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Representation of solutions of discrete linear delay systems with non permutable matrices

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Abstract

We introduce a discrete delayed exponential depending on sequence of matrices. This discrete matrix gives a representation of a solution to the Cauchy problem for a discrete linear system with pure delay with sequence of matrices. We discard the commutativity condition used in recent works related to the representation of solutions for discrete delay linear systems.

Keywords: Linear discrete systems, delay, matrix delayed exponential function;.non permutable matrices.

1 Introduction

For given $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a < b$, we set $Z_a^b := \{a, a+1, ..., b\}$. We study a discrete linear delay system with sequence of matrices of the form:

$$\begin{cases} x(k+1) = Ax(k) + B_k x(k-m) + f(k), \\ x(k) = \varphi(k), \end{cases} \tag{1}$$

where $m \geq 1$ is a fixed integer, $k \in Z_0^\infty$, $A = (a_{ij})$, $\det A \neq 0$ and $B_k = \left(b_{ij}^k\right)$ are a constant $n \times n$ matrices, $f : Z_0^\infty \to R^n$, $\varphi : Z_{-m}^0 \to R^n$, $\Delta x(k) = x(k+1) - x(k)$. Solution $x : Z_{-m}^\infty \to R^n$ of initial value problem is defined as an infinite sequence $\{\varphi(-m), \varphi(-m+1), ..., \varphi(0), x(1), ..., x(k), ...\}$ such that for any $k \in Z_0^\infty$, (1) holds.

Substituting in (1) $z(k) := A^{-k}x(k)$, $D_k := A^{-k-1}B_k A^{k-m}$, $k \in Z_{-m}^\infty$, we get an equivalent discrete linear system of the form

$$z(k+1) = z(k) + D_k z(k-m) + A^{-k-1}f(k), \quad k \in Z_0^\infty, \tag{2}$$

$$z(k) = A^{-k}\varphi(k), \quad k \in Z_{-m}^0. \tag{3}$$

Recently, Diblík and Khusainov presented in [1], [2] a solution of difference equations with linear parts with constant coefficients given by permutable matrices and constant delay via a discrete matrix delayed exponential. Advantage of discrete delayed exponential matrix is to help transferring the classical idea to represent the solution of linear ordinary differential equations into linear delay discrete equations. Although there are many continued contributions in a discrete linear system with pure delay with permutable matrices, to stability theory [5], [6], [7], [9], [11], controllability theory [12], [13], [10], delay oscillating systems [14], discrete linear system with two delays [3], [4], Fredholm integral equations [15], no results were obtained for such systems with non permutable matrices. It should be mentioned that recently non permutable case for the contiunous delay linear systems was considered in [8].

We introduce a discrete delayed exponential depending on sequence of matrices $\mathfrak{D} = \{D_1, D_2, ...\}$ and give a representation of solution to linear system of difference equations with delay parts with nonconstant coefficients given by non permutable matrices. We discard the commutativity condition used in recent works related to representation of discrete delay linear system. In particular the results are new even for the case when matrices B_k does not depend on k, that is, $B_k = B$.

2 Main results

In order to drop the commutativity condition, we introduce the following matrix

$$P^{\mathfrak{D}}(k,d) := \begin{cases} I, & l=d=0, \\ \sum\limits_{j_1=(d-1)(m+1)}^{k-1} D_{j_1} \sum\limits_{j_2=(d-1)(m+1)}^{j_1} D_{j_2-m-1} \cdots \sum\limits_{j_d=(d-1)(m+1)}^{j_{d-1}} D_{j_d-(d-1)(m+1)}, & k \in Z_{(d-1)(m+1)+1}^{l(m+1)}, \\ & l \in Z_1^{\infty},\ 1 \le d \le l. \end{cases}$$

We state and prove our first result. Note that in the proof of Lemma 1 and Theorem 2 we follow the idea of the proofs of statetmens in [1].

Lemma 1 *For any $l \in Z_1^{\infty}$, $1 \le d \le l$, $k \in Z_{(d-1)(m+1)+1}^{l(m+1)}$, the following relation holds*

$$P^{\mathfrak{D}}(k+1,d) - P^{\mathfrak{D}}(k,d) = D_k P^{\mathfrak{D}}(k-m,d-1). \tag{4}$$

Proof. We will prove the lemma in two cases.

Case 1: $(d-1)(m+1)+1 \le k < l(m+1)$: In this case $k-m \in Z_{(d-2)(m+1)+1}^{(l-1)(m+1)}$ and by definition

$$P^{\mathfrak{D}}(k-m,d-1) = \sum_{j_1=(d-2)(m+1)}^{k-m-1} D_{j_1} \sum_{j_2=(d-2)(m+1)}^{j_2} D_{j_2-(m+1)} \cdots \sum_{j_{d-1}=(d-2)(m+1)}^{j_{d-2}} D_{j_{d-1}-(d-2)(m+1)},\ 2 \le d \le l,$$
$$P^{\mathfrak{D}}(k-m,0) = I.$$

For $d=1$ we get $1 \le k < l(m+1)$ and

$$P^{\mathfrak{D}}(k+1,1) - P^{\mathfrak{D}}(k,1) = \sum_{j_1=0}^{k} D_{j_1} - \sum_{j_1=0}^{k-1} D_{j_1} = D_k = D_k P^{\mathfrak{D}}(k-m,0).$$

For $2 \le d \le l$ we get

$$\begin{aligned} & P^{\mathfrak{D}}(k+1,d) - P^{\mathfrak{D}}(k,d) \\ & = \sum_{j_1=(d-1)(m+1)}^{k} D_{j_1} \sum_{j_2=(d-1)(m+1)}^{j_1} D_{j_2-m-1} \cdots \sum_{j_d=(d-1)(m+1)}^{j_{d-1}} D_{j_d-(d-1)(m+1)} \\ & - \sum_{j_1=(d-1)(m+1)}^{k-1} D_{j_1} \sum_{j_2=(d-1)(m+1)}^{j_1} D_{j_2-m-1} \cdots \sum_{j_d=(d-1)(m+1)}^{j_{d-1}} D_{j_d-(d-1)(m+1)} \\ & = D_k \sum_{j_2=(d-1)(m+1)}^{k} D_{j_2-m-1} \cdots \sum_{j_d=(d-1)(m+1)}^{j_{d-1}} D_{j_d-(d-1)(m+1)} \\ & = D_k \sum_{j_1=(d-2)(m+1)}^{k-m-1} D_{j_1} \sum_{j_2=(d-2)(m+1)}^{j_2} D_{j_2-(m+1)} \cdots \sum_{j_{d-1}=(d-2)(m+1)}^{j_{d-2}} D_{j_{d-1}-(d-2)(m+1)} \\ & = D_k P^{\mathfrak{D}}(k-m,d-1), \end{aligned}$$

which proves the lemma for the case 1.

Case 2: $k = l(m+1)$: In this case $k+1 = l(m+1)+1$ and $k-m = l(m+1)-m \in Z_{(l-1)(m+1)+1}^{l(m+1)}$. For $d=1$ we have:

$$P^{\mathfrak{D}}(l(m+1)+1,1) - P^{\mathfrak{D}}(l(m+1),1) = \sum_{j_1=0}^{l(m+1)} D_{j_1} - \sum_{j_1=0}^{l(m+1)-1} D_{j_1} = D_{l(m+1)} = D_{l(m+1)} P^{\mathfrak{D}}(l(m+1)-m,0).$$

For $2 \le d \le l$ we get

$$\begin{aligned}
&P^{\mathfrak{D}}\left(l(m+1)+1,d\right) - P^{\mathfrak{D}}\left(l(m+1),d\right)\\
&= \sum_{j_1=(d-1)(m+1)}^{l(m+1)} D_{j_1} \sum_{j_2=(d-1)(m+1)}^{j_1} D_{j_2-m-1} \cdots \sum_{j_d=(d-1)(m+1)}^{j_{d-1}} D_{j_d-(d-1)(m+1)}\\
&- \sum_{j_1=(d-1)(m+1)}^{l(m+1)-1} D_{j_1} \sum_{j_2=(d-1)(m+1)}^{j_1} D_{j_2-m-1} \cdots \sum_{j_d=(d-1)(m+1)}^{j_{d-1}} D_{j_d-(d-1)(m+1)}\\
&= D_{l(m+1)} \sum_{j_2=(d-1)(m+1)}^{l(m+1)} D_{j_2-m-1} \sum_{j_3=(d-1)(m+1)}^{j_2} D_{j_3-2(m+1)} \cdots \sum_{j_d=(d-1)(m+1)}^{j_{d-1}} D_{j_d-(d-1)(m+1)}\\
&= D_{l(m+1)} \sum_{j_1=(d-2)(m+1)}^{(l-1)(m+1)} D_{j_1} \sum_{j_2=(d-2)(m+1)}^{j_1} D_{j_2-(m+1)} \cdots \sum_{j_{d-1}=(d-2)(m+1)}^{j_{d-2}} D_{j_{d-1}-(d-2)(m+1)}\\
&= D_{l(m+1)} P^{\mathfrak{D}}\left(l\left(m+1\right)-m, d-1\right), \ 2 \le d \le l.
\end{aligned}$$

■

Using $P\left(k,d\right)$ we may define the delayed exponential depending on sequence of matrices:

$$e_m^{\mathfrak{D}}\left(k\right) = \begin{cases} \Theta & k \in Z_{-\infty}^{-m-1}, \\ I & k \in Z_{-m}^{0}, \\ I + \sum_{d=1}^{l} P\left(k,d\right) & k \in Z_{(l-1)(m+1)+1}^{l(m+1)}, \quad l \in Z_1^{\infty}. \end{cases} \tag{5}$$

Theorem 2 *For any $k \in Z_{(l-1)(m+1)+1}^{l(m+1)}$, the following relation holds*

$$e_m^{\mathfrak{D}}\left(k+1\right) - e_m^{\mathfrak{D}}\left(k\right) = D_k e_m^{\mathfrak{D}}\left(k-m\right), \quad k \in Z_{(l-1)(m+1)+1}^{l(m+1)-1}. \tag{6}$$

Proof. The proof is based on Lemma 1. Let us consider the cases when $\left(l-1\right)\left(m+1\right)+1 \le k < l\left(m+1\right)$ and $k = l(m+1)$.

Case 1: $\left(l-1\right)\left(m+1\right)+1 \le k < l\left(m+1\right)$: By Lemma 1,

$$\begin{aligned}
e_m^{\mathfrak{D}}\left(k+1\right) - e_m^{\mathfrak{D}}\left(k\right) &= \sum_{d=1}^{l} P^{\mathfrak{D}}\left(k+1,d\right) - \sum_{d=1}^{l} P^{\mathfrak{D}}\left(k,d\right) = D_k \sum_{d=1}^{l} P^{\mathfrak{D}}\left(k-m,d-1\right)\\
&= D_k \sum_{d=0}^{l-1} P^{\mathfrak{D}}\left(k-m,d\right) = D_k e_m^{\mathfrak{D}}\left(k-m\right). \quad k-m \in Z_{(l-2)(m+1)+1}^{(l-1)(m+1)}.
\end{aligned}$$

Case 2: $k = l(m+1)$: By Lemma 1,

$$\begin{aligned}
e_m^{\mathfrak{D}}\left(k+1\right) - e_m^{\mathfrak{D}}\left(k\right) &= \sum_{d=1}^{l+1} P^{\mathfrak{D}}\left(l(m+1)+1,d\right) - \sum_{d=1}^{l} P^{\mathfrak{D}}\left(l(m+1),d\right)\\
&= \sum_{d=1}^{l} \left(P^{\mathfrak{D}}\left(l(m+1)+1,d\right) - P^{\mathfrak{D}}\left(l(m+1),d\right)\right) + P^{\mathfrak{D}}\left(l(m+1)+1,l+1\right)\\
&= D_{l(m+1)} \sum_{d=1}^{l} P^{\mathfrak{D}}\left(l\left(m+1\right)-m,d-1\right) + D_{l(m+1)} D_{(l-1)(m+1)} \ldots D_0\\
&= D_{l(m+1)} \left(I + \sum_{d=1}^{l-1} P^{\mathfrak{D}}\left(l\left(m+1\right)-m,d\right) + P^{\mathfrak{D}}\left(l\left(m+1\right)-m,l\right)\right)\\
&= D_{l(m+1)} e_m^{\mathfrak{D}}\left(k-m\right).
\end{aligned}$$

Here we used the following formula

$$P^{\mathfrak{D}}\left(l(m+1)+1,l+1\right)=\sum_{j_1=l(m+1)}^{l(m+1)}D_{j_1}\sum_{j_2=l(m+1)}^{j_1}D_{j_2-m-1}...\sum_{j_{l+1}=l(m+1)}^{j_l}D_{j_{l+1}-l(m+1)}$$
$$=D_{l(m+1)}\sum_{j_2=l(m+1)}^{l(m+1)}D_{j_2-m-1}...\sum_{j_{l+1}=l(m+1)}^{j_l}D_{j_{l+1}-l(m+1)}$$
$$=D_{l(m+1)}D_{(l-1)(m+1)}...D_0.$$

■

Now using the matrix delayed exponential we can solve the following linear matrix equation:

$$\begin{aligned}\Phi\left(k+1\right)&=A\Phi\left(k\right)+D_k\Phi\left(k-m\right),\ k\in Z_0^{\infty},\\ X\left(k\right)&=A^k,\quad k\in Z_{-m}^0.\end{aligned}\tag{7}$$

Theorem 3 *The matrix*

$$\Phi\left(k\right)=\begin{cases}\Theta & \text{if}\quad k\in Z_{-\infty}^{-m-1},\\ A^k & \text{if}\quad k\in Z_{-m}^0,\\ A^k\left(I+\sum_{j=1}^{l}P\left(k,j\right)\right) & \text{if}\quad k\in Z_{(l-1)(m+1)+1}^{l(m+1)}.\end{cases}$$

solves the problem (7).

Remark 4 *It can be shown that*

$$\sum_{j_1=(d-1)(m+1)}^{k-1}\sum_{j_2=(d-1)(m+1)}^{j_1}...\sum_{j_d=(d-1)(m+1)}^{j_{d-1}}1=\left(\begin{array}{c}k-(d-1)\,m\\ d\end{array}\right),\qquad k\in Z_{(d-1)(m+1)+1}^{l(m+1)}.$$

If B_k does not depend on k, $B_k=B$ and matrices A and B are permutable ($AB=BA$), then $D_k=A^{-k-1}BA^{k-m}=A^{-1}BA^{-m}=:D$ and

$$P^{\mathfrak{D}}\left(k,d\right)=A^{-d}B^dA^{-dm}\sum_{j_1=(d-1)(m+1)}^{k-1}\sum_{j_2=(d-1)(m+1)}^{j_1}...\sum_{j_d=(d-1)(m+1)}^{j_{d-1}}1=A^{-d}B^dA^{-dm}\left(\begin{array}{c}k-(d-1)\,m\\ d\end{array}\right).$$

In this case

$$e_m^{\mathfrak{D}}\left(k\right)=\begin{cases}\Theta & k\in Z_{-\infty}^{-m-1},\\ I & k\in Z_{-m}^0,\\ I+\sum_{d=1}^{l}A^{-d}B^dA^{-dm}\left(\begin{array}{c}k-(d-1)\,m\\ d\end{array}\right), & k\in Z_{(l-1)(m+1)+1}^{l(m+1)},\end{cases}$$

and coincides with the discrete matrix delayed exponential defined in [1].

Using $\Phi\left(k\right)$ we give the representation of solution to the homogeneous delay problem

$$x\left(k+1\right)=Ax\left(k\right)+B_kx\left(k-m\right),\quad k\in Z_0^{\infty},\tag{8}$$
$$x\left(k\right)=\varphi\left(k\right),\quad k\in Z_{-m}^0.\tag{9}$$

Theorem 5 *The solution of the problem (8), (9) can be expressed as*

$$x\left(k\right)=\Phi\left(k\right)A^{-m}\varphi\left(-m\right)+A^m\sum_{j=-m+1}^{0}\Phi\left(k-m-j\right)\left(\varphi\left(j\right)-A\varphi\left(j-1\right)\right).\tag{10}$$

Proof. Introduce a new variable $z(k) = A^{-k}x(k)$, $k \in Z_{-m}^{\infty}$. Then the problem (8), (9) is equivalent to

$$z(k+1) = z(k) + D_k z(k-m), \quad k \in Z_0^{\infty}, \tag{11}$$

$$z(k) = A^{-k}\varphi(k), \quad k \in Z_{-m}^{0}. \tag{12}$$

We are looking for the representation of solution of the problem (11), (12) in the form

$$z(k) = e_m^{\mathfrak{D}}(k)\, C + \sum_{j=-m+1}^{0} e_m^{\mathfrak{D}}(k-m-j)\,\omega(j), \quad k \in Z_{-m}^{\infty}, \tag{13}$$

where $C \in R^n$ is an unknown vector and $\omega : Z_{-m}^{0} \to R^n$ is an unknown discrete function. The representation (13) is a solution of homogeneous delay equation (11) for any C and ω and for $k \in Z_0^{\infty}$. Indeed, by the formula (6) we get

$$\begin{aligned}
\Delta z(k) &= \Delta\left[e_m^{\mathfrak{D}}(k)\right] C + \sum_{j=-m+1}^{0} \Delta\left[e_m^{\mathfrak{D}}(k-m-j)\right]\omega(j) \\
&= D_k\left[e_m^{\mathfrak{D}}(k-m)\, C + \sum_{k=-m+1}^{0} e_m^{\mathfrak{D}}(k-2m-j)\,\omega(j)\right] \\
&= D_k z(k-m), \quad k \in Z_0^{\infty}.
\end{aligned}$$

So expression (13) solves (11) for $k \in Z_0^{\infty}$. Now we determine C and ω. By definition, C and ω must satisfy the initial condition (12) for $k \in Z_{-m}^{0}$. For $k \in Z_{-m}^{0}$, (13) leads to relation

$$\begin{aligned}
z(k) = A^{-k}\varphi(k) &= e_m^{\mathfrak{D}}(k)\, C + \sum_{j=-m+1}^{0} e_m^{\mathfrak{D}}(k-m-j)\,\omega(j) \\
&= e_m^{\mathfrak{D}}(k)\, C + \sum_{j=-m+1}^{k} e_m^{\mathfrak{D}}(k-m-j)\,\omega(j) + \sum_{j=k+1}^{0} e_m^{\mathfrak{D}}(k-m-j)\,\omega(j) \\
&= C + \sum_{j=-m+1}^{k} e_m^{\mathfrak{D}}(k-m-j)\,\omega(j) + \sum_{j=k+1}^{0} e_m^{\mathfrak{D}}(k-m-j)\,\omega(j), \quad k \in Z_{-m}^{0}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&A^{m}\varphi(-m) = C, \quad k = -m, \\
&A^{-k}\varphi(k) = C + \sum_{j=-m+1,\, j\le k}^{k} \omega(j), \quad k \in Z_{-m+1}^{0},
\end{aligned}$$

and one can obtain

$$\begin{aligned}
&\omega(k) = A^{-k}\varphi(k) - A^{-k+1}\varphi(k-1), \quad k = -m, -m+1, ..., 0, \\
&C = A^{-m}\varphi(-m).
\end{aligned}$$

In order to get the formula (10), it remains to insert C and ω into (13). Indeed

$$\begin{aligned}
x(k) = A^k z(k) &= A^k\left(e_m^{\mathfrak{D}}(k)\, A^{-m}\varphi(-m) + \sum_{j=-m+1}^{0} e_m^{\mathfrak{D}}(k-m-j)\, A^{-j}\left(\varphi(j) - A\varphi(j-1)\right)\right) \\
&= \Phi(k)\, A^{-m}\varphi(-m) + A^k \sum_{j=-m+1}^{0} A^{-k+m+j} A^{k-m-j}\Phi(k-m-j)\, A^{-j}\left(\varphi(j) - A\varphi(j-1)\right) \\
&= \Phi(k)\, A^{-m}\varphi(-m) + \sum_{j=-m+1}^{0} A^{m+j}\Phi(k-m-j)\, A^{-j}\left(\varphi(j) - A\varphi(j-1)\right).
\end{aligned}$$

■

Corollary 6 *A solution $x : Z_{-m}^{\infty} \to R^n$ of initial value problem (1) has a form*

$$x(k) = \Phi(k) A^{-m} \varphi(-m) + \sum_{j=-m+1}^{0} A^{m+j} \Phi(k-m-j) A^{-j} (\varphi(j) - A\varphi(j-1))$$
$$+ \sum_{j=1}^{k} A^{m+j} \Phi(k-m-j) A^{-j} f(j-1). \tag{14}$$

Remark 7 *If B_k does not depend on k, that is, $B_k = B$ and matrices A and B are permutable ($AB = BA$), then $D_k = A^{-k-1} B A^{k-m} = A^{-1} B A^{-m} =: D$ then the presentation (14) coincides with the formula obtained in [1].*

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