# Concentration in flux-function limits of solutions to a deposition model <sup>∗</sup>

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Abstract This paper is concerned with a singular flux-function limit of the Riemann solutions to a deposition model. As a result, it is shown that the Riemann solutions to the deposition model just converge to the corresponding Riemann solutions to the limit system, which is one of typical models admitting delta-shocks. Especially, the phenomenon of concentration and the formation of delta-shocks in the limit are analyzed in detail, and the process of concentration is numerically simulated.

keywords Deposition model; Flux-function limits; Concentration; Delta-shocks; Numerical simulations.

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#### 1. Introduction

We consider the following deposition model

<span id="page-0-0"></span>
$$
\begin{cases}\nv_t + (uv)_x = 0, \\
u_t + (u^2 + \epsilon v)_x = 0,\n\end{cases}
$$
\n(1.1)

where  $v \ge 0$  is the density of the population performing the deposition,  $u = -\partial_x h$  with  $h = h(x, t)$ being the deposition height, and  $\epsilon$  is a positive parameter. The first equation describes the conservation of total population. The second one is derived from the rules governing the time evolution of the deposition system: (1) the deposition consists of population-generating deposition and self-generating deposition; (2) the population is driven by a velocity field proportional to the negative gradient of height. Besides, model  $(1.1)$  can also be derived as the hydrodynamic limit of some randomly growing interface models [\[9,](#page-16-0) [33\]](#page-17-0). The system [\(1.1\)](#page-0-0) is also called as the Leroux system in the PDE literature [\[18,](#page-16-1) [24\]](#page-17-1). For some investigations concerning [\(1.1\)](#page-0-0), see [\[25,](#page-17-2) [10,](#page-16-2) [20,](#page-16-3) [21\]](#page-16-4), etc.

One can notice that as  $\epsilon \to 0^+$ , the model [\(1.1\)](#page-0-0) formally becomes the following system

<span id="page-0-1"></span>
$$
\begin{cases}\n u_t + (u^2)_x = 0, \\
 v_t + (vu)_x = 0.\n\end{cases}
$$
\n(1.2)

This is one of very typical models in the literature with respect to delta-shocks [\[1,](#page-15-0) [28,](#page-17-3) [31,](#page-17-4) [8,](#page-16-5) [16,](#page-16-6) [15,](#page-16-7) [34,](#page-17-5) [6,](#page-16-8) [26,](#page-17-6) [4,](#page-15-1) [5\]](#page-15-2), an interesting topic. It is a mathematical simplification of Euler equations of gas dynamics and can be obtained by setting density and pressure to be constant in the momentum conservation laws. It also has some physical interpretations. For instance, it can be used to model the flow of particles with u being the velocity and v the density. In 1977, Korchinski  $[17]$  considered  $(1.2)$  in

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his unpublished Ph.D. thesis. Motivated by his numerical study, he used a kind of generalized deltafunction in the construction of his unique solution to the Riemann problem. Afterwards, in 1994, Tan et al. [\[30\]](#page-17-7) found that in the Riemann problem for [\(1.2\)](#page-0-1), no classical weak solution exists and deltashocks should be introduced for some initial data. With the delta-shocks, they solved the Riemann problem completely. Under some reasonable second order viscous approximations, the stability of delta-shocks for  $(1.2)$  was also proved in [\[30,](#page-17-7) [29,](#page-17-8) [12\]](#page-16-10).

The main purpose of this paper is to study the behaviors of solutions of system [\(1.1\)](#page-0-0) as the flux  $\epsilon v$ vanishes (that is,  $\epsilon \to 0^+$ ) by the Riemann problem. We are especially concerned with the phenomenon of concentration and the formation of delta-shocks in the limit.

Firstly, we consider the Riemann problem for [\(1.1\)](#page-0-0) with initial data

<span id="page-1-0"></span>
$$
(u, v)(x, t = 0) = (u_{\pm}, v_{\pm}), \quad \pm x > 0,
$$
\n
$$
(1.3)
$$

where  $v_{\pm} > 0$ . System [\(1.1\)](#page-0-0) is nonstrictly hyperbolic, and both characteristic fields are genuinely nonlinear. The elementary waves include shocks and rarefaction waves, and [\(1.1\)](#page-0-0) belongs to the socalled Temple class [\[32\]](#page-17-9). By the analysis method in phase plane, the unique global Riemann solution is constructed with four different kinds of structures containing shock(s) and/or rarefaction wave(s).

Secondly, we study the behaviors of solutions of system  $(1.1)$  as the flux  $\epsilon v$  vanishes by the Riemann problem. As a result, it is rigorously shown that as  $\epsilon \to 0^+$ , the Riemann solutions to [\(1.1\)](#page-0-0) just converge to the Riemann solutions to [\(1.2\)](#page-0-1) with the same initial data. Especially, when  $u_+ \leq 0 \leq u_-,$ the two-shock solution to  $(1.1)$  and  $(1.3)$  tends to the delta-shock solution to  $(1.2)$  and  $(1.3)$ , where the intermediate density between the two shocks tends to a weighted  $\delta$ -measure which forms the deltashock. Further, the process of concentration is numerically simulated. It can be seen that such a flux-function limit may be very singular: the limit functions of solutions are no longer in the spaces of functions BV or  $L^{\infty}$ , and the space of Radon measures, for which the divergences of certain entropy and entropy flux fields are also Radon measures, is a natural space in order to deal with such a limit.

Let us remark that in the past more than 10 years, more attention has been paid on the investigation of phenomenon of concentration and the formation of delta-shocks in solutions to hyperbolic systems of conservation laws. Li [\[19\]](#page-16-11) and Chen and Liu [\[2,](#page-15-3) [3\]](#page-15-4) identified and analyzed the phenomenon of concentration and the formation of delta-shocks in solutions to the Euler equations for both isentropic and nonisentropic fluids as the pressure vanishes. Yin and Sheng [\[37,](#page-17-10) [38\]](#page-17-11) extended the studies to the relativistic Euler equations. With respect to this topic, also see [\[22,](#page-16-12) [35,](#page-17-12) [36,](#page-17-13) [27\]](#page-17-14).

We arrange the rest of the paper as follows. In the following section, we recall the Riemann problem for system  $(1.2)$ . In Section 3, we solve the Riemann problem for  $(1.1)$  by the analysis method in phase-plane. In Section 4 and Section 5, we study the limits of solutions of the Riemann problem for [\(1.1\)](#page-0-0) as  $\epsilon \to 0^+$ . In Section 6, we examine the process of concentration as  $\epsilon$  decreases by some numerical results.

### 2. Solutions of the Riemann problem for [\(1.2\)](#page-0-1)

In this section, we recall the Riemann problem for  $(1.2)$  with initial data  $(1.3)$  which was solved by Tan et al. [\[30\]](#page-17-7). The characteristic roots of [\(1.2\)](#page-0-1) are  $\lambda_1 = u$  and  $\lambda_2 = 2u$ , and the corresponding right characteristic vectors are  $r_1 = (0, 1)^T$  and  $r_2 = (1, v/u)^T$ , respectively. They satisfy  $\nabla \lambda_1 \cdot r_1 \equiv 0$  and  $\nabla \lambda_2 \cdot r_2 = 2$ , where and in the following  $\nabla = (\partial/\partial u, \partial/\partial v)$  is the gradient operator. Therefore [\(1.2\)](#page-0-1) is nonstrictly hyperbolic because of  $\lambda_1 = \lambda_2$  at  $u = 0$ ,  $\lambda_1$  is linearly degenerate, and  $\lambda_2$  is genuinely nonlinear.

Since the equations and the Riemann data are invariant under uniform stretching of coordinates  $(x, t) \to (\beta x, \beta t) (\beta > 0)$ , we consider the self-similar solutions  $(u, v)(x, t) = (u, v)(\xi)$ , where  $\xi = x/t$ . Then the Riemann problem turns into

$$
\begin{cases}\n-\xi u_{\xi} + (u^2)_{\xi} = 0, \\
-\xi v_{\xi} + (vu)_{\xi} = 0,\n\end{cases}
$$
\n(2.1)

and

$$
(u, v)(\pm \infty) = (u_{\pm}, v_{\pm}). \tag{2.2}
$$

This is a two-point boundary value problem of first-order ordinary differential equations with the boundary values in the infinity.

Besides the constant states, the self-similar waves  $(u, v)(\xi)(\xi = x/t)$  of the first family are contact discontinuities

$$
J: \xi = u_l = u_r,\tag{2.3}
$$

and those of the second family are rarefaction waves

$$
R: \xi = 2u, \quad u/v = u_l/v_l, \quad u > u_l,
$$
\n(2.4)

or shocks

$$
S: \xi = u_l + u_r, \quad u_r/v_r = u_l/v_l, \quad u_l > u_r > 0 \quad \text{or} \quad 0 > u_l > u_r,\tag{2.5}
$$

where the indices l and r denote the left and right states respectively. All of  $J, R$  and  $S$  are waves with  $(u(\xi), v(\xi)) \in BV$  and are called the classical waves.

Using the classical waves, by the analysis in phase-plane, one can construct the solutions of Riemann problem  $(1.2)$  and  $(1.3)$  in the following cases

$$
R + J (u_{-} < u_{+} < 0), J + R (0 < u_{-} < u_{+}), R + R (u_{-} \le 0 \le u_{+}),
$$
  

$$
S + J (u_{+} < u_{-} < 0), J + S (0 < u_{+} < u_{-}).
$$

However, for the case  $u_+ \leq 0 \leq u_-,$  the singularity cannot be a jump with finite amplitude; that is, there is no solution which is piecewise smooth and bounded. Hence a solution containing a weighted  $\delta$ measure (i.e., delta-shock) supported on a line should be introduced in order to establish the existence in a space of measures from the mathematical point of view.

We define the weighted  $\delta$ -measure  $w(s)\delta_L$  supported on a smooth curve L parameterized as  $t =$  $t(s), x = x(s)$   $(c \leq s \leq d)$  by

$$
\langle w(s)\delta_L, \psi(x,t) \rangle = \int_c^d w(s)\psi(t(s), x(s))ds \qquad (2.6)
$$

for all test functions  $\psi(x,t) \in C_0^{\infty}((-\infty, +\infty) \times [0, +\infty)).$ 

With this definition, when  $u_+ \leq 0 \leq u_-,$  the solution of Riemann problem [\(1.2\)](#page-0-1) and [\(1.3\)](#page-1-0) is the following solution involving delta-shock in the form

$$
(u, v)(x, t) = \begin{cases} (u_-, v_-, & x < x(t), \\ (u_\delta(t), w(t)\delta(x - x(t))), & x = x(t), \\ (u_+, v_+), & x > x(t) \end{cases}
$$
(2.7)

satisfying the generalized Rankine-Hugoniot relation

<span id="page-3-0"></span>
$$
\begin{cases}\n\frac{dx(t)}{dt} = u_{\delta}(t) \\
-u_{\delta}(t)[u] + [u^2] = 0, \\
\frac{dw(t)}{dt} = -u_{\delta}(t)[v] + [uv]\n\end{cases}
$$
\n(2.8)

and the entropy condition

<span id="page-3-1"></span>
$$
\lambda_2(u_+) \leq \lambda_1(u_+) \leq u_\delta(t) \leq \lambda_1(u_-) \leq \lambda_2(u_-),\tag{2.9}
$$

where  $[g] = g_{-} - g_{+}$  is the jump of g across the discontinuity. Solving the generalized Rankine-Hugoniot relation [\(2.8\)](#page-3-0) under the entropy condition [\(2.9\)](#page-3-1) gives

$$
\begin{cases}\nx(t) = (u_{-} + u_{+})t, \\
u_{\delta}(t) = u_{-} + u_{+}, \\
w(t) = (u_{-}v_{+} - u_{+}v_{-})t.\n\end{cases}
$$
\n(2.10)

## 3. Solutions of the Riemann problem for [\(1.1\)](#page-0-0)

In this section, we solve the Riemann problem for system  $(1.1)$  with initial data  $(1.3)$ , and examine the dependence of the Riemann solutions on the parameter  $\epsilon > 0$ . Also see the paper [\[13\]](#page-16-13). The characteristic roots and corresponding right characteristic vectors of [\(1.1\)](#page-0-0) are

$$
\lambda_1^{\epsilon} = u + \frac{u - \sqrt{u^2 + 4\epsilon v}}{2}, \quad \lambda_2^{\epsilon} = u + \frac{u + \sqrt{u^2 + 4\epsilon v}}{2},
$$

$$
\overrightarrow{r}_1^{\epsilon} = \left(1, \frac{-u - \sqrt{u^2 + 4\epsilon v}}{2\epsilon}\right)^T, \quad \overrightarrow{r}_2^{\epsilon} = \left(1, \frac{-u + \sqrt{u^2 + 4\epsilon v}}{2\epsilon}\right)^T.
$$

It is easy to calculate  $\nabla \lambda_i^{\epsilon} \cdot \vec{r}_i^{\epsilon} = 2$   $(i = 1, 2)$ . So [\(1.1\)](#page-0-0) is nonstrictly hyperbolic, both characteristic fields are genuinely nonlinear. Moreover, the Riemann invariants along with the characteristic fields may be selected as, respectively,

$$
w(u,v) = \frac{-u - \sqrt{u^2 + 4\epsilon v}}{2\epsilon}, \qquad z(u,v) = \frac{-u + \sqrt{u^2 + 4\epsilon v}}{2\epsilon}.
$$
 (3.1)

As usual, we seek the self-similar solutions  $(u, v)(x, t) = (u, v)(\xi)$ , where  $\xi = x/t$ . Then the Riemann problem becomes the boundary value problem

<span id="page-3-2"></span>
$$
\begin{cases}\n-\xi u_{\xi} + \left(u^2 + \epsilon v\right)_{\xi} = 0, \\
-\xi v_{\xi} + \left(vu\right)_{\xi} = 0,\n\end{cases}
$$
\n(3.2)

and

$$
(u, v)(\pm \infty) = (u_{\pm}, v_{\pm}). \tag{3.3}
$$

For any smooth solution, [\(3.2\)](#page-3-2) becomes

$$
\begin{pmatrix} 2u - \xi & \epsilon \\ v & u - \xi \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{\xi} = 0.
$$
 (3.4)

Besides the constant states, the smooth solutions are composed of the 1-rarefaction waves

$$
\begin{cases}\n\xi = \lambda_1^{\epsilon} = u + \frac{u - \sqrt{u^2 + 4\epsilon v}}{2}, \\
u - u_0 = \left(\frac{u_0 - \sqrt{u_0^2 + 4\epsilon v_0}}{2v_0}\right)(v - v_0),\n\end{cases} (3.5)
$$

and the 2-rarefaction waves

$$
\begin{cases}\n\xi = \lambda_2^{\epsilon} = u + \frac{u + \sqrt{u^2 + 4\epsilon v}}{2}, \\
u - u_0 = \left(\frac{u_0 + \sqrt{u_0^2 + 4\epsilon v_0}}{2v_0}\right)(v - v_0),\n\end{cases} (3.6)
$$

where  $(u_0, v_0)$  is any state. For them, we have

<span id="page-4-0"></span>
$$
\frac{d\lambda_i^{\epsilon}}{du} = \frac{\partial \lambda_i^{\epsilon}}{\partial u} + \frac{\partial \lambda_i^{\epsilon}}{\partial v} \frac{dv}{du} = 2 > 0, \quad i = 1, 2.
$$
 (3.7)

Let  $(u_l, v_l)$  and  $(u_r, v_r)$  denote the states connected by a rarefaction wave on the left and right sides respectively. Then the condition  $\lambda_1^{\epsilon}(u_r, v_r) > \lambda_1^{\epsilon}(u_l, v_l)$  and  $\lambda_2^{\epsilon}(u_r, v_r) > \lambda_2^{\epsilon}(u_l, v_l)$  are required for the 1- and 2-rarefaction wave, respectively. From [\(3.7\)](#page-4-0), it is known that both the 1- and 2-rarefaction wave should satisfy

$$
u_r > u_l. \tag{3.8}
$$

For a given state  $(u_l, v_l)$ , all possible states which can connect to  $(u_l, v_l)$  on the right by a 1rarefaction wave must be located on the straight line

$$
R_1(u_l, v_l): \t u - u_l = \left(\frac{u_l - \sqrt{u_l^2 + 4\epsilon v_l}}{2v_l}\right)(v - v_l), \t u > u_l, \t (3.9)
$$

and all possible states which can connect to  $(u_l, v_l)$  on the right by a 2-rarefaction wave must be located on straight line

$$
R_2(u_l, v_l): \t u - u_l = \left(\frac{u_l + \sqrt{u_l^2 + 4\epsilon v_l}}{2v_l}\right)(v - v_l), \t u > u_l.
$$
\t(3.10)

Let us turn to the discontinuous solutions. For a bounded discontinuity at  $x = x(t)$ , the Rankine-Hugoniot relation reads

<span id="page-4-1"></span>
$$
\begin{cases}\n-\sigma[u] + [u^2 + \epsilon v] = 0, \\
-\sigma[v] + [vu] = 0,\n\end{cases}
$$
\n(3.11)

where  $\sigma = dx/dt$ ,  $[u] = u_l - u_r$  with  $u_l = u(x(t) - 0, t)$  and  $u_r = u(x(t) + 0, t)$ , and so forth.

From [\(3.11\)](#page-4-1), one easily obtains

<span id="page-4-2"></span>
$$
\epsilon \left( \frac{[v]}{[u]} \right)^2 + (u_l + u_r) \frac{[v]}{[u]} - \frac{[uv]}{[u]} = 0. \tag{3.12}
$$

By noticing

$$
\frac{[uv]}{[u]} = v_l + u_r \frac{[v]}{[u]},
$$

we solve  $(3.12)$  to obtain

$$
\frac{[v]}{[u]} = \frac{-u_l \pm \sqrt{u_l^2 + 4\epsilon v_l}}{2\epsilon}.
$$
\n(3.13)

Then we obtain two kinds of discontinuities

<span id="page-5-0"></span>
$$
\begin{cases}\n\sigma_1 = u_r + \frac{u_l - \sqrt{u_l^2 + 4\epsilon v_l}}{2}, \\
u_r - u_l = \left(\frac{u_l - \sqrt{u_l^2 + 4\epsilon v_l}}{2v_l}\right)(v_r - v_l)\n\end{cases}
$$
\n(3.14)

and

<span id="page-5-1"></span>
$$
\begin{cases}\n\sigma_2 = u_r + \frac{u_l + \sqrt{u_l^2 + 4\epsilon v_l}}{2}, \\
u_r - u_l = \left(\frac{u_l + \sqrt{u_l^2 + 4\epsilon v_l}}{2v_l}\right)(v_r - v_l).\n\end{cases} (3.15)
$$

Notice that the second equations in [\(3.14\)](#page-5-0) and [\(3.15\)](#page-5-1) are equivalent to

$$
\frac{-u_l - \sqrt{u_l^2 + 4\epsilon v_l}}{2\epsilon} = \frac{-u_r - \sqrt{u_r^2 + 4\epsilon v_r}}{2\epsilon}
$$
(3.16)

and

$$
\frac{-u_l + \sqrt{u_l^2 + 4\epsilon v_l}}{2\epsilon} = \frac{-u_r + \sqrt{u_r^2 + 4\epsilon v_r}}{2\epsilon}
$$
(3.17)

respectively.

In order to identity the admissible solution, the discontinuity [\(3.14\)](#page-5-0) associating with  $\lambda_1^{\epsilon}$  should satisfy

<span id="page-5-2"></span>
$$
\sigma_1 < \lambda_1^{\epsilon}(u_l, v_l) < \lambda_2^{\epsilon}(u_l, v_l), \quad \lambda_1^{\epsilon}(u_r, v_r) < \sigma_1 < \lambda_2^{\epsilon}(u_r, v_r), \tag{3.18}
$$

while the discontinuity [\(3.15\)](#page-5-1) associating with  $\lambda_2^\epsilon$  should satisfy

<span id="page-5-3"></span>
$$
\lambda_1^{\epsilon}(u_l, v_l) < \sigma_2 < \lambda_2^{\epsilon}(u_l, v_l), \quad \lambda_1^{\epsilon}(u_r, v_r) < \lambda_2^{\epsilon}(u_r, v_r) < \sigma_2. \tag{3.19}
$$

Then one can check that both the inequality [\(3.18\)](#page-5-2) and [\(3.19\)](#page-5-3) are equivalent to

<span id="page-5-4"></span>
$$
u_r < u_l. \tag{3.20}
$$

The discontinuity  $(3.14)$  with  $(3.20)$  is called as 1-shock and symbolized by  $S_1$ , and  $(3.15)$  with  $(3.20)$  is called as 2-shock and symbolized by  $S_2$ .

For a given state  $(u_l, v_l)$ , all possible states which can connect to  $(u_l, v_l)$  on the right by a 1-shock must be located on the straight line

$$
S_1(u_l, v_l): \t u - u_l = \left(\frac{u_l - \sqrt{u_l^2 + 4\epsilon v_l}}{2v_l}\right)(v - v_l), \t u < u_l,
$$
\t(3.21)

and all possible states which can connect to  $(u_l, v_l)$  on the right by a 2-shock must be located on the straight line

$$
S_2(u_l, v_l): \t u - u_l = \left(\frac{u_l + \sqrt{u_l^2 + 4\epsilon v_l}}{2v_l}\right)(v - v_l), \t u < u_l.
$$
\t(3.22)

Let us denote  $W_1(u_l, v_l) = R_1(u_l, v_l) \cup S_1(u_l, v_l)$  and  $W_2(u_l, v_l) = R_2(u_l, v_l) \cup S_2(u_l, v_l)$ . Draw the line  $W_1(u_-, v_-)$  and  $W_2(u_-, v_-)$  in the upper half  $(u, v)$ -plane, then the upper half  $(u, v)$ -plane is divided into four regions (see Fig.1). According to the right state  $(u_+, v_+)$  in the different regions, one can construct the unique global Riemann solution connecting two constant states  $(u_-, v_-)$  and  $(u_+, v_+)$ . To be more exact, the Riemann solutions contain (i) a 1-rarefaction wave and a 2-rarefaction wave when  $(u_+, v_+) \in R_1R_2(u_-, v_-),$  (ii) a 1-rarefaction wave and a 2-shock when  $(u_+, v_+) \in R_1S_2(u_-, v_-),$ (iii) a 1-shock and a 2-rarefaction wave when  $(u_+, v_+) \in S_1R_2(u_-, v_-)$ , (iv) a 1-shock and a 2-shock when  $(u_+, v_+) \in S_1S_2(u_-, v_-).$ 



The conclusion can be stated in the following theorem.

**Theorem 3.1.** The Riemann problem for  $(1.1)$  $(1.1)$  with initial data  $(1.3)$  $(1.3)$  has a unique piecewise smooth solution consisting of waves of constant states, shocks and rarefaction waves.

## 4. Limits of Riemann solution to [\(1.1\)](#page-0-0) for  $u_+ < u_-, u_+/v_+ < u_-/v_-$

In this section, we study the limits of the Riemann solution  $\epsilon \to 0^+$  when the initial data satisfy  $u_+ < u_-, u_+/v_+ < u_-/v_-.$  Especially, we pay more attention on the phenomenon of concentration and the formation of delta-shocks in the limit.

For  $u_+ < u_-, u_+/v_+ < u_-/v_-,$  there must exist  $\epsilon_0 > 0$  such that the Riemann solution just consists of two shocks for any  $\epsilon < \epsilon_0$ . In fact, since all states  $(u, v)$  connected with  $(u_-, v_-)$  by  $S_1$  and  $S_2$ satisfy

$$
u - u_{-} = \left(\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2v_{-}}\right)(v - v_{-}), \quad u < u_{-}, \quad v > v_{-},
$$

and

$$
u - u_- = \left(\frac{u_- + \sqrt{u_-^2 + 4\epsilon v_-}}{2v_-}\right)(v - v_-), \quad u < u_-, \quad v < v_-,
$$

respectively, then if  $v_{+} = v_{-}$ ,  $\epsilon_{0}$  may be taken any real positive number, otherwise, we have the conclusion by taking

$$
\epsilon_0 = \frac{\left(2v_-\left(\frac{u_+-u_-}{v_+-v_-}\right)-u_-\right)^2 - u_-^2}{4v_-} = \frac{(u_+-u_-)(v_-u_+-v_+u_-)}{(v_+-v_-)^2}.
$$

For fixed  $\epsilon < \epsilon_0$ , let  $U^{\epsilon}(\xi)$  denote the two-shock Riemann solution for [\(1.1\)](#page-0-0) and [\(1.3\)](#page-1-0) constructed in Section 3

<span id="page-7-2"></span>
$$
U^{\epsilon}(\xi) = (u^{\epsilon}, v^{\epsilon})(\xi) = \begin{cases} (u_{-}, v_{-}), & \xi < \sigma_1^{\epsilon}, \\ (u^{\epsilon}_{*}, v^{\epsilon}_{*}), & \sigma_1^{\epsilon} < \xi < \sigma_2^{\epsilon}, \\ (u_{+}, v_{+}), & \xi > \sigma_2^{\epsilon}, \end{cases}
$$
(4.1)

where  $(u_-, v_-)$  and  $(u_*^{\epsilon}, v_*^{\epsilon})$  are connected by a shock  $S_1$  with speed  $\sigma_1^{\epsilon}$ , and  $(u_*^{\epsilon}, v_*^{\epsilon})$  and  $(u_+, v_+)$  are connected by a shock  $S_2$  with speed  $\sigma_2^{\epsilon}$ :

<span id="page-7-0"></span>
$$
S_{1}: \begin{cases} \n\sigma_{1}^{\epsilon} = u_{*}^{\epsilon} + \frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2}, \\
u_{*}^{\epsilon} - u_{-} = \left( \frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2v_{-}} \right) (v_{*}^{\epsilon} - v_{-}), \\
S_{2}: \begin{cases} \n\sigma_{2}^{\epsilon} = u_{+} + \frac{u_{*}^{\epsilon} + \sqrt{(u_{*}^{\epsilon})^{2} + 4\epsilon v_{*}^{\epsilon}}}{2}, \\
u_{+} - u_{*}^{\epsilon} = \left( \frac{u_{*}^{\epsilon} + \sqrt{(u_{*}^{\epsilon})^{2} + 4\epsilon v_{*}^{\epsilon}}}{2v_{*}^{\epsilon}} \right) (v_{+} - v_{*}^{\epsilon}),\n\end{cases} \n\qquad \qquad (4.3)
$$

Here

<span id="page-7-1"></span>
$$
\frac{u_{-} - \sqrt{u_{-}^2 + 4\epsilon v_{-}}}{2v_{-}} = \frac{u_{*}^{\epsilon} - \sqrt{(u_{*}^{\epsilon})^2 + 4\epsilon v_{*}^{\epsilon}}}{2v_{*}^{\epsilon}}
$$
(4.4)

and

$$
\frac{u_*^{\epsilon} + \sqrt{(u_*^{\epsilon})^2 + 4\epsilon v_*^{\epsilon}}}{2v_*^{\epsilon}} = \frac{u_+ + \sqrt{u_+^2 + 4\epsilon v_+}}{2v_+}.
$$
\n(4.5)

The following Lemmas 4.1-4.2 show the limit behaviors of the states between two shocks. Lemma 4.1.

$$
\lim_{\epsilon \to 0^+} v_*^{\epsilon} = \begin{cases} (u_-/u_+)v_+, & \text{for} \quad u_- > u_+ > 0, \\ (u_+/u_-)v_-, & \text{for} \quad 0 > u_- > u_+, \\ +\infty, & \text{for} \quad u_- \ge 0 \ge u_+ . \end{cases}
$$

*Proof.* Based on [\(4.2\)](#page-7-0) and [\(4.3\)](#page-7-1),  $v_*^{\epsilon}$  can be expressed as

$$
u_{-} + \left(\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2v_{-}}\right)(v_{*}^{\epsilon} - v_{-}) = u_{+} - \left(\frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}}\right)(v_{+} - v_{*}^{\epsilon}).
$$

Solving this equation gives

$$
v_{*}^{\epsilon} = \frac{u_{+} - u_{-} - \left(\frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}}\right)v_{+} + \left(\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2v_{-}}\right)v_{-}}{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}} - \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}}}.
$$
(4.6)

Taking the limit  $\epsilon \to 0^+$  will lead to the conclusions. The proof is finished.

Lemma 4.2.

$$
\lim_{\epsilon \to 0^+} u_*^\epsilon = \left\{ \begin{array}{ll} u_-, & \quad for \quad u_- > u_+ > 0, \\ u_+, & \quad for \quad 0 > u_- > u_+, \\ u_- + u_+, & \quad for \quad u_- \geq 0 \geq u_+ . \end{array} \right.
$$

Proof. From the second equation in  $(4.2)$ , we have

$$
u_*^{\epsilon} = u_- + \left(\frac{u_- - \sqrt{u_-^2 + 4\epsilon v_-}}{2v_-}\right)(v_*^{\epsilon} - v_-).
$$

For the cases  $u_-> u_+ > 0$  and  $0 > u_- > u_+$ , the conclusions are obvious because of the Lemma 4.2. For the case  $u_-\geq 0\geq u_+$ , due to

$$
\frac{\frac{u_+ + \sqrt{u_+^2 + 4\epsilon v_+}}{2v_+}}{\frac{u_- - \sqrt{u_-^2 + 4\epsilon v_-}}{2v_-}} = \frac{u_+ + \sqrt{u_+^2 + 4\epsilon v_+}}{u_- - \sqrt{u_-^2 + 4\epsilon v_-}} \cdot \frac{v_-}{v_+} = \frac{u_- + \sqrt{u_-^2 + 4\epsilon v_-}}{u_+ - \sqrt{u_+^2 + 4\epsilon v_+}} \rightarrow \frac{u_-}{u_+} \qquad as \ \epsilon \rightarrow 0^+,
$$

we have

$$
\begin{pmatrix}\n\underline{u}_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}} \\
2v_{-}\n\end{pmatrix} v_{*}^{\epsilon} = \left(\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2v_{-}}\right) \cdot \frac{u_{+} - u_{-} - \left(\frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}}\right)v_{+} + \left(\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2v_{-}}\right)v_{-} \\
= \frac{u_{+} - u_{-} - \left(\frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}}\right)v_{+} + \left(\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2v_{-}}\right)v_{-}}{1 - \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}}} \rightarrow u_{+} \quad as \epsilon \rightarrow 0^{+},
$$

which gives the conclusion. The proof is finished.  $\Box$ 

The following Lemma 4.3 shows the limit behaviors of the speeds of two shocks.

#### Lemma 4.3.

$$
\lim_{\epsilon \to 0^+} (\sigma_1^{\epsilon}, \sigma_2^{\epsilon}) = \begin{cases} (u_-, u_- + u_+), & \text{for} \quad u_- > u_+ > 0, \\ (u_- + u_+, u_+), & \text{for} \quad 0 > u_- > u_+, \\ (u_- + u_+, u_- + u_+), & \text{for} \quad u_- \ge 0 \ge u_+ . \end{cases}
$$

*Proof.* For  $u_{-} > u_{+} > 0$ ,

$$
\lim_{\epsilon \to 0^+} \sigma_1^{\epsilon} = \lim_{\epsilon \to 0^+} \left( u_*^{\epsilon} + \frac{u - \sqrt{u_-^2 + 4\epsilon v_-}}{2} \right) = \lim_{\epsilon \to 0^+} u_*^{\epsilon} = u_-,
$$

and

$$
\lim_{\epsilon \to 0^{+}} \sigma_{2}^{\epsilon} = \lim_{\epsilon \to 0^{+}} \left( u_{+} + \frac{u_{*}^{\epsilon} + \sqrt{(u_{*}^{\epsilon})^{2} + 4\epsilon v_{*}^{\epsilon}}}{2} \right)
$$
\n
$$
= \lim_{\epsilon \to 0^{+}} \left( u_{+} + \left( \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}} \right) v_{*}^{\epsilon} \right)
$$
\n
$$
= u_{-} + u_{+}.
$$

For  $0 > u_> u_+$ , the conclusions can be proved in a similar way. For  $u_-\geq 0 \geq u_+$ ,

$$
\lim_{\epsilon \to 0^+} \sigma_1^{\epsilon} = \lim_{\epsilon \to 0^+} u_*^{\epsilon} = u_- + u_+
$$

and

$$
\lim_{\epsilon \to 0^{+}} \sigma_{2}^{\epsilon} = \lim_{\epsilon \to 0^{+}} \left( u_{+} + \frac{u_{*}^{\epsilon} + \sqrt{(u_{*}^{\epsilon})^{2} + 4\epsilon v_{*}^{\epsilon}}}{2} \right)
$$
\n
$$
= \lim_{\epsilon \to 0^{+}} \left( u_{+} + \left( \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}} \right) v_{*}^{\epsilon} \right)
$$
\n
$$
= \lim_{\epsilon \to 0^{+}} \left( u_{+} + \left( \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2v_{+}} \right) v_{+} - (u_{+} - u_{*}^{\epsilon}) \right)
$$
\n
$$
= u_{-} + u_{+},
$$

where we have used

<span id="page-9-0"></span>
$$
\left(\frac{u_+ + \sqrt{u_+^2 + 4\epsilon v_+}}{2v_+}\right)v_*^{\epsilon} = \left(\frac{u_+ + \sqrt{u_+^2 + 4\epsilon v_+}}{2v_+}\right)v_+ - (u_+ - u_*^{\epsilon})\tag{4.7}
$$

obtaining from the second equality of  $(4.3)$ . The proof is finished.

Let  $U^0(\xi) = \lim_{\epsilon \to 0^+} U^{\epsilon}(\xi)$ . Then when  $u_- > u_+ > 0, u_+/v_+ < u_-/v_-,$ 

$$
U^{0}(\xi) = \begin{cases} (u_{-}, v_{-}), & \xi < \sigma_{1}, \\ (u_{-}, v_{*}), & \sigma_{1} < \xi < \sigma_{2}, \\ (u_{+}, v_{+}), & \xi > \sigma_{2}, \end{cases}
$$

where  $\sigma_1 = u_-, \sigma_2 = u_- + u_+$  and  $v_* = (u_-/u_+)v_-$ . When  $0 > u_- > u_+, u_+/v_+ < u_-/v_-,$ 

$$
U^{0}(\xi) = \begin{cases} (u_{-}, v_{-}), & \xi < \sigma_{1}, \\ (u_{+}, v_{*}), & \sigma_{1} < \xi < \sigma_{2}, \\ (u_{+}, v_{+}), & \xi > \sigma_{2}, \end{cases}
$$

 $\sigma_1 = u_- + u_+, \sigma_2 = u_+$  and  $v_* = (u_+ / u_-) v_+$ . It can be seen that  $U^0(\xi)$  coincides with the Riemann solution for [\(1.2\)](#page-0-1) constructed in Section 2.

For the case  $u_-\geq 0\geq u_+$ , it has been shown that two shocks will coincide at  $\xi=u_-+u_+:=\sigma$ as  $\epsilon \to 0^+$ . Furthermore, for the component  $u^{\epsilon}(\xi)$ , it has been shown that

$$
\lim_{\epsilon \to 0^+} u^{\epsilon}(\xi) = \begin{cases} u_{-}, & \xi < \sigma, \\ u_{-} + u_{+}, & \xi = \sigma, \\ u_{+}, & \xi > \sigma. \end{cases}
$$
(4.8)

For the component  $v^{\epsilon}(\xi)$ , we have proven that the intermediate state  $v^{\epsilon}_{*}$  becomes infinity as  $\epsilon \to 0^{+}$ . Further, we have

Lemma 4.4.

$$
\lim_{\epsilon \to 0^+} (\sigma_2^{\epsilon} - \sigma_1^{\epsilon}) v_*^{\epsilon} = u_- v_+ - u_+ v_-.
$$

Proof. With

$$
\frac{u_*^{\epsilon} + \sqrt{(u_*^{\epsilon})^2 + 4\epsilon v_*^{\epsilon}}}{2} = \frac{u_*^{\epsilon} + \sqrt{(u_*^{\epsilon})^2 + 4\epsilon v_*^{\epsilon}}}{2v_*^{\epsilon}}v_*^{\epsilon} = \frac{u_* + \sqrt{u_*^2 + 4\epsilon v_*}}{2v_*}v_*^{\epsilon}
$$

and [\(4.7\)](#page-9-0), it follows

$$
\sigma_2^{\epsilon} - \sigma_1^{\epsilon} = u_+ + \frac{u_*^{\epsilon} + \sqrt{(u_*^{\epsilon})^2 + 4\epsilon v_*^{\epsilon}}}{2} - u_*^{\epsilon} - \frac{u_- - \sqrt{u_-^2 + 4\epsilon v_-}}{2}
$$

$$
= \frac{u_+ + \sqrt{u_+^2 + 4\epsilon v_+}}{2} - \frac{u_- - \sqrt{u_-^2 + 4\epsilon v_-}}{2}.
$$

Then

$$
\lim_{\epsilon \to 0^{+}} (\sigma_{2}^{\epsilon} - \sigma_{1}^{\epsilon}) v_{*}^{\epsilon} = \lim_{\epsilon \to 0^{+}} \left( \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2} \right) \cdot \frac{\frac{u_{+} - \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{+}}}}{\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2} \right)}{\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{+}} - (u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}})}{2} - \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2} \right)}{\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{+} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} + \sqrt{u_{-}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} + \sqrt{u_{-}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} + \sqrt{u_{+}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} + \sqrt{u_{-}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} + \sqrt{u_{-}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} + \sqrt{u_{-}^{2} + 4\epsilon v_{+}}}{2} - \frac{u_{-} + \sqrt{u_{+}^{2} + 4\epsilon v_{+
$$

The proof is finished.  $\blacksquare$ 

Take  $\phi(\xi) \in C_0^{\infty}(-\infty, +\infty)$  such that  $\phi(\xi) \equiv \phi(\sigma)$  for  $\xi$  in a neighborhood  $\Omega$  of  $\xi = \sigma$  ( $\phi$  is called a sloping test function [\[7,](#page-16-14) [11\]](#page-16-15)). Assume when  $\epsilon < \epsilon_0$ , it holds  $\sigma_1^{\epsilon} \in \Omega$  and  $\sigma_2^{\epsilon} \in \Omega$ . It is well known that the solution [\(4.1\)](#page-7-2) satisfies weak formulation

<span id="page-10-0"></span>
$$
-\int_{-\infty}^{+\infty} v^{\epsilon} (u^{\epsilon} - \xi) \phi' d\xi + \int_{-\infty}^{+\infty} v^{\epsilon} \phi d\xi = 0.
$$
 (4.9)

Since

$$
\int_{-\infty}^{+\infty} v^{\epsilon} (u^{\epsilon} - \xi) \phi' d\xi = \left( \int_{-\infty}^{\sigma_1^{\epsilon}} + \int_{\sigma_2^{\epsilon}}^{+\infty} \right) v^{\epsilon} (u^{\epsilon} - \xi) \phi' d\xi,
$$

we have

$$
\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} v^{\epsilon} (u^{\epsilon} - \xi) \phi' d\xi = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\sigma_1^{\epsilon}} v_{-}(u_{-} - \xi) \phi' d\xi + \lim_{\epsilon \to 0^+} \int_{\sigma_2^{\epsilon}}^{+\infty} v_{+}(u_{+} - \xi) \phi' d\xi
$$

$$
= (u_{-}v_{+} - u_{+}v_{-}) \phi(\sigma) + \int_{-\infty}^{+\infty} H(\xi - \sigma) \phi d\xi,
$$

where

$$
H(x) = \begin{cases} v_{-}, & x < 0, \\ v_{+}, & x > 0. \end{cases}
$$

Returning to [\(4.9\)](#page-10-0),we get

$$
\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \left( v^{\epsilon} - H(\xi - \sigma) \right) \phi d\xi = (u_{-}v_{+} - u_{+}v_{-})\phi(\sigma) \tag{4.10}
$$

for all sloping test functions  $\phi \in C_0^{\infty}(-\infty, +\infty)$ .

For an arbitrary test function  $\varphi(\xi) \in C_0^{\infty}(-\infty, +\infty)$ , we take a sloping test function  $\phi$  such that  $\phi(\sigma) = \varphi(\sigma)$  and

$$
\max_{\xi \in (-\infty, +\infty)} |\phi - \varphi| < \mu.
$$

We have

$$
\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \left( v^{\epsilon} - H(\xi - \sigma) \right) \varphi d\xi
$$
\n
$$
= \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \left( v^{\epsilon} - H(\xi - \sigma) \right) \varphi d\xi + \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \left( v^{\epsilon} - H(\xi - \sigma) \right) (\varphi - \phi) d\xi.
$$

The first limit on the right side

$$
\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \left( v^{\epsilon} - H(\xi - \sigma) \right) \phi d\xi = (u_{-}v_{+} - u_{+}v_{-})\phi(\sigma)
$$

$$
= (u_{-}v_{+} - u_{+}v_{-})\varphi(\sigma).
$$

The second limit on the right side

$$
\int_{-\infty}^{+\infty} \left( v_*^{\epsilon} - H(\xi - \sigma) \right) (\varphi - \phi) d\xi = \int_{\sigma_1^{\epsilon}}^{\sigma_2^{\epsilon}} \left( v_*^{\epsilon} - H(\xi - \sigma) \right) (\varphi - \phi) d\xi
$$
  
= 
$$
\int_{\sigma_1^{\epsilon}}^{\sigma_2^{\epsilon}} v^{\epsilon} (\varphi - \phi) d\xi + \int_{\sigma_1^{\epsilon}}^{\sigma_2^{\epsilon}} H(\xi - \sigma) (\varphi - \phi) d\xi,
$$

which converges to 0 by sending  $\mu \to 0$  and recalling Lemma 4.5. Thus we have that

<span id="page-11-0"></span>
$$
\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \left( v^{\epsilon} - H(\xi - \sigma) \right) \varphi d\xi = (u_- v_+ - u_+ v_-) \varphi(\sigma) \tag{4.11}
$$

for all test functions  $\varphi \in C_0^{\infty}(-\infty, +\infty)$ .

Let  $\psi(x,t) \in C_0^{\infty}((-\infty, +\infty) \times [0, +\infty))$  be a smooth test function, and let  $\widetilde{\psi}(\xi, t) := \psi(\xi t, t)$ . Then it follows

$$
\lim_{\epsilon \to 0^+} \int_0^{+\infty} \int_{-\infty}^{+\infty} v^{\epsilon}(x/t) \psi(x,t) dx dt = \lim_{\epsilon \to 0^+} \int_0^{+\infty} \int_{-\infty}^{+\infty} v^{\epsilon}(\xi) \psi(\xi t, t) d(\xi t) dt
$$

$$
= \lim_{\epsilon \to 0^+} \int_0^{+\infty} t \left( \int_{-\infty}^{+\infty} v^{\epsilon}(\xi) \widetilde{\psi}(\xi, t) d\xi \right) dt
$$

and from [\(4.11\)](#page-11-0)

$$
\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} v^{\epsilon}(\xi) \widetilde{\psi}(\xi, t) d\xi
$$
\n
$$
= \int_{-\infty}^{+\infty} H(\xi - \sigma) \widetilde{\psi}(\xi, t) d\xi + (u_{-}v_{+} - u_{+}v_{-}) \widetilde{\psi}(\sigma, t)
$$
\n
$$
= t^{-1} \int_{-\infty}^{+\infty} H(x - \sigma t) \psi(x, t) dx + (u_{-}v_{+} - u_{+}v_{-}) \psi(\sigma t, t).
$$

Combining the two relations above yields

$$
\lim_{\epsilon \to 0^+} \int_0^{+\infty} \int_{-\infty}^{+\infty} v^{\epsilon}(x/t) \psi(x,t) dx dt
$$
  
= 
$$
\int_0^{+\infty} \int_{-\infty}^{+\infty} H(x - \sigma t) \psi(x,t) dx dt + \int_0^{+\infty} (u_- v_+ - u_+ v_-) t \psi(\sigma t,t) dt.
$$

The last term, by the definition,

$$
\int_0^{+\infty} (u_-v_+ - u_+v_-)t\psi(\sigma t,t)dt = \langle w(t)\delta_{x=\sigma t}, \psi(x,t)\rangle
$$

with

$$
w(t) = (u_{-}v_{+} - u_{+}v_{-})t.
$$

Thus we obtain the following Theorem.

**Theorem 4.5.** Let  $u_-\geq 0 \geq u_+$ , and  $(u^{\epsilon}(x,t), v^{\epsilon}(x,t))$  is the two-shock solution to [\(1](#page-0-0).1) and (1.[3\)](#page-1-0). Then  $(u^{\epsilon}(x,t), v^{\epsilon}(x,t))$  converges in the sense of distributions. Denote the limit functions  $U^{0}(x,t)$ , then

$$
U^{0}(x,t) = \begin{cases} (u_{-}, v_{-}), & x < \sigma t, \\ (u_{-} + u_{+}, w(t)\delta(x - \sigma t)), & x = \sigma t, \\ (u_{+}, v_{+}), & x > \sigma t, \end{cases}
$$

where  $w(t) = (u_{-}v_{+} - u_{+}v_{-})t$  and  $\sigma = u_{-} + u_{+}$ , which is just the delta-shock Riemann solution of [\(1.2\)](#page-0-1) with the same initial data.

# 5. Limits of Riemann solution to [\(1.1\)](#page-0-0) for  $u_+ > u_-, u_+/v_+ > u_-/v_-$

In this section, we study the limits of the Riemann solution as  $\epsilon \to 0^+$  when the initial data satisfy  $u_+ > u_-, u_+/v_+ > u_-/v_-.$  At this moment, there must exist  $\epsilon_0 > 0$  such that the Riemann solution just consists of two rarefaction waves for any  $\epsilon < \epsilon_0$ .

For fixed  $\epsilon < \epsilon_0$ , let  $U^{\epsilon}(\xi)$  denote the two-rarefaction-wave Riemann solution for [\(1.1\)](#page-0-0) and [\(1.3\)](#page-1-0) constructed in Section 3

$$
U^{\epsilon}(\xi) = (u^{\epsilon}, v^{\epsilon})(\xi) = \begin{cases} (u_{-}, v_{-}), & -\infty < \xi \leq \lambda_{1}(u_{-}, v_{-}), \\ R_{1}, & \lambda_{1}(u_{-}, v_{-}) \leq \xi \leq \lambda_{1}(u_{*}^{\epsilon}, v_{*}^{\epsilon}), \\ (u_{*}^{\epsilon}, v_{*}^{\epsilon}), & \lambda_{1}(u_{*}^{\epsilon}, v_{*}^{\epsilon}) \leq \xi \leq \lambda_{2}(u_{*}^{\epsilon}, v_{*}^{\epsilon}), \\ R_{2}, & \lambda_{2}(u_{*}^{\epsilon}, v_{*}^{\epsilon}) \leq \xi \leq \lambda_{2}(u_{+}, v_{+}), \\ (u_{+}, v_{+}), & \lambda_{2}(u_{+}, v_{+}) \leq \xi < +\infty, \end{cases}
$$
(5.1)

where

<span id="page-12-0"></span>
$$
R_1: \begin{cases} \xi = \lambda_1^{\epsilon}(u, v) = u + \frac{u - \sqrt{u^2 + 4\epsilon v}}{2}, \\ u - u_{-} = \left(\frac{u_{-} - \sqrt{u_{-}^2 + 4\epsilon v_{-}}}{2v_{-}}\right)(v - v_{-}), \end{cases} \qquad v_{-} \ge v \ge v_*^{\epsilon}, \ u_*^{\epsilon} \ge u \ge u_{-}, \tag{5.2}
$$

and

<span id="page-12-1"></span>
$$
R_2: \begin{cases} \xi = \lambda_2^{\epsilon}(u, v) = u + \frac{u + \sqrt{u^2 + 4\epsilon v}}{2}, \\ u - u_+ = \left(\frac{u_+ + \sqrt{u_+^2 + 4\epsilon v_+}}{2v_+}\right)(v - v_+), \end{cases} v_+ \ge v \ge v_*^{\epsilon}, u_+ \ge u \ge u_*^{\epsilon}.
$$
 (5.3)

The follow lemmas describe the limit behaviors of the intermediate state  $(u^{\epsilon}_{*}, v^{\epsilon}_{*})$  between two rarefaction waves.

Lemma 5.1.

$$
\lim_{\epsilon \to 0^+} v_*^\epsilon = \left\{ \begin{array}{ll} (u_-/u_+)v_+, & for & u_+ > u_- > 0, \\ (u_+/u_-)v_-, & for & 0 > u_+ > u_-, \\ 0, & for & u_+ > 0 > u_-.\end{array} \right.
$$

*Proof.* From  $(5.2)$  and  $(5.3)$ , it follows

$$
v^\epsilon_*=\frac{u_+-u_- -\frac{u_++\sqrt{u_+^2+4\epsilon v_+}}{2v_+}v_++\frac{u_--\sqrt{u_-^2+4\epsilon v_-}}{2v_-}v_-}{\frac{u_--\sqrt{u_-^2+4\epsilon v_-}}{2v_-}-\frac{u_++\sqrt{u_+^2+4\epsilon v_+}}{2v_+}}.
$$

Then the conclusions can be obtained directly by taking the limit  $\epsilon \to 0^+$ . The proof is finished.  $\blacksquare$ 

Lemma 5.2.

$$
\lim_{\epsilon \to 0^+} u_*^{\epsilon} = \begin{cases} u_-, & \text{for} \quad u_+ > u_- > 0, \\ u_+, & \text{for} \quad 0 > u_+ > u_-, \\ 0, & \text{for} \quad u_+ \ge 0 \ge u_-. \end{cases}
$$

*Proof.* From  $(5.2)$ , we have

$$
u = u_{-} + \left(\frac{u_{-} - \sqrt{u_{-}^{2} + 4\epsilon v_{-}}}{2v_{-}}\right)(v_{*}^{\epsilon} - v_{-}).
$$

With the Lemma 5.1, we easily get the conclusions. The proof is complete.  $\blacksquare$ 

Besides, as  $\epsilon \to 0^+$ , when  $u_+ > u_- > 0$ , the rarefaction wave  $R_1$  tends to

$$
\xi = u = u_-, \tag{5.4}
$$

and the rarefaction wave  $R_2$  tends to

<span id="page-13-1"></span>
$$
\xi = 2u, \quad u/v = u_{+}/v_{+}.\tag{5.5}
$$

When  $0 > u_+ > u_-,$  the rarefaction wave  $R_1$  tends to

<span id="page-13-0"></span>
$$
\xi = 2u, \quad u/v = u_{-}/v_{-},\tag{5.6}
$$

and the rarefaction wave  $R_2$  tends to

$$
\xi = u = u_+, \tag{5.7}
$$

When  $u_+ \geq 0 \geq u_-,$  the rarefaction wave  $R_1$  tends to [\(5.6\)](#page-13-0), and the rarefaction wave  $R_2$  tends to  $(5.5).$  $(5.5).$ 

In conclusion, when  $u_+ > u_-, u_+/v_+ > u_-/v_-,$  the limits of Riemann solution of [\(1.1\)](#page-0-0) are just the solutions of [\(1.2\)](#page-0-1) with the same initial data.

In the above two sections, we have proven that when  $u_+ < u_-, u_+/v_+ < u_-/v_-$  and  $u_+ >$  $u_-, u_+/v_+ > u_-/v_-,$  the solutions to the Riemann problem for [\(1.1\)](#page-0-0) just are the solutions to the Riemann problem for [\(1.2\)](#page-0-1) with the same initial data. The same conclusions are true for the rest two cases  $u_+ > u_-, u_+/v_+ < u_-/v_-$  and  $u_+ < u_-, u_+/v_+ > u_-/v_-,$  and we omit the discussions.

#### 6. Process of concentration: Numerical simulations

To understand the phenomenon of concentration and the process of formation of delta-shocks in the Riemann solutions to  $(1.1)$  as the flux  $\epsilon v$  vanishes, in this section, we present some representative numerical results, obtained by employing the Nessyahu-Tadmor scheme [\[23,](#page-17-15) [14\]](#page-16-16) with 500 cells and  $CFL = 0.475$ . We take the initial data as follows

$$
(u, v)(x, t = 0) = \begin{cases} (1, 1), & x < 0, \\ (-1, 1.5), & x > 0. \end{cases}
$$

The numerical simulations for different choices of  $\epsilon$  are presented in Figs.2-5.



Figure 2: Velocity and density for  $\epsilon = 0.3$  at  $t = 0.4$ 



Figure 3: Velocity and density for  $\epsilon = 0.15$  at  $t = 0.4$ 

One can observe clearly from these above numerical results that, when  $\epsilon$  decreases, the location of the two shocks becomes closer and closer, and the density of the intermediate state increases dramatically, while the velocity is closer to a step function. The numerical simulations are in complete agreement with the theoretical analysis.



Figure 4: Velocity and density for  $\epsilon = 0.07$  at  $t = 0.4$ 



Figure 5: Velocity and density for  $\epsilon = 0.001$  at  $t = 0.4$ 

## <span id="page-15-0"></span>References

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