A global existence result for a semilinear wave equation with scale-invariant damping and mass in even space dimension

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Abstract

In the present article a semilinear wave equation with scale-invariant damping and mass is considered. The global (in time) existence of radial symmetric solutions in even spatial dimension n is proved using weighted $L^{\infty} - L^{\infty}$ estimates, under the assumption that the multiplicative constants, which appear in the coefficients of damping and of mass terms, fulfill an interplay condition which yields somehow a "wave-like" model. In particular, combining this existence result with a recently proved blow-up result, a suitable shift of Strauss exponent is proved to be the critical exponent for the considered model. Moreover, the still open part of the conjecture done by D'Abbicco - Lucente - Reissig in [4] is proved to be true in the massless case.

1 Introduction

In the last decade several papers have been devoted to the study of the semilinear wave equation with scale-invariant damping and power nonlinearity

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p, & x \in \mathbb{R}^n, \ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$
(1.1)

where μ is a positive constant. The damping term in (1.1) is critical, indeed, it represents a threshold between effective and non-effective dissipation (see [24, 25]). Here, roughly speaking, effective (non-effective, respectively) means that the solution behaves somehow as the solution of the classical damped wave equation (the free wave equation, respectively) from the point of view of decay estimates. Due to the limit behavior of the time dependent coefficient in the damping term, it is quite natural that the magnitude of the constant μ influences strongly the nature of the equation.

Naively speaking, we can say that for suitably large μ (1.1) and its corresponding linear Cauchy problem are "parabolic" from the point of view of the critical exponent for the power-nonlinearity and decay estimates, respectively.

More precisely, global (in time) existence results are proved in [1] for super-Fujita exponents, that is, for $p > p_{\text{Fuj}}(n) \doteq 1 + \frac{2}{n}$ in dimensions n = 1, 2 and $n \ge 3$ in the cases $\mu \ge \frac{5}{4}$, $\mu \ge 3$ and $\mu \ge n + 2$, respectively. Combining these existence results with a blow-up result from [2], it results that the critical exponent for (1.1) is the so-called *Fujita exponent* $p_{\text{Fuj}}(n)$ when μ is sufficiently large.

Simultaneously and independently, in [23] with different techniques $p_{\mathrm{Fuj}}(n)$ is proved to be critical, assumed that μ is greater than a given constant $\mu_0 \approx (p - p_{\mathrm{Fuj}}(n))^{-2}$. In particular, the test function method is employed to prove the blow-up of the solution for $1 when <math>\mu \geq 1$ and for $1 when <math>\mu \in (0, 1)$, for suitable initial data. Hence, for μ suitably large, in (1.1) the damping term has the same influence on properties of solutions as in the constant coefficients case (classical damped wave equation).

Yet, the situation is completely different when μ is small. In [4] the special value $\mu = 2$ is considered. Indeed, for this value of μ , (1.1) can be transformed in a semilinear free wave equation with nonlinearity $(1+t)^{-(p-1)}|u|^p$. Thus, by using the so-called *Kato's lemma* (see for example [26, Lemma 2.1] or [20, Lemma 2.1]), the authors prove a blow-up result for

$$1 (1.2)$$

in any spatial dimension, assuming nonnegative and compactly supported initial data; here $p_0(n)$ denotes the so-called *Strauss exponent*, that is, the critical exponent for the free wave equation with power nonlinearity, which is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Furthermore, the previous upper bound for p is optimal in the cases n = 1, 2, 3, since global existence results are prove for $p > p_2(n)$ in [1] for n = 1 and in [4] for n = 2 and n = 3in the radial symmetric case. Afterwords, in [3] the sharpness of that blow-up result is shown also in odd dimensions $n \ge 5$ for the radial symmetric case. Since the critical exponent is $p_0(n + 2)$ for n = 2 and any $n \ge 3$, n odd, we remark for the value $\mu = 2$ a "wave-like" behavior from the point of view of the critical exponent p in (1.1). Moreover, recently, in several works, namely [13, 6, 21, 22], it has been studied the blow-up of solutions to (1.1) in the case in which the constant μ is small.

Roughly speaking, in those papers it is derived $p > p_0(n + \mu)$ as a necessary condition for the global (in time) existence of solutions of (1.1), under suitable assumptions on initial data, for $0 < \mu < \frac{n^2 + n + 2}{n + 2}$. Furthermore, some upper bound estimates for the life-span of non-global (in time) solutions are proved. This necessary condition points out once again the hyperbolic nature of the model (1.1) for small μ .

The semilinear model (1.1) can be generalized, considering a further lower order term, namely a mass term, whose time dependent coefficient is chosen suitably in order to preserve the scale-invariance property of the corresponding linear model. Therefore, in this work we will focus on the Cauchy problem for semilinear wave equation with scale-invariant damping and mass and power nonlinearity

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$
(1.3)

where μ, ν are nonnegative constants. Let us define the quantity $\delta \doteq (\mu - 1)^2 - 4\nu^2$, which describes the interplay between the damping term $\frac{\mu}{1+t}u_t$ and the mass term $\frac{\nu^2}{(1+t)^2}u$. For further considerations on how the quantity δ describes the interplay between the damping term $\frac{\mu}{1+t}u_t$ and the mass term $\frac{\nu^2}{(1+t)^2}u$ one can see [19].

Recently, (1.3) has been studied in [14, 16, 15, 17, 18, 5] under different assumptions on δ .

In this article, the following relation between μ and ν is required:

$$\delta = 1. \tag{1.4}$$

We stress, that (1.4) allows to relate the solution to (1.3) with the solution to the semilinear Cauchy problem

$$\begin{cases} v_{tt} - \Delta v = (1+t)^{-\frac{\mu}{2}(p-1)} |v|^p, & x \in \mathbb{R}^n, t > 0, \\ v(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ v_t(0,x) = u_1(x) + \frac{\mu}{2} u_0(x), & x \in \mathbb{R}^n, \end{cases}$$
(1.5)

through the transformation $u(t, x) = (1+t)^{-\frac{\mu}{2}}v(t, x).$

Supposing the validity of (1.4), in [14] has been proved a blow-up result for

$$1$$

provided that data are nonnegative and compactly supported (see also [14, Theorem 2.6]). Very recently, in [18] the exponent $p_{\rm crit}(n,\mu)$ is shown to be critical in the odd dimensional case $n \ge 1$. In particular, following the approach of [7, 10, 8, 11, 9], in the odd dimensional case $n \ge 3$ the radial symmetric case is considered, but an upper bound for μ has to be required.

Since in even dimension Huygens' principle is no longer valid, it is clear that something has to be modified with respect to the approach in [18], in order to study the even case. Purpose of this work is study the sufficiency part for (1.3) under the assumption (1.4) when $n \ge 4$ is even. In other words, we want to prove that $p_0(n + \mu)$ is actually the critical exponent, by proving a global (in time) existence result for supercritical exponents. However, due to technical reasons, it is necessary to claim μ below a certain threshold (exactly as in the odd dimensional case we mentioned above). More specifically, in the treatment we will follow the approach developed in [12] for the free wave equation in the radial symmetric and even dimensional case.

Notations In the present paper we denote $\langle y \rangle = 1 + |y|$ for any $y \in \mathbb{R}$. Furthermore, $f \leq g$ means $0 \leq f \leq Cg$ for a suitable, independent of f and g constant C > 0 and $f \approx g$ stands for $f \leq g$ and $f \geq g$. Finally, as in the introduction, throughout the article $p_{\text{Fuj}}(n)$ and $p_0(n)$ denote the Fujita exponent and the Strauss exponent, respectively.

2 Main result

In this section we state the global (in time) existence result. But first, let us introduce some preparatory definitions. Using the so-called *dissipative transformation* $v(t,x) = \langle t \rangle^{\frac{\mu}{2}} u(t,x)$, thanks to (1.4) we find that v is a solution to (1.5). Due to the fact that we are looking for radial solutions, we are interested to solutions of (1.5) that solve

$$\begin{cases} v_{tt} - v_{rr} - \frac{n-1}{r} v_r = \langle t \rangle^{-\frac{\mu}{2}(p-1)} |v|^p, & r > 0, \ t > 0, \\ v(0,r) = f(r), & r > 0, \\ v_t(0,r) = g(r), & r > 0, \end{cases}$$
(2.6)

where where $f \doteq u_0$ and $g \doteq u_1 + \frac{\mu}{2}u_0$, r = |x| and a singular behavior of solutions and their *r*-derivatives is allowed as $r \to 0^+$. Let us recall some known result for the corresponding linear problem

$$\begin{cases} v_{tt} - v_{rr} - \frac{n-1}{r} v_r = 0, & r > 0, \ t > 0, \\ v(0,r) = f(r), & r > 0, \\ v_t(0,r) = g(r), & r > 0. \end{cases}$$
(2.7)

Let us begin with the linear Cauchy problem

$$\begin{cases} v_{tt} - v_{rr} - \frac{n-1}{r} v_r = 0, & r > 0, \ t > 0, \\ v(0, r) = 0, & r > 0, \\ v_t(0, r) = c_n g(r), & r > 0, \end{cases}$$
(2.8)

where we included the multiplicative constant

$$c_n \doteq \sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right) = \pi 2^{-\frac{n-2}{2}} (n-3)!!$$

in the second data in order to "normalize" the representation formula for the solution of this Cauchy problem. Let us introduce the parameter $m \doteq \frac{n-2}{2} \ge 1$.

We define for $t \ge 0$ and r > 0 the function

$$\Theta(g)(t,r) \doteq r^{-2m} \{ w_1(t,r) + w_2(t,r) \},$$
(2.9)

where we have set

$$w_1(t,r) \doteq \int_{|t-r|}^{t+r} \lambda^{2m+1} g(\lambda) K_m(\lambda,t,r) \, d\lambda, \qquad (2.10)$$

$$w_2(t,r) \doteq \int_0^{(t-r)_+} \lambda^{2m+1} g(\lambda) \,\widetilde{K}_m(\lambda,t,r) \,d\lambda, \qquad (2.11)$$

with

$$K_j(\lambda, t, r) \doteq \int_{\lambda}^{t+r} \frac{H_j(\rho, t, r)}{\sqrt{\rho^2 - \lambda^2}} d\rho \quad \text{for} \quad j = 0, 1, \cdots, m,$$
(2.12)

$$\widetilde{K}_{j}(\lambda, t, r) \doteq \int_{t-r}^{t+r} \frac{H_{j}(\rho, t, r)}{\sqrt{\rho^{2} - \lambda^{2}}} d\rho \quad \text{for} \quad j = 0, 1, \cdots, m,$$
(2.13)

and

$$H_j(\rho, t, r) \doteq \left(\left(\frac{1}{2\rho} \frac{\partial}{\partial \rho}\right)^* \right)^j H(\rho - t, r) \quad \text{for} \quad j = 0, 1, \cdots, m \quad \text{and} \quad |\rho - t| \le r,$$
$$H(\rho, r) \doteq (r^2 - \rho^2)^{m - \frac{1}{2}},$$

being $(\frac{1}{2\rho}\frac{\partial}{\partial\rho})^* = \frac{\partial}{\partial\rho}(-\frac{1}{2\rho})$ the adjoint operator of $\frac{1}{2\rho}\frac{\partial}{\partial\rho}$ (for further considerations on the representation formula (2.9) cf. [12, Section 3.2]).

Let $v^0 = v^0(t, r)$ be the function defined as follows:

$$v^{0} \doteq c_{n}^{-1} \{ \Theta(g) + \partial_{t} \Theta(f) \}.$$

$$(2.14)$$

The function v^0 is the solution to (2.7). In Section 3 we will clarify more precisely in which sense v^0 solves (2.7) (cf. Proposition 3.2). We introduce now the space for solutions. Given a positive parameter κ , we define the Banach space

$$X_{\kappa} \doteq \left\{ v \in \mathcal{C}([0,\infty), \mathcal{C}^{1}(0,\infty)) : \|v\|_{X_{\kappa} < \infty} \right\},\$$

equipped with the norm

$$\|v\|_{X_{\kappa}} \doteq \sup_{t \ge 0, r > 0} \left(r^{m-1} \langle r \rangle |v(t,r)| + r^m |\partial_r v(t,r)| \right) \phi_{\kappa}(t,r)^{-1},$$

where the weight function ϕ_{κ} is defined by

$$\phi_{\kappa}(t,r) \doteq \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\kappa}.$$
(2.15)

Let us consider the integral operator L defined for any $v \in X_{\kappa}$ by

$$Lv(t,r) \doteq c_n^{-1} \int_0^t \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \Theta(|u(\tau,\cdot)|^p)(t-\tau,r) \, d\tau,$$
(2.16)

where $\Theta(|u(\tau, \cdot)|^p)$ is defined by (2.9), replacing $g(\lambda)$ with $|u(\tau, \lambda)|^p$.

According to Duhamel's principle, we introduce the following definition:

Definition 2.1. Let $v^0 = v^0(t, r)$ be the function defined through (2.14). We say that v = v(t, r) is a radial solution to (1.5) in X_{κ} , if $v \in X_{\kappa}$ for some $\kappa > 0$ and v satisfies the integral equation

$$v(t,r) = v^{0}(t,r) + Lv(t,r)$$
 for any $t \ge 0, r > 0.$

In the following we will use the notation:

$$M(n) \doteq \frac{n-1}{2} \left(1 + \sqrt{\frac{n+7}{n-1}} \right).$$
 (2.17)

As we will see in the next result, which is the main theorem of this article, M(n) is the upper bound for the coefficient μ .

Theorem 2.2. Let $n \ge 4$ be an even integer. Let us assume $\mu \in [2, M(n))$ and $\nu \ge 0$ satisfying the relation (1.4), where M(n) is defined by (2.17), and

$$p \in \left(p_0(n+\mu), \min\left\{p_{\mathrm{Fuj}}\left(\frac{n+\mu-1}{2}\right), p_{\mathrm{Fuj}}(\mu)\right\}\right).$$
 (2.18)

Then, there exist $\varepsilon_0 > 0$ and $0 < \kappa_1 < \kappa_2 < m + \frac{1}{2}$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any radial data $u_0 \in \mathcal{C}^2(\mathbb{R}^n), u_1 \in \mathcal{C}^1(\mathbb{R}^n)$ satisfying

$$\begin{aligned} |d_r^j u_0(r)| &\leq \varepsilon \, r^{-m-j+1} \langle r \rangle^{-\bar{\kappa} - \frac{3}{2}} & \text{for} \quad j = 0, 1, 2, \\ |d_r^j (u_1(r) + \frac{\mu}{2} u_0(r))| &\leq \varepsilon \, r^{-m-j} \langle r \rangle^{-\bar{\kappa} - \frac{3}{2}} & \text{for} \quad j = 0, 1, \end{aligned}$$

for some $\bar{\kappa} \in (\kappa_1, \kappa_2]$, the Cauchy problem (1.3) admits a uniquely determined radial solution $u \in \mathcal{C}([0, \infty), \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}))$, in the sense that $v(t, r) = \langle t \rangle^{\frac{\mu}{2}} u(t, r)$ satisfies Definition 2.1 for any $\kappa \in (\kappa_1, \bar{\kappa}]$.

Furthermore, the following decay estimates hold for any $t \ge 0$, r > 0 and $\kappa \in (\kappa_1, \bar{\kappa}]$:

$$\begin{aligned} |u(t,r)| &\lesssim \varepsilon \, r^{-m+1} \langle r \rangle^{-1} \langle t \rangle^{-\frac{\mu}{2}} \langle t-r \rangle^{-\kappa} \langle t+r \rangle^{-\frac{1}{2}}, \\ |\partial_r u(t,r)| &\lesssim \varepsilon \, r^{-m} \langle t \rangle^{-\frac{\mu}{2}} \langle t-r \rangle^{-\kappa} \langle t+r \rangle^{-\frac{1}{2}}. \end{aligned}$$

For the proof of Theorem 2.2 it is necessary to modify some tools developed for the proof of the main theorem in [12].

The remain part of the article is organized as follows: in Section 3 we will recall some known results for the linear problem, following the treatment of [12]; hence, in Section 4 some preparatory results are derived; also, using these preliminary estimates the proof of Theorem 2.2 is provided in Section 5. Finally, in Section 6 some final remarks and comments on open problems are given.

3 Linear equation

In this section we recall some known estimates from [12, Sections 3-4-5], which will be useful for the proof of Theorem 2.2. In Section 2, we introduced the definition of $\Theta(g)$. Now, we will show an alternative representation for $\Theta(g)$ and a representation for its r-derivative involving the kernels $K_{m-1}(\lambda, t, r)$ and $\tilde{K}_{m-1}(\lambda, t, r)$.

Lemma 3.1. Let g be a $C^1((0,\infty))$ function such that $g^{(j)}(\lambda) = O(\lambda^{-2m-j+\varsigma})$ as $\lambda \to 0^+$ for j = 0, 1 and some $\varsigma > 0$. Then, it holds for $t \ge 0, r > 0$ and $t \ne r$

$$2r^{2m}\Theta(g)(t,r) \doteq w_3(t,r) + w_4(t,r)$$
(3.19)

with

$$w_3(t,r) \doteq \int_{|t-r|}^{t+r} \partial_\lambda(\lambda^{2m}g(\lambda)) K_{m-1}(\lambda,t,r) \, d\lambda, \tag{3.20}$$

$$w_4(t,r) \doteq \int_0^{t-r} \partial_\lambda(\lambda^{2m} g(\lambda)) \widetilde{K}_{m-1}(\lambda,t,r) \, d\lambda \qquad \text{for} \quad t > r, \tag{3.21}$$

$$w_4(t,r) \doteq (r-t)^{2m} g(r-t) K_{m-1}(r-t,t,r) \qquad for \quad t < r, \tag{3.22}$$

where K_{m-1} and \widetilde{K}_{m-1} are defined by (2.12) and (2.13), respectively.

Furthermore, it holds for $t \ge 0$, r > 0 and $t \ne r$

$$\partial_r \left(2r^{2m} \Theta(g)(t,r) \right) \doteq w_5(t,r) + w_6(t,r), \tag{3.23}$$

where we have set

$$w_5(t,r) \doteq \int_{|t-r|}^{t+r} \partial_\lambda(\lambda^{2m}g(\lambda)) \,\partial_r K_{m-1}(\lambda,t,r) \,d\lambda, \tag{3.24}$$

$$w_6(t,r) \doteq \int_0^{t-r} \partial_\lambda(\lambda^{2m} g(\lambda)) \,\partial_r \widetilde{K}_{m-1}(\lambda,t,r) \,d\lambda \qquad \text{for} \quad t > r, \tag{3.25}$$

$$w_6(t,r) \doteq (r-t)^{2m} g(r-t) \,\partial_r K_{m-1}(r-t,t,r) \qquad for \quad t < r.$$
(3.26)

Proof. See [12, Lemma 4.6].

Using the above described operator Θ , we have seen how to provide through (2.14) the representation formula for the solution to (2.7) in the case *n* even.

In the next result, which describes the properties of the solution v^0 to (2.7). A condition, which allows the data to be possibly singular as $r \to 0^+$, will be introduced. For the proof of the forthcoming proposition one can see [12, Theorem 2.1].

Proposition 3.2. Let us consider an even integer $n \ge 4$ and radial initial data $f \in C^2((0,\infty)), g \in C^1((0,\infty))$ such that

$$|f^{(j)}(r)| \le \varepsilon r^{-m-j+1} \langle r \rangle^{-\kappa - \frac{3}{2}} \quad for \quad j = 0, 1, 2,$$
 (3.27)

$$|g^{(j)}(r)| \le \varepsilon r^{-m-j} \langle r \rangle^{-\kappa - \frac{3}{2}} \qquad for \quad j = 0, 1, \tag{3.28}$$

where the parameters ε, κ satisfy $\varepsilon > 0$ and $0 < \kappa < m + \frac{1}{2}$. Let $v^0 = v^0(t, r)$ be defined by (2.14). Then, $v^0 \in \mathcal{C}^1([0, \infty) \times (\mathbb{R}^n \setminus \{0\})) \cap \mathcal{C}^2([0, \infty), \mathcal{D}'(\mathbb{R}^n))$ is the uniquely determined radial symmetric distributional solution to (2.7), in the following sense:

$$\begin{split} & \frac{d^2}{dt^2} \langle v^0(t,\cdot),\psi\rangle = \langle v^0(t,\cdot),\Delta\psi\rangle & \text{for any } t > 0, \\ & \langle v^0(t,\cdot),\psi\rangle\big|_{t=0} = \langle f,\psi\rangle, \quad \frac{d}{dt} \langle v^0(t,\cdot),\psi\rangle\big|_{t=0} = \langle g,\psi\rangle, \end{split}$$

for any $\psi \in \mathcal{C}_0^{\infty}(\mathbb{R}^n)$, here $\langle \cdot, \cdot \rangle$ denotes the real scalar product in $L^2(\mathbb{R}^n)$ and r = |x|. Besides, the solution v^0 fulfills the following decay estimates for any $t \ge 0$ and r > 0:

$$|v^{0}(t,r)| \lesssim \varepsilon r^{1-m} \langle r \rangle^{-1} \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\kappa}, |\partial_{r} v^{0}(t,r)| \lesssim \varepsilon r^{-m} \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\kappa}.$$
(3.29)

Remark 3.3. In the setting of the previous theorem it is possible to derive stronger decay estimates than those we have written in the statement, see [12, Proposition 4.9]. Nevertheless, (3.29) is enough for our purposes, in order to prove the global existence result for the semilinear radial symmetric Cauchy problem (2.6) for $n \ge 4$ even. Indeed, it follows by (3.29) that

$$\|v^0\|_{X_{\kappa}} \lesssim \varepsilon \qquad for \quad 0 < \kappa < m + \frac{1}{2}. \tag{3.30}$$

Finally, let us recall some known estimates for the kernels $K_j(\lambda, t, r)$ and $K_j(\lambda, t, r)$, which are going to be used in the treatment of the semilinear case. For the proof of the next lemmas see [12, Lemma 4.2 and Corollary 4.3].

Lemma 3.4. Let $\gamma \in \{0, \frac{1}{2}\}$ and let $t \ge 0, r > 0$. Let us consider $\alpha = 0, 1$. Then, we have for $|t - r| < \lambda < t + r$

$$|K_m(\lambda, t, r)| \lesssim r^{m+\gamma - \frac{1}{2}} \lambda^{-m-\gamma} (\lambda - t + r)^{-\frac{1}{2}}, \qquad (3.31)$$

$$\left|\partial_r^{\alpha} K_{m-1}(\lambda, t, r)\right| \lesssim r^{m+\gamma+\frac{1}{2}-\alpha} \lambda^{-m-\gamma+1} (\lambda - t + r)^{-\frac{1}{2}},\tag{3.32}$$

while for $0 < \lambda < t - r$ we get

$$|\widetilde{K}_m(\lambda, t, r)| \lesssim r^{m+\gamma - \frac{1}{2}} (t - r)^{-m-\gamma} (t - r - \lambda)^{-\frac{1}{2}},$$
(3.33)

$$|\widetilde{K}_{m}(\lambda, t, r)| \lesssim r^{m+\gamma-\frac{1}{2}}(t-r)^{-m-\gamma}(t-r-\lambda)^{-\frac{1}{2}},$$

$$\partial_{r}^{\alpha}\widetilde{K}_{m-1}(\lambda, t, r)| \lesssim r^{m+\gamma+\frac{1}{2}-\alpha}(t-r)^{-m-\gamma+1}(t-r-\lambda)^{-\frac{1}{2}}.$$
(3.33)
(3.34)

Lemma 3.5. Let $\gamma \in \{0, \frac{1}{2}\}$ and let $t \ge 0, r > 0$. Let us consider $j = 0, 1, \dots, m$ and $\alpha \in \mathbb{N}$ such that $j + \alpha \leq m$. If $0 < \lambda < t - r$, then,

$$\left|\partial_{\lambda}\partial_{r}^{\alpha}\widetilde{K}_{j}(\lambda,t,r)\right| \lesssim r^{2m-j+\gamma-\frac{1}{2}-\alpha}(t-r)^{-j-\gamma}(t-r-\lambda)^{-\frac{3}{2}}.$$
(3.35)

Preliminaries 4

In this section, we derive some estimates which will be fundamental in the proof of Theorem 2.2, making it more fluent.

Throughout this section we assume the following conditions on p > 1 and κ : for the exponent of the nonlinearity p we require

$$p_0(n+\mu)
(4.36)$$

while for the parameter κ , which appears in the definition of the norm on X_{κ} , we require as upper and lower bounds

$$0 < \kappa \le q + \frac{\mu}{2}(p-1) = \frac{n+\mu-1}{2}p - \frac{n+\mu+1}{2}, \tag{4.37}$$

$$k > \frac{-q+1}{p-1} - \frac{\mu}{2} = \frac{2}{p-1} - \frac{n+\mu-1}{2} \qquad \Leftrightarrow \qquad p\kappa + q + \frac{\mu}{2}(p-1) > \kappa + 1, \tag{4.38}$$

respectively, where

$$q \doteq \frac{n-1}{2}p - \frac{n+1}{2}.$$
 (4.39)

In particular, (4.36) implies the nonemptiness of the range for κ , since the upper bound for p provides a positive lower bound for κ in (4.37), while the lower bound for p is equivalent to require the validity of the relation $\frac{-q+1}{p-1} - \frac{\mu}{2} < q + \frac{\mu}{2}(p-1)$, which provides the compatibility between (4.37) and (4.38). Besides, the range for p is not empty since $p_0(n+\mu) < p_{\text{Fuj}}\left(\frac{n+\mu-1}{2}\right)$ is always true.

In Section 2 we defined the integral operator L. In order to estimate the integrand in (2.16), similarly to (2.9), (3.19) and (3.23), the following representations are valid for any $0 \le \tau \le t$ and r > 0 such that $t \ne r$:

$$r^{2m}\Theta(|u(\tau,\cdot)|^p)(t,r) = W_1(t,r;\tau) + W_2(t,r;\tau), \qquad (4.40)$$

$$2r^{2m}\Theta(|u(\tau,\cdot)|^p)(t,r) = W_3(t,r;\tau) + W_4(t,r;\tau), \qquad (4.41)$$

$$\partial_r \left(2r^{2m} \Theta(|u(\tau, \cdot)|^p)(t, r) \right) = W_5(t, r; \tau) + W_6(t, r; \tau), \tag{4.42}$$

where $W_i(t,r;\tau)$, for $i=1,\cdots,6$, is defined analogously to $w_i(t,r)$ by substituting $|u(\tau,\lambda)|^p$ in place of $g(\lambda)$ in (2.10), (2.11), (3.20), (3.21), (3.22), (3.24), (3.25) and (3.26).

We introduce now a quantity which prescribes somehow the decay rate we allow for the nonlinearity $|v|^p$. We define for any $v \in X_{\kappa}$, j = 0, 1 and $\nu \in \mathbb{R}$ the quantity

$$N_{j}^{\nu}(|v|^{p}) \doteq \sup_{t \ge 0, r > 0} \left| \partial_{\lambda}^{j} \left(\lambda^{2m} |v(\tau, \lambda)|^{p} \right) \right| \lambda^{-m-\nu+j} \langle \lambda \rangle^{q-\frac{p}{2}+\frac{3}{2}+\nu-j} \phi_{\kappa}(\tau, \lambda)^{-p},$$
(4.43)

where q and ϕ_{κ} are defined by (4.39) and (2.15), respectively.

Let us prove now some preliminary lemmas which are going to be useful in the proof of the last proposition of this section. A fondamental tool for their proofs is the next estimate, which is taken from [11] (cf. Lemma 4.7).

If $a, b \ge 0$ satisfy a + b > 1, then, it holds

$$\int_{\mathbb{R}} \langle x \rangle^{-a} \langle x + y \rangle^{-b} \, dx \lesssim 1 \qquad \text{for any} \quad y \in \mathbb{R}.$$
(4.44)

Lemma 4.1. Let us consider p, κ satisfying (4.36), (4.37) and (4.38) and let q be defined by (4.39). Then, we have for any $y \in \mathbb{R}$

$$\int_{\mathbb{R}} \langle x \rangle^{-p\kappa} \langle x + y \rangle^{-q - \frac{\mu}{2}(p-1)} \, dx \lesssim \langle y \rangle^{-\kappa}.$$

Proof. We follow [11, Lemma 4.8]. Let us denote $G(y) \doteq \int_{\mathbb{R}} \langle x \rangle^{-p\kappa} \langle x + y \rangle^{-q - \frac{\mu}{2}(p-1)} dx$. We consider first the case $y \ge 0$. We split G(y) as follows:

$$G(y) = \int_{-\infty}^{-\frac{y}{2}} \langle x \rangle^{-p\kappa} \langle x+y \rangle^{-q-\frac{\mu}{2}(p-1)} dx + \int_{-\frac{y}{2}}^{\infty} \langle x \rangle^{-p\kappa} \langle x+y \rangle^{-q-\frac{\mu}{2}(p-1)} dx$$
$$\doteq G_1(y) + G_2(y).$$

Since on the domain of integration of $G_1(y)$ it holds $\langle x \rangle \gtrsim \langle y \rangle$ and $\kappa > 0$, we get

$$G_1(y) \lesssim \langle y \rangle^{-\kappa} \int_{-\infty}^{-\frac{y}{2}} \langle x \rangle^{-\kappa(p-1)} \langle x+y \rangle^{-q-\frac{\mu}{2}(p-1)} dx \lesssim \langle y \rangle^{-\kappa},$$

where in the last inequality we may use (4.44) thanks to (4.37) and (4.38).

On the other hand, when $x \ge -\frac{y}{2}$ the inequality $\langle x + y \rangle \gtrsim \langle y \rangle$ is satisfied. Therefore, using again (4.44), we find

$$G_2(y) \lesssim \langle y \rangle^{-\kappa} \int_{-\frac{y}{2}}^{\infty} \langle x \rangle^{-p\kappa} \langle x + y \rangle^{\kappa-q-\frac{\mu}{2}(p-1)} dx \lesssim \langle y \rangle^{-\kappa}.$$

If $y \leq 0$, we get $G(y) = \int_{\mathbb{R}} \langle x - y \rangle^{-p\kappa} \langle x \rangle^{-q - \frac{\mu}{2}(p-1)} dx$. Then, splitting G(y) on $x \leq \frac{y}{2}$ and on $x \geq \frac{y}{2}$ and proceeding as before, we have the desired estimate.

Lemma 4.2. Let us consider p, κ satisfying (4.36), (4.37) and (4.38) and let q be defined by (4.39) such that $q \geq \frac{1}{2}$. Then, we have for any $y \geq 0$

$$\int_{-y}^{-\frac{y}{2}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1)} \frac{\langle x+y \rangle^{-q+\frac{1}{2}}}{\sqrt{x+y}} \, dx \lesssim \langle y \rangle^{-\kappa}. \tag{4.45}$$

Proof. Let H(y) be the integral that appears in the left-hand side of (4.45). Let us split H(y) in two integrals

$$H(y) = \int_{-y}^{\tilde{y}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1)} \frac{\langle x+y \rangle^{-q+\frac{1}{2}}}{\sqrt{x+y}} \, dx + \int_{\tilde{y}}^{-\frac{y}{2}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1)} \frac{\langle x+y \rangle^{-q+\frac{1}{2}}}{\sqrt{x+y}} \, dx$$

$$\doteq H_1(y) + H_2(y),$$

where $\tilde{y} \doteq \min(-y+1, -\frac{y}{2})$.

For $H_1(y)$, being $\langle x \rangle^{-\frac{\mu}{2}(p-1)}$ and $\langle x+y \rangle^{-q+\frac{1}{2}}$ bounded on [-y, -y+1], we get

$$H_1(y) \lesssim \int_{-y}^{-y+1} \frac{\langle x \rangle^{-\kappa p}}{\sqrt{x+y}} dx.$$

If $y \ge 2$, then, $\langle x \rangle \approx \langle y \rangle$ on [-y, -y+1]. Also, because of p > 1 and $\kappa > 0$, we obtain

$$H_1(y) \lesssim \langle y \rangle^{-\kappa p} \int_{-y}^{-y+1} \frac{dx}{\sqrt{x+y}} \lesssim \langle y \rangle^{-\kappa p} \lesssim \langle y \rangle^{-\kappa}.$$

Else, for $0 \le y \le 2$, since $\langle y \rangle^{\kappa}$ is bounded, we have

$$H_1(y) \lesssim \int_{-y}^{-y+1} \frac{dx}{\sqrt{x+y}} = 2 \lesssim \langle y \rangle^{-\kappa}.$$

Let us estimate $H_2(y)$. Since $\langle x + y \rangle \leq 2(x + y)$ for $x \geq -y + 1$, then,

$$H_2(y) \lesssim \int_{\widetilde{y}}^{-\frac{y}{2}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1)} \langle x + y \rangle^{-q} \, dx \le \int_{-y}^{-\frac{y}{2}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1)} \langle x + y \rangle^{-q} \, dx \doteq \widetilde{H}_2(y).$$

Being $\langle x \rangle \gtrsim \langle x + y \rangle$ and $\langle x \rangle \approx \langle y \rangle$ for $x \in [-y, -\frac{y}{2}]$, we estimate $H_2(y)$ as follows:

$$\begin{split} \widetilde{H}_2(y) &\lesssim \langle y \rangle^{-\kappa} \int_{-y}^{-\frac{y}{2}} \langle x \rangle^{-\kappa(p-1)-\frac{\mu}{2}(p-1)} \langle x+y \rangle^{-q} \, dx \\ &\lesssim \langle y \rangle^{-\kappa} \int_{-y}^{-\frac{y}{2}} \langle x+y \rangle^{-\kappa(p-1)-\frac{\mu}{2}(p-1)-q} \, dx \lesssim \langle y \rangle^{-\kappa} \end{split}$$

here we used (4.38) in order to guarantee the uniform boundedness of the integral in the last line. Combining the estimates for $H_1(y)$ and $H_2(y)$, we find (4.45).

Lemma 4.3. Let us consider p, κ satisfying (4.36), (4.37) and (4.38) and let q be defined by (4.39) such that $0 \le q < \frac{1}{2}$. Then, we have for any $y \ge 0$

$$\int_{-y}^{-\frac{y}{2}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1) + \frac{1}{2}} \frac{\langle x+y \rangle^{-q}}{\sqrt{x+y}} \, dx \lesssim \langle y \rangle^{-\kappa}. \tag{4.46}$$

Proof. We have to modify slightly the proof of Lemma 4.2. Let I(y) be the integral that appears in the left-hand side of (4.46). Even in this case we split I(y) in two integrals

$$I(y) = \int_{-y}^{\tilde{y}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1) + \frac{1}{2}} \frac{\langle x+y \rangle^{-q}}{\sqrt{x+y}} \, dx + \int_{\tilde{y}}^{-\frac{y}{2}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1) + \frac{1}{2}} \frac{\langle x+y \rangle^{-q}}{\sqrt{x+y}} \, dx$$
$$\doteq I_1(y) + I_2(y),$$

where $\tilde{y} = \min(-y+1, -\frac{y}{2})$ as before.

We begin with $I_1(y)$. Since $\langle x+y\rangle^{-q}$ is bounded on [-y, -y+1], it holds

$$I_1(y) \lesssim \int_{-y}^{-y+1} \frac{\langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1) + \frac{1}{2}}}{\sqrt{x+y}} \, dx.$$

If $y \ge 2$, then, $\langle x \rangle \approx \langle y \rangle$ on [-y, -y + 1]. Also,

$$I_1(y) \lesssim \langle y \rangle^{-\kappa} \int_{-y}^{-y+1} \frac{\langle x \rangle^{-\kappa(p-1)-\frac{\mu}{2}(p-1)+\frac{1}{2}}}{\sqrt{x+y}} \, dx \lesssim \langle y \rangle^{-\kappa} \int_{-y}^{-y+1} \frac{dx}{\sqrt{x+y}} \lesssim \langle y \rangle^{-\kappa}$$

where in the second last inequality we used the fact that the exponent of $\langle x \rangle$ is nonpositive. Indeed, $-\kappa(p-1) - \frac{\mu}{2}(p-1) + \frac{1}{2} \leq 0$ is equivalent to require

$$\kappa \ge \frac{1}{2(p-1)} - \frac{\mu}{2}.$$
(4.47)

But thanks to the assumption $q < \frac{1}{2}$ this lower bound on κ is weaker than the lower bound in (4.38). Therefore, under the assumptions we are working with, (4.47) is always satisfied. On the other hand, for $0 \le y \le 2$, since $\langle x \rangle$ is bounded on [-y, -y + 1] and $\langle y \rangle^{\kappa}$ is bounded as well, we have

$$I_1(y) \lesssim \int_{-y}^{-y+1} \frac{dx}{\sqrt{x+y}} = 2 \lesssim \langle y \rangle^{-\kappa}.$$

Using again (4.47), it is possible to show the estimate $I_2(y) \leq \langle y \rangle^{-\kappa}$ exactly as we have done in Lemma 4.2 for the term $H_2(y)$.

Summarizing, the estimates for $I_1(y)$ and $I_2(y)$ imply (4.46).

Lemma 4.4. Let us consider p, κ satisfying (4.36), (4.37) and (4.38) and let q be defined by (4.39) such that $-\frac{1}{2} \leq q < 0$. Then, we have for any $y \geq 0$

$$\int_{-y}^{-\frac{y}{2}} \langle x \rangle^{-\kappa p - \frac{\mu}{2}(p-1) + 1} \frac{\langle x + y \rangle^{-q - \frac{1}{2}}}{\sqrt{x+y}} \, dx \lesssim \langle y \rangle^{-\kappa}. \tag{4.48}$$

Proof. First of all, we point out that $\langle x \rangle \approx \langle y \rangle$ on the domain of integration. Therefore, if we denote by J(y) the integral in the left-hand side of (4.48), then,

$$J(y) \lesssim \langle y \rangle^{-\kappa} \int_{-y}^{-\frac{y}{2}} \langle x \rangle^{-\kappa(p-1)-\frac{\mu}{2}(p-1)+1} \frac{\langle x+y \rangle^{-q-\frac{1}{2}}}{\sqrt{x+y}} \, dx.$$

Using (4.38), it results

$$\langle x \rangle^{-\kappa(p-1) - \frac{\mu}{2}(p-1) + 1} \le \langle x \rangle^{q-\varepsilon},$$

for a suitably small $\varepsilon > 0$. Also,

$$J(y) \lesssim \langle y \rangle^{-\kappa} \int_{-y}^{-\frac{y}{2}} \langle x \rangle^{q-\varepsilon} \frac{\langle x+y \rangle^{-q-\frac{1}{2}}}{\sqrt{x+y}} \, dx.$$

Since $\langle x+y\rangle \lesssim \langle x\rangle$ on $[-y,-\frac{y}{2}]$ and q<0, we find $\langle x+y\rangle^{-q} \lesssim \langle x\rangle^{-q}$, which implies

$$J(y) \lesssim \langle y \rangle^{-\kappa} \int_{-y}^{-\frac{y}{2}} \langle x \rangle^{-\varepsilon} \frac{\langle x+y \rangle^{-\frac{1}{2}}}{\sqrt{x+y}} \, dx.$$

The last step is to prove the uniform boundedness of the x-integral in the right-hand side of the previous inequality. Integration by parts leads to

$$\begin{split} \int_{-y}^{-\frac{y}{2}} \langle x \rangle^{-\varepsilon} \frac{\langle x+y \rangle^{-\frac{1}{2}}}{\sqrt{x+y}} \, dx &\leq \int_{-y}^{\infty} \langle x \rangle^{-\varepsilon} \frac{\langle x+y \rangle^{-\frac{1}{2}}}{\sqrt{x+y}} \, dx \\ &\lesssim \int_{-y}^{\infty} \sqrt{x+y} \big(\langle x \rangle^{-1-\varepsilon} \langle x+y \rangle^{-\frac{1}{2}} + \langle x \rangle^{-\varepsilon} \langle x+y \rangle^{-\frac{3}{2}} \big) \, dx \\ &\lesssim \int_{-y}^{\infty} \big(\langle x \rangle^{-1-\varepsilon} + \langle x \rangle^{-\varepsilon} \langle x+y \rangle^{-1} \big) \, dx \lesssim 1, \end{split}$$

where in the last inequality we used (4.44). This concludes the proof.

Remark 4.5. Comparing the statements of Lemmas 4.2, 4.3 and 4.4, we see that we have estimated suitable integrals for $q \ge -\frac{1}{2}$. The condition $q \ge -\frac{1}{2}$ is equivalent to require $p \ge \frac{n}{n-1}$. Of course, we want to keep the lower bound in (4.36) as main lower bound for p. Therefore, we have to guarantee that

$$\frac{n}{n-1} \le p_0(n+\mu).$$

It turns out that such a condition is equivalent to require

$$\mu \le \widetilde{M}(n) \doteq \frac{3n^2 - 5n + 2}{n}.$$

In the upcoming results we will require a stronger upper bound for μ , so that, the above condition on μ will be every time fulfilled and, in turn, the condition $q \ge -\frac{1}{2}$ will be valid as well.

Lemma 4.6. Let us consider p, κ satisfying (4.36), (4.37) and (4.38) and let q be defined by (4.39). Then, we have for any $y \ge 0$

$$\int_{-2y}^{y} \langle x - y \rangle^{-\frac{\mu}{2}(p-1)} \langle x + 2y \rangle^{-q-1} \langle x \rangle^{-\kappa p} \, dx \lesssim \langle y \rangle^{-\kappa}. \tag{4.49}$$

Proof. Let K(y) denotes the integral on the left-hand side of (4.49). We split the integral in two parts

$$K(y) = \int_{-2y}^{-\frac{y}{2}} \langle x - y \rangle^{-\frac{\mu}{2}(p-1)} \langle x + 2y \rangle^{-q-1} \langle x \rangle^{-\kappa p} dx + \int_{-\frac{y}{2}}^{y} \langle x - y \rangle^{-\frac{\mu}{2}(p-1)} \langle x + 2y \rangle^{-q-1} \langle x \rangle^{-\kappa p} dx \doteq K_1(y) + K_2(y)$$

On the one hand, we can use the relations $\langle x \rangle \approx \langle y \rangle$ and $\langle x + 2y \rangle \leq \langle x - y \rangle$ when $x \in [-2y, -\frac{y}{2}]$, obtaining for $K_1(y)$

$$K_1(y) \lesssim \langle y \rangle^{-\kappa} \int_{-2y}^{-\frac{y}{2}} \langle x + 2y \rangle^{-q-1-\frac{\mu}{2}(p-1)} \langle x \rangle^{-\kappa(p-1)} \, dx \lesssim \langle y \rangle^{-\kappa},$$

where in the last inequality we can apply (4.44) because of (4.38). On the other hand, since $\langle x + 2y \rangle \geq \langle x - y \rangle, \langle x \rangle$ for $x \in [-\frac{y}{2}, y]$, employing again (4.38), we find

$$K_2(y) \lesssim \int_{-\frac{y}{2}}^{y} \langle x + 2y \rangle^{-\frac{\mu}{2}(p-1)-q-1-\kappa p} \, dx \lesssim \langle y \rangle^{-\frac{\mu}{2}(p-1)-q-\kappa p} \lesssim \langle y \rangle^{-\kappa}.$$

The desired estimate follows from the estimates for $K_1(y)$ and $K_2(y)$.

Let us introduce four auxiliary integrals, which will come into play in the treatment of the semilinear problem. Let $t \ge 0, r > 0$ and let γ be 0 or $\frac{1}{2}$, we define

$$I_{\gamma}(t,r) \doteq \int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{|\lambda_{-}|}^{\lambda_{+}} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{1}{2}-\gamma} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda-\lambda_{-}}} \, d\lambda \, d\tau, \tag{4.50}$$

$$J_{\gamma}(t,r) \doteq \int_{0}^{(t-r)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q-\frac{1}{2}-\gamma} \int_{0}^{\lambda_{-}} \frac{\langle \lambda \rangle^{\frac{p}{2}} \phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} \, d\lambda \, d\tau, \tag{4.51}$$

$$P_{\gamma}(t,r) \doteq \int_{0}^{(t-r)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-1-\gamma} \phi_{\kappa}(\tau,\frac{\lambda_{-}}{2})^{p} d\tau, \qquad (4.52)$$

$$Q_{\gamma}(t,r) \doteq \int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-1-\gamma} \phi_{\kappa}(\tau,-\lambda_{-})^{p} d\tau, \qquad (4.53)$$

where we have set $\lambda_{\pm} \doteq t - \tau \pm r$, q is defined by (4.39) and ϕ_{κ} is given by (2.15).

The next proposition provides some estimates for the above defined integrals. Let us underline explicitly that the core of the proof of Theorem 2.2 is the next result.

Proposition 4.7. Let us consider an even integer $n \ge 4$ and p, κ satisfying (4.36), (4.37) and (4.38) and let q be defined by (4.39) such that

$$-\frac{1}{2} \le q \le m - \frac{1}{2},\tag{4.54}$$

$$p < p_{\mathrm{Fuj}}(\mu). \tag{4.55}$$

Then, the following estimates are fulfilled for any $t \ge 0, r > 0$ and $\gamma \in \{0, \frac{1}{2}\}$:

$$I_{\gamma}(t,r) \lesssim \langle t-r \rangle^{-\kappa-\gamma}, \qquad (4.56)$$

$$J_{\gamma}(t,r) \lesssim \langle t-r \rangle^{-\kappa-\gamma}, \qquad (4.57)$$

$$P_{\gamma}(t,r) \lesssim \langle t-r \rangle^{-\kappa-\gamma}, \tag{4.58}$$

$$Q_{\gamma}(t,r) \lesssim \langle t-r \rangle^{-\kappa-\gamma}. \tag{4.59}$$

Remark 4.8. Let us analyze all assumptions we have done in the previous statement. The condition from above on q in (4.54) is equivalent to $p \leq 2$ and, therefore, it is always fulfilled in our setting. Indeed, we are assuming $p < p_{Fuj}\left(\frac{n+\mu-1}{2}\right)$ and this implies p < 2 for $n \geq 4$

and $\mu \geq 2$. On the other hand, the condition on p prescribed by (4.55) requires the validity of $p_0(n + \mu) < p_{Fuj}(\mu)$, in order to have a nonempty range for p. This condition is valid for $\mu < M(n)$, where M(n) is defined by (2.17). Indeed, $p_0(n + \mu) < p_{Fuj}(\mu)$ is equivalent to require

$$(n+\mu-1)p_{\mathrm{Fuj}}(\mu)^2 - (n+\mu+1)p_{\mathrm{Fuj}}(\mu) - 2 = -\frac{1}{\mu^2}(\mu^2 - (n-1)\mu - 2(n-1)) > 0,$$

which is obviously satisfied for $0 \le \mu < M(n)$. Furthermore, as we said in Remark 4.5, for $n \ge 4$ it holds $M(n) < \widetilde{M}(n)$, so that, $q \ge -\frac{1}{2}$ is valid for $\mu \in [2, M(n))$ and, then, the lower bound for q in (4.54) is satisfied.

Remark 4.9. Thanks to Remark 4.8 we can clarify now the choice of the parameters κ_1 , κ_2 and μ in Theorem 2.2. Indeed, as in (4.37) and in (4.38), in the statement of Theorem 2.2, we have

$$\kappa_1 \doteq \frac{-q+1}{p-1} - \frac{\mu}{2} = \frac{2}{p-1} - \frac{n+\mu-1}{2};$$

$$\kappa_2 \doteq q + \frac{\mu}{2}(p-1) = \frac{n+\mu-1}{2}(p-1) - 1.$$

As we have already seen, $(\kappa_1, \kappa_2]$ is not empty because of the condition $p > p_0(n + \mu)$ and $\kappa_1 > 0$ thanks to the upper bound $p < p_{Fuj}(\frac{n+\mu-1}{2})$. Besides, we can find $\kappa \in (\kappa_1, \kappa_2]$ such that $k < m + \frac{1}{2}$, since $\kappa_2 < m + \frac{1}{2}$ is equivalent to require $p < \frac{2n+\mu}{n+\mu-1}$. However, this upper bound for p is weaker than the upper bound for p in (4.36). Hence, for the range of ps considered in the statement of Theorem 2.2 the inequality $\kappa_2 < m + \frac{1}{2}$ is always satisfied. Finally, as we have just explained in Remark 4.8, the upper bound for μ is due to the fact that we want to guarantee the validity of the condition $p_0(n + \mu) < p_{Fuj}(\mu)$, which implies a not empty range of admissible values for p in Theorem 2.2, while the lower bound $\mu \ge 2$ is a necessary condition coming from (1.4) for nontrivial and nonnegative μ and ν .

Proof of Proposition 4.7. We will modify the proof of Proposition 6.6 in [12], by using the previously derived lemmas. Let us start with $I_{\gamma}(t,r)$. Since

$$I_{\gamma}(t,r) = \int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{|\lambda_{-}|}^{\lambda_{+}} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{1}{2}-\gamma} \langle \lambda - \tau \rangle^{-\kappa p} \frac{\langle \lambda + \tau \rangle^{-\frac{p}{2}}}{\sqrt{\lambda - \lambda_{-}}} \, d\lambda \, d\tau$$

performing the change of variables $\xi = \lambda + \tau, \eta = \lambda - \tau$, we get

$$I_{\gamma}(t,r) \lesssim \int_{|t-r|}^{t+r} \frac{\langle \xi \rangle^{-\frac{p}{2}}}{\sqrt{\xi+r-t}} \int_{-\xi}^{\xi} \langle \xi - \eta \rangle^{-\frac{\mu}{2}(p-1)} \langle \xi + \eta \rangle^{-q+\frac{p}{2}-\frac{1}{2}-\gamma} \langle \eta \rangle^{-\kappa p} \, d\eta \, d\xi.$$

Let us estimate the η -integral. We split the domain of integration in three subintervals.

$$\int_{-\xi}^{\xi} \langle \xi - \eta \rangle^{-\frac{\mu}{2}(p-1)} \langle \xi + \eta \rangle^{-q + \frac{p}{2} - \frac{1}{2} - \gamma} \langle \eta \rangle^{-\kappa p} \, d\eta = A_1(\xi) + A_2(\xi) + A_3(\xi),$$

where $A_1(\xi)$, $A_2(\xi)$ and $A_3(\xi)$ denote the integrals of the integrand in the left-hand side over $\left[\frac{\xi}{2},\xi\right]$, $\left[-\frac{\xi}{2},\frac{\xi}{2}\right]$ and $\left[-\xi,-\frac{\xi}{2}\right]$, respectively.

Let us begin with $A_1(\xi)$. Since $\langle \xi + \eta \rangle \approx \langle \eta \rangle \approx \langle \xi \rangle$ for $\eta \in [\frac{\xi}{2}, \xi]$, using (4.55), we have

$$A_{1}(\xi) \lesssim \langle \xi \rangle^{-q + \frac{p}{2} - \frac{1}{2} - \gamma - \kappa p} \int_{\frac{\xi}{2}}^{\xi} \langle \xi - \eta \rangle^{-\frac{\mu}{2}(p-1)} d\eta \lesssim \langle \xi \rangle^{-q + \frac{p}{2} - \frac{1}{2} - \gamma - \kappa p - \frac{\mu}{2}(p-1) + 1}.$$

We estimate now $A_2(\xi)$. Being $\langle \xi - \eta \rangle \approx \langle \xi + \eta \rangle \approx \langle \xi \rangle$ for $\eta \in [-\frac{\xi}{2}, \frac{\xi}{2}]$, we have

$$A_{2}(\xi) \lesssim \langle \xi \rangle^{-\frac{\mu}{2}(p-1)-q+\frac{p}{2}-\frac{1}{2}-\gamma} \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \langle \eta \rangle^{-\kappa p} d\eta$$

$$\lesssim \begin{cases} \langle \xi \rangle^{-\frac{\mu}{2}(p-1)-q+\frac{p}{2}-\frac{1}{2}-\gamma-\kappa p+1} & \text{if } -\kappa p+1 > 0 \\ \langle \xi \rangle^{-\frac{\mu}{2}(p-1)-q+\frac{p}{2}-\frac{1}{2}-\gamma-\kappa p+1+\varepsilon} & \text{if } -\kappa p+1 = 0 \\ \langle \xi \rangle^{-\frac{\mu}{2}(p-1)-q+\frac{p}{2}-\frac{1}{2}-\gamma} & \text{if } -\kappa p+1 < 0 \end{cases}$$

where $\varepsilon > 0$ in the logarithmic case can be chosen sufficiently small so that

$$-\frac{\mu}{2}(p-1) - q - \kappa(p-1) + 1 + \varepsilon < 0, \tag{4.60}$$

thanks to (4.38).

Eventually, we consider $A_3(\xi)$. On $\left[-\xi, -\frac{\xi}{2}\right]$ we have $\langle \xi - \eta \rangle \approx \langle \eta \rangle \approx \langle \xi \rangle$, also,

$$A_{3}(\xi) \lesssim \langle \xi \rangle^{-\frac{\mu}{2}(p-1)-\kappa p} \int_{-\xi}^{-\frac{\xi}{2}} \langle \xi + \eta \rangle^{-q+\frac{p}{2}-\frac{1}{2}-\gamma} d\eta \lesssim \langle \xi \rangle^{-\frac{\mu}{2}(p-1)-\kappa p-q+\frac{p}{2}+\frac{1}{2}-\gamma},$$

where in the last inequality we used $-q + \frac{p}{2} - \frac{1}{2} - \gamma > -1$, which is equivalent to $p < p_{\text{Fuj}}(\frac{n-2}{2})$. This condition on p is always fulfilled thanks to the upper bound in (4.36). Combining the estimates for $A_1(\xi), A_2(\xi)$ and $A_3(\xi)$, it results

$$I_{\gamma}(t,r) \lesssim \int_{|t-r|}^{t+r} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)-q-\frac{1}{2}-\gamma+\alpha(\kappa)}}{\sqrt{\xi+r-t}} d\xi \lesssim \langle t-r \rangle^{-\kappa-\gamma} \int_{|t-r|}^{t+r} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)-q-\frac{1}{2}+\kappa+\alpha(\kappa)}}{\sqrt{\xi+r-t}} d\xi,$$

where $\alpha(\kappa) = -\kappa p + 1$ if $\kappa < \frac{1}{p}$, $\alpha(\kappa) = -\kappa p + 1 + \varepsilon$ if $\kappa = \frac{1}{p}$ and $\alpha(\kappa) = 0$ if $\kappa > \frac{1}{p}$. We point out that the power for $\langle \xi \rangle$ in the last integral can be written in all three subcases as $-\beta(\kappa) - \frac{1}{2}$ for a positive constant $\beta(\kappa)$, due to (4.38), (4.60) and (4.37). Therefore,

$$I_{\gamma}(t,r) \lesssim \langle t-r \rangle^{-\kappa-\gamma} \int_{|t-r|}^{t+r} \frac{\langle \xi \rangle^{-\beta(\kappa)-\frac{1}{2}}}{\sqrt{\xi+r-t}} d\xi \lesssim \langle t-r \rangle^{-\kappa-\gamma}.$$

Indeed, using integration by parts, for $t \ge r$ we may estimate the ξ -integral as follows:

$$\begin{split} \int_{|t-r|}^{t+r} \frac{\langle \xi \rangle^{-\beta(\kappa) - \frac{1}{2}}}{\sqrt{\xi + r - t}} \, d\xi &\leq \int_{t-r}^{\infty} \frac{\langle \xi \rangle^{-\beta(\kappa) - \frac{1}{2}}}{\sqrt{\xi + r - t}} \, d\xi \lesssim \int_{t-r}^{\infty} \sqrt{\xi + r - t} \langle \xi \rangle^{-\beta(\kappa) - \frac{3}{2}} \, d\xi \\ &\lesssim \int_{t-r}^{\infty} \langle \xi \rangle^{-\beta(\kappa) - 1} \, d\xi \lesssim 1. \end{split}$$

On the other hand, employing again integration by parts, for $t \leq r$ we get

$$\begin{split} \int_{r-t}^{t+r} \frac{\langle \xi \rangle^{-\beta(\kappa) - \frac{1}{2}}}{\sqrt{\xi + r - t}} d\xi &\leq \int_{r-t}^{\infty} \frac{\langle \xi \rangle^{-\beta(\kappa) - \frac{1}{2}}}{\sqrt{\xi + r - t}} d\xi \\ &\lesssim \sqrt{r - t} \langle r - t \rangle^{-\beta(\kappa) - \frac{1}{2}} + \int_{r-t}^{\infty} \sqrt{\xi + r - t} \langle \xi \rangle^{-\beta(\kappa) - \frac{3}{2}} d\xi \\ &\lesssim \langle r - t \rangle^{-\beta(\kappa)} + \int_{r-t}^{\infty} \langle \xi \rangle^{-\beta(\kappa) - 1} d\xi \lesssim 1. \end{split}$$

Let us estimate $J_{\gamma}(t,r)$. We can assume t > r. Carrying out the same change of variable we used for $I_{\gamma}(t,r)$, we get

$$J_{\gamma}(t,r) = \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q-\frac{1}{2}-\gamma} \int_{0}^{\lambda_{-}} \frac{\langle \lambda \rangle^{\frac{p}{2}} \langle \lambda + \tau \rangle^{-\frac{p}{2}} \langle \lambda - \tau \rangle^{-\kappa p}}{\sqrt{\lambda_{-} - \lambda}} d\lambda d\tau$$
$$\lesssim \int_{0}^{t-r} \frac{\langle \xi \rangle^{-\frac{p}{2}}}{\sqrt{t-r-\xi}} \int_{-\xi}^{\xi} \langle \xi - \eta \rangle^{-\frac{\mu}{2}(p-1)} \langle t - r + \frac{\eta-\xi}{2} \rangle^{-q-\frac{1}{2}-\gamma} \langle \xi + \eta \rangle^{\frac{p}{2}} \langle \eta \rangle^{-\kappa p} d\eta d\xi$$

Let us split the domain of integration in the following three regions:

$$\begin{aligned} \Omega_1 &= \left\{ (\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < t - r, -\frac{\xi}{2} < \eta < \xi \right\}, \\ \Omega_2 &= \left\{ (\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < \frac{t - r}{2}, -\xi < \eta < -\frac{\xi}{2} \right\}, \\ \Omega_3 &= \left\{ (\xi, \eta) \in \mathbb{R}^2 : \frac{t - r}{2} < \xi < t - r, -\xi < \eta < -\frac{\xi}{2} \right\}. \end{aligned}$$

Thus, we can write $J_{\gamma}(t,r) = J_{\gamma,1}(t,r) + J_{\gamma,2}(t,r) + J_{\gamma,3}(t,r)$, where $J_{\gamma,k}(t,r)$ is the integral over Ω_k for k = 1, 2, 3. We begin with $J_{\gamma,1}(t,r)$. Since on Ω_1 we have

$$\left\langle t - r + \frac{\eta - \xi}{2} \right\rangle \approx \left\langle t - r \right\rangle, \quad \left\langle t - r + \frac{\eta - \xi}{2} \right\rangle \gtrsim \left\langle \xi + \eta \right\rangle,$$

$$(4.61)$$

then, being $\kappa - q - \frac{\mu}{2}(p-1) \leq 0$ because of (4.37), we have

$$\left\langle t - r + \frac{\eta - \xi}{2} \right\rangle^{-q - \frac{1}{2} - \gamma} \lesssim \left\langle t - r \right\rangle^{-\kappa - \gamma - \frac{1}{2} + \frac{\mu}{2}(p-1)} \left\langle \xi + \eta \right\rangle^{\kappa - q - \frac{\mu}{2}(p-1)}.$$

Besides, $\langle \xi + \eta \rangle \lesssim \langle \xi \rangle$ implies that $\langle \xi + \eta \rangle^{\frac{p}{2}} \langle \xi \rangle^{-\frac{p}{2}}$ is bounded on the domain of integration. Also, using $\langle \xi + \eta \rangle \approx \langle \xi \rangle$ for $\eta \in [-\frac{\xi}{2}, \xi]$, it follows:

$$\begin{aligned} \langle t-r \rangle^{\kappa+\gamma+\frac{1}{2}-\frac{\mu}{2}(p-1)} J_{\gamma,1}(t,r) &\lesssim \iint_{\Omega_1} \frac{\langle \xi-\eta \rangle^{-\frac{\mu}{2}(p-1)} \langle \xi+\eta \rangle^{\kappa-q-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} \langle \eta \rangle^{-\kappa p} \, d\eta \, d\xi \\ &\lesssim \int_0^{t-r} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} \int_{-\frac{\xi}{2}}^{\xi} \langle \xi-\eta \rangle^{-\frac{\mu}{2}(p-1)} \langle \xi+\eta \rangle^{\kappa-q} \langle \eta \rangle^{-\kappa p} \, d\eta \, d\xi \end{aligned}$$

Now we show that the η -integral in the last line of the previous estimate is uniformly bounded. Since $\langle \eta + \xi \rangle \approx \langle \xi - \eta \rangle$ for $\eta \in [-\frac{\xi}{2}, \frac{\xi}{2}]$ and $\langle \eta + \xi \rangle \approx \langle \eta \rangle$ for $\eta \in [\frac{\xi}{2}, \xi]$, then, using (4.44), we obtain

$$\begin{split} \int_{-\frac{\xi}{2}}^{\xi} \langle \xi - \eta \rangle^{-\frac{\mu}{2}(p-1)} \langle \xi + \eta \rangle^{\kappa-q} \langle \eta \rangle^{-\kappa p} \, d\eta \lesssim \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \langle \xi - \eta \rangle^{-\frac{\mu}{2}(p-1)+\kappa-q} \langle \eta \rangle^{-\kappa p} \, d\eta \\ &+ \int_{\frac{\xi}{2}}^{\xi} \langle \xi - \eta \rangle^{-\frac{\mu}{2}(p-1)} \langle \eta \rangle^{-\kappa(p-1)-q} \, d\eta \lesssim 1. \end{split}$$

In particular, in the last estimates we used that the exponent for $\langle \xi - \eta \rangle$ in the first integral is nonnegative thanks to (4.37) and that the exponent of $\langle \eta \rangle$ in the second integral is smaller than $\frac{\mu}{2}(p-1)-1$, due to (4.38), and, then, smaller than 0 thanks to the assumption (4.55). Thus, it follows:

$$J_{\gamma,1}(t,r) \lesssim \langle t-r \rangle^{-\kappa-\gamma-\frac{1}{2}+\frac{\mu}{2}(p-1)} \int_0^{t-r} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} d\xi.$$

For $t - r \ge 1$, we may estimate the ξ -integral in the following way:

$$\int_{0}^{t-r} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} d\xi = \int_{0}^{\frac{t-r}{2}} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} d\xi + \int_{\frac{t-r}{2}}^{t-r} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} d\xi$$
$$\lesssim (\sqrt{t-r})^{-1} \int_{0}^{\frac{t-r}{2}} \langle \xi \rangle^{-\frac{\mu}{2}(p-1)} d\xi + \langle t-r \rangle^{-\frac{\mu}{2}(p-1)} \int_{\frac{t-r}{2}}^{t-r} \frac{d\xi}{\sqrt{t-r-\xi}}$$
$$\lesssim \langle t-r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{\mu}{2}(p-1)+1} + \langle t-r \rangle^{-\frac{\mu}{2}(p-1)} \sqrt{t-r} \lesssim \langle t-r \rangle^{-\frac{\mu}{2}(p-1)+\frac{1}{2}}$$

Otherwise, if 0 < t - r < 1, then, using the fact that $\langle t - r \rangle \approx 1$, we get immediately

$$\int_0^{t-r} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} d\xi \le \int_0^{t-r} \frac{d\xi}{\sqrt{t-r-\xi}} \approx \sqrt{t-r} \le \langle t-r \rangle^{\frac{1}{2}} \approx \langle t-r \rangle^{\frac{1}{2}-\frac{\mu}{2}(p-1)}.$$

Summarizing, we got $J_{\gamma,1}(t,r) \lesssim \langle t-r \rangle^{-\kappa-\gamma}$.

Similarly, we can now estimate $J_{\gamma,2}(t,r)$. Indeed, since (4.61) is valid also in Ω_2 , proceeding as before, we find

$$\begin{split} J_{\gamma,2}(t,r) &\lesssim \langle t-r \rangle^{-\kappa-\gamma-\frac{1}{2}+\frac{\mu}{2}(p-1)} \iint_{\Omega_2} \frac{\langle \xi-\eta \rangle^{-\frac{\mu}{2}(p-1)} \langle \xi+\eta \rangle^{\kappa-q-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} \langle \eta \rangle^{-\kappa p} \, d\eta \, d\xi \\ &\lesssim \langle t-r \rangle^{-\kappa-\gamma-\frac{1}{2}+\frac{\mu}{2}(p-1)} \int_0^{\frac{t-r}{2}} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} \int_{-\xi}^{-\frac{\xi}{2}} \langle \xi+\eta \rangle^{\kappa-q-\frac{\mu}{2}(p-1)} \langle \eta \rangle^{-\kappa p} \, d\eta \, d\xi \\ &\lesssim \langle t-r \rangle^{-\kappa-\gamma-\frac{1}{2}+\frac{\mu}{2}(p-1)} \int_0^{\frac{t-r}{2}} \frac{\langle \xi \rangle^{-\frac{\mu}{2}(p-1)}}{\sqrt{t-r-\xi}} \, d\xi \lesssim \langle t-r \rangle^{-\kappa-\gamma}, \end{split}$$

where we used the relation $\langle \xi - \eta \rangle \approx \langle \xi \rangle$ in the second inequality, (4.44) in the third one and the same estimate for the ξ -integral seen before on Ω_1 in the last one.

It remains to check $J_{\gamma,3}(t,r)$ in order to show (4.57). Being $\langle t-r+\frac{\eta-\xi}{2}\rangle \geq \langle t-r-\xi\rangle$ and $\langle t-r+\frac{\eta-\xi}{2}\rangle \geq \langle \frac{\xi+\eta}{2}\rangle \gtrsim \langle \xi+\eta\rangle$ valid on Ω_3 , then,

$$\begin{split} & \left\langle t-r+\frac{\eta-\xi}{2} \right\rangle^{-q-\frac{1}{2}-\gamma} \lesssim \langle t-r-\xi \rangle^{-q+\frac{1}{2}} \langle \xi+\eta \rangle^{-1-\gamma} & \text{ if } q \geq \frac{1}{2}, \\ & \left\langle t-r+\frac{\eta-\xi}{2} \right\rangle^{-q-\frac{1}{2}-\gamma} \lesssim \langle t-r-\xi \rangle^{-q} \langle \xi+\eta \rangle^{-\frac{1}{2}-\gamma} & \text{ if } q \in [0,\frac{1}{2}), \\ & \left\langle t-r+\frac{\eta-\xi}{2} \right\rangle^{-q-\frac{1}{2}-\gamma} \lesssim \langle t-r-\xi \rangle^{-q-\frac{1}{2}} \langle \xi+\eta \rangle^{-\gamma} & \text{ if } q \in [-\frac{1}{2},0]. \end{split}$$

Moreover, $\langle \xi - \eta \rangle \approx \langle \xi \rangle \approx \langle \eta \rangle$ and $\langle \xi \rangle \lesssim \langle t - r \rangle^{-\gamma} \langle \xi \rangle^{\gamma - \frac{p}{2}}$ on Ω_3 , so, we have

where $\theta(q) = \frac{1}{2}$ if $q \ge \frac{1}{2}$, $\theta(q) = 0$ if $0 \le q < \frac{1}{2}$ and $\theta(q) = -\frac{1}{2}$ if $-\frac{1}{2} \le q < 0$ and in the second inequality we used $\frac{p}{2} - \gamma - \frac{1}{2} - \theta(q) > -1$. Thanks to Lemmas 4.2, 4.3 and 4.4, we have $J_{\gamma,3}(t,r) \lesssim \langle t-r \rangle^{-\kappa-\gamma}$. Hence, we proved (4.57).

Now, we deal with $P_{\gamma}(t, r)$. Also in this case we work with t > r. Then, since $\langle \tau + \frac{\lambda_{-}}{2} \rangle \gtrsim \langle \lambda_{-} \rangle$ and $\langle \tau + \frac{\lambda_{-}}{2} \rangle \gtrsim \langle t - r \rangle$ on the domain of integration, we can estimate $\langle \tau + \frac{\lambda_{-}}{2} \rangle^{-\frac{p}{2}} \lesssim \langle \lambda_{-} \rangle^{-\frac{p}{2} + \gamma} \langle t - r \rangle^{-\gamma}$. Then,

$$P_{\gamma}(t,r) = \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-1-\gamma} \langle \tau + \frac{\lambda_{-}}{2} \rangle^{-\frac{p}{2}} \langle \tau - \frac{\lambda_{-}}{2} \rangle^{-\kappa p} d\tau$$
$$\lesssim \langle t-r \rangle^{-\gamma} \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle t-r-\tau \rangle^{-q-1} \langle t-r-3\tau \rangle^{-\kappa p} d\tau.$$

Performing the change of variables $x = t - r - 3\tau$ and using Lemma 4.6, we find

$$\langle t-r\rangle^{\gamma} P_{\gamma}(t,r) \lesssim \int_{-2(t-r)}^{t-r} \langle t-r-x\rangle^{-\frac{\mu}{2}(p-1)} \langle 2(t-r)+x\rangle^{-q-1} \langle x\rangle^{-\kappa p} \, dx \lesssim \langle t-r\rangle^{-\kappa}.$$

Finally, we consider $Q_{\gamma}(t,r)$. Since $\langle \tau - \lambda_{-} \rangle \gtrsim \langle t - r \rangle, \langle \lambda_{-} \rangle$ on the domain of integration, then, it holds $\langle \tau - \lambda_{-} \rangle^{-\frac{p}{2}} \lesssim \langle t - r \rangle^{-\gamma} \langle \lambda_{-} \rangle^{-\frac{p}{2}+\gamma}$, which implies

$$Q_{\gamma}(t,r) = \langle t-r \rangle^{-\kappa p} \int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-1-\gamma} \langle \tau - \lambda_{-} \rangle^{-\frac{p}{2}} d\tau$$
$$\lesssim \langle t-r \rangle^{-\kappa p-\gamma} \int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q-1} d\tau \lesssim \langle t-r \rangle^{-\kappa p-\gamma} \lesssim \langle t-r \rangle^{-\kappa-\gamma}.$$

where in the second inequality we may use (4.44) to estimate the integral by a constant, due to (4.37) and q > -1. Thus, we proved also (4.59). This concludes the proof.

5 Proof of Theorem 2.2

In this Section we prove Theorem 2.2, using the estimates from Section 4.

Proposition 5.1. Let us consider p, κ, q satisfying (4.36), (4.37), (4.38), (4.54) and (4.55). Let $v \in X_{\kappa}$ and $\nu \in \mathbb{R}$. Then, the following estimates are satisfied for any $t \ge 0, r > 0$

$$|Lv(t,r)| \lesssim N_0^{\nu}(|v|^p) r^{-m} \phi_{\kappa}(t,r) \qquad \text{if} \quad \nu > -2, \qquad (5.62)$$

$$|Lv(t,r)| \lesssim \widetilde{N}_1^{\nu}(|v|^p) r^{1-m} \langle t-r \rangle^{-\kappa - \frac{1}{2}} \qquad \text{if } \nu > -1, \qquad (5.63)$$

$$\left|\partial_r Lv(t,r)\right| \lesssim \tilde{N}_1^{\nu}(|v|^p) r^{-m} \phi_{\kappa}(t,r) \qquad \qquad \text{if} \quad \nu > -1, \tag{5.64}$$

where $\phi_{\kappa}(t,r)$ is defined by (2.15) and $\widetilde{N}_{1}^{\nu}(|v|^{p}) = N_{0}^{\nu}(|v|^{p}) + N_{1}^{\nu}(|v|^{p})$, being N_{0}^{ν}, N_{1}^{ν} defined by (4.43). In particular, it holds

$$||Lv||_{X_{\kappa}} \lesssim \widetilde{N}_{1}^{\nu}(|v|^{p}) \quad if \quad \nu > -1.$$
 (5.65)

Remark 5.2. Let $v, \bar{v} \in X_{\kappa}$. If we replace $N_j^{\nu}(|v|^p)$ by $N_j^{\nu}(|v|^p - |\bar{v}|^p)$, then, we obtain for $Lv - L\bar{v}$ the estimates which correspond to (5.62), (5.63) and (5.64).

We anticipate to the proof of Proposition 5.1 some lemmas.

Lemma 5.3. Let us consider p, κ, q satisfying (4.36), (4.37), (4.38), (4.54) and (4.55) and let γ be 0 or $\frac{1}{2}$. Let $v \in X_{\kappa}$ and let W_1, W_3, W_5 be as in (4.40), (4.41) and (4.42). Then, the following estimates are valid for any $t \ge 0, r > 0$:

$$\int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{1}(t-\tau,r;\tau)| d\tau$$

$$\lesssim N_{0}^{\nu}(|v|^{p}) r^{m+\gamma-\frac{1}{2}} \Big(I_{\gamma}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big), \quad \text{if } \nu > -2,$$
(5.66)

$$\int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{3}(t-\tau,r;\tau)| d\tau$$

$$\lesssim N_{1}^{\nu}(|v|^{p}) r^{m+1} \Big(I_{\frac{1}{2}}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big), \qquad \text{if } \nu > -1,$$
(5.67)

$$\int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{5}(t-\tau,r;\tau)| d\tau$$

$$\lesssim N_{1}^{\nu}(|v|^{p}) r^{m+\gamma-\frac{1}{2}} \Big(I_{\gamma}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big), \quad \text{if} \quad \nu > -1,$$
(5.68)

where $I_{\gamma}(t,r)$ is given by (4.50).

Proof. We will follow the proof of Lemma 6.3 in [12]. We begin with the estimate for the integral that involves W_1 . Since (4.43) and (3.31) imply for j = 0, 1

$$\left|\partial_{\lambda}^{j}\left(\lambda^{2m}|v(\tau,\lambda)|^{p}\right)\right| \lesssim \lambda^{m+\nu-j}\langle\lambda\rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu+j}\phi_{\kappa}(\tau,\lambda)^{p}N_{j}^{\nu}(|v|^{p})$$
(5.69)

and

$$|K_m(\lambda, t-\tau, r)| \lesssim r^{m+\gamma-\frac{1}{2}} \lambda^{-m-\gamma} (\lambda-\lambda_-)^{-\frac{1}{2}} \quad \text{for} \quad |\lambda_-| < \lambda < \lambda_+,$$

respectively, by using the representation formula

$$\int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} W_{1}(t-\tau,r;\tau) \, d\tau = \int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{|\lambda_{-}|}^{\lambda_{+}} \lambda^{2m+1} |v(\tau,\lambda)|^{p} K_{m}(\lambda,t-\tau,r) \, d\lambda d\tau,$$

we get

$$\begin{split} &\int_0^t \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_1(t-\tau,r;\tau)| \, d\tau \\ &\lesssim N_0^{\nu}(|v|^p) \, r^{m+\gamma-\frac{1}{2}} \int_0^t \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{|\lambda_-|}^{\lambda_+} \lambda^{\nu-\gamma+1} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \, \frac{\phi_{\kappa}(\tau,\lambda)^p}{\sqrt{\lambda-\lambda_-}} \, d\lambda \, d\tau, \\ &\lesssim N_0^{\nu}(|v|^p) \, r^{m+\gamma-\frac{1}{2}} \Big(I_{\gamma}(t,r) + \int_0^t \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{|\lambda_-|}^{\min(1,\lambda_+)} \lambda^{\nu-\gamma+1} \, \frac{\phi_{\kappa}(\tau,\lambda)^p}{\sqrt{\lambda-\lambda_-}} \, d\lambda \, d\tau \Big), \end{split}$$

where in the last inequality we used $\lambda \approx \langle \lambda \rangle$ for $\lambda \geq 1$ and $\langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \approx 1$ for $\lambda \in [0,1]$ and $I_{\gamma}(t,r)$ is defined by (4.50). In order to show (5.66), it remains to prove that the second integral in the last line of the previous chain of inequalities can be estimated by $\langle t-r \rangle^{-(\kappa+\frac{1}{2})p}$.

First of all, $\langle \tau + \lambda \rangle \geq \langle \tau \rangle$, since τ and λ are nonnegative. Besides, $|\lambda| \leq 1$ implies $\langle \tau - \lambda \rangle \gtrsim \langle \tau \rangle$. Consequently, $\phi_{\kappa}(\tau, \lambda)^p \lesssim \langle t - r \rangle^{-(\kappa + \frac{1}{2})p}$ on the domain of integration. Hence, applying Fubini's theorem, since $\langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lesssim 1$, we get

$$\begin{split} \int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{|\lambda|}^{\min(1,\lambda_{+})} \lambda^{\nu-\gamma+1} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda-\lambda_{-}}} \, d\lambda \, d\tau \\ &\lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \int_{0}^{1} \lambda^{\nu-\gamma+1} \int_{t-r-\lambda}^{t-r-\lambda} \frac{d\tau}{\sqrt{\lambda-\lambda_{-}}} \, d\lambda \\ &\lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \int_{0}^{1} \lambda^{\nu-\gamma+\frac{3}{2}} \, d\lambda \lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p}, \end{split}$$

where in the last step we used $\nu - \gamma + \frac{3}{2} \ge \nu + 1 > -1$ for $\nu > -2$. The proofs of (5.67) and (5.68) are analogous. Indeed, using (5.69) for j = 1, the representation formulas

$$W_{3}(t-\tau,r;\tau) = \int_{|\lambda_{-}|}^{\lambda_{+}} \partial_{\lambda} (\lambda^{2m} | v(\tau,\lambda) |^{p}) K_{m-1}(\lambda,t-\tau,r) d\lambda,$$
$$W_{5}(t-\tau,r;\tau) = \int_{|\lambda_{-}|}^{\lambda_{+}} \partial_{\lambda} (\lambda^{2m} | v(\tau,\lambda) |^{p}) \partial_{r} K_{m-1}(\lambda,t-\tau,r) d\lambda.$$

and

$$|K_{m-1}(\lambda, t-\tau, r)| \lesssim r^{m+1} \lambda^{-m-\frac{1}{2}} (\lambda - \lambda_{-})^{-\frac{1}{2}} \qquad \text{for} \quad |\lambda_{-}| < \lambda < \lambda_{+}, \tag{5.70}$$

$$\left|\partial_{r}K_{m-1}(\lambda, t-\tau, r)\right| \lesssim r^{m+\gamma-\frac{1}{2}}\lambda^{-m-\gamma+1}(\lambda-\lambda_{-})^{-\frac{1}{2}} \quad \text{for} \quad |\lambda_{-}| < \lambda < \lambda_{+}, \quad (5.71)$$

where the last two inequalities are derived by (3.32), then, we can follow step by step the previous computations. In the end, the only difference is that we lose one order in the power for λ in the second integral, so, we have to require in this case $\nu > -1$ instead of $\nu > -2$. Hence, the proof is complete.

Lemma 5.4. Let us consider p, κ, q satisfying (4.36), (4.37), (4.38), (4.54) and (4.55) and let γ be 0 or $\frac{1}{2}$. Let $v \in X_{\kappa}$ and let W_2, W_4, W_6 be as in (4.40), (4.41) and (4.42). Then, the following estimates are valid for any $t \ge 0, r > 0$ such that t > r:

$$\int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{2}(t-\tau,r;\tau)| d\tau$$

$$\leq N_{0}^{\nu}(|v|^{p}) r^{m+\gamma-\frac{1}{2}} \Big(J_{\gamma}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big), \qquad \text{if } \nu > -2,$$
(5.72)

$$\int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_4(t-\tau,r;\tau)| \, d\tau \tag{5.73}$$

$$\leq \widetilde{N}_{1}^{\nu}(|v|^p) \, r^{m+1} \Big(J_1(t,r) + P_1(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big), \qquad \text{if } \nu > -1.$$

$$\sum_{n=1}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_6(t-\tau,r;\tau)| d\tau$$

$$\leq \widetilde{N}_1^{\nu}(|v|^p) r^{m+\gamma-\frac{1}{2}} \Big(J_{\gamma}(t,r) + P_{\gamma}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big), \quad if \quad \nu > -1,$$

$$(5.74)$$

where $J_{\gamma}(t,r)$ and $P_{\gamma}(t,r)$ are given by (4.51) and (4.52), respectively.

Proof. In this case we will modify the proof of Lemma 6.4 in [12]. Let us start with the proof

of (5.72). Using the representation formula

$$\int_0^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} W_2(t-\tau,r;\tau) d\tau$$
$$= \int_0^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_0^{\lambda-} \lambda^{2m+1} |v(\tau,\lambda)|^p \widetilde{K}_m(\lambda,t-\tau,r) d\lambda d\tau,$$

the condition (5.69) for j = 0 and

$$|\widetilde{K}_m(\lambda, t-\tau, r)| \lesssim r^{m+\gamma-\frac{1}{2}} \lambda_-^{-m-\gamma} (\lambda_- - \lambda)^{-\frac{1}{2}} \quad \text{for} \quad 0 < \lambda < \lambda_-,$$

where the previous inequality follows from (3.33), we obtain

$$\int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{2}(t-\tau,r;\tau)| d\tau \lesssim N_{0}^{\nu}(|v|^{p}) r^{m+\gamma-\frac{1}{2}}$$

$$\times \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{-m-\gamma} \int_{0}^{\lambda_{-}} \lambda^{m+1+\nu} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda d\tau.$$
(5.75)

The next step is to split the τ -integral on two intervals divided by $(t - r - 1)_+$. On $[0, (t - r - 1)_+]$, we have $\lambda_- \geq 1$ and, then, $\lambda_- \approx \langle \lambda_- \rangle$. Therefore,

$$\begin{split} \int_{0}^{(t-r-1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{-m-\gamma} \int_{0}^{\lambda_{-}} \lambda^{m+1+\nu} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda d\tau \\ &\lesssim \int_{0}^{(t-r-1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-m-\gamma} \int_{0}^{\lambda_{-}} \lambda^{m+1+\nu} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda d\tau \\ &\lesssim \int_{0}^{(t-r-1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-m-\gamma} \int_{0}^{\lambda_{-}} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{1}{2}+m} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda d\tau \\ &\lesssim \int_{0}^{(t-r-1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-\gamma-q-\frac{1}{2}} \int_{0}^{\lambda_{-}} \frac{\langle \lambda \rangle^{\frac{p}{2}} \phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda d\tau \lesssim J_{\gamma}(t,r), \end{split}$$

where in the second inequality we employed the condition $m + \nu + 1 > 0$ (we are assuming $\nu > -2$) and in the third inequality the upper bound for q in (4.54) is used to get

$$\langle \lambda_{-} \rangle^{-m-\gamma} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{1}{2}+m} \leq \langle \lambda_{-} \rangle^{-\gamma-q-\frac{1}{2}} \langle \lambda \rangle^{\frac{p}{2}} \quad \text{for} \quad \lambda \in [0, \lambda_{-}].$$

On the other hand, using Fubini's theorem, on $[(t - r - 1)_+, t - r]$, we find

$$\begin{split} \int_{(t-r-1)_{+}}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{-m-\gamma} \int_{0}^{\lambda_{-}} \lambda^{m+1+\nu} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda d\tau \\ &= \int_{0}^{1} \lambda^{m+1+\nu} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \int_{(t-r-1)_{+}}^{t-r-\lambda} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{-m-\gamma} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\tau d\lambda \\ &\lesssim \int_{0}^{1} \lambda^{m+1+\nu} \int_{(t-r-1)_{+}}^{t-r-\lambda} \lambda_{-}^{-m-\gamma} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\tau d\lambda \\ &\lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \int_{0}^{1} \lambda^{m+1+\nu} \int_{(t-r-1)_{+}}^{t-r-\lambda} \frac{\lambda_{-}^{-m-\gamma}}{\sqrt{\lambda_{-}-\lambda}} d\tau d\lambda, \end{split}$$

where in the last inequality we used the estimate

$$\phi_{\kappa}(\tau,\lambda)^p \lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \quad \text{for} \quad \tau \in [(t-r-1)_+, t-r] \quad \text{and} \quad \lambda \in [0,\lambda_-].$$
 (5.76)

Indeed, trivially $\phi_{\kappa}(\tau,\lambda)^p \leq \langle \tau - \lambda \rangle^{-(\kappa+\frac{1}{2})p}$. Moreover, if t-r > 2, then,

$$|\tau - \lambda| \ge \tau - \lambda \ge t - r - 1 - \lambda \ge t - r - 2$$

implies $\langle \tau - \lambda \rangle \gtrsim \langle t - r \rangle$ and, in turn, the desired inequality. On the other hand, for $0 < t - r \leq 2$, we have immediately $\phi_{\kappa}(\tau, \lambda)^p \lesssim \langle t - r \rangle^{-(\kappa + \frac{1}{2})p}$, being $\langle t - r \rangle \approx 1$. Let $\varepsilon > 0$ be such that $\varepsilon < \min(\frac{1}{2}, \nu + 2)$. For $0 \leq \lambda \leq \lambda_{-}$ we obtain

$$\lambda_{-}^{-m-\gamma} \leq (\lambda_{-} - \lambda)^{-\frac{1}{2} + \varepsilon} \lambda^{-m-\gamma + \frac{1}{2} - \varepsilon}$$

due to $\varepsilon < \frac{1}{2}$ and $-m - \gamma + \frac{1}{2} - \varepsilon < 0$. Hence,

$$\int_{0}^{1} \lambda^{m+1+\nu} \int_{(t-r-1)_{+}}^{t-r-\lambda} \frac{\lambda_{-}^{-m-\gamma}}{\sqrt{\lambda_{-}-\lambda}} d\tau \, d\lambda \tag{5.77}$$

$$\lesssim \int_{0}^{1} \lambda^{\nu-\gamma+\frac{3}{2}-\varepsilon} \int_{t-r-1}^{t-r-\lambda} (\lambda_{-}-\lambda)^{-1+\varepsilon} \, d\tau \, d\lambda \lesssim \int_{0}^{1} \lambda^{\nu-\gamma+\frac{3}{2}-\varepsilon} \, d\lambda \lesssim 1,$$

where in the last inequality the condition $\varepsilon < \nu + 2$ implies the boundedness of the integral. Summarizing, if we combine the estimate for the integrals over $[0, (t - r - 1)_+]$ and $[(t - r - 1)_+, t - r]$, then, it follows (5.72).

Let us prove now (5.74). We consider the representation formula

$$\int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} W_{6}(t-\tau,r;\tau) d\tau$$

$$= \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{0}^{\lambda_{-}} \partial_{\lambda} (\lambda^{2m} |v(\tau,\lambda)|^{p}) \partial_{r} \widetilde{K}_{m-1}(\lambda,t-\tau,r) d\lambda d\tau.$$
(5.78)

We will split the inner λ -integral in two parts. We begin with the integral over $\left[\frac{\lambda_{-}}{2}, \lambda_{-}\right]$. From (3.34) it follows:

$$\left|\partial_{r}\widetilde{K}_{m-1}(\lambda,t-\tau,r)\right| \lesssim r^{m+\gamma-\frac{1}{2}}\lambda_{-}^{-m-\gamma+1}(\lambda_{-}-\lambda)^{-\frac{1}{2}} \quad \text{for} \quad \lambda \in (0,\lambda_{-}).$$

Thus, combining the previous estimate with (5.69) for j = 1, we find

$$\begin{split} &\int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{\lambda_{-}/2}^{\lambda_{-}} \left| \partial_{\lambda} (\lambda^{2m} | v(\tau, \lambda) |^{p}) \, \partial_{r} \widetilde{K}_{m-1}(\lambda, t-\tau, r) \right| d\lambda \, d\tau \\ &\lesssim N_{1}^{\nu}(|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{-m-\gamma+1} \int_{\lambda_{-}/2}^{\lambda_{-}} \lambda^{m+\nu-1} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{1}{2}-\nu} \, \frac{\phi_{\kappa}(\tau, \lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} \, d\lambda d\tau. \end{split}$$

In the last integral we consider a further division of the domain of integration, in this case with respect to the τ -integral. On the one hand, it holds

$$\begin{split} \int_{0}^{(t-r-1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{-m-\gamma+1} \int_{\lambda_{-}/2}^{\lambda_{-}} \lambda^{m+\nu-1} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{1}{2}-\nu} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda \, d\tau \\ \lesssim \int_{0}^{(t-r-1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q-\frac{1}{2}-\gamma} \int_{\lambda_{-}/2}^{\lambda_{-}} \frac{\langle \lambda \rangle^{\frac{p}{2}} \phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda \, d\tau \lesssim J_{\gamma}(t,r), \end{split}$$

where we use $\lambda \approx \lambda_{-} \approx \langle \lambda_{-} \rangle$ thanks to $\lambda \in \left[\frac{\lambda_{-}}{2}, \lambda_{-}\right]$ and $\tau \leq t - r - 1$. On the other hand, using Fubini's theorem, for the second part we get

$$\begin{split} \int_{(t-r-1)_{+}}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{-m-\gamma+1} \int_{\lambda_{-}/2}^{\lambda_{-}} \lambda^{m+\nu-1} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{1}{2}-\nu} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\lambda d\tau \\ &\lesssim \int_{0}^{1} \lambda^{m+\nu} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{1}{2}-\nu} \int_{(t-r-1)_{+}}^{t-r-\lambda} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{-m-\gamma} \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} d\tau d\lambda \\ &\lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \int_{0}^{1} \lambda^{m+\nu} \int_{(t-r-1)_{+}}^{t-r-\lambda} \frac{\lambda_{-}^{-m-\gamma}}{\sqrt{\lambda_{-}-\lambda}} d\tau d\lambda, \end{split}$$

where in the last inequality we used (5.76). Choosing $\varepsilon < \min(\frac{1}{2}, \nu+1)$, we can repeat exactly the same estimate seen in (5.77) for the last integral, requiring $\nu > -1$. Summarizing, we have shown

$$\begin{split} \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{\lambda_{-}/2}^{\lambda_{-}} \left| \partial_{\lambda} (\lambda^{2m} | v(\tau, \lambda) |^{p}) \, \partial_{r} \widetilde{K}_{m-1}(\lambda, t-\tau, r) \right| d\lambda \, d\tau \\ \lesssim N_{1}^{\nu} (|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \Big(J_{\gamma}(t, r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big). \end{split}$$

Let us deal with the second term coming from the λ -integral in (5.78). Integrating by parts, we have

$$\int_{0}^{\lambda_{-}/2} \partial_{\lambda} (\lambda^{2m} | v(\tau, \lambda) |^{p}) \partial_{r} \widetilde{K}_{m-1}(\lambda, t - \tau, r) d\lambda$$

= $\lambda^{2m} | v(\tau, \lambda) |^{p} \partial_{r} \widetilde{K}_{m-1}(\lambda, t - \tau, r) \Big|_{\lambda = \lambda_{-}/2}$
- $\int_{0}^{\lambda_{-}/2} \lambda^{2m} | v(\tau, \lambda) |^{p} \partial_{\lambda} \partial_{r} \widetilde{K}_{m-1}(\lambda, t - \tau, r) d\lambda \doteq W_{6,1} + W_{6,2}.$

Let us begin with $W_{6,1}$. Using (5.69) and (3.34), one gets

$$\int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{6,1}| \, d\tau$$

$$\lesssim N_{0}^{\nu}(|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{\nu-\gamma+\frac{1}{2}} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \phi_{\kappa}(\tau, \frac{\lambda_{-}}{2})^{p} \, d\tau.$$

We split now the τ -integral as usual. On the one hand,

$$\int_{0}^{(t-r-1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{\nu-\gamma+\frac{1}{2}} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \phi_{\kappa}(\tau,\frac{\lambda_{-}}{2})^{p} d\tau$$
$$\lesssim \int_{0}^{(t-r-1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-1-\gamma} \phi_{\kappa}(\tau,\frac{\lambda_{-}}{2})^{p} d\tau \leq P_{\gamma}(t,r).$$

On the other hand, (5.76) yields

$$\int_{(t-r-1)_{+}}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \lambda_{-}^{\nu-\gamma+\frac{1}{2}} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \phi_{\kappa}(\tau,\frac{\lambda_{-}}{2})^{p} d\tau$$
$$\lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \int_{(t-r-1)_{+}}^{t-r} \lambda_{-}^{\nu-\gamma+\frac{1}{2}} d\tau \lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p},$$

where in the first inequality we also employed $\langle \lambda_{-} \rangle \approx 1$ and in the second one the assumption $\nu > -1$ is necessary to guarantee the finiteness of the integral. So, we proved

$$\int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{6,1}| \, d\tau \lesssim N_{0}^{\nu}(|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \Big(P_{\gamma}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big).$$

We consider now the integral involving $W_{6,2}$. From (3.35), we have

$$|\partial_{\lambda}\partial_{r}\widetilde{K}_{m-1}(\lambda,t-\tau,r)| \lesssim r^{m+\gamma-\frac{1}{2}}\lambda_{-}^{-m+1-\gamma}(\lambda_{-}-\lambda)^{-\frac{3}{2}} \quad \text{for} \quad \lambda \in (0,\lambda_{-}).$$

Combining the previous estimate with (5.69) for j = 0, we obtain

$$\begin{split} &\int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{6,2}| \, d\tau \\ &\lesssim N_{0}^{\nu}(|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{0}^{\lambda_{-}/2} \lambda^{m+\nu} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \lambda_{-}^{-m+1-\gamma} \, \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{(\lambda_{-}-\lambda)^{\frac{3}{2}}} \, d\lambda \, d\tau \\ &\lesssim N_{0}^{\nu}(|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \int_{0}^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_{0}^{\lambda_{-}/2} \lambda^{m+\nu} \langle \lambda \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \lambda_{-}^{-m-\gamma} \, \frac{\phi_{\kappa}(\tau,\lambda)^{p}}{\sqrt{\lambda_{-}-\lambda}} \, d\lambda \, d\tau, \end{split}$$

where in the last step the relation $\lambda_{-} - \lambda \geq \frac{\lambda_{-}}{2}$ is used. The right-hand side of the previous chain of inequality may be estimated exactly as the right-hand side in (5.75). The only difference is the power for λ , so that, in this case we have to require $\nu > -1$ instead of $\nu > -2$. Therefore, it holds

$$\int_0^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{6,2}| \lesssim N_0^{\nu}(|v|^p) \, r^{m+\gamma-\frac{1}{2}} \Big(J_{\gamma}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big).$$

Combining the estimates for the integrals involving $W_{6,1}$ and $W_{6,2}$, it follows (5.74).

Finally, (3.34) and (3.35) imply for $\gamma = \frac{1}{2}$

$$|\widetilde{K}_{m-1}(\lambda, t-\tau, r)| \lesssim r^{m+1} \lambda_{-}^{-m+\frac{1}{2}} (\lambda_{-} - \lambda)^{-\frac{1}{2}} \qquad \text{for} \quad \lambda \in (0, \lambda_{-}),$$

$$|\partial_{\lambda} \widetilde{K}_{m-1}(\lambda, t, r)| \lesssim r^{m+1} \lambda_{-}^{-m+\frac{1}{2}} (\lambda_{-} - \lambda)^{-\frac{3}{2}} \qquad \text{for} \quad \lambda \in (0, \lambda_{-}).$$

Thus, using these estimates and the representation formula

$$\int_0^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} W_4(t-\tau,r;\tau) d\tau$$
$$= \int_0^{t-r} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \int_0^{\lambda_-} \partial_\lambda (\lambda^{2m} |v(\tau,\lambda)|^p) \widetilde{K}_{m-1}(\lambda,t-\tau,r) d\lambda d\tau,$$

one can prove (5.73) exactly as (5.74) has just been proved.

Lemma 5.5. Let us consider p, κ, q satisfying (4.36), (4.37), (4.38), (4.54) and (4.55) and let γ be 0 or $\frac{1}{2}$. Let $v \in X_{\kappa}$ and let W_2, W_4, W_6 be as in (4.40), (4.41) and (4.42). Then, the following estimates are valid for any $t \ge 0, r > 0$:

$$\int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} W_{2}(t-\tau,r;\tau) d\tau = 0$$
(5.79)

$$\int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{4}(t-\tau,r;\tau)| d\tau$$
(5.80)

$$\lesssim N_{0}^{\nu}(|v|^{p}) r^{m+1} \Big(Q_{\frac{1}{2}}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big), \qquad \text{if} \quad \nu > -1,$$

$$\int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{6}(t-\tau,r;\tau)| d\tau \qquad (5.81)$$

$$\lesssim N_{0}^{\nu}(|v|^{p}) r^{m+\gamma-\frac{1}{2}} \Big(Q_{\gamma}(t,r) + \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \Big), \qquad \text{if} \quad \nu > -1,$$

where $Q_{\gamma}(t,r)$ is given by (4.53).

Proof. We will adapt the proof of Lemma 6.5 in [12] to our case. From (2.11) we get immediately (5.79), being $t - \tau - r \leq 0$.

Now we prove (5.81). Using (5.71), (5.69) for j = 0 and the representation formula

$$\int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} W_{6}(t-\tau,r;\tau) d\tau$$
$$= \int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \left(\lambda^{2m} |v(\tau,\lambda)|^{p} \partial_{r} K_{m-1}(\lambda,t-\tau,r) \right) \Big|_{\lambda=-\lambda_{-}} d\tau,$$

we find

$$\int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{6}(t-\tau,r;\tau)| d\tau \\ \lesssim N_{0}^{\nu}(|v|^{p}) r^{m+\gamma-\frac{1}{2}} \int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |\lambda_{-}|^{\nu+\frac{1}{2}-\gamma} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \phi_{\kappa}(\tau,-\lambda_{-})^{p} d\tau.$$

We divide the integral in two parts. Firstly,

$$\int_{(t-r)_{+}}^{(t-r+1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |\lambda_{-}|^{\nu+\frac{1}{2}-\gamma} \langle \lambda_{-} \rangle^{-q+\frac{p}{2}-\frac{3}{2}-\nu} \phi_{\kappa}(\tau,-\lambda_{-})^{p} d\tau \\
\lesssim \int_{(t-r)_{+}}^{(t-r+1)_{+}} |\lambda_{-}|^{\nu+\frac{1}{2}-\gamma} \phi_{\kappa}(\tau,-\lambda_{-})^{p} d\tau \lesssim \langle t-r \rangle^{-(\kappa+\frac{1}{2})p} \int_{(t-r)_{+}}^{(t-r+1)_{+}} |\lambda_{-}|^{\nu+\frac{1}{2}-\gamma} d\tau$$

here in the first inequality $\langle \lambda_{-} \rangle \approx 1$ is used, while in the second inequality we used

$$\phi_{\kappa}(\tau, -\lambda_{-})^p \lesssim \langle t - r \rangle^{-(\kappa + \frac{1}{2})p}$$

Indeed, $\tau + \lambda_{-} = t - r$ and $\tau - \lambda_{-} = (r - t) + 2\tau \ge |t - r|$ for $\tau \ge (t - r)_{+}$ imply the previous inequality. Since $\nu > -1$, then,

$$\int_{(t-r)_{+}}^{(t-r+1)_{+}} |\lambda_{-}|^{\nu+\frac{1}{2}-\gamma} d\tau \leq \int_{t-r}^{t-r+1} |\lambda_{-}|^{\nu+\frac{1}{2}-\gamma} d\tau = \int_{0}^{1} \tau^{\nu+\frac{1}{2}-\gamma} d\tau \lesssim 1.$$

So, we proved,

$$\int_{(t-r)_{+}}^{(t-r+1)_{+}} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} W_{6}(t-\tau,r;\tau) \, d\tau \lesssim N_{0}^{\nu}(|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \langle t-r \rangle^{-(\kappa+\frac{1}{2})p}.$$

Finally,

being $|\lambda_{-}| \approx \langle \lambda_{-} \rangle$ on the domain of integration. Also, we showed (5.81).

Analogously, by the representation formula

$$\int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} W_4(t-\tau,r;\tau) d\tau$$
$$= \int_{(t-r)_{+}}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \Big(\lambda^{2m} |v(\tau,\lambda)|^p K_{m-1}(\lambda,t-\tau,r)\Big)\Big|_{\lambda=-\lambda_{-}} d\tau$$

it is possible to show (5.80) exactly as we have proved (5.81). In particular, one has to employ the inequality (5.70). This concludes the proof. $\hfill \Box$

Proof of Proposition 5.1. In order to represent Lv and $\partial_r Lv$, we will use (4.40), (4.41) and (4.42). Since $-(\kappa + \frac{1}{2})p \leq -\kappa - \gamma$, combining Lemmas 5.3, 5.4 and 5.5 and Proposition 4.7, we get for $\gamma \in \{0, \frac{1}{2}\}$

$$\int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{i}(t-\tau,r;\tau)| \, d\tau \lesssim N_{0}^{\nu}(|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \langle t-r \rangle^{-\kappa-\gamma} \quad \text{for } \nu > -2 \,, \, i = 1, 2,$$

$$\int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{i}(t-\tau,r;\tau)| \, d\tau \lesssim \widetilde{N}_{1}^{\nu}(|v|^{p}) \, r^{m+\gamma-\frac{1}{2}} \langle t-r \rangle^{-\kappa-\gamma} \quad \text{for } \nu > -1 \,, \, i = 5, 6,$$

and for $\nu > -1$, i = 3, 4.

$$\int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{i}(t-\tau,r;\tau)| \, d\tau \lesssim \widetilde{N}_{1}^{\nu}(|v|^{p}) \, r^{m+1} \langle t-r \rangle^{-\kappa-\frac{1}{2}}.$$
(5.82)

We note that for $\gamma = 0$ or $\gamma = \frac{1}{2}$ it holds

$$r^{\gamma - \frac{1}{2}} \langle t - r \rangle^{-\gamma} \lesssim \langle t + r \rangle^{-\frac{1}{2}}.$$
(5.83)

Indeed, for $t \ge 2r > 0$ or $r \le 1$ we have $\langle t + r \rangle \approx \langle t - r \rangle$, so (5.83) is valid for $\gamma = \frac{1}{2}$. On the other hand, for $0 \le t \le 2r$ and $r \ge 1$ we have $r \approx \langle r \rangle \gtrsim \langle t + r \rangle$, thus (5.83) is satisfied for $\gamma = 0$. Hence, from the previous first two estimates, we find

$$\int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{i}(t-\tau,r;\tau)| \, d\tau \lesssim N_{0}^{\nu}(|v|^{p}) \, r^{m} \phi_{\kappa}(t,r) \quad \text{for } \nu > -2 \,, \, i = 1, 2, \quad (5.84)$$

$$\int_{0}^{\infty} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_{i}(t-\tau,r;\tau)| \, d\tau \lesssim \widetilde{N}_{1}^{\nu}(|v|^{p}) \, r^{m} \phi_{\kappa}(t,r) \quad \text{for } \nu > -1, \, i = 5, 6.$$
(5.85)

Let us prove now (5.62). Combining (2.16) and (4.40) and employing (5.84), we have for $\nu > -2$

$$|Lv(t,r)| \lesssim r^{-2m} \int_0^t \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} |W_1(t-\tau,r;\tau) + W_2(t-\tau,r;\tau)| d\tau \lesssim N_0^{\nu}(|v|^p) r^{-m} \phi_{\kappa}(t,r).$$

Similarly, using (4.41) and (5.82) instead of (4.40) and (5.84), respectively, it follows (5.63). Let us show now (5.64). By using (2.16) and (4.42), we have

$$\begin{aligned} |\partial_{r}Lv(t,r)| &\approx \left|\partial_{r} \left(r^{-2m} \int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} \left(2r^{2m} \Theta(|v(\tau,\cdot)|^{p})(t-\tau,r)\right) d\tau\right| \\ &\lesssim |r^{-1}Lv(t,r)| + \left|r^{-2m} \int_{0}^{t} \langle \tau \rangle^{-\frac{\mu}{2}(p-1)} (W_{5}(t-\tau,r;\tau) + W_{6}(t-\tau,r;\tau)) d\tau\right|. \end{aligned}$$

We can estimate the second term in the last line with $\widetilde{N}_1^{\nu}(|v|^p)r^{-m}\phi_{\kappa}(t,r)$ thanks to (5.85). Also, in order to prove (5.64), it remains to show that $|r^{-1}Lv(t,r)|$ can be controlled by the same quantity as the second term. Let us distinguish two subcases. When $r \leq 1$, then, since $\langle t+r \rangle \approx \langle t-r \rangle$, by using (5.63), we obtain

$$|r^{-1}Lv(t,r)| \lesssim r^{-1}\widetilde{N}_{1}^{\nu}(|v|^{p}) r^{1-m} \langle t-r \rangle^{-(\kappa+\frac{1}{2})} \approx \widetilde{N}_{1}^{\nu}(|v|^{p}) r^{-m} \phi_{\kappa}(t,r).$$

Else, for $r \ge 1$ from (5.62) we get immediately

$$|r^{-1}Lv(t,r)| \lesssim r^{-1}N_0^{\nu}(|v|^p) r^{-m}\phi_{\kappa}(t,r) \lesssim N_0^{\nu}(|v|^p) r^{-m}\phi_{\kappa}(t,r).$$

Finally, we prove (5.65). We can rewrite (5.64) as

$$r^m \phi_\kappa(t,r)^{-1} |\partial_r Lv(t,r)| \lesssim \widetilde{N}_1(|v|^p).$$
(5.86)

Therefore, if we show that

$$r^{m-1}\langle r \rangle \phi_{\kappa}(t,r)^{-1} |Lv(t,r)| \lesssim \widetilde{N}_1(|v|^p), \tag{5.87}$$

then, we are done. We distinguish again two subcases. If $r \leq 1$, then, by the estimate (5.63) we find

$$r^{m-1} \langle r \rangle \phi_{\kappa}(t,r)^{-1} | Lv(t,r) | \lesssim \langle r \rangle \langle t-r \rangle^{-(\kappa+\frac{1}{2})} \phi_{\kappa}(t,r)^{-1} \widetilde{N}_{1}^{\nu}(|v|^{p})$$

= $\langle r \rangle \left(\frac{\langle t+r \rangle}{\langle t-r \rangle} \right)^{\frac{1}{2}} \widetilde{N}_{1}^{\nu}(|v|^{p}) \lesssim \widetilde{N}_{1}^{\nu}(|v|^{p}),$

where in the last inequality we used the fact that $\frac{\langle t+r \rangle}{\langle t-r \rangle}$ is bounded in this case. On the other hand, if $r \geq 1$, since $r \approx \langle r \rangle$, (5.62) implies

$$r^{m-1}\langle r\rangle\phi_{\kappa}(t,r)^{-1}|Lv(t,r)| \lesssim r^{-1}\langle r\rangle\widetilde{N}_{1}^{\nu}(|v|^{p}) \approx \widetilde{N}_{1}^{\nu}(|v|^{p})$$

Combining (5.86) and (5.87), we got (5.65). This concludes the proof.

In the next result we take a closer look to the relation between $\|\cdot\|_{X_{\kappa}}$ and $N_{j}^{\nu}(\cdot)$.

Lemma 5.6. Let us consider p, κ, q satisfying (4.36), (4.37), (4.38), (4.54) and (4.55). Then, for any $v \in X_{\kappa}$

$$N_j^{\nu}(|v|^p) \lesssim \|v\|_{X_{\kappa}}^p \quad \text{with} \quad \nu \doteq m - (m-1)p$$

In particular, for any $v \in X_{\kappa}$

$$\|Lv\|_{X_{\kappa}} \lesssim \|v\|_{X_{\kappa}}^{p}. \tag{5.88}$$

Proof. Let $v \in X_{\kappa}$. We start with $N_0^{\nu}(|v|^p)$. If $\tau \ge 0$ and $\lambda > 0$, then, by using the definition of $\|\cdot\|_{X_{\kappa}}$, we obtain

$$\lambda^{2m} |v(\tau,\lambda)|^p \lambda^{-m-\nu} \langle \lambda \rangle^{q-\frac{p}{2}+\frac{3}{2}+\nu} \phi_{\kappa}(\tau,\lambda)^{-p} \lesssim \lambda^{m-(m-1)p-\nu} \langle \lambda \rangle^{q-\frac{3}{2}p+\frac{3}{2}+\nu} \|v\|_{X_{\kappa}}^p = \|v\|_{X_{\kappa}}^p.$$

Let us remark that we used $\nu = m - (m - 1)p$ in the last line, which is equivalent to $\nu = -q + \frac{3}{2}(p-1)$. So, the supremum of the left-hand side, also known as $N_0^{\nu}(|v|^p)$, can be estimated by $\|v\|_{X_{\kappa}}^{p}$ with this choice of ν . Similarly, for $\tau \geq 0$ and $\lambda > 0$

$$\begin{aligned} \left| \partial_{\lambda} (\lambda^{2m} | v(\tau, \lambda) |^{p}) \right| &\lesssim \lambda^{2m} | \partial_{\lambda} v(\tau, \lambda) | | v(\tau, \lambda) |^{p-1} + \lambda^{2m-1} | v(\tau, \lambda) |^{p} \\ &\lesssim \lambda^{2m-1-(m-1)p} \langle \lambda \rangle^{-p+1} \phi_{\kappa}(\tau, \lambda)^{p} \| v \|_{X_{r}}^{p} \end{aligned}$$

implies

$$\left|\partial_{\lambda}(\lambda^{2m}|v(\tau,\lambda)|^{p})\right|\lambda^{-m-\nu+1}\langle\lambda\rangle^{q-\frac{p}{2}+\frac{1}{2}+\nu}\phi_{\kappa}(\tau,\lambda)^{-p}\lesssim \|v\|_{X_{\kappa}}^{p}$$

Taking the supremum of the left-hand side in the previous inequality, we obtain the inequality $N_0^{\nu}(|v|^p) \lesssim ||v||_{X_{\kappa}}^p$, for ν as before.

Finally, we prove (5.88). It is sufficient to use (5.65), provided that

$$\nu = m - (m - 1)p > -1.$$

The previous condition is equivalent to require $p < \frac{m+1}{m-1} = \frac{n}{n-4}$ for $n \ge 6$ (in the case n = 4 the condition $\nu > -1$ is always true, being $\nu = m$). However, the upper bound for p in (4.36) is smaller than $\frac{n}{n-4} = p_{\text{Fuj}}(\frac{n-4}{2})$. Therefore, m - (m-1)p > -1 is fulfilled under the assumptions of this lemma. The proof is completed.

The next step is to prove the Hölder continuity of L and the Lipschitz continuity of Lwith respect to a different norm. For this purpose we introduce an auxiliary norm on X_{κ} . For any $v \in X_{\kappa}$ we define

$$|||v|||_{X_{\kappa}} \doteq \sup_{t>0, r>0} r^{m} |v(t,r)| \phi_{\kappa}(t,r)^{-1}.$$

We note that $|||v|||_{X_{\kappa}} \leq ||v||_{X_{\kappa}}$ for $v \in X_{\kappa}$.

Lemma 5.7. Let us consider p, κ, q satisfying (4.36), (4.37), (4.38), (4.54) and (4.55). Then, for any $v, \bar{v} \in X_{\kappa}$

$$N_0^{\nu_0}(|v|^p - |\bar{v}|^p) \lesssim |||v - \bar{v}|||_{X_{\kappa}} (||v||_{X_{\kappa}}^{p-1} + ||\bar{v}||_{X_{\kappa}}^{p-1}),$$
(5.89)

$$N_0^{\nu_1}(|v|^p - |\bar{v}|^p) \lesssim \|v - \bar{v}\|_{X_\kappa} (\|v\|_{X_\kappa}^{p-1} + \|\bar{v}\|_{X_\kappa}^{p-1}),$$
(5.90)

$$N_{1}^{\nu_{2}}(|v|^{p} - |\bar{v}|^{p}) \lesssim \|v - \bar{v}\|_{X_{\kappa}} (\|v\|_{X_{\kappa}}^{p-1} + \|\bar{v}\|_{X_{\kappa}}^{p-1}) + \|v - \bar{v}\|_{X_{\kappa}}^{p-1} (\|v\|_{X_{\kappa}} + \|\bar{v}\|_{X_{\kappa}}), \quad (5.91)$$

where $\nu_0 \doteq (m-1)(1-p)$, $\nu_1 \doteq m - (m-1)p$ and $\nu_2 \doteq m + 1 - (m-1)p$ and $N_i^{\nu}(|v|^p - |\bar{v}|^p)$ is defined analogously to (4.43) with $|v|^p - |\bar{v}|^p$ in place of $|v|^p$.

In particular, the following inequalities are satisfied for any $v, \bar{v} \in X_{\kappa}$:

$$\|Lv - L\bar{v}\|_{X_{\kappa}} \lesssim \||v - \bar{v}\||_{X_{\kappa}} (\|v\|_{X_{\kappa}}^{p-1} + \|\bar{v}\|_{X_{\kappa}}^{p-1}),$$
(5.92)

$$\|Lv - L\bar{v}\|_{X_{\kappa}} \lesssim \|v - \bar{v}\|_{X_{\kappa}} \left(\|v\|_{X_{\kappa}}^{p-1} + \|\bar{v}\|_{X_{\kappa}}^{p-1} \right) + \|v - \bar{v}\|_{X_{\kappa}}^{p-1} \left(\|v\|_{X_{\kappa}} + \|\bar{v}\|_{X_{\kappa}} \right).$$
(5.93)

Proof. Let $v, \bar{v} \in X_{\kappa}$. For the sake of brevity, we use throughout the proof the notations $\widetilde{G}(\tau, \lambda) \doteq |v(\tau, \lambda)|^p - |\bar{v}(\tau, \lambda)|^p$ and

$$M_{1} \doteq \|v - \bar{v}\|_{X_{\kappa}} (\|v\|_{X_{\kappa}}^{p-1} + \|\bar{v}\|_{X_{\kappa}}^{p-1}), \qquad M_{2} \doteq \|v - \bar{v}\|_{X_{\kappa}}^{p-1} (\|v\|_{X_{\kappa}} + \|\bar{v}\|_{X_{\kappa}}), M_{3} \doteq \|v - \bar{v}\|_{X_{\kappa}} (\|v\|_{X_{\kappa}}^{p-1} + \|\bar{v}\|_{X_{\kappa}}^{p-1}).$$

Using the definitions of $\|\cdot\|_{X_{\kappa}}$ and $\|\cdot\|_{X_{\kappa}}$, we arrive at

$$|\lambda^{2m}\widetilde{G}(\tau,\lambda)| \lesssim \lambda^{2m-1-(m-1)p} \langle \lambda \rangle^{-p+1} \phi_{\kappa}(\tau,\lambda)^{p} M_{3}, \qquad (5.94)$$

$$|\lambda^{2m}\widetilde{G}(\tau,\lambda)| \lesssim \lambda^{2m-(m-1)p} \langle \lambda \rangle^{-p} \phi_{\kappa}(\tau,\lambda)^{p} M_{1}, \qquad (5.95)$$

for any $\tau \ge 0, \lambda > 0$.

In a similar way, since the derivative of $|v|^p$ is a (p-1)-Hölder continuous function, then,

$$\begin{aligned} \left| \lambda^{2m} \partial_{\lambda} \widetilde{G}(\tau, \lambda) \right| &\lesssim \lambda^{2m} |v(\tau, \lambda)|^{p-1} |\partial_{\lambda} v(\tau, \lambda) - \partial_{\lambda} \overline{v}(\tau, \lambda)| \\ &+ \lambda^{2m} |v(\tau, \lambda) - \overline{v}(\tau, \lambda)|^{p-1} |\partial_{\lambda} \overline{v}(\tau, \lambda)| \\ &\lesssim \lambda^{2m - (m-1)p-1} \langle \lambda \rangle^{-p+1} \phi_{\kappa}(\tau, \lambda)^{p} M_{1} + \lambda^{2m - mp} \phi_{\kappa}(\tau, \lambda)^{p} M_{2}. \end{aligned}$$

$$(5.96)$$

Let us derive (5.89). Using (5.94), we get immediately

$$\lambda^{2m} |\widetilde{G}(\tau,\lambda)| \lambda^{-m-\nu_0} \langle \lambda \rangle^{m(p-1)+\nu_0} \phi_{\kappa}(\tau,\lambda)^{-p} \lesssim M_3.$$

Here we used the equality $m(p-1) = q - \frac{p}{2} + \frac{3}{2}$. Thus, taking the supremum of the left-hand side for $\tau \ge 0, \lambda > 0$, we get (5.89).

Analogously, one can prove (5.90), by using (5.95). Let us derive now (5.91). By (5.95) and (5.96), it follows:

$$\begin{aligned} \left| \partial_{\lambda} (\lambda^{2m} \partial_{\lambda} \widetilde{G}(\tau, \lambda)) \right| \lambda^{-m-\nu_{2}+1} \langle \lambda \rangle^{m(p-1)+\nu_{2}-1} \phi_{\kappa}(\tau, \lambda)^{-p} &\lesssim \lambda^{p-1} \langle \lambda \rangle^{-(p-1)} M_{1} + M_{2} \\ &\leq M_{1} + M_{2}, \end{aligned}$$

which implies (5.91).

It remains to prove (5.92) and (5.93). First of all, let us remark that $\nu_2 > -1$ if and only if $p < \frac{m+2}{m} = \frac{n+2}{n-2} = p_{\text{Fuj}}\left(\frac{n-2}{2}\right)$. Nevertheless, the upper bound for p in (4.36) is smaller then $p_{\text{Fuj}}\left(\frac{n-2}{2}\right)$, therefore, the condition $\nu_2 > -1$ is always fulfilled under the assumptions of this lemma. Secondly, $\nu_0 = \nu_1 - 1$ and $\nu_1 > \nu_2$. Hence, $\nu_2 > -1$ implies $\nu_1 > -1$ and $\nu_0 > -2$.

According to Remark 5.2, analogously to (5.62), it is possible to show that

$$|Lv(t,r) - L\bar{v}(t,r)| \leq N_0^{\nu}(|v|^p - |\bar{v}|^p) r^{-m} \phi_{\kappa}(t,r) \quad \text{for} \quad \nu > -2$$

In particular, the previous condition for $\nu_0 > -2$ implies, together with (5.89), the Lipschitz condition (5.92).

Similarly, replacing the source term $|v|^p$ with the difference $|v|^p - |\bar{v}|^p$, analogously to (5.65) we obtain

$$||Lv - L\bar{v}||_{X_{\kappa}} \lesssim N_0^{\nu}(|v|^p - |\bar{v}|^p) + N_1^{\nu}(|v|^p - |\bar{v}|^p) \quad \text{for} \quad \nu > -1.$$

Also, because of $\nu_2 > -1$, employing (5.90) and (5.91) and using the fact N_j^{ν} is increasing with respect to ν , we have

$$\begin{aligned} \|Lv - L\bar{v}\|_{X_{\kappa}} &\lesssim N_0^{\nu_2} (|v|^p - |\bar{v}|^p) + N_1^{\nu_2} (|v|^p - |\bar{v}|^p) \\ &\lesssim N_0^{\nu_1} (|v|^p - |\bar{v}|^p) + N_1^{\nu_2} (|v|^p - |\bar{v}|^p) \lesssim M_1 + M_2, \end{aligned}$$

since $\nu_1 > \nu_2$. This condition is exactly (5.93), so, the proof is over.

Proof of Theorem 2.2. Let κ_1 and κ_2 be defined as in Remark 4.9. Let us fix a κ in $(\kappa_1, \bar{\kappa}]$. Considering the transformed Cauchy problem (2.6), according to our setting it is enough to prove that the operator

$$Fv = v^0 + Lv$$
 for any $v \in X_{\kappa}$

admits a uniquely determined fixed point on a ball in X_{κ} around 0 with sufficiently small radius, where v^0 is defined by (2.14). Thanks to Proposition 3.2, we get $\|v^0\|_{X_{\kappa}} \lesssim \varepsilon$.

Since L satisfies (5.88), (5.92) and (5.93), following the approach used in the proof of Proposition 5.4 in [11], we find that F has a uniquely determined fixed point, which is the desired solution, provided that $\varepsilon < \varepsilon_0$ for a suitably small $\varepsilon_0 > 0$.

6 Concluding remarks and open problems

Finally, we point out the main consequences of Theorem 2.2. When $\mu = 2$ and $\nu = 0$, then, (1.3) coincides with the semilinear model studied in [4]. Therefore, combining the result proved in Theorem 2.2 with the blow-up result [4, Theorem 1] and the global (in time) existence results [1, Theorem 2], [4, Theorems 2 and 3] and [3, Theorem 2.1], we find that $p_2(n)$, defined as in (1.2), is the critical exponent for the semilinear model (1.1) when $\mu = 2$ as it is conjectured in [4].

Similarly, combining Theorem 2.2 with the blow-up result [14, Theorem 2.6] and the global existence results from [18], we have that $p_{\text{crit}}(n,\mu)$ is critical exponent for the model (1.3) assuming (1.4) for n = 1 and for $n \ge 3$ in the radial symmetric case with $\mu \le M(n)$. The two-dimensional case is still open, even though from the necessity part we expect $p_{\text{Fuj}}(1+\frac{\mu}{2})$ to be critical.

In this paper and in [18], we restrict our consideration to the case in which μ and ν satisfy (1.4). However, recently in [17] a blow-up result has been shown for $\delta \in (0, 1]$ for

$$1$$

excluding the case $p = p_0(n + \mu)$ for n = 1. Consequently, a challenging open problem is to study the necessary part also in the case $\delta \in (0, 1)$, showing that the previous upper bound is actually critical.

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