

MOMENTS AND REGULARITY FOR A BOLTZMANN EQUATION VIA WIGNER TRANSFORM

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ABSTRACT. In this paper, we continue our study of the Boltzmann equation by use of tools originating from the analysis of dispersive equations in quantum dynamics. Specifically, we focus on properties of solutions to the Boltzmann equation with collision kernel equal to a constant in the spatial domain \mathbb{R}^d , $d \geq 2$, which we use as a model in this paper. Local well-posedness for this equation has been proven using the Wigner transform when $\langle v \rangle^\beta f_0 \in L_v^2 H_x^\alpha$ for $\min(\alpha, \beta) > \frac{d-1}{2}$. We prove that if α, β are large enough, then it is possible to propagate moments in x and derivatives in v (for instance, $\langle x \rangle^k \langle \nabla_v \rangle^\ell f \in L_T^\infty L_{x,v}^2$ if f_0 is nice enough). The mechanism is an exchange of regularity in return for moments of the (inverse) Wigner transform of f . We also prove a persistence of regularity result for the scale of Sobolev spaces $H^{\alpha,\beta}$; and, continuity of the solution map in $H^{\alpha,\beta}$. Altogether, these results allow us to conclude non-negativity of solutions, conservation of energy, and the H -theorem for sufficiently regular solutions constructed via the Wigner transform. Non-negativity in particular is proven to hold in $H^{\alpha,\beta}$ for any $\alpha, \beta > \frac{d-1}{2}$, without any additional regularity or decay assumptions.

1. INTRODUCTION

We are interested in the local Cauchy theory for the full Boltzmann equation:

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = Q(f, f)(t, x, v). \quad (1)$$

Here $t \geq 0$, $x, v \in \mathbb{R}^d$ with $d \geq 2$, and the collision operator $Q(f, f)$ is defined as follows:

$$Q(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_* \mathbf{b} \left(|v - v_*|, \omega \cdot \frac{v - v_*}{|v - v_*|} \right) (f' f'_* - f f_*). \quad (2)$$

We have defined $f' = f(t, x, v')$, $f'_* = f(t, x, v'_*)$, $f_* = f(t, x, v_*)$; the velocities (v', v'_*) are defined in terms of (v, v_*) and $\omega \in \mathbb{S}^{d-1}$ by the following relation:

$$\begin{aligned} v' &= v + (\omega \cdot (v_* - v))\omega \\ v'_* &= v_* - (\omega \cdot (v_* - v))\omega. \end{aligned} \quad (3)$$

We will assume throughout this paper that the collision kernel \mathbf{b} is a bounded function or, for some results, identically constant. These restrictive assumptions are for technical simplicity; they can certainly be relaxed for most

(and probably all) of our results. We refer to [10, 30] for background on Boltzmann’s equation.

Although in this paper we focus on properties of local in time solutions for Boltzmann’s equation, we recall that there are two known theories of global solutions for Boltzmann’s equation (apart from small data solutions, which are closer to the local theory). One is the theory of renormalized solutions, which applies for arbitrary large data having finite mass, second moments, and entropy. [14] Very little is known about renormalized solutions; in particular, it is not known whether they are unique. The other theory of global solutions concerns solutions near a Maxwellian equilibrium of some fixed temperature, [1, 15, 18, 19, 28, 29]. The construction of such solutions is intimately tied to the properties of the linearized collision operator. We are ultimately motivated by certain applications for which it seems better to view Boltzmann’s equation as a perturbation of free transport, rather than a perturbation of the linearized equation. For example, in the derivation of Boltzmann’s equation from Hamiltonian particle systems [16, 21, 23], it is very hard to make use of the structure arising from the linearized collision operator (but see [6, 7]); the transport structure is still available in this context and that motivates us to employ it. For now, we do that in the setting of local in time solutions.

There are a number of theories of *local* solutions for Boltzmann’s equation currently available; a local well-posedness (LWP) theory is a theory of existence and uniqueness which allows data of arbitrary size in some norm, but which may break down after a short time depending (solely) on the norm of the data. We refer the reader to [2, 4, 20] for several of the LWP theories which are currently known for Boltzmann’s equation. A new LWP theory for Boltzmann’s equation has recently been developed in [11] using the Wigner transform and tools including the bilinear Strachartz estimate that are inspired by techniques for the treatment of nonlinear Schrödinger equations and the Gross-Pitaevskii hierarchy (see e.g. [12]). In the present paper, we show that the solutions of Boltzmann’s equation constructed¹ in [11] propagate higher regularity and moments if they are available at the initial time. We give a complete description of this phenomenon when the collision kernel is identically constant; the reasons for this limitation seem to be purely technical.

The problem of regularity for Boltzmann’s equation has been studied by various authors. We refer especially to [8, 26]; both of these works discuss the persistence of regularity of small solutions near vacuum, as in [20]. It is also proven in [8] that, for some Boltzmann equations with Grad cut-off, in the case of small solutions, the solution at positive times propagates the singularities of the initial data. This shows that, in the Grad cut-off case,

¹We emphasize that, contrary to [4], the theory of [11] has not been optimized to take advantage of the regularizing properties of the Boltzmann “gain” collision term. We expect such optimizations to be available through the use of more general $X^{s,b}$ norms than considered in this paper; such refinements are the subject of ongoing research.

the Boltzmann flow does not smooth out irregular initial data. For this reason, in the present work we cannot hope to prove that the solution is *smoother* than the data; we can only hope to show that the solution is *as regular* as the data and that is what we achieve. More precisely, we obtain a fairly complete description of the persistence of regularity for local-in-time solutions using the functional framework of [11], at least when the collision kernel is constant.

The study of moments for the Boltzmann equation has a long history, particularly in the space homogeneous case. We refer to [30] for a review of the classical results, as well as to e.g. [3, 5, 17, 25, 27] and references therein. The rough picture is that, for hard potentials (including hard spheres), the space homogeneous Boltzmann equation generates higher-order moments *instantaneously* as soon as the initial energy is finite. For Maxwell molecules or soft potentials, however, only those moments which are initially finite are finite at positive times. In particular, in the case of *bounded* collision kernels (which is the only case studied in this paper), one does not expect any generation of moments effect. Instead, one should seek to prove *propagation* of moments, which is precisely the type of result we can prove. Our techniques could be applied in the case of hard potentials with Grad cut-off, but at present we cannot expect to capture the generation of moments effect in our estimates. We also remark that we address L^2 moments, whereas the space homogeneous theory is primarily concerned with L^1 moments.²

As is typically the case when one proves a propagation result, the proofs in this paper are based on the following idea: we write an equation for a desired moment or derivative, and then solve that equation by a fixed point argument as in [11].³ In all cases we must pay a cost in terms of regularity/moments in order to propagate regularity/moments in some other variable. For example, we can propagate moments in x by paying with moments in v until we run out of currency to exchange; this results in a natural limit to the number of moments we can propagate. Similarly, for an *identically constant* collision kernel, we can propagate derivatives in v by paying with derivatives in x , until we run out of derivatives to trade. Now it is *a priori* possible that solutions which are initially smooth in x with rapid decay in v lose this property after some short time, only to persist for a longer time in a less regular space. We can rule this out, to some extent, by our methods as well. In particular, we prove persistence of regularity results for the scale of Sobolev spaces which arise naturally from the analysis of [11]. Effectively, as soon as the solution has enough regularity and decay to apply the methods of [11], we can prove that *more regular* data leads to more regular solutions for as long as the solution exists in the less regular

²There is nothing particularly special about L^1 moments; for example, a theorem of Lanford on the derivation of Boltzmann's equation [23] relies upon exponential L^∞ moments.

³Technically one must show that the solution of the new fixed point problem is related to the solution the original fixed point problem in the expected way; this is formally trivial but inconvenient to prove rigorously. We shall not address this issue in full detail.

space. The higher Sobolev norms may grow much more rapidly in time than the lower Sobolev norms, however.

As an application of our results, we are able to prove that for constant collision kernels, the solutions constructed in [11] propagate *non-negativity* assuming only that the data itself is non-negative. This is true under minimal assumptions for which the theory of [11] applies; in particular, we *do not* require higher moment or regularity estimates to prove non-negativity. The reason is that our persistence of regularity results allow us to approximate low-regularity solutions by higher regularity solutions for a short time interval; even though the higher Sobolev norms may be very large, they will at least be finite on a time interval bounded from below uniformly with respect to the mollification parameter. Once we have enough regularity and decay, we can apply a theorem of [24] directly (that theorem is based on a Gronwall type argument to control the negative part of the solution). The non-negativity is preserved upon passage to the limit.

Organization of the paper. In Section 2, we outline the basic notation and the main results we prove in this paper. In Section 3, we quote a slightly refined version of a key proposition from [11] which we require to prove our main results; we also prove a couple of useful lemmas. Section 4 is dedicated to proving propagation of moments in x and derivatives in v . In Section 5, we prove persistence of regularity in the scale of Sobolev spaces $H^{\alpha,\beta}$; this corresponds to derivatives in x and moments in v . In Section 6, we address regularity with respect to the time variable. Section 7 contains a proof of continuity of the solution map. A new bilinear estimate with loss is proven in Appendix A; this estimate is used in Section 6.

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2. NOTATION AND MAIN RESULTS

In this section we present the notation, followed by a brief review of the relevant previous works and the statement of main results of this paper.

2.1. Notation. We will employ the Wigner transform as in [11]. For any function $f(x, v) \in L^2_{x,v}$, we define $\gamma(x, x') \in L^2_{x,x'}$ as follows:

$$\gamma(x, x') = \int_{\mathbb{R}^d} f\left(\frac{x+x'}{2}, v\right) e^{iv \cdot (x-x')} dv. \quad (4)$$

The inverse of this formula is the Wigner transform, given by:

$$f(x, v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \gamma\left(x + \frac{y}{2}, x - \frac{y}{2}\right) e^{-iv \cdot y} dy. \quad (5)$$

We may write $f = \mathcal{W}[\gamma]$ and $\gamma = \mathcal{W}^{-1}[f]$. The map $\mathcal{W} : L^2_{x,x'} \rightarrow L^2_{x,v}$ is an isometric linear isomorphism by Plancharel's theorem. We will assume throughout this paper that

$$\|\mathbf{b}\|_\infty = \sup_{u \in \mathbb{R}^d, \omega \in \mathbb{S}^{d-1}} \left| \mathbf{b} \left(|u|, \omega \cdot \frac{u}{|u|} \right) \right| < \infty. \quad (6)$$

For some results, we will need to assume (for technical reasons) that \mathbf{b} is identically constant. We define the Fourier transform of the collision kernel, namely:

$$\hat{\mathbf{b}}^\omega(\xi) = \int_{\mathbb{R}^d} \mathbf{b} \left(|u|, \omega \cdot \frac{u}{|u|} \right) e^{-iu \cdot \xi} du. \quad (7)$$

The functional setting is the same as that of [11], and it is defined via the Fourier transform of $\gamma = \mathcal{W}^{-1}[f]$:

$$\hat{\gamma}(\xi, \xi') = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-ix \cdot \xi} e^{-ix' \cdot \xi'} \gamma(x, x') dx dx'. \quad (8)$$

For any $\alpha, \beta \geq 0$ we define

$$\|\gamma(x, x')\|_{H^{\alpha, \beta}} = \left\| \langle \xi + \xi' \rangle^\alpha \langle \xi - \xi' \rangle^\beta \hat{\gamma}(\xi, \xi') \right\|_{L^2_{\xi, \xi'}}. \quad (9)$$

This norm is equivalent to the following norm for $f(x, v)$:

$$\left\| \langle 2v \rangle^\beta (1 - \Delta_x)^{\frac{\alpha}{2}} f(x, v) \right\|_{L^2_{x,v}}. \quad (10)$$

2.2. Previous results. If $f(t, x, v)$ is a smooth and rapidly decaying solution of Boltzmann's equation (1) with $\|\mathbf{b}\|_\infty < \infty$, it is possible to show that $\gamma(t) = \mathcal{W}^{-1}[f(t)]$ solves the following equation: (see Appendix A of [11] for a proof)

$$\left(i\partial_t + \frac{1}{2}(\Delta_x - \Delta_{x'}) \right) \gamma(t) = B(\gamma(t), \gamma(t)) \quad (11)$$

$$B(\gamma_1, \gamma_2) = B^+(\gamma_1, \gamma_2) - B^-(\gamma_1, \gamma_2) \quad (12)$$

$$B^-(\gamma_1, \gamma_2)(x, x') = \frac{i}{2^{2d}\pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega \left(\frac{z}{2} \right) \times \quad (13)$$

$$\times \gamma_1 \left(x - \frac{z}{4}, x' + \frac{z}{4} \right) \gamma_2 \left(\frac{x+x'}{2} + \frac{z}{4}, \frac{x+x'}{2} - \frac{z}{4} \right)$$

$$B^+(\gamma_1, \gamma_2)(x, x') = \frac{i}{2^{2d}\pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega \left(\frac{z}{2} \right) \times$$

$$\times \gamma_1 \left(x - \frac{1}{2}P_\omega(x-x') - \frac{R_\omega(z)}{4}, x' + \frac{1}{2}P_\omega(x-x') + \frac{R_\omega(z)}{4} \right) \times$$

$$\times \gamma_2 \left(\frac{x+x'}{2} + \frac{1}{2}P_\omega(x-x') + \frac{R_\omega(z)}{4}, \frac{x+x'}{2} - \frac{1}{2}P_\omega(x-x') - \frac{R_\omega(z)}{4} \right) \quad (14)$$

Here we have

$$P_\omega(x) = (\omega \cdot x)\omega \quad (15)$$

$$R_\omega(x) = (\mathbb{I} - 2P_\omega)(x) \quad (16)$$

and $\mathbb{I}(x) = x$.

We will refer to f as a solution of Boltzmann's equation if $\gamma = \mathcal{W}^{-1}[f]$ solves the Duhamel-type integral formula:

$$\gamma(t) = e^{\frac{1}{2}it\Delta_\pm}\gamma(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_\pm} B(\gamma(t_1), \gamma(t_1)) dt_1 \quad (17)$$

We may also say in this case that γ solves Boltzmann's equation. Note that if $\mathbf{b} \equiv \text{cst.}$ then $\hat{\mathbf{b}}^\omega$ is a δ -function concentrated at $z = 0$ (cf. Bobylev's formula).

Remark 2.1. *We anticipate that some parts of the analysis of Boltzmann's equation via bilinear Strichartz estimates can be presented in terms of the kinetic transport operator*

$$\partial_t + v \cdot \nabla_x \quad (18)$$

Indeed, starting from (18) and taking the Fourier transform in t and x , with dual variables τ and η respectively, yields the weight

$$\tau + v \cdot \eta \quad (19)$$

To see the equivalence, replace $v \mapsto (\xi - \xi')/2$ and $\eta \mapsto \xi + \xi'$, and thereby recover the difference of squares

$$v \cdot \eta \mapsto \frac{1}{2} (|\xi|^2 - |\xi'|^2) \quad (20)$$

which appears in the spacetime Fourier analysis of the density matrix formulation of Schrödinger's equation. This observation would be especially helpful for problems with periodic boundary conditions, where the interpretation of the Wigner transform is less clear.

Compared to the kinetic formulation of Boltzmann's equation, the density matrix formulation has some unique advantages; most importantly, it facilitates a direct comparison to the substantial literature on Schrödinger's equation in density matrix formulation.⁴ This is particularly evident in the case of a constant collision kernel $\mathbf{b} \equiv \text{cst.}$, where the Boltzmann loss operator becomes (up to a real constant)

$$B^-(\gamma, \gamma) = i\rho_\gamma \left(\frac{x + x'}{2} \right) \gamma(x, x') \quad (21)$$

where $\rho_\gamma(x) = \gamma(x, x)$ is the diagonal of a density matrix. Though this is unfortunately not a commutator, it does look very similar to the bilinear operator

$$B(\gamma, \gamma) = (\rho_\gamma(x) - \rho_\gamma(x')) \gamma(x, x') = [\rho_\gamma, \gamma](x, x') \quad (22)$$

⁴A detailed analysis of the spectral properties of γ (viewed as a linear operator on $L^2(\mathbb{R}^d)$), or the connections to Bobylev's formula, may also provide interesting avenues for research; however, we do not explore these directions in this work.

which defines the density matrix formulation of cubic NLS (see [22]). Moreover, just as $(d-1)/2$ is L^2 -based Sobolev regularity threshold for our proof (in both α and β for a constant collision kernel), the regularity threshold for the proof of [22] in the cubic NLS case is exactly $(d-1)/2$. This is not surprising because our argument is a translation of the arguments of [22] to the Boltzmann equation. Note that, in the Boltzmann case, a somewhat better regularity class has been obtained for some collision kernels with Grad cut-off using the Strichartz estimates of Castella and Perthame [4, 9], but it was necessary to use the convoluting effects of the Boltzmann gain operator. We have not made essential use of the special properties of the gain term in the present work.

The following LWP result was proven in [11]:

Theorem 2.1. *Let $\alpha, \beta \in (\frac{d-1}{2}, \infty)$ and consider the Boltzmann equation with $\|\mathbf{b}\|_\infty < \infty$. For any $\gamma_0 \in H^{\alpha, \beta}$ there exists a unique solution $\gamma(t)$ of Boltzmann's equation on a small time interval $[0, T]$ such that*

$$\|\gamma\|_{L_T^\infty H^{\alpha, \beta}} < \infty \quad (23)$$

and

$$\|B(\gamma, \gamma)\|_{L_T^1 H^{\alpha, \beta}} < \infty \quad (24)$$

both hold, and $\gamma(0) = \gamma_0$. Moreover, if $\|\gamma_0\|_{H^{\alpha, \beta}} \leq M$ then for all small enough T depending only on α, β and M , there holds:

$$T^{\frac{1}{2}} \|\gamma\|_{L_T^\infty H^{\alpha, \beta}} + \|B(\gamma, \gamma)\|_{L_T^1 H^{\alpha, \beta}} \leq C(M, \alpha, \beta) T^{\frac{1}{2}} \|\gamma_0\|_{H^{\alpha, \beta}} \quad (25)$$

Remark 2.2. *Note carefully that, as a consequence of Theorem 2.1,*

$$\|B(\gamma, \gamma)\|_{L_T^1 H^{\alpha, \beta}}$$

scales at worst like a power of T , namely $T^{\frac{1}{2}}$, when T is small. We will make use of this fact repeatedly in this work.

The functional spaces $H^{\alpha, \beta}$ automatically guarantee some regularity in space and decay in velocity variables. Our first result, to be proven in Section 4, states that it is possible to trade moments in v for moments in x ; and, if $\mathbf{b} \equiv \text{cst.}$, it is also possible to trade derivatives in x for derivatives in v . For these results to hold, we must always assume that enough decay or regularity is available in the initial data. We remind the reader that, for bounded collision kernels, the Boltzmann equation is not expected to generate derivatives or moments in any variable at positive times; hence, any regularity or decay estimate must have been present at the initial time for it to be available at positive times.

2.3. Main results of this paper. Now we are ready to state main results of this paper which are formulated in Theorem 2.2, Theorem 2.3 and Theorem 2.6.

Theorem 2.2. *Let $\gamma(t)$ be a solution of Boltzmann's equation with bounded collision kernel, $\|\mathbf{b}\|_\infty < \infty$, satisfying the following bounds on some time interval $[0, T]$:*

$$\|\gamma(t)\|_{L_T^\infty H^{\alpha, \beta}} < \infty \quad (26)$$

$$\|B(\gamma(t), \gamma(t))\|_{L_T^1 H^{\alpha, \beta}} < \infty \quad (27)$$

with $\alpha, \beta > \frac{d-1}{2}$. Then we have the following:

(i) *Suppose that, for some integer $K > 0$ with $K < \beta - \frac{d-1}{2}$, for any integer k with $1 \leq k \leq K$ there holds*

$$\langle x + x' \rangle^k \gamma(0) \in H^{\alpha, \beta - k} \quad (28)$$

Then for all $1 \leq k \leq K$ we have

$$\langle x + x' \rangle^k \gamma(t) \in L_T^\infty H^{\alpha, \beta - k} \quad (29)$$

(ii) *Assume that $\mathbf{b} \equiv \text{cst.}$; then, suppose that, for some integer $K > 0$ with $2K < \alpha - \frac{d-1}{2}$, for all $1 \leq k \leq K$ there holds*

$$\langle x - x' \rangle^{2k} \gamma(0) \in H^{\alpha - 2k, \beta} \quad (30)$$

Then for all $1 \leq k \leq K$ we have

$$\langle x - x' \rangle^{2k} \gamma(t) \in L_T^\infty H^{\alpha - 2k, \beta} \quad (31)$$

Remark 2.3. *Our proof provides quantitative estimates for $\langle x + x' \rangle^k \gamma(t)$ and $\langle x - x' \rangle^{2k} \gamma(t)$ in the relevant function spaces. However, these bounds may grow very rapidly with time and we make no effort to prove optimal bounds on the growth rate.*

In order to apply Theorem 2.2, we generally require solutions $\gamma \in H^{\alpha, \beta}$ for large values of α, β . This naturally leads us to inquire whether a solution may blow up in $H^{\alpha, \beta}$ only to persist longer in some less regular space. This question is particularly relevant if we want to approximate some irregular initial data by some other, very regular, data for the purpose of formal computation. Our next result, proven in Section 5, rules out such pathological behavior within the scale of Sobolev spaces $H^{\alpha, \beta}$ with $\alpha, \beta > \frac{d-1}{2}$.

Theorem 2.3. *Let $\gamma(t)$ be a solution of Boltzmann's equation with $\|\mathbf{b}\|_\infty < \infty$, and suppose $\gamma \in L_T^\infty H^{\alpha, \beta}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha, \beta}$ for some $\alpha, \beta > \frac{d-1}{2}$. Then we have the following:*

(i) *If $\gamma(0) \in H^{\alpha+r, \beta}$ for some $r \in \mathbb{N}$, then $\gamma \in L_T^\infty H^{\alpha+r, \beta}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha+r, \beta}$.*

(ii) *If $\gamma(0) \in H^{\alpha, \beta+r}$ for some (real) $r > 0$, then $\gamma \in L_T^\infty H^{\alpha, \beta+r}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha, \beta+r}$.*

Remark 2.4. *As with Theorem 2.2, we can extract quantitative estimates from the proof of Theorem 2.3, but they may grow very rapidly with time.*

We also prove, in Sections 6 and 7 respectively, results on regularity in time and also the continuity of the solution map.⁵ We quote those results here for the convenience of the reader.

Proposition 2.4. *Let $\gamma(t)$ be a solution of Boltzmann's equation with $\|\mathbf{b}\|_\infty < \infty$, and suppose $\gamma \in L_T^\infty H^{\alpha,\beta}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha,\beta}$ for some $\alpha, \beta > \frac{d-1}{2}$. Further suppose that $K > 0$ is an integer with $K < \min(\alpha, \beta) - \frac{d}{2}$. Then for any integer k with $1 \leq k \leq K$ there holds $\partial_t^k \gamma \in L_T^\infty H^{\alpha-k, \beta-k}$.*

Proposition 2.5. *Let $\gamma^j(t)$ be a solution of Boltzmann's equation with $\|\mathbf{b}\|_\infty < \infty$, for $j = 1, 2$, with $\gamma^j \in L_T^\infty H^{\alpha,\beta}$ and $B(\gamma^j, \gamma^j) \in L_T^1 H^{\alpha,\beta}$ for $j = 1, 2$ and some $\alpha, \beta > \frac{d-1}{2}$. Furthermore, suppose that $\|\gamma^j\|_{L_T^\infty H^{\alpha,\beta}} \leq M$ for $j = 1, 2$. Then we have*

$$\|\gamma^1 - \gamma^2\|_{L_T^\infty H^{\alpha,\beta}} \leq C_{M,T} \|\gamma^1(0) - \gamma^2(0)\|_{H^{\alpha,\beta}} \quad (32)$$

where the constant may depend on α, β .

Remark 2.5. *Most likely, the proofs of the preceding theorems can be combined to prove propagation of mixed derivatives and moments; for example,*

$$\langle x + x' \rangle^{k_1} \langle x - x' \rangle^{k_2} \langle \nabla_x + \nabla_{x'} \rangle^{k_3} \langle \nabla_x - \nabla_{x'} \rangle^{k_4} \partial_t^{k_5} \gamma(t) \quad (33)$$

We will not address these mixed estimates in detail. For what follows, it will suffice to notice that by Fourier transforming in time (only), we can always estimate

$$\langle x + x' \rangle^k \langle x - x' \rangle^k \partial_t^k \lesssim \langle x + x' \rangle^{3k} + \langle x - x' \rangle^{3k} + \partial_t^{3k} \quad (34)$$

This is obvious in $L_t^2 L_{x,x'}^2$, but the same estimate holds in $L_t^2 H^{\alpha,\beta}$ as well, at least for integer values of α and β . In order to apply this estimate in practice, we must use a smooth compactly supported cut-off in the time variable; to this end, it is helpful to solve Boltzmann's equation backwards in time on a short time interval. This allows us to perform estimates on $[0, T_1]$ (with $T_1 < T$) by choosing a cut-off which is supported on $[-\Delta, T]$ for sufficiently small $\Delta > 0$. Due to the Grad cut-off condition, there is no difficulty in solving Boltzmann's equation backwards for a short time interval.

Using Theorem 2.2 and Proposition 2.4, we can construct solutions $\gamma(t)$ which have very strong regularity and decay properties on a short time interval. For such solutions, we can reverse the steps from Appendix A of [11] and thereby show that $f = \mathcal{W}[\gamma]$ is a *classical* solution of Boltzmann's equation. In particular, conservation of mass, momentum, and energy follow by the usual computations. In view of the next result on non-negativity, we can also prove the H -theorem for solutions having enough regularity and decay, under the additional assumption that $f(0) = \mathcal{W}[\gamma(0)] \geq ce^{-c(|x|^2 + |v|^2)}$

⁵Note in particular that the proof in Section 6 relies upon the bilinear estimates *with loss* proven in Appendix A; we emphasize that those bilinear estimates cannot replace Proposition 3.1 elsewhere in this paper, regardless of the size of α, β .

for some $c > 0$. Obviously we could optimize the spaces in which energy conservation holds by density arguments; we will not state a precise result along these lines.

One very important issue which was not addressed in [11] was the *non-negativity* of solutions. Only non-negative solutions of Boltzmann's equation are considered to have physical meaning. Moreover, the conserved mass and energy only supply useful control for non-negative solutions; and, the entropy is only *defined* for non-negative solutions. Combining all of the results quoted in this section, we can prove the following:

Theorem 2.6. *Let $\gamma(t)$ be a solution of Boltzmann's equation, with $\mathbf{b} \equiv \text{cst.}$; furthermore, suppose that $\gamma \in L_T^\infty H^{\alpha,\beta}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha,\beta}$ for some $\alpha, \beta \in (\frac{d-1}{2}, \infty)$. Then if $f(0, x, v) = \mathcal{W}[\gamma(0)](x, v) \geq 0$ for almost every $x, v \in \mathbb{R}^d$, then for all $t \in [0, T]$ we have $f(t, x, v) = \mathcal{W}[\gamma(t)](x, v) \geq 0$ for almost every $x, v \in \mathbb{R}^d$.*

Remark 2.6. *Note that in Theorem 2.6 we do not require γ to have any higher regularity or moment bounds.*

We omit a complete proof of Theorem 2.6, but we will state a few remarks about the proof. The first important point is that $\gamma(t)$ is actually continuous in $H^{\alpha,\beta}$, so we can evaluate $\gamma(t)$ for *any* $t \in [0, T]$. Fixing a solution $\gamma \in L_T^\infty H^{\alpha,\beta}$ with $B(\gamma, \gamma) \in L_T^1 H^{\alpha,\beta}$, we define

$$T_1 = \sup \{t_1 \in [0, T] : f(t) = \mathcal{W}[\gamma(t)] \geq 0 \forall t \in [0, t_1]\} \quad (35)$$

Assume by way of contradiction that $T_1 < T$. Since $\gamma(t)$ is continuous in time and non-negativity is preserved under passage to the limit in L^2 , we know that $f(T_1) \geq 0$. Therefore, it suffices to propagate non-negativity on a *small* time interval (possibly much smaller than T). We pick a sequence of very regular functions (say, in the Schwartz class) which converge to $\gamma(T_1)$ in $H^{\alpha,\beta}$; we use these approximate functions as initial data in Boltzmann's equation. Note carefully that the approximate solutions *may not exist* on the full time interval $[T_1, T]$, but they will have a time of existence which is bounded uniformly from below due to uniform boundedness in $H^{\alpha,\beta}$. Since $f(T_1) \geq 0$, we can arrange for the approximating functions to be non-negative at time $t = T_1$. We can apply Theorem 2.3, Theorem 2.2, and Proposition 2.4 to conclude that the approximating functions are smooth and rapidly decaying for as long as they exist in $H^{\alpha,\beta}$; in particular, inverting the steps from Appendix A of [11], we have a sequence of classical solutions of Boltzmann's equation. We can apply the results of [24] to conclude that the approximating sequence remains non-negative on a short time interval. Now we can pass to the limit, applying Theorem 2.5, to reach the desired contradiction.

3. A PROPOSITION AND TWO LEMMAS

Proposition 3.1. *Suppose $\alpha, \beta \in (\frac{d-1}{2}, \infty)$ and let $\delta \geq 0$ be chosen sufficiently small (with smallness depending continuously on α, β, d). Then*

there exists a constant C (depending on d, α, β) such that if $\|\mathbf{b}\|_\infty < \infty$ then for any $\gamma_1, \gamma_2 \in H^{\alpha, \beta}$, **both** the following estimates hold:

$$\begin{aligned} \left\| B^- \left(e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right) \right\|_{L_t^2 H^{\alpha, \beta + \delta}} &\leq \\ &\leq C \|\mathbf{b}\|_\infty \|\gamma_1\|_{H^{\alpha, \beta + \delta}} \|\gamma_2\|_{H^{\alpha, \beta}} \end{aligned} \quad (36)$$

$$\begin{aligned} \left\| B^+ \left(e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right) \right\|_{L_t^2 H^{\alpha, \beta + \delta}} &\leq \\ &\leq C \|\mathbf{b}\|_\infty \|\gamma_1\|_{H^{\alpha, \beta}} \|\gamma_2\|_{H^{\alpha, \beta}} \end{aligned} \quad (37)$$

Remark 3.1. Note carefully that the gain term B^+ regularizes in the β index; the loss term, by contrast, exhibits no such regularization.

Proof. (case $\delta > 0$)

The case $\delta = 0$ is proven in [11], or see Appendix B, so we only have to consider $\delta > 0$. It turns out that the proofs are almost identical to the proof from [11] so we only sketch the ideas.

For the loss estimate (36), we have (for example) the following commutativity:

$$(\nabla_x - \nabla_{x'}) B^-(\gamma_1, \gamma_2) = B^-((\nabla_x - \nabla_{x'}) \gamma_1, \gamma_2) \quad (38)$$

and moreover $(\nabla_x - \nabla_{x'})$ commutes with the free propagator $e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})}$. Hence the result is immediately obtained for $\delta = 1$ once it is known for $\delta = 0$. The same result can be proven for any $0 < \delta < 1$ using Fourier analysis, as in [11]; the required modifications to the proof presented there are completely trivial.

For the gain estimate (37), we note that all the estimates for the gain term from [11], or Appendix B, are stable with respect to a small perturbation of the *target* regularity index β (keeping the regularities of γ_1, γ_2 fixed). In fact, due to the *angular averaging* effect, the entire argument boils down to proving the convergence of certain geometric series of the form $\sum_{k=1}^{\infty} 2^{-bk}$ for some $b > 0$; obviously, the series will still converge if we perturb b slightly. \square

Lemma 3.2. Consider the Boltzmann equation with arbitrary bounded collision kernel. Then for any real numbers $a, b \geq 0$, there holds

$$\langle x + x' \rangle^{a+b} B^-(\gamma_1, \gamma_2) = B^- \left(\langle x + x' \rangle^a \gamma_1, \langle x + x' \rangle^b \gamma_2 \right) \quad (39)$$

$$\langle x + x' \rangle^{a+b} B^+(\gamma_1, \gamma_2) = B^+ \left(\langle x + x' \rangle^a \gamma_1, \langle x + x' \rangle^b \gamma_2 \right) \quad (40)$$

In the case that the collision kernel $\mathbf{b} \equiv \text{cst.}$, we also have for any positive integer k , and any $a, b \geq 0$,

$$\langle x - x' \rangle^a B^-(\gamma_1, \gamma_2) = B^- \left(\langle x - x' \rangle^a \gamma_1, \langle x - x' \rangle^b \gamma_2 \right) \quad (41)$$

$$(x - x') B^-(\gamma_1, \gamma_2) = B^-((x - x') \gamma_1, \gamma_2) \quad (42)$$

$$\begin{aligned}
\langle x - x' \rangle^{2k} B^+(\gamma_1, \gamma_2) &= \\
&= \sum_{j_1 + j_2 + j_3 = k} \binom{k}{j_1, j_2, j_3} (-1)^{j_3} B^+ \left(\langle x - x' \rangle^{2j_1} \gamma_1, \langle x - x' \rangle^{2j_2} \gamma_2 \right)
\end{aligned} \tag{43}$$

$$(x - x') B^+(\gamma_1, \gamma_2) = B^+((x - x')\gamma_1, \gamma_2) + B^+(\gamma_1, (x - x')\gamma_2) \tag{44}$$

Proof. Only (43) requires some explanation. The difficulty is that the action of B^+ involves the projection $P_\omega(x - x')$, which does not disappear when taking *differences* of x and x' . This is easily dealt with, however, by using the following orthogonality property:

$$\langle x - x' \rangle^{2k} = \left(\langle (\mathbb{I} - P_\omega)(x - x') \rangle^2 + \langle P_\omega(x - x') \rangle^2 - 1 \right)^k \tag{45}$$

and expanding terms using the multinomial formula. For (44), we use the simpler decomposition:

$$x - x' = (\mathbb{I} - P_\omega)(x - x') + P_\omega(x - x') \tag{46}$$

and conclude by linearity of the collision integral. \square

Lemma 3.3. *Consider the Boltzmann equation with arbitrary bounded collision kernel; then there holds*

$$(\nabla_x + \nabla_{x'}) B^-(\gamma_1, \gamma_2) = B^-((\nabla_x + \nabla_{x'})\gamma_1, \gamma_2) + B^-(\gamma_1, (\nabla_x + \nabla_{x'})\gamma_2) \tag{47}$$

$$(\nabla_x + \nabla_{x'}) B^+(\gamma_1, \gamma_2) = B^+((\nabla_x + \nabla_{x'})\gamma_1, \gamma_2) + B^+(\gamma_1, (\nabla_x + \nabla_{x'})\gamma_2) \tag{48}$$

Proof. This is an elementary computation. \square

4. MOMENT BOUNDS FOR DENSITY MATRICES

In this section we treat propagation of moments of γ in $x+x'$ and moments in $x-x'$ in turn. Combining these results allows us to control mixed moments as well, e.g. if we want to place $\langle x \rangle \langle \nabla_v \rangle f$ in $L_T^\infty L_{x,v}^2$ we could use

$$\langle x + x' \rangle \langle x - x' \rangle \lesssim \langle x + x' \rangle^2 + \langle x - x' \rangle^2 \tag{49}$$

More precise results for mixed moments may be available by combining the proofs given in this section, but we do not pursue this issue in detail.

4.1. Moments in $x + x'$. In this subsection we propagate moments of γ in $x + x'$, which is equivalent to propagating moments of the distribution $f(x, v)$ in the spatial variable. The idea of the proof is to write an equation for the k th moment and use the existence of a solution $\gamma(t)$ in $H^{\alpha, \beta}$ for large enough β .

Lemma 4.1. *Consider a distributional solution $\gamma(t)$, $t \in [0, T]$, of the Boltzmann equation. Then for any $k \in \mathbb{N}$, $k \geq 1$, there holds*

$$\begin{aligned} \left(i\partial_t + \frac{1}{2} (\Delta_x - \Delta_{x'}) \right) \left(\langle x + x' \rangle^k \gamma(t) \right) &= B \left(\langle x + x' \rangle^k \gamma(t), \gamma(t) \right) + \\ &+ k \frac{x + x'}{\langle x + x' \rangle} \cdot (\nabla_x - \nabla_{x'}) \left(\langle x + x' \rangle^{k-1} \gamma(t) \right). \end{aligned} \quad (50)$$

in the sense of distributions.

Proof. This computation follows by using (39) and (40). \square

Let us introduce the following convenient notation:

$$\zeta(t) = B(\gamma(t), \gamma(t)) \quad (51)$$

$$\gamma_{k,+}(t, x, x') = \langle x + x' \rangle^k \gamma(t, x, x') \quad (52)$$

$$\zeta_{k,+}(t) = B(\gamma_{k,+}(t), \gamma(t)). \quad (53)$$

Proposition 4.2. *Let $\gamma(t)$ be a solution of Boltzmann's equation with bounded collision kernel, $\|\mathbf{b}\|_\infty < \infty$, satisfying the following bounds on some time interval $[0, T]$:*

$$\|\gamma(t)\|_{L_T^\infty H^{\alpha,\beta}} < \infty \quad (54)$$

$$\|B(\gamma(t), \gamma(t))\|_{L_T^1 H^{\alpha,\beta}} < \infty \quad (55)$$

with $\alpha, \beta > \frac{d-1}{2}$. Further assume that, for some integer $K > 0$ with $K < \beta - \frac{d-1}{2}$, for all $1 \leq k \leq K$ there holds

$$\|\gamma_{k,+}(0)\|_{H^{\alpha,\beta-k}} < \infty \quad (56)$$

Then for all $1 \leq k \leq K$ we have

$$\|\gamma_{k,+}(t)\|_{L_T^\infty H^{\alpha,\beta-k}} < \infty. \quad (57)$$

Moreover there is an explicit bound on $\sup_{1 \leq k \leq K} \|\gamma_{k,+}(t)\|_{L_T^\infty H^{\alpha,\beta-k}}$ that only depends on $\|\gamma(t)\|_{L_T^\infty H^{\alpha,\beta}}$, $\|B(\gamma(t), \gamma(t))\|_{L_T^1 H^{\alpha,\beta}}$, and $\sup_{1 \leq k \leq K} \|\gamma_{k,+}(0)\|_{H^{\alpha,\beta-k}}$.

Proof. We will prove the result assuming T is small. To prove the general result, it suffices to split the whole time interval $[0, T]$ into small sub-intervals (whose size depends only on the bounds (54) and (55)) and iterate the same argument as many times as needed.⁶

⁶Note in particular that $\|B(\gamma(t), \gamma(t))\|_{L_T^1 H^{\alpha,\beta}}$ scales at worst like $T^{\frac{1}{2}}$ for T small under the hypotheses of the proposition, so this norm will certainly be small if T is chosen small. By a time translation argument the same control holds on any small interval $[t_0, t_0 + T]$ for as long as $\gamma(t)$ remains bounded in $H^{\alpha,\beta}$.

Let us denote by $\Delta_{\pm}^{(2)} = \sum_{i=1}^2 (\Delta_{x_i} - \Delta_{x'_i})$ the Laplace operator acting on two particles. Using Lemma 4.1, we easily obtain:

$$\begin{aligned} \left(i\partial_t + \frac{1}{2}\Delta_{\pm}^{(2)} \right) (\gamma_{k,+} \otimes \gamma) &= \gamma_{k,+} \otimes B(\gamma, \gamma) + B(\gamma_{k,+}, \gamma) \otimes \gamma + \\ &+ \left(k \frac{x + x'}{\langle x + x' \rangle} \cdot (\nabla_x - \nabla_{x'}) \gamma_{k-1,+} \right) \otimes \gamma \end{aligned} \quad (58)$$

In integral form, this is:

$$\begin{aligned} (\gamma_{k,+} \otimes \gamma)(t) &= e^{\frac{1}{2}it\Delta_{\pm}^{(2)}} (\gamma_{k,+} \otimes \gamma)(0) \\ &- i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}^{(2)}} (\gamma_{k,+} \otimes B(\gamma, \gamma))(t_1) dt_1 \\ &- i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}^{(2)}} (B(\gamma_{k,+}, \gamma) \otimes \gamma)(t_1) dt_1 \\ &- i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}^{(2)}} \left\{ \left(k \frac{x + x'}{\langle x + x' \rangle} \cdot (\nabla_x - \nabla_{x'}) \gamma_{k-1,+} \right) \otimes \gamma \right\} (t_1) dt_1. \end{aligned} \quad (59)$$

The key step is to apply the collision integral to each side of (59) to obtain a (nearly) closed equation for $\zeta_{k,+}$; this idea is adapted from [12]. To fully close the system we need to incorporate the equation for $\gamma_{k,+}(t)$ which comes directly by re-writing Lemma 4.1 in integral form. Altogether we need to solve the following *system* of equations, where $\Delta_{\pm} = \Delta_x - \Delta_{x'}$:

$$\begin{aligned} \gamma_{k,+}(t) &= e^{\frac{1}{2}it\Delta_{\pm}} \gamma_{k,+}(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta_{k,+}(t_1) dt_1 \\ &- i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \left(k \frac{x + x'}{\langle x + x' \rangle} \cdot (\nabla_x - \nabla_{x'}) \gamma_{k-1,+}(t_1) \right) dt_1 \end{aligned} \quad (60)$$

$$\begin{aligned} \zeta_{k,+}(t) &= B \left(e^{\frac{1}{2}it\Delta_{\pm}} \gamma_{k,+}(0), e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0) \right) \\ &- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma_{k,+}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta(t_1) \right) dt_1 \\ &- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta_{k,+}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) dt_1 \\ &- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \left\{ k \frac{x + x'}{\langle x + x' \rangle} \cdot (\nabla_x - \nabla_{x'}) \gamma_{k-1,+}(t_1) \right\}, \right. \\ &\quad \left. e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) dt_1. \end{aligned} \quad (61)$$

We can solve (60)-(61) on $[0, T]$ for sufficiently small T by applying a Picard iteration using Proposition 3.1; we omit the details. In any case the only fact to be drawn from the fixed point iteration is that the moments $\gamma_{k,+}$

do not instantaneously diverge in $H^{\alpha, \beta-k}$, so the *quantitative* estimates we prove next are justified.

First, using the fact that the propagator $e^{\frac{1}{2}it\Delta_{\pm}}$ preserves the spaces $H^{\alpha, \beta}$, we easily obtain from (60) the following bound:

$$\begin{aligned} \|\gamma_{k,+}\|_{L_T^\infty H^{\alpha, \beta-k}} &\leq \\ &\leq \|\gamma_{k,+}(0)\|_{H^{\alpha, \beta-k}} + \|\zeta_{k,+}\|_{L_T^1 H^{\alpha, \beta-k}} + C_{\alpha, \beta} T \|\gamma_{k-1,+}\|_{L_T^\infty H^{\alpha, \beta-(k-1)}}. \end{aligned} \quad (62)$$

For the second estimate, we take the $L_T^1 H^{\alpha, \beta-k}$ norm on both sides of (61). We obtain:

$$\begin{aligned} \|\zeta_{k,+}(t)\|_{L_T^1 H^{\alpha, \beta-k}} &\leq \left\| B \left(e^{\frac{1}{2}it\Delta_{\pm}} \gamma_{k,+}(0), e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0) \right) \right\|_{L_T^1 H^{\alpha, \beta-k}} \\ &+ \int_0^T \int_0^t \left\| B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma_{k,+}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta(t_1) \right) \right\|_{H^{\alpha, \beta-k}} dt_1 dt \\ &+ \int_0^T \int_0^t \left\| B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta_{k,+}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) \right\|_{H^{\alpha, \beta-k}} dt_1 dt \\ &+ \int_0^T \int_0^t dt_1 dt \times \\ &\quad \times \left\| B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \left\{ k \frac{x+x'}{\langle x+x' \rangle} \cdot (\nabla_x - \nabla_{x'}) \gamma_{k-1,+}(t_1) \right\}, \right. \right. \\ &\quad \left. \left. e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) \right\|_{H^{\alpha, \beta-k}}. \end{aligned} \quad (63)$$

Now we bound $\int_0^t dt_1(\dots)$ by $\int_0^T dt_1(\dots)$ and apply Fubini.

$$\begin{aligned} \|\zeta_{k,+}(t)\|_{L_T^1 H^{\alpha, \beta-k}} &\leq \left\| B \left(e^{\frac{1}{2}it\Delta_{\pm}} \gamma_{k,+}(0), e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0) \right) \right\|_{L_T^1 H^{\alpha, \beta-k}} \\ &+ \int_0^T \left\| B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma_{k,+}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta(t_1) \right) \right\|_{L_T^1 H^{\alpha, \beta-k}} dt_1 \\ &+ \int_0^T \left\| B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta_{k,+}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) \right\|_{L_T^1 H^{\alpha, \beta-k}} dt_1 \\ &+ \int_0^T dt_1 \times \\ &\quad \times \left\| B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \left\{ k \frac{x+x'}{\langle x+x' \rangle} \cdot (\nabla_x - \nabla_{x'}) \gamma_{k-1,+}(t_1) \right\}, \right. \right. \\ &\quad \left. \left. e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) \right\|_{L_T^1 H^{\alpha, \beta-k}}. \end{aligned} \quad (64)$$

Finally we apply Cauchy-Schwarz to bound $\|\dots\|_{L_T^1}$ by $T^{\frac{1}{2}}\|\dots\|_{L_T^2}$; then, we are able to apply Proposition 3.1 to deduce the following bound:

$$\begin{aligned} \|\zeta_{k,+}\|_{L_T^1 H^{\alpha,\beta-k}} &\leq CT^{\frac{1}{2}} \|\gamma_{k,+}(0)\|_{H^{\alpha,\beta-k}} \|\gamma(0)\|_{H^{\alpha,\beta-k}} + \\ &+ CT^{\frac{1}{2}} \|\gamma_{k,+}\|_{L_T^\infty H^{\alpha,\beta-k}} \|\zeta\|_{L_T^1 H^{\alpha,\beta-k}} + CT^{\frac{1}{2}} \|\zeta_{k,+}\|_{L_T^1 H^{\alpha,\beta-k}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta-k}} + \\ &+ C_{\alpha,\beta} k T^{\frac{3}{2}} \|\gamma_{k-1,+}\|_{L_T^\infty H^{\alpha,\beta-(k-1)}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta-k}}. \end{aligned} \quad (65)$$

Since $H^{\alpha,\beta} \subset H^{\alpha,\beta-k}$, this implies:

$$\begin{aligned} \|\zeta_{k,+}\|_{L_T^1 H^{\alpha,\beta-k}} &\leq CT^{\frac{1}{2}} \|\gamma_{k,+}(0)\|_{H^{\alpha,\beta-k}} \|\gamma(0)\|_{H^{\alpha,\beta}} + \\ &+ CT^{\frac{1}{2}} \|\gamma_{k,+}\|_{L_T^\infty H^{\alpha,\beta-k}} \|\zeta\|_{L_T^1 H^{\alpha,\beta}} + CT^{\frac{1}{2}} \|\zeta_{k,+}\|_{L_T^1 H^{\alpha,\beta-k}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} + \\ &+ C_{\alpha,\beta} k T^{\frac{3}{2}} \|\gamma_{k-1,+}\|_{L_T^\infty H^{\alpha,\beta-(k-1)}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}}. \end{aligned} \quad (66)$$

Let us define

$$M_T = T^{\frac{1}{2}} \|\gamma_{k,+}\|_{L_T^\infty H^{\alpha,\beta-k}} + \|\zeta_{k,+}\|_{L_T^1 H^{\alpha,\beta}}. \quad (67)$$

Then combining (62) and (66), we obtain:

$$\begin{aligned} M_T &\leq C \left(T^{\frac{1}{2}} + T^{\frac{1}{2}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} + \|\zeta\|_{L_T^1 H^{\alpha,\beta}} \right) M_T + \\ &+ T^{\frac{1}{2}} \|\gamma_{k,+}(0)\|_{H^{\alpha,\beta-k}} + T^{\frac{1}{2}} \|\gamma_{k,+}(0)\|_{H^{\alpha,\beta-k}} \|\gamma(0)\|_{H^{\alpha,\beta}} + \\ &+ C_{\alpha,\beta} T^{\frac{3}{2}} \|\gamma_{k-1,+}\|_{L_T^\infty H^{\alpha,\beta-(k-1)}} + \\ &+ C_{\alpha,\beta} k T^{\frac{3}{2}} \|\gamma_{k-1,+}\|_{L_T^\infty H^{\alpha,\beta-(k-1)}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}}. \end{aligned} \quad (68)$$

Since $\|\zeta\|_{L_T^1 H^{\alpha,\beta}}$ is $\mathcal{O}(T^{\frac{1}{2}})$ for small T , we find that the prefactor of M_T on the right-hand side is small if T is small. The smallness of T depends only on the underlying solution $\gamma(t)$ of Boltzmann's equation. \square

4.2. Moments in $x - x'$. In this subsection we propagate moments of γ in $x - x'$, which is equivalent to propagating derivatives of the distribution $f(x, v)$ in the velocity variable. As in the previous subsection, we will write an equation for the $(2k)$ th moment of γ and use the existence of a solution $\gamma(t)$ in $H^{\alpha,\beta}$ for large enough α . However, as we will see, the proof is much more technical both because (43) introduces many new terms and because we can only close the estimate for moments of *even* order.

Lemma 4.3. *Consider a distributional solution $\gamma(t)$, $t \in [0, T]$, of the Boltzmann equation. Then for any $k \in \mathbb{N}$, $k \geq 1$, there holds*

$$\begin{aligned}
& \left(i\partial_t + \frac{1}{2} (\Delta_x - \Delta_{x'}) \right) \left(\langle x - x' \rangle^{2k} \gamma(t) \right) = \\
& = B \left(\langle x - x' \rangle^{2k} \gamma(t), \gamma(t) \right) + B^+ \left(\gamma(t), \langle x - x' \rangle^{2k} \gamma(t) \right) + \\
& + \sum_{\substack{j_1+j_2+j_3=k \\ j_1 \neq k \\ j_2 \neq k}} \binom{k}{j_1, j_2, j_3} (-1)^{j_3} B^+ \left(\langle x - x' \rangle^{2j_1} \gamma(t), \langle x - x' \rangle^{2j_2} \gamma(t) \right) + \\
& + 2k (\nabla_x + \nabla_{x'}) \cdot \left((x - x') \langle x - x' \rangle^{2k-2} \gamma(t) \right)
\end{aligned} \tag{69}$$

Additionally, for any $k \in \mathbb{N}$, $k \geq 1$, there holds

$$\begin{aligned}
& \left(i\partial_t + \frac{1}{2} (\Delta_x - \Delta_{x'}) \right) \left((x - x') \langle x - x' \rangle^{2k-2} \gamma(t) \right) = \\
& = B \left((x - x') \langle x - x' \rangle^{2k-2} \gamma(t), \gamma(t) \right) + \\
& + B^+ \left(\gamma(t), (x - x') \langle x - x' \rangle^{2k-2} \gamma(t) \right) + \\
& + \sum_{\substack{j_1+j_2+j_3=k-1 \\ j_1 \neq k-1}} \binom{k-1}{j_1, j_2, j_3} (-1)^{j_3} B^+ \left(\begin{array}{c} (x - x') \langle x - x' \rangle^{2j_1} \gamma(t), \\ \langle x - x' \rangle^{2j_2} \gamma(t) \end{array} \right) + \\
& + \sum_{\substack{j_1+j_2+j_3=k-1 \\ j_2 \neq k-1}} \binom{k-1}{j_1, j_2, j_3} (-1)^{j_3} B^+ \left(\begin{array}{c} \langle x - x' \rangle^{2j_1} \gamma(t), \\ (x - x') \langle x - x' \rangle^{2j_2} \gamma(t) \end{array} \right) + \\
& + (\nabla_x + \nabla_{x'}) \left(\langle x - x' \rangle^{2k-2} \gamma(t) \right) + \\
& + (2k - 2) \left(\frac{x - x'}{\langle x - x' \rangle} \cdot (\nabla_x + \nabla_{x'}) \right) \left((x - x') \langle x - x' \rangle^{2k-3} \gamma(t) \right)
\end{aligned} \tag{70}$$

Proof. This computation follows by using (41)-(44). \square

Remark 4.1. *Note carefully that $(x - x') \langle x - x' \rangle^{2k-2} \gamma(t)$ is a complex vector field.*

Let us introduce the following notation:

$$\zeta(t) = B(\gamma(t), \gamma(t)) \tag{71}$$

$$\gamma_{k,-}(t, x, x') = \langle x - x' \rangle^k \gamma(t, x, x') \tag{72}$$

$$\check{\zeta}_{k,-}(t) = B(\gamma_{k,-}(t), \gamma(t)) + B^+(\gamma(t), \gamma_{k,-}(t)) \tag{73}$$

$$\check{\gamma}_{k,-} = (x - x') \langle x - x' \rangle^{k-1} \gamma(t, x, x') \tag{74}$$

$$\check{\zeta}_{k,-}(t) = B(\check{\gamma}_{k,-}(t), \gamma(t)) + B^+(\gamma(t), \check{\gamma}_{k,-}(t)) \tag{75}$$

Proposition 4.4. *Let $\gamma(t)$ be a solution of Boltzmann's equation with **constant** collision kernel, $\mathbf{b} \equiv \text{cst.}$, satisfying the following bounds on some time interval $[0, T]$:*

$$\|\gamma(t)\|_{L_T^\infty H^{\alpha,\beta}} < \infty \quad (76)$$

$$\|B(\gamma(t), \gamma(t))\|_{L_T^1 H^{\alpha,\beta}} < \infty \quad (77)$$

with $\alpha, \beta > \frac{d-1}{2}$. Further assume that, for some integer $K > 0$ with $2K < \alpha - \frac{d-1}{2}$, for all $1 \leq k \leq K$ there holds

$$\|\gamma_{2k,-}(0)\|_{H^{\alpha-2k,\beta}} < \infty \quad (78)$$

Then for all $1 \leq k \leq K$ we have

$$\|\gamma_{2k,-}(t)\|_{L_T^\infty H^{\alpha-2k,\beta}} < \infty. \quad (79)$$

Moreover there is an explicit bound on $\sup_{1 \leq k \leq K} \|\gamma_{2k,-}(t)\|_{L_T^\infty H^{\alpha-2k,\beta}}$ that only depends on $\|\gamma(t)\|_{L_T^\infty H^{\alpha,\beta}}$, $\|B(\gamma(t), \gamma(t))\|_{L_T^1 H^{\alpha,\beta}}$, and $\sup_{1 \leq k \leq K} \|\gamma_{2k,-}(0)\|_{H^{\alpha-2k,\beta}}$.

Proof. We will prove the result assuming T is small. To prove the general result, it suffices to split the whole time interval $[0, T]$ into small sub-intervals and iterate the argument, as in Proposition 4.2.

Similar to Proposition 4.2, the main idea is to write a closed equation for the system $\{\gamma_{2k,-}, \zeta_{2k,-}, \check{\gamma}_{2k-1,-}, \check{\zeta}_{2k-1,-}\}$. Since the computations are quite involved, we only write down the main steps. We will *assume* for the induction that, if $j_1, j_2 \leq k-1$ and $j_1 + j_2 \leq k$, then

$$B^+(\gamma_{2j_1,-}, \gamma_{2j_2,-}) \in L_T^1 H^{\alpha-2k,\beta}, \quad (80)$$

and that if $j_1 \leq k-2$ and $j_1 + j_2 \leq k-1$, then

$$B^+(\check{\gamma}_{2j_1+1,-}, \gamma_{2j_2,-}) \in L_T^1 H^{\alpha-2k+1,\beta}, \quad (81)$$

and that if $j_2 \leq k-2$ and $j_1 + j_2 \leq k-1$, then

$$B^+(\gamma_{2j_1,-}, \check{\gamma}_{2j_2+1,-}) \in L_T^1 H^{\alpha-2k+1,\beta}. \quad (82)$$

These assumptions will, of course, have to be verified later (the case $k=1$ is easily checked using the facts that $\gamma \in L_T^\infty H^{\alpha,\beta}$ and $\zeta \in L_T^1 H^{\alpha,\beta}$).

We observe first that (69) is equivalent to the following system of equations (this system is *not* closed due to the presence of $\check{\gamma}_{2k-1,-}$, which is not

given to us by the inductive hypothesis):

$$\begin{aligned}
 \gamma_{2k,-}(t) &= e^{\frac{1}{2}it\Delta_{\pm}}\gamma_{2k,-}(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\zeta_{2k,-}(t_1)dt_1 \\
 &- i \sum_{\substack{j_1+j_2+j_3=k \\ j_1 \neq k \\ j_2 \neq k}} \binom{k}{j_1, j_2, j_3} (-1)^{j_3} \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} B^+ (\gamma_{2j_1,-}(t_1), \gamma_{2j_2,-}(t_1)) dt_1 \\
 &- 2ki \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} ((\nabla_x + \nabla_{x'}) \cdot \check{\gamma}_{2k-1,-}(t_1)) dt_1.
 \end{aligned} \tag{83}$$

Also using a Duhamel expression for $(\gamma_{2k,-} \otimes \gamma)(t)$, which can be obtained in a similar way as (59), we obtain:

$$\begin{aligned}
 \zeta_{2k,-}(t) &= B \left(e^{\frac{1}{2}it\Delta_{\pm}}\gamma_{2k,-}(0), e^{\frac{1}{2}it\Delta_{\pm}}\gamma(0) \right) + B^+ \left(e^{\frac{1}{2}it\Delta_{\pm}}\gamma(0), e^{\frac{1}{2}it\Delta_{\pm}}\gamma_{2k,-}(0) \right) \\
 &- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\gamma_{2k,-}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\zeta(t_1) \right) dt_1 \\
 &- i \int_0^t B^+ \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\zeta(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\gamma_{2k,-}(t_1) \right) dt_1 \\
 &- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\zeta_{2k,-}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\gamma(t_1) \right) dt_1 \\
 &- i \int_0^t B^+ \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\zeta_{2k,-}(t_1) \right) dt_1 \\
 &- i \sum_{\substack{j_1+j_2+j_3=k \\ j_1 \neq k \\ j_2 \neq k}} \binom{k}{j_1, j_2, j_3} (-1)^{j_3} \times \\
 &\quad \times \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} B^+ (\gamma_{2j_1,-}(t_1), \gamma_{2j_2,-}(t_1)), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\gamma(t_1) \right) dt_1 \\
 &- i \sum_{\substack{j_1+j_2+j_3=k \\ j_1 \neq k \\ j_2 \neq k}} \binom{k}{j_1, j_2, j_3} (-1)^{j_3} \times \\
 &\quad \times \int_0^t B^+ \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} B^+ (\gamma_{2j_1,-}(t_1), \gamma_{2j_2,-}(t_1)) \right) dt_1 \\
 &- 2ki \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} ((\nabla_x + \nabla_{x'}) \cdot \check{\gamma}_{2k-1,-}(t_1)), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\gamma(t_1) \right) dt_1 \\
 &- 2ki \int_0^t B^+ \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}}\gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} ((\nabla_x + \nabla_{x'}) \cdot \check{\gamma}_{2k-1,-}(t_1)) \right) dt_1
 \end{aligned} \tag{84}$$

Arguing as in Proposition 4.2, and applying Proposition 3.1, we deduce the following estimates:

$$\begin{aligned}
\|\gamma_{2k,-}\|_{L_T^\infty H^{\alpha-2k,\beta}} &\leq \|\gamma_{2k,-}(0)\|_{H^{\alpha-2k,\beta}} + \|\zeta_{2k,-}\|_{L_T^1 H^{\alpha-2k,\beta}} + \\
&+ C_k \sup_{\substack{j_1+j_2 \leq k \\ j_1 \neq k \\ j_2 \neq k}} \|B^+(\gamma_{2j_1,-}, \gamma_{2j_2,-})\|_{L_T^1 H^{\alpha-2k,\beta}} + \\
&+ 2kT \|\check{\gamma}_{2k-1,-}\|_{L_T^\infty H^{\alpha-(2k-1),\beta}},
\end{aligned} \tag{85}$$

$$\begin{aligned}
\|\zeta_{2k,-}\|_{L_T^1 H^{\alpha-2k,\beta}} &\leq CT^{\frac{1}{2}} \|\gamma_{2k,-}(0)\|_{H^{\alpha-2k,\beta}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} + \\
&+ CT^{\frac{1}{2}} \|\gamma_{2k,-}\|_{L_T^\infty H^{\alpha-2k,\beta}} \|\zeta\|_{L_T^1 H^{\alpha,\beta}} + \\
&+ CT^{\frac{1}{2}} \|\zeta_{2k,-}\|_{L_T^1 H^{\alpha-2k,\beta}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} + \\
&+ C_k T^{\frac{1}{2}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} \sup_{\substack{j_1+j_2 \leq k \\ j_1 \neq k \\ j_2 \neq k}} \|B^+(\gamma_{2j_1,-}, \gamma_{2j_2,-})\|_{L_T^1 H^{\alpha-2k,\beta}} + \\
&+ C_k T^{\frac{3}{2}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} \|\check{\gamma}_{2k-1,-}\|_{L_T^\infty H^{\alpha-(2k-1),\beta}}.
\end{aligned} \tag{86}$$

Observe that (85)-(86) does *not* yield a closed estimate in terms of $\gamma_{j,-}$ (with $j \leq 2k-2$) precisely because of the terms involving $\check{\gamma}_{2k-1,-}$. This is why we have to solve (70) simultaneously with (69).

Obviously the equation (70) yields a system of equations for the pair $\{\check{\gamma}_{2k-1,-}, \check{\zeta}_{2k-1,-}\}$, but this system is very cumbersome to write down. Instead, we will simply note the resulting estimates:

$$\begin{aligned}
\|\check{\gamma}_{2k-1,-}\|_{L_T^\infty H^{\alpha-(2k-1),\beta}} &\leq \|\check{\gamma}_{2k-1,-}(0)\|_{H^{\alpha-(2k-1),\beta}} + \\
&+ \|\check{\zeta}_{2k-1,-}\|_{L_T^1 H^{\alpha-(2k-1),\beta}} + \\
&+ C_k \sup_{\substack{j_1+j_2 \leq k-1 \\ j_1 \neq k-1}} \|B^+(\check{\gamma}_{2j_1+1,-}, \gamma_{2j_2,-})\|_{L_T^1 H^{\alpha-(2k-1),\beta}} + \\
&+ C_k \sup_{\substack{j_1+j_2 \leq k-1 \\ j_2 \neq k-1}} \|B^+(\gamma_{2j_1,-}, \check{\gamma}_{2j_2+1,-})\|_{L_T^1 H^{\alpha-(2k-1),\beta}} + \\
&+ CT \|\gamma_{2k-2,-}\|_{L_T^\infty H^{\alpha-(2k-2),\beta}} + \\
&+ C_{\alpha,\beta}(2k-2)T \|\check{\gamma}_{2k-2,-}\|_{L_T^\infty H^{\alpha-(2k-2),\beta}}.
\end{aligned} \tag{87}$$

$$\begin{aligned}
& \left\| \check{\zeta}_{2k-1,-} \right\|_{L_T^1 H^{\alpha-(2k-1),\beta}} \leq CT^{\frac{1}{2}} \|\check{\gamma}_{2k-1,-}(0)\|_{H^{\alpha-(2k-1),\beta}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} + \\
& + CT^{\frac{1}{2}} \|\check{\gamma}_{2k-1,-}\|_{L_T^\infty H^{\alpha-(2k-1),\beta}} \|\zeta\|_{L_T^1 H^{\alpha,\beta}} + \\
& + CT^{\frac{1}{2}} \left\| \check{\zeta}_{2k-1,-} \right\|_{L_T^1 H^{\alpha-(2k-1),\beta}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} + \\
& + C_k T^{\frac{1}{2}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} \sup_{\substack{j_1+j_2 \leq k-1 \\ j_1 \neq k-1}} \|B^+(\check{\gamma}_{2j_1+1,-}, \gamma_{2j_2,-})\|_{L_T^1 H^{\alpha-(2k-1),\beta}} + \\
& + C_k T^{\frac{1}{2}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} \sup_{\substack{j_1+j_2 \leq k-1 \\ j_2 \neq k-1}} \|B^+(\gamma_{2j_1,-}, \check{\gamma}_{2j_2+1,-})\|_{L_T^1 H^{\alpha-(2k-1),\beta}} + \\
& + CT^{\frac{3}{2}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} \|\gamma_{2k-2,-}\|_{L_T^\infty H^{\alpha-(2k-2),\beta}} + \\
& + C_{\alpha,\beta}(2k-2)T^{\frac{3}{2}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} \|\check{\gamma}_{2k-2,-}\|_{L_T^\infty H^{\alpha-(2k-2),\beta}}.
\end{aligned} \tag{88}$$

Combining (87)-(88) and using (81)-(82), we can conclude the for sufficiently small T depending only on $\|\gamma\|_{L_T^\infty H^{\alpha,\beta}}$ and $\|B(\gamma, \gamma)\|_{L_T^1 H^{\alpha,\beta}}$, we have

$$\check{\gamma}_{2k-1,-} \in L_T^\infty H^{\alpha-(2k-1),\beta} \tag{89}$$

$$\check{\zeta}_{2k-1,-} \in L_T^1 H^{\alpha-(2k-1),\beta}. \tag{90}$$

Now we can combine (85)-(86) with (89) and (80) to conclude that

$$\gamma_{2k,-} \in L_T^\infty H^{\alpha-2k,\beta} \tag{91}$$

$$\zeta_{2k,-} \in L_T^1 H^{\alpha-2k,\beta}. \tag{92}$$

Finally, we must verify the assumptions (80-82) which were used in the inductive process. The point here is that it is not enough to prove that $\gamma_{2k,-} \in L_T^\infty H^{\alpha-2k,\beta}$ and $\check{\gamma}_{2k-1,-} \in L_T^\infty H^{\alpha-(2k-1),\beta}$, because we do not have continuity bounds for the operator B^+ itself. Rather it is essential to use the facts that $\zeta_{2k,-} \in L_T^1 H^{\alpha-2k,\beta}$ and $\check{\zeta}_{2k-1,-} \in L_T^1 H^{\alpha-(2k-1),\beta}$; fortunately, these bounds are provided to us by the induction itself, so we may conclude. \square

5. PERSISTENCE OF REGULARITY

Now the question is as follows: suppose we have a solution $\gamma(t)$ in $H^{\alpha,\beta}$ on a maximal time interval $[0, T)$, and suppose further that $\gamma(0) \in H^{\alpha_1,\beta_1}$ for some $\alpha_1 \geq \alpha$ and $\beta_1 \geq \beta$. Then there is a maximal solution $\gamma_1(t)$ in H^{α_1,β_1} which exists on a time interval $[0, T_1)$ with $\gamma_1(0) = \gamma(0)$. Clearly γ_1 coincides with γ on $[0, T_1)$, and in particular $T_1 \leq T$. Can we say that $T_1 = T$? The results in this section answer this question in the affirmative when \mathbf{b} is bounded and $(\alpha_1 - \alpha)$ is an integer.

Proposition 5.1. *Let $\gamma(t)$ be a solution of Boltzmann's equation with $\|\mathbf{b}\|_\infty < \infty$, and suppose $\gamma \in L_T^\infty H^{\alpha,\beta}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha,\beta}$ for some $\alpha, \beta >$*

$\frac{d-1}{2}$, and further suppose that $\gamma(0) \in H^{\alpha+r,\beta}$ for some $r \in \mathbb{N}$. Then $\gamma \in L_T^\infty H^{\alpha+r,\beta}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha+r,\beta}$.

Proof. To begin, notice that we have local well-posedness in $H^{\alpha+r,\beta}$, and $\gamma(0) \in H^{\alpha+r,\beta}$; this suffices to justify our formal computations. All we have to show is that $\gamma(t)$ remains bounded in $H^{\alpha+r,\beta}$ on the full time interval $[0, T]$. By an iteration in time, it suffices to prove the result for T small enough depending only on $\|\gamma\|_{L_T^\infty H^{\alpha,\beta}}$ and $\|B(\gamma, \gamma)\|_{L_T^1 H^{\alpha,\beta}}$.

The proof follows by a simple induction using Lemma 3.3, combined with Proposition 3.1 and the fact that $(\nabla_x + \nabla_{x'})$ commutes with the free propagator $e^{it\Delta_\pm}$. For notational convenience, we denote by $\partial_*^{\mathbf{k}}$ a multi-index of the following form:

$$\left(\partial_{x_1} + \partial_{x'_1}\right)^{k_1} \left(\partial_{x_2} + \partial_{x'_2}\right)^{k_2} \dots \left(\partial_{x_d} + \partial_{x'_d}\right)^{k_d} \quad (93)$$

with $k_1 + k_2 + \dots + k_d = |\mathbf{k}|$. Differentiating Boltzmann's equation and using Lemma 3.3, we have

$$\begin{aligned} \left(i\partial_t + \frac{1}{2}(\Delta_x - \Delta_{x'})\right) \left(\partial_*^{\mathbf{k}}\gamma\right) &= B\left(\partial_*^{\mathbf{k}}\gamma, \gamma\right) + B\left(\gamma, \partial_*^{\mathbf{k}}\gamma\right) + \\ &+ \sum_{\substack{\mathbf{c} \leq \mathbf{k} \\ \mathbf{c} \neq 0 \\ \mathbf{c} \neq \mathbf{k}}} \frac{\mathbf{k}!}{\mathbf{c}!(\mathbf{k}-\mathbf{c})!} B\left(\partial_*^{\mathbf{c}}\gamma, \partial_*^{\mathbf{k}-\mathbf{c}}\gamma\right) \end{aligned} \quad (94)$$

where $\mathbf{k}! = k_1!k_2!\dots k_d!$. We assume for the induction that if $\mathbf{c} \leq \mathbf{k}$, $\mathbf{c} \neq 0$, and $\mathbf{c} \neq \mathbf{k}$, then

$$B\left(\partial_*^{\mathbf{c}}\gamma, \partial_*^{\mathbf{k}-\mathbf{c}}\gamma\right) \in L_T^1 H^{\alpha,\beta} \quad (95)$$

This assertion will eventually be justified by the induction (note that the base case $|\mathbf{k}| = 1$ is trivial).

Let us define

$$\zeta_{\mathbf{k}}(t) = B\left(\partial_*^{\mathbf{k}}\gamma(t), \gamma(t)\right) + B\left(\gamma(t), \partial_*^{\mathbf{k}}\gamma(t)\right) \quad (96)$$

Then (94) is equivalent to the following system of equations:

$$\begin{aligned} \partial_*^{\mathbf{k}}\gamma(t) &= e^{\frac{1}{2}it\Delta_\pm} \partial_*^{\mathbf{k}}\gamma(0) - i \int_0^T e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta_{\mathbf{k}}(t_1) dt_1 \\ &- i \sum_{\substack{\mathbf{c} \leq \mathbf{k} \\ \mathbf{c} \neq 0 \\ \mathbf{c} \neq \mathbf{k}}} \frac{\mathbf{k}!}{\mathbf{c}!(\mathbf{k}-\mathbf{c})!} \int_0^T e^{\frac{1}{2}i(t-t_1)\Delta_\pm} B\left(\partial_*^{\mathbf{c}}\gamma(t_1), \partial_*^{\mathbf{k}-\mathbf{c}}\gamma(t_1)\right) dt_1 \end{aligned} \quad (97)$$

$$\begin{aligned}
\zeta_{\mathbf{k}}(t) &= \\
&= B \left(e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0), e^{\frac{1}{2}it\Delta_{\pm}} \partial_{*}^{\mathbf{k}} \gamma(0) \right) + B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \partial_{*}^{\mathbf{k}} \gamma(0), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(0) \right) \\
&- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \partial_{*}^{\mathbf{k}} \gamma(t_1) \right) dt_1 \\
&- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \partial_{*}^{\mathbf{k}} \gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta(t_1) \right) dt_1 \\
&- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta_{\mathbf{k}}(t_1) \right) dt_1 \\
&- i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta_{\mathbf{k}}(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) dt_1 \\
&- i \sum_{\substack{\mathbf{c} \leq \mathbf{k} \\ \mathbf{c} \neq 0 \\ \mathbf{c} \neq \mathbf{k}}} \frac{\mathbf{k}!}{\mathbf{c}!(\mathbf{k}-\mathbf{c})!} \times \\
&\quad \times \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} B \left(\partial_{*}^{\mathbf{c}} \gamma(t_1), \partial_{*}^{\mathbf{k}-\mathbf{c}} \gamma(t_1) \right) \right) dt_1 \\
&- i \sum_{\substack{\mathbf{c} \leq \mathbf{k} \\ \mathbf{c} \neq 0 \\ \mathbf{c} \neq \mathbf{k}}} \frac{\mathbf{k}!}{\mathbf{c}!(\mathbf{k}-\mathbf{c})!} \times \\
&\quad \times \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} B \left(\partial_{*}^{\mathbf{c}} \gamma(t_1), \partial_{*}^{\mathbf{k}-\mathbf{c}} \gamma(t_1) \right), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) dt_1.
\end{aligned} \tag{98}$$

We deduce the following estimates using Proposition 3.1:

$$\begin{aligned}
\left\| \partial_{*}^{\mathbf{k}} \gamma \right\|_{L_T^{\infty} H^{\alpha, \beta}} &\leq \left\| \partial_{*}^{\mathbf{k}} \gamma(0) \right\|_{H^{\alpha, \beta}} + \|\zeta_{\mathbf{k}}\|_{L_T^1 H^{\alpha, \beta}} + \\
&\quad + C_{\mathbf{k}} \sup_{\substack{\mathbf{c} \leq \mathbf{k} \\ \mathbf{c} \neq 0 \\ \mathbf{c} \neq \mathbf{k}}} \left\| B \left(\partial_{*}^{\mathbf{c}} \gamma, \partial_{*}^{\mathbf{k}-\mathbf{c}} \gamma \right) \right\|_{L_T^1 H^{\alpha, \beta}}
\end{aligned} \tag{99}$$

$$\begin{aligned}
\|\zeta_{\mathbf{k}}\|_{L_T^1 H^{\alpha, \beta}} &\leq CT^{\frac{1}{2}} \|\gamma\|_{L_T^{\infty} H^{\alpha, \beta}} \left\| \partial_{*}^{\mathbf{k}} \gamma(0) \right\|_{H^{\alpha, \beta}} + \\
&\quad + CT^{\frac{1}{2}} \left\| \partial_{*}^{\mathbf{k}} \gamma \right\|_{L_T^{\infty} H^{\alpha, \beta}} \|\zeta\|_{L_T^1 H^{\alpha, \beta}} + \\
&\quad + CT^{\frac{1}{2}} \|\gamma\|_{L_T^{\infty} H^{\alpha, \beta}} \|\zeta_{\mathbf{k}}\|_{L_T^1 H^{\alpha, \beta}} + \\
&\quad + C_{\mathbf{k}} T^{\frac{1}{2}} \|\gamma\|_{L_T^{\infty} H^{\alpha, \beta}} \sup_{\substack{\mathbf{c} \leq \mathbf{k} \\ \mathbf{c} \neq 0 \\ \mathbf{c} \neq \mathbf{k}}} \left\| B \left(\partial_{*}^{\mathbf{c}} \gamma, \partial_{*}^{\mathbf{k}-\mathbf{c}} \gamma \right) \right\|_{L_T^1 H^{\alpha, \beta}}.
\end{aligned} \tag{100}$$

Combining (99)-(100) with (95), and choosing T sufficiently small depending only on $\|\gamma\|_{L_T^{\infty} H^{\alpha, \beta}}$ and $\|\zeta\|_{L_T^1 H^{\alpha, \beta}}$ (and noting that $\|\zeta\|_{L_T^1 H^{\alpha, \beta}}$ scales at worst

like $T^{\frac{1}{2}}$ for T small), we obtain:

$$\partial_*^{\mathbf{k}}\gamma \in L_T^\infty H^{\alpha,\beta} \quad (101)$$

$$\zeta_{\mathbf{k}} \in L_T^1 H^{\alpha,\beta} \quad (102)$$

Letting \mathbf{k} range over multi-indices of magnitude $|\mathbf{k}|$, we are able to conclude that $\gamma \in L_T^\infty H^{\alpha+|\mathbf{k}|,\beta}$.

Finally we must verify the assertion (95) to use for the next step of the induction. This follows immediately from Proposition 3.1 combined with the facts that $\partial_*^{\mathbf{k}}\gamma \in L_T^\infty H^{\alpha,\beta}$ and $\zeta_{\mathbf{k}} \in L_T^1 H^{\alpha,\beta}$. \square

Proposition 5.2. *Let $\gamma(t)$ be a solution of Boltzmann's equation with $\|\mathbf{b}\|_\infty < \infty$, and suppose $\gamma \in L_T^\infty H^{\alpha,\beta}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha,\beta}$ for some $\alpha, \beta > \frac{d-1}{2}$, and further suppose that $\gamma(0) \in H^{\alpha,\beta+r}$ for some $r > 0$. Then $\gamma \in L_T^\infty H^{\alpha,\beta+r}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha,\beta+r}$.*

Proof. As usual, it suffices to prove the result for a small time depending on $\|\gamma\|_{L_T^\infty H^{\alpha,\beta}}$ and $\|B(\gamma, \gamma)\|_{L_T^1 H^{\alpha,\beta}}$. Furthermore, we will only prove the result for small values of r (with smallness only depending on d, α, β), as allowed by Proposition 3.1; the general result then follows immediately.

We know γ solves Boltzmann's equation,

$$\left(i\partial_t + \frac{1}{2}(\Delta_x - \Delta_{x'}) \right) \gamma = B(\gamma, \gamma) \quad (103)$$

This equation is equivalent to the following system of equations, where we write $\zeta(t) = B(\gamma(t), \gamma(t))$:

$$\gamma(t) = e^{\frac{1}{2}it\Delta_\pm} \gamma(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta(t_1) dt_1 \quad (104)$$

$$\begin{aligned} \zeta(t) &= B \left(e^{\frac{1}{2}it\Delta_\pm} \gamma(0), e^{\frac{1}{2}it\Delta_\pm} \gamma(0) \right) \\ &\quad - i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta(t_1) \right) dt_1 \\ &\quad - i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \gamma(t_1) \right) dt_1 \end{aligned} \quad (105)$$

Applying Proposition 3.1 with a small $\delta > 0$, we obtain:

$$\|\gamma\|_{L_T^\infty H^{\alpha,\beta+\delta}} \leq \|\gamma(0)\|_{H^{\alpha,\beta+\delta}} + \|\zeta\|_{L_T^1 H^{\alpha,\beta+\delta}} \quad (106)$$

$$\begin{aligned} \|\zeta\|_{L_T^1 H^{\alpha,\beta+\delta}} &\leq CT^{\frac{1}{2}} \|\gamma(0)\|_{H^{\alpha,\beta+\delta}} \|\gamma(0)\|_{H^{\alpha,\beta}} + \\ &\quad + CT^{\frac{1}{2}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta+\delta}} \|\zeta\|_{L_T^1 H^{\alpha,\beta}} + CT^{\frac{1}{2}} \|\zeta\|_{L_T^1 H^{\alpha,\beta+\delta}} \|\gamma\|_{L_T^\infty H^{\alpha,\beta}} \end{aligned} \quad (107)$$

Since $\|\zeta\|_{L_T^1 H^{\alpha,\beta}}$ scales at worst like $\mathcal{O}(T^{\frac{1}{2}})$, we easily deduce that $\gamma \in L_T^\infty H^{\alpha,\beta+\delta}$ and $\zeta \in L_T^1 H^{\alpha,\beta+\delta}$, as soon as T is chosen sufficiently small depending only on the norm of γ in $H^{\alpha,\beta}$. \square

6. REGULARITY IN TIME

In this section we discuss regularity in the *time* variable. This should seem to be a very simple matter due to the obvious formula

$$\partial_t B(\gamma, \gamma) = B(\partial_t \gamma, \gamma) + B(\gamma, \partial_t \gamma) \quad (108)$$

and the fact that ∂_t commutes with $(i\partial_t + \frac{1}{2}(\Delta_x - \Delta_{x'}))$. The difficulty which arises can already be seen when we try to control $\partial_t \gamma$ in $L_T^\infty H^{\alpha, \beta}$ (in fact we will need to control higher derivatives $\partial_t^k \gamma$ to eventually prove that γ is locally C^r in (t, x, x')). Indeed we have the following closed equation for $\partial_t \gamma$:

$$\left(i\partial_t + \frac{1}{2}(\Delta_x - \Delta_{x'}) \right) (\partial_t \gamma) = B(\partial_t \gamma, \gamma) + B(\gamma, \partial_t \gamma) \quad (109)$$

We can re-cast this system as a closed pair of integral equations for the functions $\{\partial_t \gamma, \partial_t \zeta\}$ where $\zeta = B(\gamma, \gamma)$, exactly as we have done previously to propagate regularity and moments in all other variables.

However, to solve the system, we will at the very least need to know that $(\partial_t \gamma)(0) \in H^{\alpha, \beta}$, which means

$$\frac{i}{2}(\Delta_x - \Delta_{x'})\gamma(0) - iB(\gamma(0), \gamma(0)) \in H^{\alpha, \beta} \quad (110)$$

For the first term it is enough to suppose that $\gamma(0) \in H^{\alpha+1, \beta+1}$ (say), but the second term is tricky because Proposition 3.1 does not provide bounds for $B(\gamma, \gamma)$ in the spaces $H^{\alpha, \beta}$. Only quantities involving the free propagator, such as $B\left(e^{\frac{1}{2}it\Delta_\pm}\gamma(0), e^{\frac{1}{2}it\Delta_\pm}\gamma(0)\right)$, are controlled via Proposition 3.1. To resolve this problem, in Appendix A we prove bilinear estimates for $B(\gamma, \gamma)$ *without* a free propagator; the price we must pay is a small loss in the β index for the gain term (and we must also assume $\alpha, \beta > \frac{d}{2}$). Due to the loss coming from Proposition A.1, it is not possible to use that bound in place of Proposition 3.1 elsewhere in this paper.

Proposition 6.1. *Let $\gamma(t)$ be a solution of Boltzmann's equation with $\|\mathbf{b}\|_\infty < \infty$, and suppose $\gamma \in L_T^\infty H^{\alpha, \beta}$ and $B(\gamma, \gamma) \in L_T^1 H^{\alpha, \beta}$ for some $\alpha, \beta > \frac{d-1}{2}$. Further suppose that $K > 0$ is an integer with $K < \min(\alpha, \beta) - \frac{d}{2}$. Then for all $1 \leq k \leq K$ there holds $\partial_t^k \gamma \in L_T^\infty H^{\alpha-k, \beta-k}$.*

Proof. We will prove the desired result on a small time interval $T > 0$ depending only on the underlying solution $\gamma(t)$ of Boltzmann's equation; the general result then follows immediately.

For any integer $k \geq 1$ we have

$$\begin{aligned} \left(i\partial_t + \frac{1}{2}(\Delta_x - \Delta_{x'}) \right) \left(\partial_t^k \gamma \right) &= B(\partial_t^k \gamma, \gamma) + B(\gamma, \partial_t^k \gamma) + \\ &+ \sum_{0 < j < k} \binom{k}{j} B(\partial_t^{k-j} \gamma, \partial_t^j \gamma) \end{aligned} \quad (111)$$

Let us define

$$\zeta_k(t) = B(\partial_t^k \gamma(t), \gamma(t)) + B(\gamma(t), \partial_t^k \gamma(t)) \quad (112)$$

Then (111) is equivalent to the following system of equations:

$$\begin{aligned} \partial_t^k \gamma(t) &= e^{\frac{1}{2}it\Delta_\pm} \partial_t^k \gamma(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta_k(t_1) dt_1 \\ &\quad - i \sum_{0 < j < k} \binom{k}{j} \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_\pm} B\left(\partial_t^{k-j} \gamma(t_1), \partial_t^j \gamma(t_1)\right) dt_1 \end{aligned} \quad (113)$$

$$\begin{aligned} \zeta_k(t) &= B\left(e^{\frac{1}{2}it\Delta_\pm} \partial_t^k \gamma(0), e^{\frac{1}{2}it\Delta_\pm} \gamma(0)\right) + B\left(e^{\frac{1}{2}it\Delta_\pm} \gamma(0), e^{\frac{1}{2}it\Delta_\pm} \partial_t^k \gamma(0)\right) \\ &\quad - i \int_0^t B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta_k(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \gamma(t_1)\right) dt_1 \\ &\quad - i \int_0^t B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \partial_t^k \gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta(t_1)\right) dt_1 \\ &\quad - i \int_0^t B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \partial_t^k \gamma(t_1)\right) dt_1 \\ &\quad - i \int_0^t B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta_k(t_1)\right) dt_1 \\ &\quad - i \sum_{0 < j < k} \binom{k}{j} \times \\ &\quad \quad \times \int_0^t B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm} B\left(\partial_t^{k-j} \gamma(t_1), \partial_t^j \gamma(t_1)\right), e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \gamma(t_1)\right) dt_1 \\ &\quad - i \sum_{0 < j < k} \binom{k}{j} \times \\ &\quad \quad \times \int_0^t B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm} B\left(\partial_t^{k-j} \gamma(t_1), \partial_t^j \gamma(t_1)\right)\right) dt_1 \end{aligned} \quad (114)$$

Hence by applying Proposition (3.1), we obtain the following estimates:

$$\begin{aligned} \left\| \partial_t^k \gamma(t) \right\|_{L_T^\infty H^{\alpha-k, \beta-k}} &\leq \left\| \partial_t^k \gamma(0) \right\|_{H^{\alpha-k, \beta-k}} + \|\zeta_k\|_{L_T^1 H^{\alpha-k, \beta-k}} \\ &\quad + \sum_{0 < j < k} \binom{k}{j} \left\| B\left(\partial_t^{k-j} \gamma(t_1), \partial_t^j \gamma(t_1)\right) \right\|_{L_T^1 H^{\alpha-k, \beta-k}} \end{aligned} \quad (115)$$

$$\begin{aligned} \|\zeta_k\|_{L_T^1 H^{\alpha-k, \beta-k}} &\leq CT^{\frac{1}{2}} \left\| \partial_t^k \gamma(0) \right\|_{H^{\alpha-k, \beta-k}} \|\gamma(0)\|_{H^{\alpha, \beta}} + \\ &\quad + CT^{\frac{1}{2}} \|\zeta_k\|_{L_T^1 H^{\alpha-k, \beta-k}} \|\gamma\|_{L_T^\infty H^{\alpha, \beta}} + CT^{\frac{1}{2}} \left\| \partial_t^k \gamma \right\|_{L_T^\infty H^{\alpha-k, \beta-k}} \|\zeta\|_{L_T^1 H^{\alpha, \beta}} + \\ &\quad + C_k T^{\frac{1}{2}} \|\gamma\|_{L_T^\infty H^{\alpha, \beta}} \sup_{0 < j < k} \left\| B\left(\partial_t^{k-j} \gamma(t), \partial_t^j \gamma(t)\right) \right\|_{L_T^1 H^{\alpha-k, \beta-k}} \end{aligned} \quad (116)$$

Now if we assume that $\partial_t^k \gamma(0) \in H^{\alpha-k, \beta-k}$, $\partial_t^j \gamma \in L_T^\infty H^{\alpha-j, \beta-j}$ for $0 \leq j < k$, and $\zeta_j \in L_T^1 H^{\alpha-j, \beta-j}$ for $0 \leq j < k$, then we can show that $\partial_t^k \gamma \in L_T^\infty H^{\alpha-k, \beta-k}$ and $\zeta_k \in L_T^1 H^{\alpha-k, \beta-k}$.

It only remains to verify that $\partial_t^k \gamma(0) \in H^{\alpha-k, \beta-k}$. To see this, observe that

$$i\partial_t^k \gamma + \frac{1}{2}(\Delta_x - \Delta_{x'})\partial_t^{k-1} \gamma = \sum_{0 \leq j \leq k-1} \binom{k-1}{j} B(\partial_t^{k-1-j} \gamma, \partial_t^j \gamma) \quad (117)$$

Now since $\partial_t^{k-1} \gamma \in L_T^\infty H^{\alpha-k+1, \beta-k+1}$ (in fact it is continuous in time in this functional space), we have $(\Delta_x - \Delta_{x'})\left(\partial_t^{k-1} \gamma\right)(0) \in H^{\alpha-k, \beta-k}$. Additionally, since $\left(\partial_t^j \gamma(0)\right) \in H^{\alpha-j, \beta-j}$ for $0 \leq j < k$, by Proposition A.1, we find that $B\left(\partial_t^{k-1-j} \gamma(0), \partial_t^j \gamma(0)\right) \in H^{\alpha-k, \beta-k}$ when $0 \leq j \leq k-1$. \square

7. CONTINUITY OF THE SOLUTION MAP

It is sometimes useful to be able to approximate a given initial data by some other, better behaved, initial data for the purpose of formal computations. In order to pass the results of computations to the limit and reach a non-void conclusion about the original data, it is necessary to know that the solution map is sufficiently smooth with respect to perturbations. Such an argument is apparently necessary to prove the non-negativity of solutions to Boltzmann's equation in the spaces $H^{\alpha, \beta}$ for arbitrary $\alpha, \beta > \frac{d-1}{2}$, because we have no convenient characterization of non-negativity of f just from looking at the (inverse) Wigner transform γ .

Proposition 7.1. *Let $\gamma^j(t)$ be a solution of Boltzmann's equation with $\|\mathbf{b}\|_\infty < \infty$, for $j = 1, 2$, with $\gamma^j \in L_T^\infty H^{\alpha, \beta}$ and $B(\gamma^j, \gamma^j) \in L_T^1 H^{\alpha, \beta}$ for $j = 1, 2$ and some $\alpha, \beta > \frac{d-1}{2}$. Furthermore, assume that $\|\gamma^j\|_{L_T^\infty H^{\alpha, \beta}} \leq M$ for $j = 1, 2$. Then we have*

$$\|\gamma^1 - \gamma^2\|_{L_T^\infty H^{\alpha, \beta}} \leq C_{M, T} \|\gamma^1(0) - \gamma^2(0)\|_{H^{\alpha, \beta}} \quad (118)$$

where the constant may depend on α, β .

Proof. We will prove the result for T small enough depending only on the upper bound M ; the full result then follows by iterating in time.

We recall that each γ^i solves Boltzmann's equation:

$$\left(i\partial_t + \frac{1}{2}(\Delta_x - \Delta_{x'})\right) \gamma^j = B(\gamma^j, \gamma^j) \quad (119)$$

This is equivalent to the following system of equations, where $\zeta^j(t) = B(\gamma^j(t), \gamma^j(t))$:

$$\gamma^j(t) = e^{\frac{1}{2}it\Delta_\pm} \gamma^j(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_\pm} \zeta^j(t_1) dt_1 \quad (120)$$

$$\begin{aligned}
\zeta^j(t) &= B \left(e^{\frac{1}{2}it\Delta_{\pm}} \gamma^j(0), e^{\frac{1}{2}it\Delta_{\pm}} \gamma^j(0) \right) \\
&\quad - i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma^j(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta^j(t_1) \right) dt_1 \\
&\quad - i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta^j(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma^j(t_1) \right) dt_1
\end{aligned} \tag{121}$$

Let us define

$$\gamma^r(t) = \gamma^1(t) - \gamma^2(t) \tag{122}$$

$$\zeta^r(t) = \zeta^1(t) - \zeta^2(t) \tag{123}$$

Then we can write the following system of equations for γ^r, ζ^r :

$$\begin{aligned}
\gamma^r(t) &= e^{\frac{1}{2}it\Delta_{\pm}} \gamma^r(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta^r(t_1) dt_1 \\
\zeta^r(t) &= B \left(e^{\frac{1}{2}it\Delta_{\pm}} \gamma^1(0), e^{\frac{1}{2}it\Delta_{\pm}} \gamma^r(0) \right) + B \left(e^{\frac{1}{2}it\Delta_{\pm}} \gamma^r(0), e^{\frac{1}{2}it\Delta_{\pm}} \gamma^2(0) \right) \\
&\quad - i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma^1(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta^r(t_1) \right) dt_1 \\
&\quad - i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma^r(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta^2(t_1) \right) dt_1 \\
&\quad - i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta^1(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma^r(t_1) \right) dt_1 \\
&\quad - i \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta^r(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma^2(t_1) \right) dt_1
\end{aligned} \tag{124}$$

Using Proposition 3.1, we deduce the following estimates:

$$\begin{aligned}
\|\gamma^r\|_{L_T^{\infty} H^{\alpha, \beta}} &\leq \|\gamma^r(0)\|_{H^{\alpha, \beta}} + \|\zeta^r\|_{L_T^1 H^{\alpha, \beta}} \\
\|\zeta^r\|_{L_T^1 H^{\alpha, \beta}} &\leq CT^{\frac{1}{2}} \left(\|\gamma^1\|_{L_T^{\infty} H^{\alpha, \beta}} + \|\gamma^2\|_{L_T^{\infty} H^{\alpha, \beta}} \right) \|\gamma^r(0)\|_{H^{\alpha, \beta}} + \\
&\quad + CT^{\frac{1}{2}} \|\gamma^1\|_{L_T^{\infty} H^{\alpha, \beta}} \|\zeta^r\|_{L_T^1 H^{\alpha, \beta}} + CT^{\frac{1}{2}} \|\gamma^r\|_{L_T^{\infty} H^{\alpha, \beta}} \|\zeta^2\|_{L_T^1 H^{\alpha, \beta}} + \\
&\quad + CT^{\frac{1}{2}} \|\zeta^1\|_{L_T^1 H^{\alpha, \beta}} \|\gamma^r\|_{L_T^{\infty} H^{\alpha, \beta}} + CT^{\frac{1}{2}} \|\zeta^r\|_{L_T^1 H^{\alpha, \beta}} \|\gamma^2\|_{L_T^{\infty} H^{\alpha, \beta}}
\end{aligned} \tag{126}$$

Hence if we define

$$A_T = T^{\frac{1}{2}} \|\gamma^r\|_{L_T^{\infty} H^{\alpha, \beta}} + \|\zeta^r\|_{L_T^1 H^{\alpha, \beta}} \tag{127}$$

then we easily deduce

$$A_T \leq C_M T^{\frac{1}{2}} A_T + C_M T^{\frac{1}{2}} \|\gamma^r(0)\|_{H^{\alpha, \beta}} \tag{128}$$

Choosing T sufficiently small (depending only on M), the conclusion follows. \square

APPENDIX A. BILINEAR ESTIMATES WITH LOSS

The main difficulty in proving time regularity estimates in Section 6 is the fact that Proposition 3.1 only controls the collision operator in L_T^1 , whereas we must make sense of $B(\gamma, \gamma)$ at a *fixed* time (namely $t = 0$) just to write down $(\partial_t \gamma)(0)$. In order to resolve this issue, in this Appendix we prove instantaneous bounds on the operator $B(\gamma, \gamma)$ in the Sobolev spaces $H^{\alpha, \beta}$ when $\alpha, \beta > \frac{d}{2}$. The proof involves a small loss in the β index; for this reason, these estimates (as stated in this Appendix) *cannot* replace Proposition 3.1 elsewhere in this paper, regardless of the size of α, β . The idea of the proof is drawn from Theorem 4.3 of [13].

Proposition A.1. *Let $\alpha, \beta \in (\frac{d}{2}, \infty)$. Then for any $\delta > 0$ we have for all $\gamma_1, \gamma_2 \in H^{\alpha, \beta + \delta}$ the following estimates:*

$$\|B^\pm(\gamma_1, \gamma_2)\|_{H^{\alpha, \beta}} \leq C_\delta \|\mathbf{b}\|_\infty \|\gamma_1\|_{H^{\alpha, \beta + \delta}} \|\gamma_2\|_{H^{\alpha, \beta + \delta}} \quad (130)$$

Proof. We treat the loss term and gain term separately.

Loss term. For any function $F(x, x')$ we denote the Fourier transform,

$$\hat{F}(\xi, \xi') = \int dx dx' e^{-i\xi \cdot x} e^{-i\xi' \cdot x'} F(x, x') \quad (131)$$

A routine computation shows that, if $\|\mathbf{b}\|_\infty < \infty$, then

$$\left| (B^-(\gamma_1, \gamma_2))^\wedge(\xi, \xi') \right| \leq C \|\mathbf{b}\|_\infty \int d\eta d\eta' \left| \hat{\gamma}_1 \left(\xi - \frac{\eta + \eta'}{2} \right) \right| |\hat{\gamma}_2(\eta, \eta')| \quad (132)$$

Therefore we have

$$\begin{aligned} \|B^-(\gamma_1, \gamma_2)\|_{H^{\alpha, \beta}}^2 &= \int \left| (B^-(\gamma_1, \gamma_2))^\wedge(\xi, \xi') \right|^2 \langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta} d\xi d\xi' \\ &\leq C^2 \|\mathbf{b}\|_\infty^2 \int d\xi d\xi' d\eta_1 d\eta'_1 d\eta_2 d\eta'_2 \langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta} \times \\ &\quad \times \left| \hat{\gamma}_1 \left(\xi - \frac{\eta_1 + \eta'_1}{2}, \xi' - \frac{\eta_1 + \eta'_1}{2} \right) \right| |\hat{\gamma}_2(\eta_1, \eta'_1)| \times \\ &\quad \times \left| \hat{\gamma}_1 \left(\xi - \frac{\eta_2 + \eta'_2}{2}, \xi' - \frac{\eta_2 + \eta'_2}{2} \right) \right| |\hat{\gamma}_2(\eta_2, \eta'_2)| \end{aligned} \quad (133)$$

Now we multiply and divide under the integral sign by the following factor,

$$\prod_{i=1,2} \left\{ \langle \xi + \xi' - \eta_i - \eta'_i \rangle^\alpha \langle \xi - \xi' \rangle^\beta \langle \eta_i + \eta'_i \rangle^\alpha \langle \eta_i - \eta'_i \rangle^\beta \right\} \quad (134)$$

and apply the Cauchy-Schwarz inequality *pointwise* under the integral. This gives us

$$\begin{aligned}
\|B^-(\gamma_1, \gamma_2)\|_{H^{\alpha, \beta}}^2 &\leq C^2 \|\mathbf{b}\|_\infty^2 \int d\xi d\xi' d\eta_1 d\eta_1' d\eta_2 d\eta_2' \times \\
&\quad \times \frac{\langle \xi + \xi' \rangle^{2\alpha}}{\langle \xi + \xi' - \eta_2 - \eta_2' \rangle^{2\alpha} \langle \eta_2 + \eta_2' \rangle^{2\alpha} \langle \eta_2 - \eta_2' \rangle^{2\beta}} \times \\
&\quad \times \left| \langle \xi + \xi' - \eta_1 - \eta_1' \rangle^\alpha \langle \xi - \xi' \rangle^\beta \hat{\gamma}_1 \left(\xi - \frac{\eta_1 + \eta_1'}{2}, \xi' - \frac{\eta_1 + \eta_1'}{2} \right) \right|^2 \times \\
&\quad \times \left| \langle \eta_1 + \eta_1' \rangle^\alpha \langle \eta_1 - \eta_1' \rangle^\beta \hat{\gamma}_2(\eta_1, \eta_1') \right|^2
\end{aligned} \tag{135}$$

Observe now that if

$$I \equiv \sup_{\xi, \xi'} \int d\eta d\eta' \frac{\langle \xi + \xi' \rangle^{2\alpha}}{\langle \xi + \xi' - \eta_2 - \eta_2' \rangle^{2\alpha} \langle \eta_2 + \eta_2' \rangle^{2\alpha} \langle \eta_2 - \eta_2' \rangle^{2\beta}} < \infty \tag{136}$$

then we conclude

$$\|B^-(\gamma_1, \gamma_2)\|_{H^{\alpha, \beta}} \leq C \|\mathbf{b}\|_\infty \|\gamma_1\|_{H^{\alpha, \beta}} \|\gamma_2\|_{H^{\alpha, \beta}} \tag{137}$$

The bound (136) is equivalent to the following estimate:

$$\sup_{W \in \mathbb{R}^d} \int_{\mathbb{R}^d} dw \frac{\langle W \rangle^{2\alpha}}{\langle W - w \rangle^{2\alpha} \langle w \rangle^{2\alpha}} \int_{\mathbb{R}^d} dz \frac{1}{\langle z \rangle^{2\beta}} < \infty \tag{138}$$

It is easy to verify that the bound (138) holds whenever $\alpha, \beta > \frac{d}{2}$.

Gain term. By a routine calculation, if $\|\mathbf{b}\|_\infty < \infty$, we have:

$$\begin{aligned}
&\left| (B^+(\gamma_1, \gamma_2))^\wedge(\xi, \xi') \right| \leq \\
&\leq C \|\mathbf{b}\|_\infty \int_{\mathbb{S}^{d-1}} d\omega \int d\eta_1 d\eta_1' d\eta_2 d\eta_2' \hat{\gamma}_1(\eta_1, \eta_1') \hat{\gamma}_2(\eta_2, \eta_2') \times \\
&\quad \times \delta \left(-\xi + \eta_1 + \frac{\eta_2 + \eta_2'}{2} - \frac{1}{2} P_\omega(\eta_1 - \eta_1') + \frac{1}{2} P_\omega(\eta_2 - \eta_2') \right) \times \\
&\quad \times \delta \left(-\xi' + \eta_1' + \frac{\eta_2 + \eta_2'}{2} + \frac{1}{2} P_\omega(\eta_1 - \eta_1') - \frac{1}{2} P_\omega(\eta_2 - \eta_2') \right)
\end{aligned} \tag{139}$$

Performing changes of variables as in [11], this gives:

$$\begin{aligned}
&\left| (B^+(\gamma_1, \gamma_2))^\wedge(\xi, \xi') \right| \leq \\
&\leq C \|\mathbf{b}\|_\infty \int_{\mathbb{S}^{d-1}} d\omega \int ds_1 ds_2 \times \\
&\quad \times \hat{\gamma}_1 \left(s_1 + 2s_2^\parallel + \frac{3\xi - \xi'}{4}, s_1 - 2s_2^\parallel + \frac{3\xi' - \xi}{4} \right) \times \\
&\quad \times \hat{\gamma}_2 \left(-s_1 - 2s_2^\perp + \frac{3\xi - \xi'}{4}, -s_1 + 2s_2^\perp + \frac{3\xi' - \xi}{4} \right)
\end{aligned} \tag{140}$$

where $s_2^\parallel = P_\omega(s_2)$ and $s_2^\perp = (\mathbb{I} - P_\omega)(s_2)$.

Reasoning as for the loss term, if we can show that the integral

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d \times \mathbb{R}^d} ds_1 ds_2 \times \\ & \times \frac{\langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta}}{\langle 2s_1 + \frac{\xi + \xi'}{2} \rangle^{2\alpha} \langle 4s_2^\parallel + \xi - \xi' \rangle^{2(\beta+\delta)} \langle -2s_1 + \frac{\xi + \xi'}{2} \rangle^{2\alpha} \langle -4s_2^\perp + \xi - \xi' \rangle^{2(\beta+\delta)}} \end{aligned} \quad (141)$$

is bounded uniformly with respect to $\xi, \xi' \in \mathbb{R}^d$, then we will have the estimate

$$\|B^+(\gamma_1, \gamma_2)\|_{H^{\alpha, \beta}} \leq C \|\mathbf{b}\|_\infty \|\gamma_1\|_{H^{\alpha, \beta+\delta}} \|\gamma_2\|_{H^{\alpha, \beta+\delta}} \quad (142)$$

In fact it suffices to show the following two bounds:

$$\sup_{W \in \mathbb{R}^d} \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\alpha}}{\langle s \rangle^{2\alpha} \langle W - s \rangle^{2\alpha}} < \infty \quad (143)$$

$$\sup_{W \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{\langle s^\parallel + W^\perp \rangle^{2(\beta+\delta)} \langle s^\perp + W^\parallel \rangle^{2(\beta+\delta)}} < \infty \quad (144)$$

It is easy to verify that (143) holds whenever $\alpha > \frac{d}{2}$, so we only have to prove (144).

We easily bound the integral (144) with respect to $s \in \mathbb{R}^d$ when $\beta > \frac{d-1}{2}$ by decomposing $ds = ds^\parallel ds^\perp$. Then all we must show is that

$$\sup_{W \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega \langle W \rangle^{2\beta} \langle W^\parallel \rangle^{d-1-2(\beta+\delta)} \langle W^\perp \rangle^{1-2(\beta+\delta)} < \infty \quad (145)$$

Now we may decompose

$$\langle W \rangle^{2\beta} \lesssim \langle W^\parallel \rangle^{2\beta} + \langle W^\perp \rangle^{2\beta} \quad (146)$$

Therefore it suffices to bound the following two integrals:

$$I = \int_{\mathbb{S}^{d-1}} d\omega \langle W^\parallel \rangle^{d-1-2\delta} \langle W^\perp \rangle^{1-2(\beta+\delta)} \quad (147)$$

$$I' = \int_{\mathbb{S}^{d-1}} d\omega \langle W^\parallel \rangle^{d-1-2(\beta+\delta)} \langle W^\perp \rangle^{1-2\delta} \quad (148)$$

We can bound both integrals using a dyadic decomposition, as in [11], whenever $\beta > \frac{d}{2}$, as follows:

$$\begin{aligned} I & \lesssim \sum_{k=1}^{\infty} \int_{\omega: 2^{-k-1}|W^\parallel| \leq |W^\perp| < 2^{-k}|W^\parallel|} d\omega \langle W^\parallel \rangle^{d-1-2\delta} \langle W^\perp \rangle^{1-2(\beta+\delta)} \\ & \lesssim \sum_{k=1}^{\infty} 2^{-k-1} \times (2^{-k})^{d-2} \times (2^{k+1})^{d-1-2\delta} < \infty \end{aligned} \quad (149)$$

$$\begin{aligned}
I' &\lesssim \sum_{k=1}^{\infty} \int_{\omega: 2^{-k-1}|W^\perp| \leq |W^\parallel| < 2^{-k}|W^\perp|} d\omega \langle W^\parallel \rangle^{d-1-2(\beta+\delta)} \langle W^\perp \rangle^{1-2\delta} \\
&\lesssim \sum_{k=1}^{\infty} 2^{-k-1} \times (2^{k+1})^{1-2\delta} < \infty
\end{aligned} \tag{150}$$

Hence we may conclude. \square

APPENDIX B. PROOF OF PROPOSITION 3.1 WHEN $\delta = 0$

In this appendix we will provide a proof of Proposition 3.1; it is based on bilinear Strichartz estimates, following the strategy of [22]. In fact, we will improve on the results of [11] by allowing exponents $\beta > \frac{d-1}{2}$ in the case of bounded collision kernels (this was claimed without proof in [11]). It is straightforward (from the proof in this Appendix) to obtain the claimed improvement in moments for the *gain term* (only) in Proposition 3.1 (i.e. $\delta > 0$).

The proof presented in this Appendix is adapted from an early manuscript of [11]. However, the proof presented here diverges from that of [11] in many details; in particular, only constant or bounded collision kernels are considered here (with a corresponding improvement in the available range of regularity in the β exponent). In fact we include only the case of *constant* collision kernel in this Appendix; the only difference with the bounded case is that we bound \mathbf{b} by its L^∞ norm after passing to the spacetime Fourier transform.

Proof. (case $\delta = 0$)

Sobolev Estimates for the Loss Term

Consider the loss term, which is as follows for a constant collision kernel:

$$B^-[\gamma_1, \gamma_2](x, x') = \gamma_1(x, x') \gamma_2\left(\frac{x+x'}{2}, \frac{x+x'}{2}\right) \tag{151}$$

We will fix some *initial data* $\gamma_1(x, x')$, $\gamma_2(x, x')$, and consider the following function (it is the action of the nonlinearity upon the free Schrödinger flow):

$$B^- \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] (t, x, x') \tag{152}$$

The spacetime Fourier transform of a function $F(t, x, x')$ is

$$\tilde{F}(\tau, \xi, \xi') = \int dt dx dx' e^{-it\tau} e^{-ix \cdot \xi} e^{-ix' \cdot \xi'} F(t, x, x') \tag{153}$$

The spacetime Fourier transform of $e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_0$ is

$$\hat{\gamma}_0(\xi, \xi') \delta\left(\tau + \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi'|^2\right) \tag{154}$$

where

$$\hat{\gamma}_0(\xi, \xi') = \int dx dx' e^{-ix \cdot \xi} e^{-ix' \cdot \xi'} \gamma_0(x, x') \tag{155}$$

We also have

$$\begin{aligned}
& \left(B^- \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] \right)^\sim (\tau, \xi, \xi') = \\
& = \int d\eta d\eta' \delta \left(\tau + \frac{1}{2} \left| \xi - \frac{\eta + \eta'}{2} \right|^2 - \frac{1}{2} \left| \xi' - \frac{\eta + \eta'}{2} \right|^2 + \frac{1}{2} |\eta|^2 - \frac{1}{2} |\eta'|^2 \right) \times \\
& \quad \times \hat{\gamma}_1 \left(\xi - \frac{\eta + \eta'}{2}, \xi' - \frac{\eta + \eta'}{2} \right) \hat{\gamma}_2(\eta, \eta')
\end{aligned} \tag{156}$$

We want to estimate the following integral, for suitable $\alpha, \beta > 0$:

$$\begin{aligned}
I_{\alpha, \beta}^- & = \int \langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta} \times \\
& \quad \times \left| \left(B^- \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] (\tau, \xi, \xi') \right)^\sim \right|^2 d\tau d\xi d\xi'
\end{aligned} \tag{157}$$

To start, observe that

$$\begin{aligned}
I_{\alpha, \beta}^- & \leq \int d\tau d\xi d\xi' d\eta_1 d\eta'_1 d\eta_2 d\eta'_2 \langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta} \times \\
& \quad \times \delta \left(\tau + \frac{1}{2} \left| \xi - \frac{\eta_1 + \eta'_1}{2} \right|^2 - \frac{1}{2} \left| \xi' - \frac{\eta_1 + \eta'_1}{2} \right|^2 + \frac{1}{2} |\eta_1|^2 - \frac{1}{2} |\eta'_1|^2 \right) \times \\
& \quad \times \delta \left(\tau + \frac{1}{2} \left| \xi - \frac{\eta_2 + \eta'_2}{2} \right|^2 - \frac{1}{2} \left| \xi' - \frac{\eta_2 + \eta'_2}{2} \right|^2 + \frac{1}{2} |\eta_2|^2 - \frac{1}{2} |\eta'_2|^2 \right) \times \\
& \quad \times \left| \hat{\gamma}_1 \left(\xi - \frac{\eta_1 + \eta'_1}{2}, \xi' - \frac{\eta_1 + \eta'_1}{2} \right) \right| |\hat{\gamma}_2(\eta_1, \eta'_1)| \times \\
& \quad \times \left| \hat{\gamma}_1 \left(\xi - \frac{\eta_2 + \eta'_2}{2}, \xi' - \frac{\eta_2 + \eta'_2}{2} \right) \right| |\hat{\gamma}_2(\eta_2, \eta'_2)|
\end{aligned}$$

Multiply and divide the integrand by the following factor:

$$\prod_{j=1}^2 \left\{ \langle \xi + \xi' - \eta_j - \eta'_j \rangle^\alpha \langle \xi - \xi' \rangle^\beta \langle \eta_j + \eta'_j \rangle^\alpha \langle \eta_j - \eta'_j \rangle^\beta \right\} \tag{158}$$

Then group terms together and apply Cauchy-Schwarz *pointwise* under the integral sign. We obtain two different terms that are equal due to symmetry

under re-labeling coordinates; hence,

$$\begin{aligned}
I_{\alpha,\beta}^- &\leq \int d\tau d\xi d\xi' d\eta_1 d\eta_1' d\eta_2 d\eta_2' \times \\
&\times \frac{\langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta}}{\langle \xi + \xi' - \eta_1 - \eta_1' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta} \langle \eta_1 + \eta_1' \rangle^{2\alpha} \langle \eta_1 - \eta_1' \rangle^{2\beta}} \times \\
&\times \delta \left(\tau + \frac{1}{2} \left| \xi - \frac{\eta_1 + \eta_1'}{2} \right|^2 - \frac{1}{2} \left| \xi' - \frac{\eta_1 + \eta_1'}{2} \right|^2 + \frac{1}{2} |\eta_1|^2 - \frac{1}{2} |\eta_1'|^2 \right) \times \\
&\times \delta \left(\tau + \frac{1}{2} \left| \xi - \frac{\eta_2 + \eta_2'}{2} \right|^2 - \frac{1}{2} \left| \xi' - \frac{\eta_2 + \eta_2'}{2} \right|^2 + \frac{1}{2} |\eta_2|^2 - \frac{1}{2} |\eta_2'|^2 \right) \times \\
&\times \left| \langle \xi + \xi' - \eta_2 - \eta_2' \rangle^\alpha \langle \xi - \xi' \rangle^\beta \hat{\gamma}_1 \left(\xi - \frac{\eta_2 + \eta_2'}{2}, \xi' - \frac{\eta_2 + \eta_2'}{2} \right) \right|^2 \times \\
&\times \left| \langle \eta_2 + \eta_2' \rangle^\alpha \langle \eta_2 - \eta_2' \rangle^\beta \hat{\gamma}_2(\eta_2, \eta_2') \right|^2
\end{aligned}$$

The integral completely factorizes in the following way:

$$\begin{aligned}
I_{\alpha}^- &\leq \int d\tau d\xi d\xi' \left(\int d\eta_1 d\eta_1' \dots \right) \left(\int d\eta_2 d\eta_2' \dots \right) \\
&\leq \left(\sup_{\tau, \xi, \xi'} \int d\eta_1 d\eta_1' \dots \right) \times \int d\tau d\xi d\xi' \left(\int d\eta_2 d\eta_2' \dots \right)
\end{aligned}$$

Finally we are able to conclude that if the following integral,

$$\begin{aligned}
&\int d\eta d\eta' \delta \left(\tau + \frac{1}{2} \left| \xi - \frac{\eta + \eta'}{2} \right|^2 - \frac{1}{2} \left| \xi' - \frac{\eta + \eta'}{2} \right|^2 + \frac{1}{2} |\eta|^2 - \frac{1}{2} |\eta'|^2 \right) \times \\
&\quad \times \frac{\langle \xi + \xi' \rangle^{2\alpha}}{\langle \xi + \xi' - \eta - \eta' \rangle^{2\alpha} \langle \eta + \eta' \rangle^{2\alpha} \langle \eta - \eta' \rangle^{2\beta}}
\end{aligned} \tag{159}$$

is bounded *uniformly* with respect to τ, ξ, ξ' , then the following estimate holds:

$$\left\| B^- \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] \right\|_{L_t^2 H^{\alpha,\beta}} \leq C \prod_{j=1,2} \|\gamma_j\|_{H^{\alpha,\beta}} \tag{160}$$

Let us make the change of variables $w = \frac{\eta + \eta'}{2}$, $z = \frac{\eta - \eta'}{2}$ in (159); then, up to a constant, the integral becomes:

$$\begin{aligned}
&\int dw dz \delta \left(\tau + \frac{1}{2} |\xi - w|^2 - \frac{1}{2} |\xi' - w|^2 + \frac{1}{2} |w + z|^2 - \frac{1}{2} |w - z|^2 \right) \times \\
&\quad \times \frac{\langle \xi + \xi' \rangle^{2\alpha}}{\langle \xi + \xi' - 2w \rangle^{2\alpha} \langle 2w \rangle^{2\alpha} \langle 2z \rangle^{2\beta}}
\end{aligned} \tag{161}$$

This is the same as:

$$K = \int dw dz \delta \left(\tau + \frac{1}{2} (|\xi|^2 - |\xi'|^2) - (\xi - \xi' - 2z) \cdot w \right) \times \frac{\langle \xi + \xi' \rangle^{2\alpha}}{\langle \xi + \xi' - 2w \rangle^{2\alpha} \langle 2w \rangle^{2\alpha} \langle 2z \rangle^{2\beta}} \quad (162)$$

Hence, one way to parametrize the integral is to let $z \in \mathbb{R}^d$ be arbitrary and let w range over a codimension one hyperplane in \mathbb{R}^d ; the hyperplane is determined by τ, ξ, ξ', z . Alternatively, we can let $w \in \mathbb{R}^d$ be arbitrary and let z range over a different codimension one hyperplane in \mathbb{R}^d . We will choose the second option.

We have

$$K = \langle \xi + \xi' \rangle^{2\alpha} \int_{\mathbb{R}^d} \frac{dw}{2|w| \langle \xi + \xi' - 2w \rangle^{2\alpha} \langle 2w \rangle^{2\alpha}} \int_P \frac{dS(z)}{\langle 2z \rangle^{2\beta}} \quad (163)$$

where

$$P = \left\{ z \in \mathbb{R}^d \mid \tau + \frac{1}{2} (|\xi|^2 - |\xi'|^2) - (\xi - \xi') \cdot w + 2w \cdot z = 0 \right\} \quad (164)$$

and $dS(z)$ is the surface measure on P .

The integral over P is no larger than the integral over a parallel hyperplane running through the origin, for which the evaluation is very easy. We find that as long as $\beta > \frac{d-1}{2}$ the integral over P converges, uniformly in w, ξ, ξ', τ . Hence we are left with

$$K \lesssim \langle \xi + \xi' \rangle^{2\alpha} \int_{\mathbb{R}^d} \frac{dw}{2|w| \langle \xi + \xi' - 2w \rangle^{2\alpha} \langle 2w \rangle^{2\alpha}} \quad (165)$$

The integral over the set $|w| \leq 1$ is trivially bounded uniformly in ξ, ξ' . Therefore the boundedness of K uniformly in ξ, ξ' is equivalent to the boundedness of the following integral

$$K' = \langle W \rangle^{2\alpha} \int_{\mathbb{R}^d} \frac{dw}{\langle w \rangle^{2\alpha+1} \langle W - w \rangle^{2\alpha}} \quad (166)$$

uniformly in W .

The integral over $|w| < \frac{1}{2}|W|$ is trivially bounded uniformly in W if $\alpha > \frac{d-1}{2}$. Similarly the integral over $|w| > 2|W|$ is bounded uniformly in W if $\alpha > \frac{d-1}{2}$. Hence we are left with the following integral:

$$\langle W \rangle^{2\alpha} \int_{\frac{1}{2}|W| \leq |w| \leq 2|W|} \frac{dw}{\langle w \rangle^{2\alpha+1} \langle W - w \rangle^{2\alpha}} \quad (167)$$

which is obviously bounded by the following integral:

$$\langle W \rangle^{-1} \int_{\frac{1}{2}|W| \leq |w| \leq 2|W|} \frac{dw}{\langle W - w \rangle^{2\alpha}} \quad (168)$$

Shifting w to $W - w$ this is bounded by

$$\langle W \rangle^{-1} \int_{|w| \leq 3|W|} \frac{dw}{\langle w \rangle^{2\alpha}} \quad (169)$$

or even

$$\langle W \rangle^{d-1} \int_{|u| \leq 3} \frac{du}{\langle |W|u \rangle^{2\alpha}} \quad (170)$$

Splitting the last integral into the pieces $|u| \leq \frac{1}{|W|}$ and $\frac{1}{|W|} \leq |u| \leq 3$, we find that the integral K' is uniformly bounded in W if $\alpha > \frac{d-1}{2}$.

Summarizing, we have the bound:

$$\left\| B^- \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] \right\|_{L_t^2 H^{\alpha, \beta}} \leq C \prod_{j=1,2} \|\gamma_j\|_{H^{\alpha, \beta}} \quad (171)$$

as long as $\min(\alpha, \beta) > \frac{d-1}{2}$. The endpoint estimates are not achieved, with respect to either α or β , by the above argument.

Sobolev Estimates for the Gain Term

Consider the gain term, which is the following for a constant collision kernel:

$$B^+[\gamma_1, \gamma_1](t, x, x') = \int_{\mathbb{S}^{d-1}} d\omega B_\omega^+[\gamma, \gamma](t, x, x') \quad (172)$$

where

$$\begin{aligned} & B_\omega^+[\gamma_1, \gamma_2](t, x, x') = \\ & = \gamma_1 \left(t, x - \frac{1}{2}P_\omega(x - x'), x' + \frac{1}{2}P_\omega(x - x') \right) \times \\ & \quad \times \gamma_2 \left(t, \frac{x + x'}{2} + \frac{1}{2}P_\omega(x - x'), \frac{x + x'}{2} - \frac{1}{2}P_\omega(x - x') \right) \end{aligned} \quad (173)$$

The spacetime Fourier transform of the function

$$B^+ \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] (t, x, x') \quad (174)$$

is the following, up to a constant:

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} d\omega \int d\eta_1 d\eta'_1 d\eta_2 d\eta'_2 \delta \left(\tau + \frac{1}{2}|\eta_1|^2 - \frac{1}{2}|\eta'_1|^2 + \frac{1}{2}|\eta_2|^2 - \frac{1}{2}|\eta'_2|^2 \right) \times \\ & \quad \times \delta \left(-\xi + \eta_1 + \frac{\eta_2 + \eta'_2}{2} - \frac{1}{2}P_\omega(\eta_1 - \eta'_1) + \frac{1}{2}P_\omega(\eta_2 - \eta'_2) \right) \times \\ & \quad \times \delta \left(-\xi' + \eta'_1 + \frac{\eta_2 + \eta'_2}{2} + \frac{1}{2}P_\omega(\eta_1 - \eta'_1) - \frac{1}{2}P_\omega(\eta_2 - \eta'_2) \right) \times \\ & \quad \times \hat{\gamma}_1(\eta_1, \eta'_1) \hat{\gamma}_2(\eta_2, \eta'_2) \end{aligned} \quad (175)$$

Introduce the change of variables $w_1 = \frac{\eta_1 + \eta'_1}{2}$, $z_1 = \frac{\eta_1 - \eta'_1}{2}$, $w_2 = \frac{\eta_2 + \eta'_2}{2}$, $z_2 = \frac{\eta_2 - \eta'_2}{2}$. Then (175) becomes

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} d\omega \int dw_1 dz_1 dw_2 dz_2 \times \\
& \times \delta \left(\tau + \frac{1}{2}|w_1 + z_1|^2 - \frac{1}{2}|w_1 - z_1|^2 + \frac{1}{2}|w_2 + z_2|^2 - \frac{1}{2}|w_2 - z_2|^2 \right) \times \\
& \times \delta(-\xi + w_1 + z_1 + w_2 - P_\omega(z_1 - z_2)) \times \\
& \times \delta(-\xi' + w_1 - z_1 + w_2 + P_\omega(z_1 - z_2)) \times \\
& \times \hat{\gamma}_1(w_1 + z_1, w_1 - z_1) \hat{\gamma}_2(w_2 + z_2, w_2 - z_2)
\end{aligned} \tag{176}$$

Introduce yet another change of variables $r_1 = \frac{w_1 + w_2}{2}$, $s_1 = \frac{w_1 - w_2}{2}$, $r_2 = \frac{z_1 + z_2}{2}$, $s_2 = \frac{z_1 - z_2}{2}$. Then (176) becomes

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} d\omega \int dr_1 ds_1 dr_2 ds_2 \times \\
& \times \delta \left(\tau + \frac{1}{2}|r_1 + s_1 + r_2 + s_2|^2 - \frac{1}{2}|r_1 + s_1 - r_2 - s_2|^2 + \right. \\
& \quad \left. + \frac{1}{2}|r_1 - s_1 + r_2 - s_2|^2 - \frac{1}{2}|r_1 - s_1 - r_2 + s_2|^2 \right) \times \\
& \times \delta(-\xi + 2r_1 + r_2 + s_2 - 2P_\omega s_2) \times \\
& \times \delta(-\xi' + 2r_1 - r_2 - s_2 + 2P_\omega s_2) \times \\
& \times \hat{\gamma}_1(r_1 + s_1 + r_2 + s_2, r_1 + s_1 - r_2 - s_2) \times \\
& \times \hat{\gamma}_2(r_1 - s_1 + r_2 - s_2, r_1 - s_1 - r_2 + s_2)
\end{aligned} \tag{177}$$

Replace r_1 with $\frac{r_1}{2}$ throughout:

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} d\omega \int dr_1 ds_1 dr_2 ds_2 \times \\
& \times \delta \left(\tau + \frac{1}{2} \left| \frac{r_1}{2} + s_1 + r_2 + s_2 \right|^2 - \frac{1}{2} \left| \frac{r_1}{2} + s_1 - r_2 - s_2 \right|^2 + \right. \\
& \quad \left. + \frac{1}{2} \left| \frac{r_1}{2} - s_1 + r_2 - s_2 \right|^2 - \frac{1}{2} \left| \frac{r_1}{2} - s_1 - r_2 + s_2 \right|^2 \right) \times \\
& \times \delta(-\xi + r_1 + r_2 + s_2 - 2P_\omega s_2) \times \\
& \times \delta(-\xi' + r_1 - r_2 - s_2 + 2P_\omega s_2) \times \\
& \times \hat{\gamma}_1 \left(\frac{r_1}{2} + s_1 + r_2 + s_2, \frac{r_1}{2} + s_1 - r_2 - s_2 \right) \times \\
& \times \hat{\gamma}_2 \left(\frac{r_1}{2} - s_1 + r_2 - s_2, \frac{r_1}{2} - s_1 - r_2 + s_2 \right)
\end{aligned} \tag{178}$$

Finally perform the change of variables $\zeta_1 = r_1 + r_2$, $\zeta_2 = r_1 - r_2$:

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} d\omega \int d\zeta_1 d\zeta_2 ds_1 ds_2 \times \\
& \times \delta \left(\tau + \frac{1}{2} \left| \frac{3\zeta_1}{4} - \frac{\zeta_2}{4} + s_1 + s_2 \right|^2 - \frac{1}{2} \left| -\frac{\zeta_1}{4} + \frac{3\zeta_2}{4} + s_1 - s_2 \right|^2 + \right. \\
& \quad \left. + \frac{1}{2} \left| \frac{3\zeta_1}{4} - \frac{\zeta_2}{4} - s_1 - s_2 \right|^2 - \frac{1}{2} \left| -\frac{\zeta_1}{4} + \frac{3\zeta_2}{4} - s_1 + s_2 \right|^2 \right) \times \\
& \times \delta(-\xi + \zeta_1 + s_2 - 2P_\omega s_2) \times \\
& \times \delta(-\xi' + \zeta_2 - s_2 + 2P_\omega s_2) \times \\
& \times \hat{\gamma}_1 \left(\frac{3\zeta_1}{4} - \frac{\zeta_2}{4} + s_1 + s_2, -\frac{\zeta_1}{4} + \frac{3\zeta_2}{4} + s_1 - s_2 \right) \times \\
& \times \hat{\gamma}_2 \left(\frac{3\zeta_1}{4} - \frac{\zeta_2}{4} - s_1 - s_2, -\frac{\zeta_1}{4} + \frac{3\zeta_2}{4} - s_1 + s_2 \right)
\end{aligned} \tag{179}$$

Now we can integrate out the variables ζ_1, ζ_2 to obtain:

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} d\omega \int ds_1 ds_2 \times \\
& \times \delta \left(\tau + \frac{1}{2} \left| s_1 + 2s_2^\parallel + \frac{3\xi - \xi'}{4} \right|^2 - \frac{1}{2} \left| s_1 - 2s_2^\parallel + \frac{3\xi' - \xi}{4} \right|^2 + \right. \\
& \quad \left. + \frac{1}{2} \left| -s_1 - 2s_2^\perp + \frac{3\xi - \xi'}{4} \right|^2 - \frac{1}{2} \left| -s_1 + 2s_2^\perp + \frac{3\xi' - \xi}{4} \right|^2 \right) \times \\
& \times \hat{\gamma}_1 \left(s_1 + 2s_2^\parallel + \frac{3\xi - \xi'}{4}, s_1 - 2s_2^\parallel + \frac{3\xi' - \xi}{4} \right) \times \\
& \times \hat{\gamma}_2 \left(-s_1 - 2s_2^\perp + \frac{3\xi - \xi'}{4}, -s_1 + 2s_2^\perp + \frac{3\xi' - \xi}{4} \right)
\end{aligned} \tag{180}$$

where $s_2^\parallel = P_\omega s_2$ and $s_2^\perp = s_2 - P_\omega s_2$.

We want to estimate the following integral, for suitable $\alpha, \beta > 0$:

$$\begin{aligned}
I_{\alpha, \beta}^+ &= \int \langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta} \times \\
& \times \left| \left(B^+ \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] (\tau, \xi, \xi') \right) \right|^2 d\tau d\xi d\xi'
\end{aligned} \tag{181}$$

Reasoning as for the loss term, if we can show that the following integral

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} d\omega \int ds_1 ds_2 \times \\
& \times \delta \left(\tau + \frac{1}{2} \left| s_1 + 2s_2^\parallel + \frac{3\xi - \xi'}{4} \right|^2 - \frac{1}{2} \left| s_1 - 2s_2^\parallel + \frac{3\xi' - \xi}{4} \right|^2 + \right. \\
& \quad \left. + \frac{1}{2} \left| -s_1 - 2s_2^\perp + \frac{3\xi - \xi'}{4} \right|^2 - \frac{1}{2} \left| -s_1 + 2s_2^\perp + \frac{3\xi' - \xi}{4} \right|^2 \right) \times \\
& \times \frac{\langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta}}{\left\langle 2s_1 + \frac{\xi + \xi'}{2} \right\rangle^{2\alpha} \left\langle 4s_2^\parallel + \xi - \xi' \right\rangle^{2\beta} \left\langle -2s_1 + \frac{\xi + \xi'}{2} \right\rangle^{2\alpha} \left\langle -4s_2^\perp + \xi - \xi' \right\rangle^{2\beta}}
\end{aligned} \tag{182}$$

is bounded uniformly in τ, ξ, ξ' , then we will have the following estimate:

$$\left\| B^+ \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] \right\|_{L_t^2 H^{\alpha, \beta}} \leq C \prod_{j=1,2} \|\gamma_j\|_{H^{\alpha, \beta}} \tag{183}$$

The integral (182) is equivalent to the following integral:

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} d\omega \int ds_1 ds_2 \times \\
& \times \delta \left(\tau + \frac{1}{2} (|\xi|^2 - |\xi'|^2) + (4s_1 - R_\omega(\xi + \xi')) \cdot s_2 \right) \times \\
& \times \frac{\langle \xi + \xi' \rangle^{2\alpha} \langle \xi - \xi' \rangle^{2\beta}}{\left\langle 2s_1 + \frac{\xi + \xi'}{2} \right\rangle^{2\alpha} \left\langle 4s_2^\parallel + \xi - \xi' \right\rangle^{2\beta} \left\langle -2s_1 + \frac{\xi + \xi'}{2} \right\rangle^{2\alpha} \left\langle -4s_2^\perp + \xi - \xi' \right\rangle^{2\beta}}
\end{aligned} \tag{184}$$

where $R_\omega(u) = u - 2P_\omega u$ is reflection about the plane perpendicular to ω . This is in turn equivalent to the following integral:

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} d\omega \int ds_2 \frac{\langle \xi - \xi' \rangle^{2\beta}}{|4s_2| \left\langle 4s_2^\parallel + \xi - \xi' \right\rangle^{2\beta} \left\langle -4s_2^\perp + \xi - \xi' \right\rangle^{2\beta}} \times \\
& \quad \times \int_P dS(s_1) \frac{\langle \xi + \xi' \rangle^{2\alpha}}{\left\langle 2s_1 + \frac{\xi + \xi'}{2} \right\rangle^{2\alpha} \left\langle -2s_1 + \frac{\xi + \xi'}{2} \right\rangle^{2\alpha}}
\end{aligned} \tag{185}$$

where $P \subset \mathbb{R}^d$ is the following codimension one hyperplane:

$$P = \left\{ s_1 \in \mathbb{R}^d \mid \tau + \frac{1}{2} (|\xi|^2 - |\xi'|^2) + (4s_1 - R_\omega(\xi + \xi')) \cdot s_2 = 0 \right\} \tag{186}$$

Therefore we only need to show the boundedness of the following two quantities uniformly in ξ, ξ', τ :

$$I_1 = \sup_{P \subset \mathbb{R}^d: \dim P = d-1} \int_P dS(s) \frac{\langle \xi + \xi' \rangle^{2\alpha}}{\langle 2s + \frac{\xi + \xi'}{2} \rangle^{2\alpha} \langle -2s + \frac{\xi + \xi'}{2} \rangle^{2\alpha}} \quad (187)$$

$$I_2 = \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle \xi - \xi' \rangle^{2\beta}}{|4s| \langle 4s^\parallel + \xi - \xi' \rangle^{2\beta} \langle -4s^\perp + \xi - \xi' \rangle^{2\beta}} \quad (188)$$

Let us first consider the integral I_2 ; here we will assume that $\beta > \frac{d-1}{2}$. Clearly, I_2 is equivalent to the following quantity:

$$I_2 \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle \xi - \xi' \rangle^{2\beta}}{|s| \langle s^\parallel + \xi - \xi' \rangle^{2\beta} \langle s^\perp + \xi - \xi' \rangle^{2\beta}} \quad (189)$$

Setting $W = \xi - \xi'$, this gives:

$$I_2 \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{|s| \langle s^\parallel + W \rangle^{2\beta} \langle s^\perp + W \rangle^{2\beta}} \quad (190)$$

Moreover, since the integral for $|s| \leq 1$ is obviously uniformly bounded in W , we may instead bound the following integral:

$$I_2' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{\langle s \rangle \langle s^\parallel + W \rangle^{2\beta} \langle s^\perp + W \rangle^{2\beta}} \quad (191)$$

Since $|s^\parallel| \leq |s|$ we have:

$$I_2' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{\langle s^\parallel \rangle \langle s^\parallel + W \rangle^{2\beta} \langle s^\perp + W \rangle^{2\beta}} \quad (192)$$

Therefore, for all large enough $|W|$,

$$\begin{aligned} I_2' &\lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{\langle s^\parallel \rangle \langle s^\parallel + W \rangle^{2\beta} \langle s^\perp + W \rangle^{2\beta}} \\ &= \int_{\mathbb{S}^{d-1}} d\omega \langle W \rangle^{2\beta} \left(\int \frac{ds^\parallel}{\langle s^\parallel \rangle \langle s^\parallel + W \rangle^{2\beta}} \right) \left(\int \frac{ds^\perp}{\langle s^\perp + W \rangle^{2\beta}} \right) \\ &\lesssim \int_{\mathbb{S}^{d-1}} d\omega \langle W \rangle^{2\beta} \left(\langle W \rangle^{-1} \langle W^\perp \rangle^{1-2\beta} \log \langle W \rangle \right) \left(\langle W^\parallel \rangle^{d-1-2\beta} \right) \end{aligned} \quad (193)$$

Note that the integral over s^\perp is estimated by a trivial computation, whereas the integral over s^\parallel may be estimated by considering separately the regions $|s^\parallel| \leq \frac{1}{2}|W|$, $|s^\parallel| > 2|W|$, and $\frac{1}{2}|W| < |s^\parallel| \leq 2|W|$. The integral over $|s^\parallel| \leq \frac{1}{2}|W|$ yields the logarithmic divergence, whereas the integral over $\frac{1}{2}|W| < |s^\parallel| \leq 2|W|$ provides the explicit dependence on W^\perp .

We find that, for any fixed $\varepsilon > 0$, I'_2 obeys the following estimate:

$$I'_2 \lesssim \int_{\mathbb{S}^{d-1}} d\omega \langle W \rangle^{2\beta-1+\varepsilon} \langle W^\perp \rangle^{1-2\beta} \langle W^\parallel \rangle^{d-1-2\beta} \quad (194)$$

Then we have

$$\langle W \rangle^{2\beta-1+\varepsilon} \lesssim \langle W^\parallel \rangle^{2\beta-1+\varepsilon} + \langle W^\perp \rangle^{2\beta-1+\varepsilon} \quad (195)$$

Hence $I'_2 \lesssim I''_2 + I'''_2$ where

$$I''_2 = \int_{\mathbb{S}^{d-1}} d\omega \langle W^\perp \rangle^{1-2\beta} \langle W^\parallel \rangle^{d-2+\varepsilon} \quad (196)$$

$$I'''_2 = \int_{\mathbb{S}^{d-1}} d\omega \langle W^\perp \rangle^\varepsilon \langle W^\parallel \rangle^{d-1-2\beta} \quad (197)$$

Then for any $\beta > \frac{d-1}{2}$, for some sufficiently small $\varepsilon > 0$ depending on β , both I''_2 and I'''_2 may be bounded using dyadic decompositions in the angular parameter ω , as follows: neglecting additive constants,

$$\begin{aligned} I''_2 &\lesssim \sum_{k=1}^{\infty} \int_{\omega: 2^{-k-1}|W^\parallel| \leq |W^\perp| < 2^{-k}|W^\parallel|} d\omega \langle W^\perp \rangle^{1-2\beta} \langle W^\parallel \rangle^{d-2+\varepsilon} \\ &\lesssim \sum_{k=1}^{\infty} 2^{-k-1} \times (2^{-k})^{d-2} \times (2^{k+1})^{d-2+\varepsilon} < \infty \end{aligned} \quad (198)$$

$$\begin{aligned} I'''_2 &\lesssim \sum_{k=1}^{\infty} \int_{\omega: 2^{-k-1}|W^\perp| \leq |W^\parallel| < 2^{-k}|W^\perp|} d\omega \langle W^\perp \rangle^\varepsilon \langle W^\parallel \rangle^{d-1-2\beta} \\ &\lesssim \sum_{k=1}^{\infty} 2^{-k-1} \times (2^{k+1})^\varepsilon < \infty \end{aligned} \quad (199)$$

The factor of $(2^{-k})^{d-2}$ in I''_2 comes from the Jacobian for spherical coordinates in \mathbb{R}^d .

We now turn to I_1 , which is clearly bounded by the following quantity:

$$I_1 \lesssim \sup_{W \in \mathbb{R}^d} \sup_{P \subset \mathbb{R}^d: \dim P = d-1} \int_P dS(s) \frac{\langle W \rangle^{2\alpha}}{\langle s \rangle^{2\alpha} \langle s + W \rangle^{2\alpha}} \quad (200)$$

The integrals over $P \cap \{|s| < \frac{1}{2}|W|\}$, $P \cap \{|s| > 2|W|\}$, and $P \cap \{\frac{1}{2}|W| \leq |s| \leq 2|W|\}$ are each easily bounded uniformly in W as long as $\alpha > \frac{d-1}{2}$.

To conclude, for any parameters α, β such that $\min(\alpha, \beta) > \frac{d-1}{2}$, we have the following estimate:

$$\left\| B^+ \left[e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_1, e^{\frac{1}{2}it(\Delta_x - \Delta_{x'})} \gamma_2 \right] \right\|_{L_t^2 H^{\alpha, \beta}} \leq C \prod_{j=1,2} \|\gamma_j\|_{H^{\alpha, \beta}} \quad (201)$$

The endpoint estimates are not achieved, with respect to either α or β , by the above argument. \square

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