

Existence of multiple solutions to an elliptic problem with measure data

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Abstract

In this paper we prove the existence of multiple nontrivial solutions of the following equation.

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u + f(x, u) + \mu \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega; \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 3$, $1 < q' < q < p - 1$; λ , and f satisfies certain conditions, $\mu > 0$ is a Radon measure, $q' = \frac{q}{q-1}$ is the conjugate of q .

Keywords: p -laplacian, Cerami sequence, Ekeland Variational principle, Radon measure.

1. Introduction

For many years now, the problem

$$\begin{aligned} -\Delta_p u &= g(x, u), \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

has been studied extensively using the celebrated critical point theory which was introduced by Ambrosetti & Rabinowitz in the Mountain pass theorem [1]. In order to apply the Mountain pass theorem one needs the Ambrosetti-Rabinowitz (AR) type condition on the nonlinear term g which is as follows.

For $\theta > p$, $R > 0$, we have

$$0 < \theta G(x, t) \leq g(x, t)t \tag{1.2}$$

$\forall |t| \geq R$ a.e. in Ω , where $G(x, t) = \int_0^t g(x, s) ds$. The (AR) condition also implies that there exists positive constants a, a_1, a_2 such that $G(x, t) \geq a_1 |t|^a - a_2$, $\forall (x, t) \in \Omega \times \mathbb{R}$.

Thus g is p -superlinear at infinity, in the sense that $\lim_{|t| \rightarrow \infty} \frac{G(x,t)}{|t|^p} = \infty$. Of late, the problem in (1.1) has been tackled without the AR condition by [7, 14, 16, 19, 20, 21, 23, 24] and the references therein. Miyagaki [21] studied (1.1) with a Laplacian by using the following condition on g : $\exists t_0 > 0$ such that $\frac{g(x,t)}{t}$ is increasing for $t \geq t_0$ and decreasing for $t \leq -t_0 \forall x \in \Omega$. The author in [21] guaranteed the existence of a nontrivial solution by using the Mountain Pass theorem with the Palais-Smale condition. Li et al [20] have extended this result, due to Miyagaki [21], by replacing $-\Delta$ with $-\Delta_p$. In Li [20], the authors needed the following subcritical growth condition $|g(x,t)| \leq C(1 + |t|^{r-1}) \forall t \in \mathbb{R}$ and for almost all x in Ω , $r \in [1, p^*)$, if $1 < p < N$ and $p^* = \infty$ if $p \geq N$. A further generalized subcritical type growth condition was introduced by Lan [17, 18], where $r = p^*$, to prove the existence of atleast one nontrivial weak solution to (1.1) using the Mountain Pass theorem but without using the AR condition.

Motivated by the work due to Chung et al [6] who have studied the existence of multiple solution for the problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u + f(x, u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega; \end{aligned} \tag{1.3}$$

with concave-convex nonlinearities in bounded domains, we consider the following problem.

$$\begin{aligned} (P) : \quad -\Delta_p u &= \lambda |u|^{q-2} u + f(x, u) + \mu \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $1 < q' < q < p - 1 < p^*$, $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate of p , $\mu > 0$ is a Radon measure. We will prove the existence of multiple nontrivial weak solutions to the problem (1.4). The conditions we assume on the continuous function $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ - is slightly different from that assumed in [6] - are as follows.

$$(f_0) \lim_{|t| \rightarrow \infty} \frac{f(x,t)}{|t|^{p^*-1}} = 0 \text{ uniformly a.e. } x \in \Omega.$$

$$(f_1) \text{ Let } F \text{ be the primitive of } f. \text{ There exists a positive constant } \bar{t} > 0 \text{ such that } F(x, t) \geq 0 \text{ a.e. } x \in \Omega \text{ and all } t \in [0, \bar{t}], \text{ where } F(x, t) = \int_0^t f(x, s) ds.$$

$$(f_2) \limsup_{|t| \rightarrow 0} \frac{F(x,t)}{|t|^p} < \lambda_1 \text{ uniformly a.e. } x \in \Omega, \lambda_1 \text{ being the first eigenvalue of } -\Delta_p.$$

$$(f_3) \lim_{|t| \rightarrow \infty} \frac{F(x,t)}{|t|^p} = \infty \text{ uniformly a.e. } x \in \Omega.$$

$$(f_4) \text{ There exists } \tilde{t} > 0 \text{ such that for any } x \in \Omega, \text{ the function } t \mapsto \frac{f(x,t)}{|t|^{p-2}t} \text{ is increasing if } t \geq \tilde{t} \text{ and decreasing if } t \leq -\tilde{t}, \forall x \in \Omega.$$

We will denote the Sobolev space as $W_0^{1,p}(\Omega) := \{u : \nabla u \in L^p(\Omega), u|_{\partial\Omega} = 0\}$ equipped

with the norm $\|\cdot\|_{1,p}$ which is defined as $\|u\|_{1,p}^p = \int_{\Omega} |\nabla u|^p dx$. We will denote $\|\cdot\|_{1,p}$ as $\|\cdot\|$ throughout this manuscript. We now state the main result of the paper which is as follows.

Theorem 1.1. *Suppose that $(f_0) - (f_4)$ hold. Then problem (P) in (1.4) possesses more than one nontrivial weak solution.*

2. Preliminary definitions

We now discuss a few definitions, notations and essential results which will be used in this paper.

Definition 2.1. (Cerami condition) A functional Φ is said to satisfy the Cerami condition at a level $c \in \mathbb{R}$ if any sequence $(u_n) \subseteq X$ such that $\Phi(u_n) \rightarrow c$ and $(1 + \|u_n\|)\Phi'(u_n) \rightarrow 0$ has a convergent subsequence.

In critical point theory, there are some situations in which a Palais-Smale sequence does not lead to a critical point, but a Cerami sequence can lead to a critical point. This whole thing based on the concept of ‘linking’ (refer [22]), for more details and examples. Cerami condition implies Palais-Smale condition and hence Cerami condition is a weaker condition than Palais-Smale.

Definition 2.2. Let (μ_n) be a bounded sequence of measures in $\mathfrak{M}(\Omega)$. We say that (μ_n) converges to a measure $\mu \in \mathfrak{M}(\Omega)$ in the sense of measure if

$$\int_{\Omega} \phi d\mu_n \rightarrow \int_{\Omega} \phi d\mu \quad \forall \phi \in C_0(\bar{\Omega}).$$

We denote this convergence by $\mu_n \rightharpoonup \mu$. The topology defined via this weak convergence is metrizable and a bounded sequence with respect to this topology is pre-compact.

Definition 2.3. (Ekeland Variational Principle) Let Φ be a lower semicontinuous bounded below function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$. For every $\epsilon > 0$, there is $x_0 \in X$ such that $\Phi(x) \geq \Phi(x_0) - \epsilon\|x - x_0\|$ for every $x \in X$ (refer [8]).

Throughout the article, we will denote the measure of a set E in the sigma algebra of Ω as where $|E|$ and the absolute value of any real number, say a , as $|a|$. We will use the Marcinkiewicz space $M^q(\Omega)$ [11] (or the weak $L^q(\Omega)$ space) defined for every $0 < q < \infty$, as the space of all measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that the corresponding distribution satisfy an estimate of the form

$$|\{x \in \Omega : |f(x)| > t\}| \leq \frac{C}{t^q}, \quad t > 0, C < \infty.$$

For bounded Ω we have $M^q \subset M^{\bar{q}}$ if $q \geq \bar{q}$, for some fixed positive \bar{q} . We recall here the following useful continuous embeddings

$$L^q(\Omega) \hookrightarrow M^q(\Omega) \hookrightarrow L^{q-\epsilon}(\Omega), \quad (2.1)$$

for every $1 < q < \infty$ and $0 < \epsilon < q - 1$.

We first consider a sequence of problems (P_n) which are as follows

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u + f(x, u) + \mu_n \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \quad (2.2)$$

where $\mu_n \rightharpoonup \mu$ in measure. From here onwards we will denote $\int_{\Omega} f dx = \int_{\Omega} f$. The corresponding energy functional of the sequence of problems (P_n) is written as

$$I_n(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \int_{\Omega} F(x, u) dx - \int_{\Omega} u d\mu_n \quad (2.3)$$

and its Fréchet derivative is defined as

$$\langle I'_n(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} u v dx - \int_{\Omega} f(x, u) v dx - \int_{\Omega} \mu_n v dx \quad (2.4)$$

$\forall u, v \in T$ where $T = W^{1,p}(\Omega) \cap C_0(\bar{\Omega})$, $C_0(\bar{\Omega}) = \{\varphi \in C(\bar{\Omega}) : \varphi|_{\partial\Omega} = 0\}$ and $C(\bar{\Omega})$ will denote the space of continuous functions over $\bar{\Omega}$. We now define the corresponding energy functional of the problem (P) as

$$I(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \int_{\Omega} F(x, u) dx - \int_{\Omega} u d\mu \quad (2.5)$$

and its Fréchet derivative is defined as

$$\langle I'(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} u v dx - \int_{\Omega} f(x, u) v dx - \int_{\Omega} v d\mu \quad (2.6)$$

for every $u, v \in T$.

Definition 2.4. $u \in S = \{u \in W_0^{1,s}(\Omega) : \|u\|_{p^*} = 1\}$, $s < \frac{N(p-1)}{N-1}$, is said to be a weak solution of the problem (P) if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi dx dy - \lambda \int_{\Omega} |u|^{q-2} u \varphi dx - \int_{\Omega} f(x, u) \varphi dx - \int_{\Omega} \varphi d\mu = 0,$$

$\forall \varphi \in T$.

3. Existence Results

In order to prove the main result of this paper, given in the form of Theorem (1.1), we first prove a few lemmas related to the mountain pass theorem and the Cerami condition. We first develop the necessary tools for the mountain pass theorem. Observe that I_n 's are C^1 functionals defined over $W_0^{1,p}(\Omega)$.

Lemma 3.1. *There exists λ' such that for all $\lambda \in (0, \lambda')$, we can choose $\rho > 0, \eta > 0$ with $I_n(u) > \eta \forall u \in W_0^{1,p}(\Omega)$ and $\|u\| = \rho$.*

Proof. From the assumption (f2) we have, $\forall \epsilon > 0 \exists \delta > 0$ such that $F(x, t) \leq (\lambda_1 - \epsilon)|t|^p, \forall |t| < \delta$. Hence,

$$\begin{aligned} I_n(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \int_{\Omega} F(x, u) dx - \int_{\Omega} u d\mu_n \\ &\geq \frac{1}{p} \|u\|^p - \frac{\lambda c_1}{q} \|u\|^q - \frac{(\lambda_1 - \epsilon)}{p} \|u\|_p^p - c_2 \|u\|_p \|\mu_n\|_{p'} \\ &\geq \frac{1}{p} \|u\|^p - \frac{\lambda c_1}{q} \|u\|^q - \frac{(\lambda_1 - \epsilon)}{p \lambda_1} \|u\|_p^p - c_2 \|u\|_p^{p^*} \|\mu_n\|_{p'} \\ &= \left[\frac{\epsilon}{p \lambda_1} - \left\{ \frac{\lambda c_1}{q} \|u\|^{q-p} + c_2 \|u\|^{p^*-p} \|\mu_n\|_{p'} \right\} \right] \|u\|^p, \end{aligned}$$

where we have used the Rayleigh constant $\lambda_1 = \min_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \left\{ \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx} \right\}$. Consider a

continuous function $\tau_{\lambda} : (0, \infty) \rightarrow \mathbb{R}$ defined as $\tau_{\lambda}(t) = \frac{\lambda c_1}{q} |t|^{q-p} + c_2 \|\mu_n\|_{p'} |t|^{p^*-p}$. Since, $1 < q < p < p^*$, so it can be seen that $\lim_{t \rightarrow \infty} \tau_{\lambda}(t) = \lim_{t \rightarrow 0^+} \tau_{\lambda}(t) = +\infty$. Hence, it is possible to find a ' t_* ' such that $0 < \tau_{\lambda}(t_*) = \min_{t \in (0, \infty)} \tau_{\lambda}(t)$. On solving for t_* such that

$\tau'_{\lambda}(t_*) = 0$, we get $t_* = \left[\frac{\lambda c_1 (p-q)}{q c_2 (p^*-p) \|\mu_n\|_{p'}} \right]^{\frac{1}{p^*-q}}$. This implies

$$\tau_{\lambda}(t_*) = k \lambda^{\frac{p^*-p}{p^*-q}} \rightarrow 0 \text{ as } \lambda \rightarrow 0. \quad (3.1)$$

Thus, by choosing $\|u\| = \rho$ and from (3.1) there exists a λ' such that $\forall \lambda \in (0, \lambda')$ we have $I_n(u) > \eta$. \square

The following lemma guarantees the existence of a function v such that $I_n(v) < 0$.

Lemma 3.2. *There exists $e_1 \in W_0^{1,p}(\Omega), \|e_1\| > 0$, such that $I_n(te_1) < 0$ for sufficiently large t .*

Proof. Let $e_1 \in W_0^{1,p}(\Omega)$ with $\|e_1\| > 0$. From the assumption (f3), we have $\forall M > 0, \exists k(M) > 0$ such that $F(x, t) \geq M|t|^p - k(M)$ a.e. in $\Omega, \forall t \in \mathbb{R}$.

$$\begin{aligned} I_n(te_1) &= \frac{1}{p} \int_{\Omega} |\nabla(te_1)|^p dx - \frac{\lambda}{q} \int_{\Omega} |te_1|^q dx - \int_{\Omega} F(x, te_1) dx - \int_{\Omega} (te_1)\mu_n dx \\ &\leq \frac{|t|^p}{p} \|e_1\|^p - \frac{\lambda|t|^q}{q} \int_{\Omega} |e_1|^q dx - M|t|^p \int_{\Omega} |e_1|^p dx + k(M)|\Omega| - t \int_{\Omega} e_1 \mu_n dx \end{aligned}$$

Choose M large enough such that the whole quantity becomes negative. Hence, $I_n(te_1) < 0$ for t sufficiently large. \square

Lemma 3.3. *There exists $e_2 \in W_0^{1,p}(\Omega), \|e_2\| > 0$ such that $I_n(te_2) < 0; \forall t > 0$ in a small neighborhood of 0.*

Proof. Let $e_2 \in W_0^{1,p}(\Omega)$ with $\|e_2\| > 0$. From the assumption in (f₁), we have, $\forall t \in [0, \bar{t}]$ and $x \in \Omega$ a.e., $F(x, t) \geq 0$. So, for $t \in \left(0, \frac{\bar{t}}{\|e_2\|_{L^\infty(\Omega)}}\right)$, we have

$$\begin{aligned} I_n(te_2) &= \frac{1}{p} \int_{\Omega} |\nabla(te_2)|^p dx - \frac{\lambda}{q} \int_{\Omega} |te_2|^q dx - \int_{\Omega} F(x, te_2) dx - \int_{\Omega} (te_2)\mu_n dx \\ &\leq \frac{t^p}{p} \|e_2\|^p - \frac{\lambda t^q}{q} \int_{\Omega} |e_2|^q dx - t \int_{\Omega} e_2 \mu_n dx \\ &\leq \frac{t^p}{p} \|e_2\|^p - \frac{\lambda t^q}{q} \int_{\Omega} |e_2|^q dx. \end{aligned}$$

We need to find a $t > 0$ for which $I_n(te_2)$ is less than 0. For this we consider,

$$\begin{aligned} 0 &> \frac{t^p}{p} \|e_2\|^p - \frac{\lambda t^q}{q} \int_{\Omega} |e_2|^q dx \\ &= t^q \left[\frac{t^{p-q}}{p} \|e_2\|^p - \frac{\lambda}{q} \int_{\Omega} |e_2|^q dx \right]. \end{aligned}$$

Thus,

$$\frac{t^{p-q}}{p} \|e_2\|^p < \frac{\lambda}{q} \int_{\Omega} |e_2|^q dx$$

and so

$$t < \left(\frac{\lambda p \int_{\Omega} |e_2|^q dx}{q \|e_2\|^p} \right)^{\frac{1}{p-q}}.$$

Thus, if $0 < t < \min \left\{ \left(\frac{\lambda p \int_{\Omega} |e_2|^q dx}{q \|e_2\|^p} \right)^{\frac{1}{p-q}}, \frac{\bar{t}}{\|e_2\|_{L^\infty(\Omega)}} \right\}$ then $I_n(te_2) < 0$. \square

Lemma 3.4. *The functional I_n satisfies the Cerami condition.*

Proof. Let $(u_{m,n})$ be a sequence in $W_0^{1,p}(\Omega)$ such that $I_n(u_{m,n}) \rightarrow c$, $(1+\|u_{m,n}\|)\|I'_n(u_{m,n})\| \rightarrow 0$ as $m \rightarrow \infty$, where $\|I'_n(u_{m,n})\| = \sup \{|\langle I'_n(u_{m,n}), \phi \rangle| : \phi \in W_0^{1,p}(\Omega), \|\phi\| = 1\}$. We first show that $(u_{m,n})$ is bounded. For if not, i.e. $\|u_{m,n}\| \rightarrow \infty$ as $m \rightarrow \infty$, define $v_{m,n} = \frac{u_{m,n}}{\|u_{m,n}\|}$ so that $\|v_{m,n}\| = 1$. Since, $W_0^{1,p}(\Omega)$ is a reflexive space, so $(v_{m,n})$ has a weakly convergent subsequence in $W_0^{1,p}(\Omega)$. Let $v_{m,n} \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$. Due to the compact embedding we have

$$v_{m,n} \rightarrow v_0 \text{ in } L^r(\Omega) \text{ for } r \in [1, p^*) \text{ and hence upto a subsequence} \quad (3.2)$$

$$v_{m,n}(x) \rightarrow v_0(x) \text{ a.e. in } \Omega \text{ as } m \rightarrow \infty. \quad (3.3)$$

We now have two cases.

Case (i): When $v_0 \neq 0$.

Let $\Omega' = \{x \in \Omega : v_0(x) \neq 0\}$. If $x \in \Omega'$, then

$$|u_{m,n}(x)| = |v_{m,n}(x)|\|u_{m,n}\| \rightarrow \infty \text{ a.e. in } \Omega'. \quad (3.4)$$

Since, $I_n(u_{m,n}) \rightarrow c$, we have $\frac{I_n(u_{m,n})}{\|u_{m,n}\|^p} \rightarrow 0$. Hence, as $m \rightarrow \infty$

$$o(1) = \frac{1}{p} - \frac{\lambda}{q} \int_{\Omega} \frac{|u_{m,n}|^q}{\|u_{m,n}\|^p} dx - \int_{\Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \int_{\Omega \setminus \Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \int_{\Omega} \frac{u_{m,n} \mu_n}{\|u_{m,n}\|^p} dx.$$

Using the Rayleigh constant $\lambda_1 = \min_{\substack{u_{m,n} \in W_0^{1,p}(\Omega) \\ u_{m,n} \neq 0}} \left\{ \frac{\int_{\Omega} |\nabla u_{m,n}|^p dx}{\int_{\Omega} |u_{m,n}|^p dx} \right\}$, we get $\int_{\Omega} |u_{m,n}|^p \leq \frac{\|u_{m,n}\|^p}{\lambda_1}$. This implies that $c \int_{\Omega} |u_{m,n}|^q \leq \frac{\|u_{m,n}\|^p}{\lambda_1}$, since $q < p$.

Thus,

$$\begin{aligned} o(1) &\leq \frac{1}{q} - \frac{\lambda}{q} \int_{\Omega} \frac{|u_{m,n}|^q}{\|u_{m,n}\|^p} dx - \int_{\Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \int_{\Omega \setminus \Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \int_{\Omega} \frac{u_{m,n} \mu_n}{\|u_{m,n}\|^p} dx \\ &\leq \frac{1}{q} \max \left\{ 1, 1 - \frac{\lambda}{c\lambda_1} \right\} - \int_{\Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \int_{\Omega \setminus \Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \frac{\int_{\Omega} u_{m,n} \mu_n}{\|u_{m,n}\|^p} dx \\ &\leq \frac{1}{q} \max \left\{ 1, 1 - \frac{\lambda}{c\lambda_1} \right\} - \int_{\Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \int_{\Omega \setminus \Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx + \left| \frac{\int_{\Omega} u_{m,n} \mu_n}{\|u_{m,n}\|^p} dx \right| \\ &\leq \frac{1}{q} \max \left\{ 1, 1 - \frac{\lambda}{c\lambda_1} \right\} - \int_{\Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \int_{\Omega \setminus \Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx + \frac{\|\mu_n\|_{p'}}{\|u_{m,n}\|^p} \|u_{m,n}\|_p \\ &\leq \frac{1}{q} \max \left\{ 1, 1 - \frac{\lambda}{c\lambda_1} \right\} - \int_{\Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx - \int_{\Omega \setminus \Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx + \frac{\|\mu_n\|_{p'} \|u_{m,n}\| (\lambda_1)^{\frac{-1}{p}}}{\|u_{m,n}\|^p} \end{aligned} \quad (3.5)$$

Also,

$$\begin{aligned}\frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} &= \frac{F(x, u_{m,n})}{|u_{m,n}(x)|^p} \cdot \frac{|u_{m,n}(x)|^p}{\|u_{m,n}\|^p} \\ &= \frac{F(x, u_{m,n})}{|u_{m,n}(x)|^p} \cdot |v_{m,n}(x)|^p\end{aligned}$$

Since $\lim_{|t| \rightarrow \infty} \frac{F(x,t)}{|t|^p} = \infty$ and $v_{m,n} \rightarrow v_0$ in $L^p(\Omega)$ with $v_0(x) \neq 0$, then $\frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} \rightarrow \infty$ a.e.

in Ω' . Using the Fatou's lemma, we have $\lim_{m \rightarrow \infty} \int_{\Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx = \infty$.

From the assumption in (f_3) , $\lim_{|t| \rightarrow \infty} F(x, t) = \infty$ uniformly in $\bar{\Omega}$ and hence, there exists two positive constants \bar{t} and M such that $F(x, t) \geq M$ for every $x \in \bar{\Omega}$ and for all t such that $|t| > \bar{t}$. Since F is continuous on $\bar{\Omega} \times \mathbb{R}$, so $|F(x, t)| \leq c_1$ for every $x \in \bar{\Omega}$ and $|t| \leq \bar{t}$. Therefore, there exists a k such that

$$F(x, t) \geq k \text{ for any } (x, t) \in \bar{\Omega} \times \mathbb{R}. \quad (3.6)$$

By our assumption that $\|u_{m,n}\|$ is unbounded in $W_0^{1,p}(\Omega)$ and using (3.5),

$\lim_{m \rightarrow \infty} \int_{\Omega \setminus \Omega'} \frac{F(x, u_{m,n})}{\|u_{m,n}\|^p} dx \geq \lim_{m \rightarrow \infty} \frac{k|\Omega \setminus \Omega'|}{\|u_{m,n}\|^p} = 0$. The last term in (3.5) converges to 0 owing to $p > 1$. This yields a contradiction that $0 \leq -\infty$. Hence, $\|u_{m,n}\|$ is bounded in $W_0^{1,p}(\Omega)$.

Case(ii): When $v_0 = 0$.

Since, $t \mapsto I_n(tu_{m,n})$ is continuous in $t \in [0, 1]$, hence for each m there exists $t_m \in [0, 1]$ such that $I_n(t_m u_{m,n}) = \max_{t \in [0,1]} I_n(tu_{m,n})$. For any $k \in \mathbb{N}$, choose $r_{k,n} = (2p\|u_{m,n}\|^p)^{\frac{1}{p}}$

such that $r_{k,n}\|u_{m,n}\|^{-1} \in (0, 1)$ for any fixed big integer k . Using the dominated convergence theorem and the fact that $v_0 = 0$, we get $\lim_{m \rightarrow \infty} \int_{\Omega} |r_{k,n}v_{m,n}(x)|^q dx = 0$ and

$\lim_{m \rightarrow \infty} \int_{\Omega} |r_{k,n}v_{m,n}(x)|^p dx = 0$. Since, $v_{m,n}(x) \rightarrow v_0(x)$ a.e. Ω and F is continuous so $F(x, r_{k,n}v_{m,n}(x)) \rightarrow F(x, r_{k,n}v_0(x))$ a.e. in Ω .

From $(f_0) \forall \epsilon > 0, \exists c(\epsilon) > 0$ such that $|F(x, t)| \leq \frac{\epsilon}{c_1}|t|^{p^*} + c(\epsilon), \forall t \in \mathbb{R}$, a.e. in Ω . Using the dominated convergence theorem, $\int_{\Omega} F(x, r_{k,n}v_{m,n}(x)) \rightarrow 0$ as $m \rightarrow \infty, \forall k \in \mathbb{N}$

since $F(x, 0) = 0$.

$$\begin{aligned}
I_n(t_m u_{m,n}) &\geq I_n(r_{k,n} \|u_{m,n}\|^{-1} u_{m,n}) \\
&= I_n(r_{k,n} v_{m,n}) \\
&= \frac{1}{p} \int_{\Omega} |\nabla r_{k,n} v_{m,n}|^p dx - \frac{\lambda}{q} \int_{\Omega} |r_{k,n} v_{m,n}|^q dx - \int_{\Omega} F(x, r_{k,n} v_{m,n}) dx - \int_{\Omega} r_{k,n} v_{m,n} \mu_n dx \\
&\geq \frac{1}{p} \int_{\Omega} (\|u_{k,n}\|^{2p} |\nabla v_{m,n}|^p) dx - \frac{\lambda}{q} \int_{\Omega} |r_{k,n} v_{m,n}|^q dx - \int_{\Omega} F(x, r_{k,n} v_{m,n}) dx \\
&\quad - \int_{\Omega} |r_{k,n} v_{m,n} \mu_n| dx \\
&\geq 2 \|u_{k,n}\|^p \|v_{m,n}\|^p - \frac{\lambda}{q} \int_{\Omega} |r_{k,n} v_{m,n}|^q dx - \int_{\Omega} F(x, r_{k,n} v_{m,n}) dx - \|\mu_n\|_{p'} \|r_{k,n} v_{m,n}\|_p
\end{aligned}$$

Since the last three term tends to zero as $n \rightarrow \infty$ so

$$I_n(t_m u_{m,n}) \geq \|u_{k,n}\|^p \quad (3.7)$$

As $\|u_{k,n}\| \rightarrow \infty$ as $k \rightarrow \infty$ so $I_n(t_m u_{m,n}) \rightarrow \infty$ as $m \rightarrow \infty$ for any large integer k . Since $I_n(u_{m,n}) \rightarrow c$ and $I_n(0) = 0$. So, for $t_m \in (0, 1)$, $I'_n(t_m u_{m,n}) = 0$ for any $n \in \mathbb{N}$ and $\langle I'_n(t_m u_{m,n}), t_m u_{m,n} \rangle = t_m \frac{d}{dt} |_{t=t_m} I_n(t u_{m,n}) = 0$.

$$\begin{aligned}
I_n(t_m u_{m,n}) &= I_n(t_m u_{m,n}) - \frac{1}{p} \langle I'_n(t_m u_{m,n}), t_m u_{m,n} \rangle \\
&= \frac{1}{p} \int_{\Omega} |\nabla t_m u_{m,n}|^p dx - \frac{\lambda}{q} \int_{\Omega} |t_m u_{m,n}|^q dx - \int_{\Omega} F(x, t_m u_{m,n}) dx - \int_{\Omega} t_m u_{m,n} \mu_n dx \\
&\quad - \left\{ \frac{1}{p} \int_{\Omega} |\nabla t_m u_{m,n}|^p dx \right. \\
&\quad \left. - \frac{\lambda}{p} \int_{\Omega} |t_m u_{m,n}|^q dx - \frac{1}{p} \int_{\Omega} f(x, t_m u_{m,n})(t_m u_{m,n}) dx - \frac{1}{p} \int_{\Omega} t_m u_{m,n} \mu_n dx \right\} \\
&= \lambda \left(\frac{1}{p} - \frac{1}{q} \right) \int_{\Omega} |t_m u_{m,n}|^q dx + \frac{1}{p} \int_{\Omega} f(x, t_m u_{m,n})(t_m u_{m,n}) dx - \int_{\Omega} F(x, t_m u_{m,n}) dx \\
&\quad - \left(1 - \frac{1}{p} \right) \int_{\Omega} t_m u_{m,n} \mu_n dx
\end{aligned}$$

This implies

$$\begin{aligned}
I_n(t_m u_{m,n}) + A \int_{\Omega} t_m u_{m,n} \mu_n dx &\leq \frac{1}{p} \int_{\Omega} f(x, t_m u_{m,n})(t_m u_{m,n}) dx - \int_{\Omega} F(x, t_m u_{m,n}) dx \\
&= \frac{1}{p} \int_{\Omega} \tilde{F}(x, t_m u_{m,n}) dx,
\end{aligned}$$

where $A = \left(1 - \frac{1}{p}\right)$.

Using the Lemma 2.3 from [19], which states that

Lemma 3.5. *If (f_4) holds, then for any $x \in \Omega$, $\tilde{F}(x, t)$ is increasing in $t \geq \bar{t}$ and decreasing in $t \leq -\bar{t}$, where $\tilde{F}(x, t) = f(x, t)t - pF(x, t)$. In particular, there exists $C_1 > 0$ such that $\tilde{F}(x, s) \leq \tilde{F}(x, t) + C_1$ for $x \in \Omega$ and $0 \leq s \leq t$ or $t \leq s \leq 0$,*

we get

$$\begin{aligned}
\frac{1}{p} \int_{\Omega} \tilde{F}(x, t_m u_{m,n}) dx &= \frac{1}{p} \int_{\{u_{m,n} \geq 0\}} \tilde{F}(x, t_m u_{m,n}) dx + \frac{1}{p} \int_{\{u_{m,n} < 0\}} \tilde{F}(x, t_m u_{m,n}) dx \\
&\leq \frac{1}{p} \int_{\{u_{m,n} \geq 0\}} [\tilde{F}(x, u_{m,n}) + c_1] dx + \frac{1}{p} \int_{\{u_{m,n} < 0\}} [\tilde{F}(x, u_{m,n}) + c_1] dx \\
&= \frac{1}{p} \int_{\Omega} \tilde{F}(x, u_{m,n}) dx + \frac{1}{p} c_1 |\Omega| \\
&= I_n(u_{m,n}) - \frac{1}{p} \langle I'_n(u_{m,n}), u_{m,n} \rangle + \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \int_{\Omega} |u_{m,n}|^q dx \\
&\quad + A \int_{\Omega} u_{m,n} \mu_n dx + \frac{1}{p} c_1 |\Omega| \\
&\leq I_n(u_{m,n}) - \frac{1}{p} \langle I'_n(u_{m,n}), u_{m,n} \rangle + \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \int_{\Omega} |u_{m,n}|^q dx \\
&\quad + A \int_{\Omega} |u_{m,n} \mu_n| dx + \frac{1}{p} c_1 |\Omega| \\
&\leq I_n(u_{m,n}) - \frac{1}{p} \langle I'_n(u_{m,n}), u_{m,n} \rangle + \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \int_{\Omega} |u_{m,n}|^q dx \\
&\quad + A \|\mu_n\|_{p'} \|u_{m,n}\| + \frac{1}{p} c_1 |\Omega| \\
&\leq I_n(u_{m,n}) - \frac{1}{p} \langle I'_n(u_{m,n}), u_{m,n} \rangle + \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \int_{\Omega} |u_{m,n}|^q dx \\
&\quad + A \|\mu_n\|_{p'} \|u_{m,n}\|^q + \frac{1}{p} c_1 |\Omega| \\
&\leq I_n(u_{m,n}) - \frac{1}{p} \langle I'_n(u_{m,n}), u_{m,n} \rangle + c_0 \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \|u_{m,n}\|^q dx \\
&\quad + A \|\mu_n\|_{p'} \|u_{m,n}\|^q + \frac{1}{p} c_1 |\Omega|
\end{aligned}$$

Hence

$$\begin{aligned}
I_n(t_m u_{m,n}) + A \int_{\Omega} t_m u_{m,n} \mu_n dx &\leq I_n(u_{m,n}) - \frac{1}{p} \langle I'_n(u_{m,n}), u_{m,n} \rangle \\
&\quad + c_0 \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \|u_{m,n}\|^q dx + A \|\mu_n\|_{p'} \|u_{m,n}\|^q + \frac{1}{p} c_1 |\Omega|.
\end{aligned}$$

This implies

$$\begin{aligned} I_n(t_m u_{m,n}) &\leq I_n(u_{m,n}) - \frac{1}{p} \langle I'_n(u_{m,n}), u_{m,n} \rangle + c_0 \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \|u_{m,n}\|^q dx \\ &\quad + A \|\mu_n\|_{p'} \|u_{m,n}\|^q + \frac{1}{p} c_1 |\Omega| - A \int_{\Omega} t_m u_{m,n} \mu_n dx \end{aligned}$$

Using (3.7), we get

$$\begin{aligned} \|u_{k,n}\|^p &\leq I_n(u_{m,n}) - \frac{1}{p} \langle I'_n(u_{m,n}), u_{m,n} \rangle + c_0 \lambda \left(\frac{1}{q} - \frac{1}{p} \right) \|u_{m,n}\|^q dx \\ &\quad + A \|\mu_n\|_{p'} \|u_{m,n}\|^q + \frac{c_1}{p} |\Omega| - A \int_{\Omega} t_m u_{m,n} \mu_n dx \end{aligned} \quad (3.8)$$

Now, $\left| \frac{A \int_{\Omega} t_m u_{m,n} \mu_n dx}{\|u_{k,n}\|^p} \right| \leq \frac{A t_m \int_{\Omega} |u_{m,n} \mu_n| dx}{\|u_{k,n}\|^p} \leq \frac{A t_m \|u_{m,n}\|_p \|\mu_n\|_{p'}}{\|u_{k,n}\|^p} \leq \frac{A t_m \|u_{m,n}\| \|\mu_n\|_{p'}}{\|u_{k,n}\|^p} \rightarrow 0$ as $\|u_{k,n}\| \rightarrow \infty \forall k \geq m$.

Since $q < p-1$, hence on dividing (3.8) by $\|u_{k,n}\|^p$ and letting $\|u_{k,n}\|^p \rightarrow \infty$, as $k \rightarrow \infty$, we get $1 \leq 0$ which is a contradiction. Hence, $(u_{m,n})$ is bounded in $W_0^{1,p}(\Omega)$.

The next step is to show that $(u_{m,n})$ admits a strongly convergent subsequence in $W_0^{1,p}(\Omega)$. Since $W_0^{1,p}(\Omega)$ is a reflexive space so $(u_{m,n})$ has a subsequence which converges weakly to u_n in $W_0^{1,p}(\Omega)$ and strongly in $L^r(\Omega)$ for $r \in [1, p^*)$ due to Rellich's compact embedding. We also have that $\|u_{m,n}\|_{p^*}^{p^*} \leq c_2$.

$$\begin{aligned} \langle I'_n(u_{m,n}), u_{m,n} - u_n \rangle &= \int_{\Omega} |\nabla u_{m,n}|^{p-2} \nabla u_{m,n} \cdot (\nabla u_{m,n} - \nabla u) dx - \int_{\Omega} |u_{m,n}|^{q-2} u_{m,n} (u_{m,n} - u_n) dx \\ &\quad - \int_{\Omega} f(x, u_{m,n}) (u_{m,n} - u_n) - \int_{\Omega} \mu_n (u_{m,n} - u_n) dx \end{aligned} \quad (3.9)$$

From the assumption in (f_0) , we have $\forall \epsilon > 0 \exists m(\epsilon) > 0$ such that $|f(x, t)t| \leq \frac{\epsilon}{2c_2} |t|^{p^*} + m(\epsilon)$, $\forall t \in \mathbb{R}$, a.e. in Ω . Choose $\delta = \frac{\epsilon}{2m(\epsilon)} > 0$, $F \subseteq \Omega$ such that $\mu(F) < \delta$ then

$$\begin{aligned} \left| \int_F f(x, u_{m,n}) u_{m,n} dx \right| &\leq \int_F |f(x, u_{m,n}) u_{m,n}| dx \\ &\leq \int_F m(\epsilon) dx + \frac{\epsilon}{2c_2} \int_F |u_{m,n}|^{p^*} dx \\ &\leq \epsilon \end{aligned} \quad (3.10)$$

Hence, $\left\{ \int_{\Omega} f(x, u_{m,n}) u_{m,n} dx : m \in \mathbb{N} \right\}$ is equiabsolutely continuous and therefore from the Vitali convergence theorem we get

$$\int_{\Omega} f(x, u_{m,n}) u_{m,n} dx \rightarrow \int_{\Omega} f(x, u_n) u_n dx \text{ as } m \rightarrow \infty. \quad (3.11)$$

Based on exactly the same line of argument using the assumption in (f_0) , it can be shown that, $\{\int_{\Omega} f(x, u_{m,n})u_n dx : m \in \mathbb{N}\}$ is equiabsolutely continuous and therefore from the Vitali convergence theorem we get

$$\int_{\Omega} f(x, u_{m,n})u_n dx \rightarrow \int_{\Omega} f(x, u_n)u_n dx \text{ as } m \rightarrow \infty \quad (3.12)$$

From (3.11) and (3.12), we get

$$\int_{\Omega} f(x, u_{m,n})(u_{m,n} - u_n)dx \rightarrow 0 \text{ as } m \rightarrow \infty \quad (3.13)$$

Again by using the Hölder's inequality and compact embedding results, we have

$$\begin{aligned} \int_{\Omega} |u_{m,n}|^{q-2}u_{m,n}(u_{m,n} - u_n)dx &\leq \int_{\Omega} | |u_{m,n}|^{q-2}u_{m,n}(u_{m,n} - u_n) | dx \\ &= \int_{\Omega} |u_{m,n}|^{q-1}|u_{m,n} - u_n|dx \\ &\leq \left(\int_{\Omega} |u_{m,n}|^q dx \right)^{\frac{q-1}{q}} \left(\int_{\Omega} |u_{m,n} - u_n|^q dx \right)^{\frac{1}{q}} \rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned} \quad (3.14)$$

Since, $u_{m,n} \rightarrow u_n$ in $L^p(\Omega)$, so

$$\int_{\Omega} (u_{m,n} - u_n)\mu_n \leq \|u_{m,n} - u_n\|_p \|\mu_n\|_{p'} \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.15)$$

We know that, $\langle I'_n(u_{m,n}), u_{m,n} - u_n \rangle \rightarrow 0$ as $m \rightarrow \infty$. Hence, from (3.13), (3.14) and (3.15) we obtain

$$\int_{\Omega} |\nabla u_{m,n}|^{p-2} \nabla u_{m,n} \cdot (\nabla u_{m,n} - \nabla u_n) dx \rightarrow 0 \text{ as } m \rightarrow \infty.$$

This implies that $(u_{m,n})$ converges strongly to u_n in $W_0^{1,p}(\Omega)$. Thus we have proved that the functional I_n satisfies the Cerami condition. \square

By the lemmas in Lemma 3.1, 3.2, 3.3, we can conclude that there exists λ' such that for every $\lambda \in (0, \lambda')$ the functional I_n satisfies the assumption of the Mountain-Pass theorem [10]. Hence, there exists critical point $u_n \in \{u \in W_0^{1,p}(\Omega) : \|u\|_{p^*} = 1\}$ corresponding to each μ_n such that $I_n(u_n) = c > 0$. So, u_n will satisfy its weak formulation, i.e. $\langle I'_n(u_n), v \rangle = 0 \forall v \in W_0^{1,p}(\Omega)$. This implies

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla v dx - \lambda \int_{\Omega} |u_n|^{q-2} u_n v dx - \int_{\Omega} f(x, u_n) v - \int_{\Omega} \mu_n v dx = 0 \quad (3.16)$$

Proof of Theorem 1.1. We will now show that there exists another distinct nontrivial solution of the problem using the Ekeland's variational method. Since I_n is a C^1 functional hence it is bounded below on the ball $\bar{B}_r(0)$. We can thus apply Ekeland variational principle (refer definition 2.3). Applying this principle to $I_n : \bar{B}_r(0) \rightarrow \mathbb{R}$ we find that to each $\delta > 0$ there exists $u_\delta \in \bar{B}_r(0)$ such that $I_n(u_\delta) < \inf_{u \in B_r(0)} I_n(u) + \delta$

and $I_n(u_\delta) < I_n(u) + \delta \|u - u_\delta\|, u \neq u_\delta$. From Lemma 3.1 and 3.3, we know that $\inf_{u \in \partial B_r(0)} I_n(u) \geq M > 0$ and $\inf_{u \in \bar{B}_r(0)} I_n(u) < 0$.

Choose $\delta > 0$ such that $0 < \delta < \inf_{u \in \partial B_r(0)} I_n(u) - \inf_{u \in \bar{B}_r(0)} I_n(u)$.

Hence, $I_n(u_\delta) < \inf_{u \in \partial B_r(0)} I_n(u)$ and so by the choice of u_δ we have $u_\delta \in B_r(0)$. We define

another functional $J_n : \bar{B}_r(0) \rightarrow \mathbb{R}$ by $J_n(u) = I_n(u) + \delta \|u - u_\delta\|$. Due to the Ekeland variational principle, Definition 2.3, we have that u_δ is a minimum point of J_n . So $\frac{J_n(u_\delta + t\phi) - J_n(u_\delta)}{t} \geq 0 \forall t > 0$ small and $\forall \phi \in B_r(0)$.

Hence, $\frac{I_n(u_\delta + t\phi) - I_n(u_\delta)}{t} + \delta \|\phi\| \geq 0$ and $\langle I'_n(u_\delta), -\phi \rangle \geq -\delta \|\phi\|$ as $t \rightarrow 0^+$. Since, $-\phi \in B_r(0)$ we replace ϕ with $-\phi$ to get $\langle I'_n(u_\delta), -\phi \rangle \geq -\epsilon \|\phi\|$ and hence $\langle I'_n(u_\delta), -\phi \rangle \leq \epsilon \|\phi\|$. This implies that $\|I'_n(u_\delta)\| \leq \delta$.

Therefore there exists a sequence $(w_{m,n}) \subset B_r(0)$ such that

$$I_n(w_{m,n}) \rightarrow c' = \inf_{u \in \bar{B}_r(0)} I_n(u) < 0$$

and $I'_n(w_{m,n}) \rightarrow 0$ in $W_0^{1,p}(\Omega)$ as $m \rightarrow \infty$. From Lemma 3.4, the sequence $(w_{m,n}) \rightarrow v_n$ in $W_0^{1,p}(\Omega)$ as $m \rightarrow \infty$.

Hence, $I_n(v_n) = c', I'_n(v_n) = 0$. So, v_n is a non-trivial weak solution of the considered problem. Since, $I_n(u_n) = c > 0 > c' = I_n(v_n)$ so u_n and v_n are distinct nontrivial solutions of the problem (P_n) . Hence, the Theorem 1.1 is proved for the sequence of problems (P_n) .

Choose a test function $v = T_k(u_n)$, where T_k is a truncation operator defined as

$$T_k(t) = \begin{cases} t, & |t| < k \\ k, & |t| \geq k. \end{cases}$$

Clearly $T_k(u_n) \in W_0^{1,p}(\Omega)$. Define $A = \{x : |u_n(x)| \geq k\}$. We have

$$\begin{aligned} \{|\nabla u_n| > t\} &= \{|\nabla u_n| > t, |u_n| < k\} \cup \{|\nabla u_n| > t, |u_n| \geq k\} \\ &\subset \{|\nabla u_n| > t, |u_n| < k\} \cup \{|u_n| \geq k\} \subset \Omega. \end{aligned}$$

Hence, by the subadditivity of Lebesgue measure, we have

$$|\{|\nabla u_n| > t\}| \leq |\{|\nabla u_n| > t, |u_n| < k\}| + |\{|u_n| \geq k\}|. \quad (3.17)$$

Hence we have

$$\begin{aligned}
\int_{\Omega} |\nabla T_k(u_n)|^p &\leq \lambda \int_{\Omega} |u_n|^{q-2} u_n T_k(u_n) + \int_{\Omega} f(x, u_n) T_k(u_n) + \int_{\Omega} \mu_n T_k(u_n) \\
&\leq k\lambda |\Omega|^{1/q} \|u_n\|_q^{q/q'} + \epsilon \int_{\{|u_n|>T\}} |u_n|^{p^*-1} T_k(u_n) + \int_{\Omega \times [-T, T]} f(x, u_n) T_k(u_n) \\
&\quad + \int_{\Omega} \mu_n T_k(u_n) \\
&\leq C_1(\lambda, q, \Omega)k + C_2(\epsilon, \Omega)k + k \int_{\Omega} \mu_n \\
&\leq Ck,
\end{aligned}$$

where we have used the condition (f_0) to bound the second integral and the L^1 bound of the sequence (μ_n) , due to $\mu_n \rightharpoonup \mu$, to bound the third integral. Restricting the above integral on $A_1 = \{x : |u_n| < k\}$ we get,

$$\begin{aligned}
\int_{\{|u_n|<k\}} |\nabla T_k(u_n)|^p &= \int_{\{|u_n|<k\}} |\nabla u_n|^p \\
&\geq \int_{\{|\nabla u_n|>t, |u_n|<k\}} |\nabla u_n|^p \\
&\geq t^p |\{|u_n|>t, |u_n|<k\}|
\end{aligned}$$

so that,

$$|\{|u_n|>t, |u_n|<k\}| \leq \frac{Ck}{t^p} \quad \forall k \geq 1.$$

Therefore, from the Sobolev inequality

$$\lambda_1 \left(\int_{\Omega} |T_k(u_n)|^{p^*} \right)^{\frac{p}{p^*}} \leq \int_{\Omega} |\nabla T_k(u_n)|^p \leq Ck,$$

where, λ_1 is the first eigen value of the p -laplacian operator. Now, if we restrict the integral on the left hand side on $A_2 = \{x : |u_n(x)| \geq k\}$, on which $T_k(u_n) = k$, we then obtain

$$k^p |\{|u_n| \geq k\}|^{\frac{p}{p^*}} \leq Ck,$$

so that

$$|\{|u_n| \geq k\}| \leq \frac{C}{k^{\frac{N(p-1)}{N-p}}} \quad \forall k \geq 1.$$

So, (u_n) is bounded in $M^{\frac{N(p-1)}{N-p}}(\Omega)$. Now (3.17) becomes

$$\begin{aligned}
|\{|\nabla u_n|>t\}| &\leq |\{|\nabla u_n|>t, |u_n|<k\}| + |\{|u_n| \geq k\}| \\
&\leq \frac{Ck}{t^p} + \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \quad \forall k > 1.
\end{aligned}$$

We then choose $k = t^{\frac{N-p}{N-1}}$ and we get

$$|\{|\nabla u_n| > t\}| \leq \frac{C}{t^{\frac{N(p-1)}{N-1}}} \quad \forall t \geq 1,$$

We have thus shown that (∇u_n) is bounded in $M^{\frac{N(p-1)}{N-1}}$ and hence bounded in $W_0^{1,s}(\Omega)$ for $s < \frac{N(p-1)}{N-1}$.

From the Definition 2.2 we have $\mu_n \rightharpoonup \mu$. Since (u_n) is bounded in $W_0^{1,s}(\Omega)$, which is a reflexive space, we have a subsequence such that $u_n \rightharpoonup u$ in $W_0^{1,s}(\Omega)$. Since $p-1 < \frac{N(p-1)}{N-1}$ always holds and according to the assumption $q < p-1$, we have that $u_n \rightarrow u$ in $L^q(\Omega)$ by Rellich's compact embedding. Further, since $u_n \rightarrow u$ in $L^q(\Omega)$, hence by the Egoroff's theorem there exists a subsequence such that $u_n(x) \rightarrow u(x)$ almost everywhere in Ω . Thus, by the continuity of f we have $f(x, u_n(x)) \rightarrow f(x, u(x))$ in Ω almost everywhere. Summing up these results we find that

$$\begin{aligned} \int_{\Omega} \lambda |u_n|^{q-2} u_n \cdot v &\rightarrow \int_{\Omega} \lambda |u|^{q-2} u v \\ \int_{\Omega} f(x, u_n) v &\rightarrow \int_{\Omega} f(x, u) v \\ \int_{\Omega} \mu_n v &\rightarrow \int_{\Omega} v d\mu \end{aligned} \quad (3.18)$$

$\forall v \in W^{1,p}(\Omega) \cap C_0(\bar{\Omega})$. Since $\lambda |u|^{q-2} u + f(x, u) + \mu = \mu_u$, say, is a bounded Radon measure, we look at the following problem

$$\begin{aligned} -\Delta_p z &= \mu_u, \quad \text{in } \Omega \\ v &= 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (3.19)$$

From [4], there exists a solution to (3.19). It can be guaranteed (refer Appendix A) that the sequence (u_n) is compact in $W_0^{1,s}(\Omega)$, for $s \in \left[q, \frac{N(p-1)}{N-1} \right)$. In the course of the proof we have shown that $(\nabla u_n(x))$ is a Cauchy sequence over Ω (Claim 4.1 in Appendix A).

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v &= \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v \\ &= \int_{\Omega} |\nabla z|^{p-2} \nabla z \cdot \nabla v \end{aligned}$$

$\forall v \in C_0^\infty(\bar{\Omega})$. Thus $z = u$ and hence the given problem has a solution. The argument can be repeated for the sequence of solution (v_n) to produce a nontrivial solution, say, w . Thus we have shown the existence of two nontrivial solutions to the problem in (1.4).

4. Appendix A

The proof is motivated from the Lemma 1 in [4].

(u_n) is bounded in $W_0^{1,s}(\Omega)$, for $s \in \left[q, \frac{N(p-1)}{N-1} \right)$ which implies that there exists a subsequence, which we will still denote as (u_n) , $u_n \rightharpoonup u$. This further implies that $u_n \rightarrow u$ in $L^s(\Omega)$ and hence there exists a subsequence such that $u_n(x) \rightarrow u(x)$ almost everywhere in Ω . This further implies that $(u_n), (\nabla u_n)$ is bounded in $L^1(\Omega)$.

claim 4.1. $\nabla u_n(x) \rightarrow \nabla u(x)$ in Ω

Given $\eta > 0, \epsilon > 0$ and set $B > 1, k > 0$ ($n, m \in \mathbb{N}$). Define the following measurable sets.

$$\begin{aligned} E_1 &= \{x \in \Omega : |\nabla u_n(x)| > B\} \cup \{x \in \Omega : |\nabla u_m(x)| > B\} \cup \{x \in \Omega : |u_n(x)| > B\} \\ &\quad \cup \{x \in \Omega : |u_m(x)| > B\} \\ E_2 &= \{x \in \Omega : |u_n(x) - u_m(x)| > k\} \\ E_3 &= \{x \in \Omega : |\nabla u_n(x)| \leq k, |\nabla u_m(x)| \leq k, |u_n(x)| \leq k, |u_m(x)| \leq k, |u_n(x) - u_m(x)| \leq k, \\ &\quad |\nabla(u_n - u_m)(x)| \geq \eta\}. \end{aligned}$$

We remark here that $\{x \in \Omega : |\nabla(u_n - u_m)| \geq \eta\} \subset E_1 \cup E_2 \cup E_3$. Since $(u_n), (\nabla u_n)$ is bounded in $L^1(\Omega)$, hence we can choose a sufficiently large B , independent of n, m , such that $|E_1| < \epsilon$. By the inequality [9]

$$\begin{aligned} (|X|^{p-2}X - |Y|^{p-2}Y).(X - Y) &\geq C_p |X - Y|^p, \text{ if } p \geq 2 \\ &\geq C_p \frac{|X - Y|^2}{(|X| + |Y|)^{2-p}} \text{ if } 1 < p < 2 \end{aligned}$$

we have $(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) > 0$. So there exists a measurable function γ such that $(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v).(u - v) \geq \gamma(x) > 0$. Let $a(x, s, \xi) = |\xi|^{p-2}\xi$. Thus we have $[a(x, s, \xi) - a(x, s, \psi)].(\xi - \psi) \geq \gamma(x), \forall s \in \mathbb{R}, \xi, \psi \in \mathbb{R}^N$ such that $|s|, |\xi|, |\psi| \leq B, |\xi - \psi| \geq \eta, \forall x \in \Omega$.

In fact there exists a set $C \subset \Omega$ such that $|C| = 0$ and $a(x, s, \xi)$ is continuous over $\Omega \setminus C$. Define

$$K = \{(s, \xi, \psi) \in \mathbb{R}^{2N+1} : |s| \leq B, |\xi| \leq B, |\psi| \leq B, |\xi - \psi| \geq \eta\}$$

which is a compact set in \mathbb{R}^{2N+1} . Then

$$\inf\{[a(x, s, \xi) - a(x, s, \psi)].(\xi - \psi) : (s, \xi, \psi) \in K\} = \gamma(x) > 0. \quad (4.1)$$

by the compactness of K . Thus by (4.1) we have

$$\begin{aligned}
\int_{E_3} \gamma &\leq \int_{E_3} [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u_m)] \cdot \nabla(u_n - u_m) \\
&\leq \int_{E_3} [a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m)] \cdot \nabla(u_n - u_m) \\
&\quad + \int_{E_3} [a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m)] \cdot \nabla(u_n - u_m) \\
&\leq \int_{E_3} [a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m)] \cdot \nabla(u_n - u_m) + 2kM \quad (4.2)
\end{aligned}$$

where M is the L^1 bound of the sequence of measures $(|u_n|^{q-2}u_n + f(x, u_n) + \mu_n)$. Due to the continuity of $a(x, s, \xi)$ with respect to (s, ξ) almost everywhere in Ω we thus have for each $\bar{\epsilon} > 0 \exists \delta(x, \bar{\epsilon}) \geq 0$ (with $|\{x \in \Omega : \delta(x, \bar{\epsilon}) = 0\}| = 0$) such that $|s - s'| \leq \delta(x, \bar{\epsilon})$, $|s|, |s'|, |\xi| \leq B$ which implies $|a(x, s, \xi) - a(x, s', \psi)| \leq \bar{\epsilon}$. We remark here that $\lim_{k \rightarrow 0} |\{x \in \Omega : \delta(x, \bar{\epsilon}) < k\}| = 0$.

Let $\delta > 0$ be from Lemma 2 of [4] which is as follows.

Lemma 4.2. *Let $(X, \Sigma, |\cdot|)$ be a measurable space such that $|X| < \infty$. Let $f : X \rightarrow [0, \infty]$ such that $|\{x \in X : f(x) = 0\}| = 0$. Then for any $\epsilon > 0 \exists \delta > 0$ such that $\int_A f dm \leq \delta$ implies $|A| \leq \epsilon$.*

We now choose $\bar{\epsilon}$ such that $\bar{\epsilon} < \frac{\delta}{3}$ and $k > 0$ such that $|E_3 \cap \{x : \delta(x, \bar{\epsilon}) < k\}| < \frac{\delta}{3}$ and $2kM < \frac{\delta}{3}$. Then we finally have

$$\int_{E_3} \gamma < \delta. \quad (4.3)$$

Thus from the Lemma 2 in [4] (stated above in (4.2)) we have $|E_3| < \epsilon$ independent of n, m . This guarantess our claim that (∇u_n) is a Cauchy sequence in Ω . Now since (∇u_n) is a Cauchy sequence in Ω we have that $\nabla u_n(x) \rightarrow v(x)$ almost everywhere in Ω . Therefore, it is also Cauchy in $W_0^{1,2}$ topology and hence convergent to, v , say. In addition to this, we also have that (u_n) is weakly convergent in $W_0^{1,s}(\Omega)$ for $s \in \left[q, \frac{N(p-1)}{N-1} \right)$ to u , i.e. $\nabla u_n \rightharpoonup \nabla u$. These two results implies that $v = \nabla u$ and therefore, (u_n) is compact and $\nabla u_n \rightarrow \nabla u$ in $W_0^{1,s}(\Omega)$.

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