word strictly greater than \mathbf{u} or strictly smaller than $\bar{\mathbf{u}}$ can occur in $y_{i+m+1} \dots y_k$. To prove the lemma, therefore, it is necessary and sufficient to show that for each $j < i + (n/2)$,

$$\bar{\mathbf{u}} \prec y_{j+1} \dots y_{j+n} \prec \mathbf{u}.$$

Note by (3.5) that $y_1 \dots y_{i+m} = x_1 \dots x_{i+m}$. By the minimality of i it follows that $y_{j+1} \dots y_{j+n} = x_{j+1} \dots x_{j+n} \prec \mathbf{u}$ for all $0 \leq j < i + m - n$. Furthermore, for $i + m - n \leq j < i + m$ we have $y_{j+1} \dots y_{j+n} \prec x_{j+1} \dots x_{j+n} \preceq \mathbf{u}$. So, $y_{j+1} \dots y_{j+n} \prec \mathbf{u}$ for all $j < i + m$. And for $i + m \leq j < i + (n/2)$, we have $j + m < i + n$ since $n > 2m$. Then the same inequality follows since \mathbf{v} is primitive.

Proving the other inequality,

$$(3.6) \quad y_{j+1} \dots y_{j+n} \succ \bar{\mathbf{u}} \quad \text{for all } j < i + \frac{n}{2},$$

is more involved. First, by the minimality of i it follows that

$$y_{j+1} \dots y_{j+n} = x_{j+1} \dots x_{j+n} \succ \bar{\mathbf{u}} \quad \text{for all } 0 \leq j < i + m - n.$$

So it remains to prove (3.6) for $j \geq i + m - n$. We consider four cases (see Figure 1):

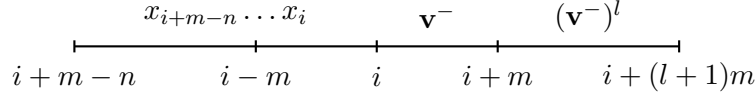


FIGURE 1. The presentation of $y_{i+m-n} \dots y_{i+(l+1)m} = x_{i+m-n} \dots x_i (\mathbf{v}^-)^{l+1}$.

- (I). $j = i + m - n$. Then $y_{j+1} \dots y_{j+n} = x_{j+1} \dots x_{i+m} = x_{j+1} \dots x_i a_1 \dots a_m^-$. Note that $y_{j+1} \dots y_i = x_{j+1} \dots x_i \succcurlyeq \overline{a_1 \dots a_{i-j}} = \overline{a_1 \dots a_{n-m}}$. Furthermore, by Lemma 3.5 (i) it follows that

$$y_{i+1} \dots y_{j+n} = a_1 \dots a_m^- \succcurlyeq \overline{a_{n-m+1} \dots a_n}.$$

This proves (3.6) for $j = i + m - n$.

- (II). $i + m - n < j \leq i - m$. Then $n > 2m$. Write $j + n = i + tm + s$ with $t \in \{1, \dots, l\}$ and $s \in \{1, \dots, m\}$. Then it follows from (3.5) that

$$(3.7) \quad \begin{aligned} y_{j+1} \dots y_i &= x_{j+1} \dots x_i, \\ y_{i+1} \dots y_{i+tm} &= (a_1 \dots a_m^-)^t, \end{aligned}$$

and

$$y_{i+tm+1} \dots y_{j+n} = \begin{cases} a_1 \dots a_s & \text{if } s < m, \\ a_1 \dots a_m^- & \text{if } s = m. \end{cases}$$

Note that $y_{j+1} \dots y_i = x_{j+1} \dots x_i \succcurlyeq \overline{a_1 \dots a_{i-j}}$. Observe that $\mathbf{u} = a_1 \dots a_n = a_1 \dots a_m (\overline{a_1 \dots a_m^+})^l \mathbf{w}$ and $m \leq i - j < n - m$. Then by (3.3) $\overline{a_{i-j+1} \dots a_{n-s}}$ is a subword of $(a_1 \dots a_m^-)^l$, so using (3.7) and the primitivity of $a_1 \dots a_m$ it follows that

$$(3.8) \quad y_{j+1} \dots y_{i+tm} = x_{j+1} \dots x_i (a_1 \dots a_m^-)^t \succcurlyeq \overline{a_1 \dots a_{i-j+tm}} = \overline{a_1 \dots a_{n-s}}.$$

Since by Lemma 3.5 (ii) $\mathbf{u} = a_1 \dots a_n$ is primitive, we also have that when $s < m$,

$$y_{i+tm+1} \dots y_{j+n} = a_1 \dots a_s \succcurlyeq \overline{a_{n-s+1} \dots a_n}.$$

On the other hand, when $s = m$ we have by Lemma 3.5 (i) that

$$y_{i+tm+1} \dots y_{j+n} = a_1 \dots a_m^- \succcurlyeq \overline{a_{n-m+1} \dots a_n}.$$

Combining this with (3.8) gives $y_{j+1} \dots y_{j+n} \succcurlyeq \overline{\mathbf{u}}$. This proves (3.6) for $i + m - n < j \leq i - m$.

- (III). $i - m < j < i$. Then $y_{j+1} \dots y_i = x_{j+1} \dots x_i \succcurlyeq \overline{a_1 \dots a_{i-j}}$. Note that $i - j < m$. Then by the primitivity of $a_1 \dots a_m$ it follows that

$$y_{i+1} \dots y_{j+m} = a_1 \dots a_{j+m-i} \succcurlyeq \overline{a_{i-j+1} \dots a_m}.$$

This proves (3.6) for $i - m < j < i$.

- (IV). $i \leq j < i + (n/2)$. Recall that we are assuming $l > 0$. Then $j + m < i + n$. By (3.5) we have

$$y_{j+1} \dots y_{j+m} = a_{t+1} \dots a_m^- a_1 \dots a_t \quad \text{for some } 0 \leq t < m.$$

By the primitivity of $a_1 \dots a_m$ it follows that $y_{j+1} \dots y_{j+m} \succcurlyeq \overline{a_1 \dots a_m}$. Hence (3.6) holds for $i \leq j < i + (n/2)$.

We have now shown (3.6) for all $j < i + (n/2)$. The proof is complete. \square

As a result of Lemma 3.8, for some large enough j (with $j < k$) we have $F^k(\mathbf{x}) = \dots = F^{j+1}(\mathbf{x}) = F^j(\mathbf{x})$.

Definition 3.9. We define

$$f_{n,k}(\mathbf{x}) := F^k(\mathbf{x}), \quad \mathbf{x} \in B_k(\mathbf{V}_{q,n}).$$

Observe that $F(f_{n,k}(\mathbf{x})) = f_{n,k}(\mathbf{x})$, so $f_{n,k}(\mathbf{x})$ does not contain the word \mathbf{u} or $\bar{\mathbf{u}}$. Hence, $f_{n,k}$ maps $B_k(\mathbf{V}_{q,n})$ into $B_k(\mathbf{U}_{q,n})$.

Proposition 3.10. Let $q \in \mathcal{B}^L$. If $l(m) > 0$ for infinitely many $m \in \mathcal{N}(q)$, then

$$\lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}) = \lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n}).$$

Proof. Let $\alpha(q) = (a_i)$ and $m \in \mathcal{N}(q)$ such that $l = l(m) > 0$. Write $n = m(l+1) + r$ as in (3.3), and $\mathbf{u} = a_1 \dots a_n = \mathbf{v}(\bar{\mathbf{v}}^+)^l \mathbf{w}$. For $k > n$ we take an arbitrary word $\mathbf{y} := y_1 \dots y_k$ in $B_k(\mathbf{U}_{q,n})$, and a subword $y_{i+1} \dots y_{i+N}$ of length $N := \lfloor n/2 \rfloor$. For convenience, and without loss of generality, we assume that k is a multiple of N . Let us consider the possible subwords $x_{i+1} \dots x_{i+N}$ of words $\mathbf{x} = x_1 \dots x_k$ with $f_{n,k}(\mathbf{x}) = \mathbf{y}$. Two such words are of course $y_{i+1} \dots y_{i+N}$ and $\overline{y_{i+1} \dots y_{i+N}}$. However, it is also possible that $x_{i+1} \dots x_{i+N}$ contains an occurrence of \mathbf{v} (or $\bar{\mathbf{v}}$) that is the beginning of an occurrence of \mathbf{u} (or $\bar{\mathbf{u}}$) and is therefore replaced by the map F with \mathbf{v}^- (or $\bar{\mathbf{v}}^+$). Note that later iterations of F do not change this block, in view of Lemma 3.8. Since there are at most $N - m$ possible starting points for \mathbf{v} (or $\bar{\mathbf{v}}$) and $m \geq 1$, it follows that there are at most $2N$ possible subwords $x_{i+1} \dots x_{i+N}$ which get mapped by $f_{n,k}$ to $y_{i+1} \dots y_{i+N}$.

Applying this argument to each of the k/N blocks $y_1 \dots y_N, y_{N+1} \dots y_{2N}, \dots, y_{k-N+1} \dots y_k$, we conclude that there are at most $(2N)^{k/N}$ different words $\mathbf{x} \in B_k(\mathbf{V}_{q,n})$ with $f_{n,k}(\mathbf{x}) = \mathbf{y}$. Thus, the map $f_{n,k}$ is at most $(2N)^{k/N}$ -to-one. It follows that

$$\#B_k(\mathbf{V}_{q,n}) \leq (2N)^{k/N} \#B_k(\mathbf{U}_{q,n}),$$

and so

$$\frac{\log \#B_k(\mathbf{V}_{q,n})}{k} \leq \frac{\log \#B_k(\mathbf{U}_{q,n})}{k} + \frac{\log 2N}{N}.$$

Letting $k \rightarrow \infty$ we get

$$(3.9) \quad h(\mathbf{V}_{q,n}) \leq h(\mathbf{U}_{q,n}) + \frac{\log 2N}{N} \leq h(\mathbf{U}_{q,n}) + \frac{\log n}{\lfloor n/2 \rfloor}.$$

Hence, if there are infinitely many $m \in \mathcal{N}(q)$ with $l(m) > 0$, then there are also infinitely many $n \in \mathcal{N}(q)$ such that (3.9) holds. We can then let $n \rightarrow \infty$ along a suitable subsequence in $\mathcal{N}(q)$, and conclude that

$$\lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}) \leq \lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n}),$$

using the fact that $h(\mathbf{U}_{q,n})$ is nondecreasing in n , and $h(\mathbf{V}_{q,n})$ is nonincreasing in n . \square

3.2. Construction of $f_{n,k}$: the second case. Next, we assume that $q \in \mathcal{B}^L$ and $l(m) = 0$ for all but finitely many $m \in \mathcal{N}(q)$. Let

$$m_1 := \min \{m \in \mathcal{N}(q) : l(m') = 0 \text{ for all } m' \in \mathcal{N}(q) \text{ with } m' \geq m\}.$$

Note that $m_1 \in \mathcal{N}(q)$. Write $\mathbf{v}_1 := a_1 \dots a_{m_1}$. Then \mathbf{v}_1 is primitive, and $\mathbf{v}_1(\overline{\mathbf{v}_1^+})^\infty \prec \alpha(q)$. So, since $l(m_1) = 0$, in view of (3.3) there exists a word \mathbf{w}_1 of shortest length $r_1 := |\mathbf{w}_1| \geq 1$ such that $\mathbf{v}_1 \mathbf{w}_1 0^\infty \succ \mathbf{v}_1(\overline{\mathbf{v}_1^+})^\infty$. Note that $1 \leq r_1 = |\mathbf{w}_1| \leq |\mathbf{v}_1| = m_1$. Define $\mathbf{v}_2 := \mathbf{v}_1 \mathbf{w}_1$. Then by Lemma 3.5 (ii) with $m = m_1, l = 0$ and $r = r_1$ it follows that $m_2 := |\mathbf{v}_2| \in \mathcal{N}(q) \cap [m_1, \infty)$. By Lemma 3.3 this implies that \mathbf{v}_2 is primitive and $\mathbf{v}_2(\overline{\mathbf{v}_2^+})^\infty \prec \alpha(q)$.

Repeating the above argument we construct a sequence of words (\mathbf{v}_i) such that for each $i \geq 1$ the word \mathbf{v}_i is primitive and $\mathbf{v}_i(\overline{\mathbf{v}_i^+})^\infty \prec (a_i)$. Furthermore, for each $i \geq 1$,

$$\mathbf{v}_{i+1} = \mathbf{v}_i \mathbf{w}_i \quad \text{with} \quad 1 \leq r_i := |\mathbf{w}_i| \leq |\mathbf{v}_i| =: m_i,$$

and

$$(3.10) \quad \mathbf{w}_i 0^\infty \succ (\overline{\mathbf{v}_i^+})^\infty.$$

Therefore,

$$(3.11) \quad \alpha(q) = (a_i) = \mathbf{v}_1 \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3 \dots = \mathbf{v}_i \mathbf{w}_i \mathbf{w}_{i+1} \mathbf{w}_{i+2} \dots, \quad i \geq 1.$$

Clearly $|\mathbf{w}_i| \geq 1$ for all $i \geq 1$. Hence there are infinitely many integers i such that $|\mathbf{w}_i| \leq |\mathbf{w}_{i+1}|$. Observe also by Lemma 3.5 (ii) that $\mathbf{v}_{i+1} = \mathbf{v}_i \mathbf{w}_i$ is primitive. This implies that $\mathbf{w}_i^- \succ \overline{a_1 \dots a_{r_i}}$ for each $i \in \mathbb{N}$. It follows that one of the following cases must hold:

- (i) $|\mathbf{w}_i| < |\mathbf{w}_{i+1}|$ for infinitely many i ; or
- (ii) $\mathbf{w}_i \succ \overline{a_1 \dots a_{r_i}}$ for infinitely many i ; or
- (iii) there is $s \in \mathbb{N}$ such that $\alpha(q) = \mathbf{v}_s \mathbf{w}_s^\infty = a_1 \dots a_{m_s} (\overline{a_1 \dots a_{r_s}})^\infty$.

We consider the first two cases together; the third case, however, requires a different approach.

Case A: $|\mathbf{w}_i| < |\mathbf{w}_{i+1}|$ for infinitely many i , or $\mathbf{w}_i \succ \overline{a_1 \dots a_{r_i}}$ for infinitely many i .

Fix an integer s such that $|\mathbf{w}_s| < |\mathbf{w}_{s+1}|$ or $\mathbf{w}_s \succ \overline{a_1 \dots a_{r_s}}$. Set

$$n := m_{s+2} = m_{s+1} + r_{s+1},$$

and write

$$\mathbf{u} := a_1 \dots a_n = \mathbf{v}_s \mathbf{w}_s \mathbf{w}_{s+1} = \mathbf{v}_{s+1} \mathbf{w}_{s+1}, \quad \mathbf{v} := \mathbf{v}_{s+1}.$$

Fix an integer $k > n$. With \mathbf{u} and \mathbf{v} as above, set $F_A := F_{\mathbf{u}, \mathbf{v}}$ (see Definition 3.7). We first show that F_A maps $B_k(\mathbf{V}_{q,n})$ into itself.

Lemma 3.11. *For any $\mathbf{x} \in B_k(\mathbf{V}_{q,n})$ we have $F_A(\mathbf{x}) \in B_k(\mathbf{V}_{q,n})$, and*

$$i_{\mathbf{u}}(F_A(\mathbf{x})) \geq i_{\mathbf{u}}(\mathbf{x}) + m_{s+1} \geq i_{\mathbf{u}}(\mathbf{x}) + \frac{n}{2}.$$

Proof. The proof is similar to that of Lemma 3.8. Let $i := i_{\mathbf{u}}(\mathbf{x})$. By symmetry we may assume that $x_{i+1} \dots x_{i+n} = \mathbf{u}$. Then $\mathbf{x} = x_1 \dots x_i \mathbf{v}_{s+1} \mathbf{w}_{s+1} x_{i+n+1} \dots x_k$. Write $F_A(\mathbf{x}) = y_1 \dots y_k$. By Definition 3.7 we have

$$(3.12) \quad \begin{aligned} y_1 \dots y_k &= x_1 \dots x_i \mathbf{v}_{s+1}^- \overline{\mathbf{w}_{s+1} x_{i+n+1} \dots x_k} \\ &= x_1 \dots x_i \mathbf{v}_s \mathbf{w}_s^- \overline{\mathbf{w}_{s+1} x_{i+n+1} \dots x_k}. \end{aligned}$$

Since the entire block $x_{i+m_{s+1}+1} \dots x_k$ is being reflected by F_A , it suffices to show that for each $0 \leq j < i + m_{s+1}$ we have $\bar{\mathbf{u}} \prec y_{j+1} \dots y_{j+n} \prec \mathbf{u}$. On one hand, by (3.12) we have $y_1 \dots y_{i+m_{s+1}} = x_1 \dots x_{i+m_{s+1}}^-$. Then using the minimality of i it follows that $y_{j+1} \dots y_{j+n} \prec \mathbf{u}$ for all $j < i + m_{s+1}$. So, it remains to prove

$$(3.13) \quad y_{j+1} \dots y_{j+n} \succ \bar{\mathbf{u}} \quad \text{for all } 0 \leq j < i + m_{s+1}.$$

First, the minimality of i implies that (3.13) holds for all $j < i + m_{s+1} - n = i - r_{s+1}$. The verification of (3.13) for $i - r_{s+1} \leq j < i + m_{s+1}$ is split into the following four cases (see Figure 2).

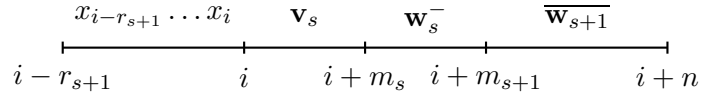


FIGURE 2. The presentation of $y_{i-r_{s+1}} \dots y_{i+n} = x_{i-r_{s+1}} \dots x_i \mathbf{v}_s \mathbf{w}_s^- \overline{\mathbf{w}_{s+1}}$.

- (I). $j = i - r_{s+1}$. Then $y_{j+1} \dots y_{j+n} = x_{j+1} \dots x_i a_1 \dots a_{m_{s+1}}^-$. Note that $x_{j+1} \dots x_i \succ \overline{a_1 \dots a_{i-j}} = \overline{a_1 \dots a_{r_{s+1}}}$. By Lemma 3.5 (i) it follows that

$$y_{i+1} \dots y_{j+n} = a_1 \dots a_{m_{s+1}}^- \succ \overline{a_{r_{s+1}+1} \dots a_n}.$$

This establishes (3.13) for $j = i - r_{s+1}$.

- (II). $i - r_{s+1} < j < i$. Then $y_{j+1} \dots y_i = x_{j+1} \dots x_i \succ \overline{a_1 \dots a_{i-j}}$. Note that $i - j < r_{s+1} \leq m_{s+1}$. Then by the primitivity of $\mathbf{v}_{s+1} = a_1 \dots a_{m_{s+1}}$ it follows that

$$y_{i+1} \dots y_{j+m_{s+1}} = a_1 \dots a_{j+m_{s+1}-i} \succ \overline{a_{i-j+1} \dots a_{m_{s+1}}}.$$

This proves (3.13) for $i - r_{s+1} < j < i$.

- (III). $i \leq j < i + m_s$. Then (3.13) follows from the primitivity of $\mathbf{v}_s = a_1 \dots a_{m_s}$, which implies

$$y_{j+1} \dots y_{i+m_s} = a_{j-i+1} \dots a_{m_s} \succ \overline{a_1 \dots a_{m_s-j+i}}.$$

- (IV). $i + m_s \leq j < i + m_{s+1}$. Let $t = j - (i + m_s)$. Then $0 \leq t < m_{s+1} - m_s = r_s$. Note that $y_{j+1} \dots y_{i+m_{s+1}}$ is a suffix of \mathbf{w}_s^- . Then by (3.12) and (3.3) it follows that

$$(3.14) \quad y_{j+1} \dots y_{i+m_{s+1}} \succ \overline{a_{t+1} \dots a_{r_s}} \succ \overline{a_1 \dots a_{r_s-t}},$$

where the second inequality follows since $a_1 \dots a_{r_s}$ is primitive by Lemma 3.5 (iii). Note that, in view of (3.3), the first inequality in (3.14) is in fact strict if $\mathbf{w}_s \succ \overline{a_1 \dots a_{r_s}}^+$, so in this case we are done. Otherwise, we have $r_{s+1} = |\mathbf{w}_{s+1}| > |\mathbf{w}_s| = r_s$, so by (3.12) it follows that

$$(3.15) \quad y_{i+m_{s+1}+1} \dots y_{j+r_{s+1}} = a_1 \dots a_{r_{s+1}-r_s+t} \succ \overline{a_{r_s-t+1} \dots a_{r_{s+1}}}.$$

Here the inequality in (3.15) follows since $a_1 \dots a_{r_{s+1}}$ is primitive by Lemma 3.5 (iii). Combining (3.14) and (3.15) we obtain (3.13) for $i + m_s \leq j < i + m_{s+1}$.

We have now shown (3.13) for all $0 \leq j < i + m_{s+1}$. Hence, the proof is complete. \square

As a result of Lemma 3.11, for some large enough j (with $j < k$) we have $F_A^k(\mathbf{x}) = \dots = F_A^{j+1}(\mathbf{x}) = F_A^j(\mathbf{x})$. We now define

$$f_{n,k}^A(\mathbf{x}) := F_A^k(\mathbf{x}), \quad \mathbf{x} \in B_k(\mathbf{V}_{q,n}).$$

By Lemma 3.11, $f_{n,k}^A$ maps $B_k(\mathbf{V}_{q,n})$ into $B_k(\mathbf{U}_{q,n})$. The next proposition now follows from a similar argument as in the proof of Proposition 3.10.

Proposition 3.12. *Let $q \in \mathcal{B}^L$ with $\alpha(q) = \mathbf{v}_1 \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3 \dots$ satisfying (3.10). If $|\mathbf{w}_i| < |\mathbf{w}_{i+1}|$ for infinitely many i or $\mathbf{w}_i \succ \overline{a_1 \dots a_{r_i}^+}$ for infinitely many i , then $\lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}) = \lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n})$.*

Case B: There is $s \in \mathbb{N}$ such that $\alpha(q) = \mathbf{v}_s \mathbf{w}_s^\infty = a_1 \dots a_{m_s} (\overline{a_1 \dots a_{r_s}^+})^\infty$.

Note by the definition of (\mathbf{v}_i) that $\mathbf{v}_{s+j} = \mathbf{v}_s \mathbf{w}_s^j$ for any $j \in \mathbb{N}$. Then by (3.11),

$$(3.16) \quad \alpha(q) = \mathbf{v}_s \mathbf{w}_s^\infty = \mathbf{v}_t \mathbf{w}_s^\infty = a_1 \dots a_{m_t} (\overline{a_1 \dots a_{r_s}^+})^\infty \quad \text{for any } t > s.$$

Lemma 3.13. *Let $q \in \mathcal{B}^L$ such that $\alpha(q) = \mathbf{v}_s \mathbf{w}_s^\infty = a_1 \dots a_{m_s} (\overline{a_1 \dots a_{r_s}^+})^\infty$. Then*

$$\alpha(q) \succcurlyeq a_1 \dots a_{r_s} (\overline{a_1 \dots a_{r_s}^+})^\infty.$$

Proof. Suppose on the contrary that $\alpha(q) = (a_i) \prec a_1 \dots a_{r_s} (\overline{a_1 \dots a_{r_s}^+})^\infty$. Then

$$(3.17) \quad (a_1 \dots a_{r_s}^-)^\infty \prec \alpha(q) \prec a_1 \dots a_{r_s} (\overline{a_1 \dots a_{r_s}^+})^\infty,$$

where the first inequality follows since $m_s \geq r_s$. By Lemma 3.5 (iii), $a_1 \dots a_{r_s}$ is primitive. Thus, the same argument as in the proof of [5, Proposition 3.9] shows that $\alpha(q) \in X_{\mathcal{G}}$, where $X_{\mathcal{G}}$ is the subshift of finite type represented by the labeled graph \mathcal{G} in Figure 3 (with $r := r_s$).

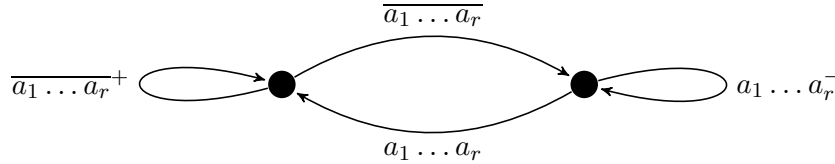


FIGURE 3. The picture of the labeled graph \mathcal{G} .

Since $\alpha(q) = \mathbf{v}_s \mathbf{w}_s^\infty$ ends with $\mathbf{w}_s^\infty = (\overline{a_1 \dots a_{r_s}^+})^\infty$, it follows from Figure 3 that

$$\sigma^j(\alpha(q)) = a_1 \dots a_{r_s} (\overline{a_1 \dots a_{r_s}^+})^\infty$$

for some $j \geq 1$. But then (3.17) gives $\sigma^j(\alpha(q)) \succcurlyeq \alpha(q)$, contradicting Lemma 2.1. \square

In view of Lemma 3.13 we first consider the case $\alpha(q) = a_1 \dots a_{r_s} (\overline{a_1 \dots a_{r_s}^+})^\infty$. Here we could not find a suitable mapping $f_{n,k}$; instead we use a different method, based on ideas from [5].

Lemma 3.14. *Let $q \in \mathcal{B}^L$ with $\alpha(q) = a_1 \dots a_r (\overline{a_1 \dots a_r^+})^\infty$, where $a_1 \dots a_r$ is primitive. Then*

$$(3.18) \quad \lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}) = \lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n}).$$

Proof. We claim that

$$(3.19) \quad h(\mathbf{V}_q) = \frac{\log 2}{r}.$$

First, observe by Figure 3 that $X_{\mathcal{G}} \subseteq \mathbf{V}_q$, and hence

$$(3.20) \quad h(\mathbf{V}_q) \geq h(X_{\mathcal{G}}) = \frac{\log 2}{r}.$$

On the other hand, by Lemma 3.3 (ii) it follows that $(a_1 \dots a_r^-)^\infty \preceq \alpha(q_{KL})$, so $h(\mathbf{U}_{q,r}) = 0$. Furthermore, by the argument from the proof of [5, Proposition 3.9], any sequence in \mathbf{V}_q is either itself in $\mathbf{U}_{q,r}$ or else consists of a finite (possibly empty) prefix from $\mathbf{U}_{q,r}$ followed by a sequence from $X_{\mathcal{G}}$. Hence, by a standard argument,

$$h(\mathbf{V}_q) \leq h(X_{\mathcal{G}}).$$

Combined with (3.20), this yields (3.19).

Next, for $n \in \mathbb{N}$ let $q_n < q$ be the base such that $\alpha(q_n) = (a_1 \dots a_r (\overline{a_1 \dots a_r^+})^n \overline{a_1 \dots a_r})^\infty$. Then $q_n \nearrow q$ as $n \rightarrow \infty$. Let $X_{\mathcal{G},n}$ be the set of those sequences $(x_i) \in X_{\mathcal{G}}$ containing neither the word $a_1 \dots a_r (\overline{a_1 \dots a_r^+})^n$ nor its reflection. Then $X_{\mathcal{G},n} \subseteq \mathbf{U}_{q_n}$, and (see [5, Lemma 4.2])

$$h(X_{\mathcal{G},n}) = \frac{\log \varphi_n}{r},$$

where φ_n is the unique positive root of $1 + x + \dots + x^{n-1} = x^n$. Since $\varphi_n \nearrow 2$ as $n \rightarrow \infty$, by (3.19) it follows that

$$h(\mathbf{U}_{q_n}) \geq h(X_{\mathcal{G},n}) = \frac{\log \varphi_n}{r} \rightarrow \frac{\log 2}{r} = h(\mathbf{V}_q).$$

This establishes the left-continuity of H at q . To obtain the stronger result (3.18), note that $\mathbf{U}_{q_n} = \mathbf{U}_{q,(n+2)r}$. Hence, $\lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n}) \geq h(\mathbf{V}_q) \geq h(\mathbf{U}_q)$. The reverse inequality is obvious, since $\mathbf{U}_{q,n} \subseteq \mathbf{V}_q$ for all $n \geq 1$. We conclude that

$$\lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n}) = h(\mathbf{V}_q) = h(\mathbf{U}_q) = \lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}),$$

where the last equality follows from the right continuity of H (see Proposition 2.8). \square

Finally, we consider the case that $(a_i) = \alpha(q) \succ a_1 \dots a_{r_s} (\overline{a_1 \dots a_{r_s^+}})^\infty$. Then there exists an integer $\ell \geq 1$ such that

$$(3.21) \quad a_1 \dots a_{(\ell+1)r_s} \succ a_1 \dots a_{r_s} (\overline{a_1 \dots a_{r_s^+}})^\ell.$$

Note by (3.16) that $\alpha(q) = \mathbf{v}_t \mathbf{w}_s^\infty = a_1 \dots a_{m_t} (\overline{a_1 \dots a_{r_s^+}})^\infty$ for any $t > s$. Take $t \in \mathbb{N}$ such that $m_t > \ell r_s$. Write for $n := m_t + \ell + 1 = m_t + (\ell + 1)r_s$ that

$$\mathbf{u} := a_1 \dots a_n = \mathbf{v}_t \mathbf{w}_s^{\ell+1} = a_1 \dots a_{m_t} (\overline{a_1 \dots a_{r_s^+}})^{\ell+1}.$$

Furthermore, put

$$\mathbf{v} := \mathbf{v}_t \mathbf{w}_s = \mathbf{v}_{t+1}.$$

With \mathbf{u} and \mathbf{v} as above, define $F_B := F_{\mathbf{u},\mathbf{v}}$ (see Definition 3.7).

Using (3.21) and by a similar reasoning as in the proof of Lemma 3.11 it can be shown that F_B maps $B_k(\mathbf{V}_{q,n})$ into $B_k(\mathbf{V}_{q,n})$. Furthermore, the earliest possible occurrence of \mathbf{u} or $\bar{\mathbf{u}}$ in

$F_B(\mathbf{x})$ starts later than the earliest occurrence of \mathbf{u} or $\bar{\mathbf{u}}$ in \mathbf{x} . This implies that, for some large enough j (with $j < k$) we have $F_B^k(\mathbf{x}) = \cdots = F_B^{j+1}(\mathbf{x}) = F_B^j(\mathbf{x})$. We now define

$$f_{n,k}^B(\mathbf{x}) := F_B^k(\mathbf{x}), \quad \mathbf{x} \in B_k(\mathbf{V}_{q,n}).$$

By the above argument, $f_{n,k}^B$ maps $B_k(\mathbf{V}_{q,n})$ into $B_k(\mathbf{U}_{q,n})$. Note that the length $|\mathbf{v}| = |\mathbf{v}_t \mathbf{w}_s| = m_t + r_s \geq n/2$ (since $m_t > \ell r_s$). As in the proof of Proposition 3.10 we can now prove that the map $f_{n,k}^B$ is at most $(2N)^{k/N}$ -to-one, where $N := \lceil n/2 \rceil$. This gives

Lemma 3.15. *Let $q \in \mathcal{B}^L$ with $\alpha(q) = \mathbf{v}_s \mathbf{w}_s^\infty = a_1 \dots a_{m_s} (\overline{a_1 \dots a_{r_s}})^+$ for some $s \geq 1$. If $\alpha(q) \succ a_1 \dots a_{r_s} (\overline{a_1 \dots a_{r_s}})^+$, then $\lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}) = \lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n})$.*

Combining Lemmas 3.13, 3.14 and 3.15 we obtain:

Proposition 3.16. *Let $q \in \mathcal{B}^L$ with $\alpha(q) = \mathbf{v}_s \mathbf{w}_s^\infty = a_1 \dots a_{m_s} (\overline{a_1 \dots a_{r_s}})^+$ for some $s \geq 1$. Then $\lim_{n \rightarrow \infty} h(\mathbf{V}_{q,n}) = \lim_{n \rightarrow \infty} h(\mathbf{U}_{q,n})$.*

Proof of Theorem 3.1. The theorem follows from Propositions 3.10, 3.12 and 3.16. \square

4. PROOF OF THEOREM 1.2

We will use the following lemma for the Hausdorff dimension under Hölder continuous maps (cf. [13]).

Lemma 4.1. *Let $f : (X, \rho_X) \rightarrow (Y, \rho_Y)$ be a Hölder map between two metric spaces, i.e., there exist constants $C > 0$ and $\xi > 0$ such that*

$$\rho_Y(f(x), f(x')) \leq C \rho_X(x, x')^\xi \quad \text{for any } x, x' \in X.$$

Then $\dim_H f(X) \leq \frac{1}{\xi} \dim_H X$.

It will be convenient to introduce a family of (mutually equivalent) metrics $\{\rho_q : q > 1\}$ on Ω defined by

$$\rho_q((c_i), (d_i)) := q^{-\inf\{i \geq 1 : c_i \neq d_i\}}, \quad q > 1.$$

Then (Ω, ρ_q) is a compact metric space. Let $\dim_H^{(q)}$ denote Hausdorff dimension on Ω with respect to the metric ρ_q . For $p > 1$ and $q > 1$,

$$\rho_q((c_i), (d_i)) = \rho_p((c_i), (d_i))^{\log q / \log p},$$

and by Lemma 4.1 this gives the useful relationship

$$(4.1) \quad \dim_H^{(p)} E = \frac{\log q}{\log p} \dim_H^{(q)} E, \quad E \subseteq \Omega.$$

The following result is well known (see [15, Lemma 2.7] or [4, Lemma 2.2]):

Lemma 4.2. *For each $q \in (1, M + 1)$, the map π_q is Lipschitz on (Ω, ρ_q) , and the restriction*

$$\pi_q : (\mathbf{U}_q, \rho_q) \rightarrow (\mathcal{U}_q, |\cdot|); \quad \pi_q((x_i)) = \sum_{i=1}^{\infty} \frac{x_i}{q^i}$$

is bi-Lipschitz, where $|\cdot|$ denotes the Euclidean metric on \mathbb{R} . In particular,

$$\dim_H \mathcal{U}_q = \dim_H^{(q)} \mathbf{U}_q.$$

Lastly, we need an analog of Proposition 2.6 for Hausdorff dimension.

Lemma 4.3. *For every $q \in (1, M + 1]$, $\dim_H^{(q)} \mathbf{U}_q = \dim_H^{(q)} \mathbf{V}_q$.*

Proof. The proof is similar to that of Proposition 2.6, but easier: By (2.1) and the countable stability of Hausdorff dimension, we have $\dim_H^{(q)} \mathbf{U}_q \leq \dim_H^{(q)} \mathbf{V}_q$. The reverse inequality follows since $\mathbf{V}_q \setminus \mathbf{U}_q$ is countable. \square

Proof of Theorem 1.2. Consider first the case when $\alpha(q) = (a_1 \dots a_m^-)^\infty$ for a primitive word $a_1 \dots a_m$. Here \mathbf{V}_q can be written in finite terms as

$$\mathbf{V}_q = \{(x_i) \in \Omega : \overline{a_1 \dots a_m} \preceq x_{n+1} \dots x_{n+m} \preceq a_1 \dots a_m \text{ for all } n \geq 0\},$$

so \mathbf{V}_q is a subshift of finite type. It is well known (see, for instance, [17]) that the Hausdorff dimension of a subshift of finite type is given by its topological entropy. Thus, using Lemmas 4.1 and 4.2, we obtain

$$\dim_H \mathcal{U}_q = \dim_H^{(q)} \mathbf{U}_q = \dim_H^{(q)} \mathbf{V}_q = \frac{h(\mathbf{V}_q)}{\log q} = \frac{h(\mathbf{U}_q)}{\log q},$$

where we also used Proposition 2.6 and Lemma 4.3.

Next, let $q \in \overline{\mathcal{W}}$ and write $\alpha(q) = (a_i)$. Then by Lemma 2.5 there is a sequence of points $(q_n : n \in \mathbb{N})$ such that q_n increases to q and for each n , $\alpha(q_n) = (a_1 \dots a_{m_n}^-)^\infty$ for some integer m_n such that $a_1 \dots a_{m_n}$ is primitive. So by the first case above and Theorem 1.1,

$$\dim_H^{(q)} \mathbf{U}_q \geq \dim_H^{(q)} \mathbf{U}_{q_n} = \frac{h(\mathbf{U}_{q_n})}{\log q} \rightarrow \frac{h(\mathbf{U}_q)}{\log q}.$$

On the other hand, for any set $E \subseteq \Omega^{\mathbb{N}}$ we have $\dim_H^{(q)} \leq h(E)/\log q$, and so

$$\dim_H^{(q)} \mathbf{U}_q \leq \frac{h(\mathbf{U}_q)}{\log q}.$$

Hence, by Lemma 4.2,

$$\dim_H \mathcal{U}_q = \dim_H^{(q)} \mathbf{U}_q = \frac{h(\mathbf{U}_q)}{\log q}.$$

Finally, let $q \in (q_{KL}, M + 1] \setminus \overline{\mathcal{W}}$. Then q lies in a connected component (q_0, q_1) of $(q_{KL}, M + 1] \setminus \overline{\mathcal{W}}$. It was shown in [21] that $h(\mathbf{U}_q)$ is constant on $[q_0, q_1)$ and

$$\dim_H \mathcal{U}_q = \frac{h(\mathbf{V}_{q_0})}{\log q} = \frac{h(\mathbf{U}_{q_0})}{\log q} = \frac{h(\mathbf{U}_q)}{\log q}.$$

This completes the proof. \square

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