CONSTANT NONLOCAL MEAN CURVATURES SURFACES AND RELATED PROBLEMS

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1. Introduction

The concept of *mean curvature* of a surface goes back to Sophie Germain's work on elasticity theory in the seventeenth century. The mathematical formulation of the mean curvature was first derived by Young and then by Laplace in the eighteenth, see [40]. The mean curvature of a surface is an extrinsic measure of curvature which locally describes the curvature of surface in some ambient space. The notion of *nonlocal mean curvature* appeared recently and for the first time in the work of Caffarelli and Souganidis [13] on cellular automata. It is also an extrinsic geometric quantity that is invariant under global reparameterization of a surface. While Constant Mean Curvature (CMC) surfaces are stationary points for the area functional under some constraints, Constant Nonlocal Mean Curvature (CNMC) surfaces are also critical points of a fractional order area functional, called fractional or nonlocal perimeter, under some constraints. For simplicity, the fractional perimeter of a bounded set is given by a Sobolev fractional seminorm of its indicator function. Moreover, up to a normalization, as the fractional parameter tends to a maximal value, it approaches the classical perimeter functional. In some phase transition problems driven by Lèvy-type diffusion, the sharp-interface energy is given by the fractional perimeter functional, see [70]. Moreover, the rich structure of CNMC surfaces captures the geometry and distributions of some periodic equilibrium patterns observed in physical systems involving short and long range competitions, see [29].

In this note, we survey recent results on embedded surfaces with nonzero constant nonlocal mean curvature together with their counterparts within the classical theory of constant mean curvature surfaces such as the Alexandrov's classification theorem, rotationally symmetric and periodic surfaces. Unlike the study of CMC surfaces, where we can rely on the theory ordinary differential equations in certain cases, the nonlocal case does not provide such advantage. Neither does it provide, in general, an analytic verification of parameterized surfaces to have CNMC. Up to now, almost all nontrivial surfaces with CNMC appearing in the literature are either derived by variational methods or by means of topological bifurcation theory. The study of solutions to partial differential equations with overdetermined boundary values is surprisingly intimately related with the question of finding CNMC surfaces. We also describe a number of such phenomena, notably those that were found recently as formal consequences of the highly nonlinear and nonlocal equations which are involved.

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2. Constant mean curvature hypersurfaces

Let Σ be an orientable C^2 hypersurface of $\mathbb{R}^N$ and denote by $\mathcal{V}_\Sigma : \Sigma \to \mathbb{R}^N$ the unit normal vector field on Σ. For every $p \in \Sigma$, we let $\{e_1; \dots; e_{N-1}\}$ be an orthonormal basis of the tangent plane $T_p\Sigma$ of Σ at p. The (normalized) *mean curvature* at p of Σ is given by

$$H(\Sigma; p) := \frac{1}{N-1} \sum_{i=1}^{N-1} \langle D\mathcal{V}_\Sigma(p) e_i, e_i \rangle. \tag{2.1}$$

Here and in the following, $\langle , \rangle$ and "$\cdot$" denote scalar product on $\mathbb{R}^N$. As a consequence, for a C^1-extension of $\mathcal{V}_\Sigma$ by a unit vector field $\widetilde{\mathcal{V}}_\Sigma$ in a neighborhood of p in $\mathbb{R}^N$, we have

$$H(\Sigma; p) := \frac{1}{N-1} \mathrm{div}_{\mathbb{R}^N} \widetilde{\mathcal{V}}_\Sigma(p).$$

Let Ω and E be two open subsets of $\mathbb{R}^N$ with $E \subset \Omega$. Then the *perimeter functional of E relative to* Ω (total variation of 1_E in Ω) is given by

$$P(E, \Omega) = |D1_E|(\Omega) := \sup \left\{ \int_E \mathrm{div}\xi(x)\, dx \, : \, \xi \in C^\infty_c(\Omega; \mathbb{R}^N), \, |\xi| \leq 1 \right\}.$$

In the following, we will simply write $P(E) := P(E, \mathbb{R}^N)$.

Definition 2.1. *We consider a vector field $\zeta \in C^\infty_c(\mathbb{R}^N, \mathbb{R}^N)$ and define the flow $(Y_t)_{t\in\mathbb{R}}$ induced by ζ given by*

$$\begin{cases} \partial_t Y_t(x) = \zeta(Y_t(x)) & t \in \mathbb{R} \\ Y_0(x) = x & \text{for all } x \in \mathbb{R}^N. \end{cases} \tag{2.2}$$

For $E \subset \mathbb{R}^N$, we call the family of sets $E_t := Y_t(E)$, $t \in \mathbb{R}$, the variation of E with respect to the vector field ζ.

Physically, constant mean curvatures surfaces appear also when looking at surface at equilibrium enclosing a given volume in the absence of gravity. We have the well know formula for first variation of area, see [76].

Proposition 2.2. *Let Ω and E be two bounded domains of $\mathbb{R}^N$, with E of class C^2. Let $\lambda \in \mathbb{R}$ and $(E_t)_t$ be a variation of E with respect to $\zeta \in C^\infty_c(\Omega; \mathbb{R}^N)$. Then the map $t \mapsto J(t) := P(E_t, \Omega) - \lambda |E_t \cap \Omega|$ is differentiable at zero. Moreover*

$$J'(0) = (N-1) \int_{\partial E} \{H(\partial E; p) - \lambda\}\, v(p)\, dV(p),$$

where $v(p) := \langle \zeta(p), \mathcal{V}_{\partial E}(p) \rangle$ and $\mathcal{V}_{\partial E}$ is the unit exterior normal vector field of E.

The first classification result of compact CMC surfaces is due to Jellet in the 18$^{\text{th}}$ in [51], showing that a compact *CMC* surface, enclosing a star-shaped domain in $\mathbb{R}^3$ must be the standard sphere. An other classification is due to Alexandrov in [2], which we record in the following

Theorem 2.3. *An embedded closed C^2 hypersurface in $\mathbb{R}^N$, with nonzero constant mean curvature is a finite union of disjoint round spheres with same radius.*

Alexandrov's result for embedded CMC hypersurfaces provides, in particular, a positive partial answer to a conjecture of H. Hopf, stating that: a compact orientable CMC surface must be a sphere. Hopf gave a positive answer to his conjecture in [47] for the case of CMC immersions of S^2 into $\mathbb{R}^3$. As what concerns immersed hypersurfaces with genus larger than 1, Hsiang found in [49] a first counterexample in $\mathbb{R}^4$ to Hopf's conjecture. Later Wente constructed in [79] an immersion of the torus $\mathbb{T}^2$ in $\mathbb{R}^3$ having constant mean curvature. While Wente's CMC torus has genus $g = 1$, compact CMC immersions with any genus $g > 1$ in $\mathbb{R}^3$ were obtained by N. Kapouleas [52]. For a recent survey on constant mean curvature surfaces, we refer the reader to [58] and the references therein.
Alexandrov introduced the method of moving plane in the proof of Theorem 2.3. Formally, the argument goes as follows. Consider a connected hypersurface Σ of class C^2 and pick a unit vector $e \in \mathbb{R}^N$ together with a hyperplane P_e which is perpendicular to e and not intersecting Σ. Then slide the plane along the e-direction, toward Σ, until one of the two properties occurs for the first time:

1. *interior touching:* the reflection of Σ with respect to P_e, called Σ_e, intersects Σ at a point $p_0 \notin P_e$,
2. *non-transversal intersection:* the vector $e \in T_{p_0}\Sigma$, for some $p_0 \in \Sigma \cap \Sigma_e$.

In either case, local comparison principles for elliptic equations applied to the graphs u and u_e, locally representing Σ and Σ_e, show that there exists an $r > 0$ such that $\Sigma \cap B_r(p_0) = \Sigma_e \cap B_r(p_0)$. Now by unique continuations property of elliptic PDEs, we find that $\Sigma = \Sigma_e$. Since e is arbitrary, this implies that Σ must be a round sphere. Indeed, letting H the mean curvature of Σ, then u and u_e satisfies

$$\mathrm{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \mathrm{div}\frac{\nabla u_e}{\sqrt{1+|\nabla u_e|^2}} = H \qquad \text{on } B,$$

for some centered ball $B \subset \mathbb{R}^{N-1} \simeq T_{p_0}\Sigma$. Moreover, making the ball smaller if necessary, we have $u \geq u_e$ on B. The main point here is that $w = u - u_e$ solves an elliptic equation of the form

$$\partial_i(a_{ij}(x)\partial_j w) = 0 \qquad \text{on } B,$$

for a symmetric matrix $(a_{ij}(x))_{1\leq i,j\leq N-1}$ with positive and C^1 coefficients, only depending on the partial derivatives of u and u_e. The comparison principles that is need is resumed in the following result.

Lemma 2.4. *Let $w \in C^2(B) \cap C^1(\overline{B})$ be a nonegative function on $\overline{B}$ and satisfy*

$$\partial_i(a_{ij}(x)\partial_j w) = 0 \qquad \text{on } B,$$

for some positive definite matrix a_{ij} of class C^1. Then the following holds.

(i) If $w(x_0) = 0$, for some $x_0 \in \partial B$ then $w = 0$ in B.
(ii) If $w(x_0) = \nabla w(x_0) \cdot \nu = 0$, for some $x_0 \in \partial B$ and ν a unit vector normal to $T_{x_0}\partial B$, then $w = 0$ in B.

The proof of Lemma 2.4 can be found in many text books, see e.g. [44]. Lemma 2.4(i) and (ii) are now used, to deal with cases *1.* and *2.*, respectively, yielding $w = 0$ in B.

2.1. **Rotationally symmetric constant mean curvature surfaces.** An important class of CMC surfaces can be found in the family of unbounded rotationnaly symmetric ones. They were first studied in 1841 by Delaunay in [25], which he described explicitly as surfaces of revolution of *roulettes of the conics*[1]. These surfaces are the *catenoids, unduloids, nodoids and right circular cylinders*. They have the form

$$\Sigma = \{(x(s), y(s)\sigma) \,:\, s \in \mathbb{R},\, \sigma \in S^1\} \subset \mathbb{R} \times \mathbb{R}^2.$$

where the rotating plane curve $s \mapsto (x(s), y(s))$ is parmeterized by arclength and y a positive function on $\mathbb{R}$. The explicit form of the mean curvature of Σ at a point $q(s,\sigma) := (x(s), y(s)\sigma)$ is given by

$$H(\Sigma; q(s,\sigma)) = \frac{1}{2}\frac{-x'(s) + y(s)\{x'(s)y''(s) - x''(s)y'(s)\}}{y(s)}. \tag{2.3}$$

Of particular interest to us in this note are the *embedded* surfaces with nonzero CMC, the so called *unduloids*. They constitute a smooth 1-parameter family of surfaces $(\Sigma_b)_{b\in(0,1)}$ varying from the straight cylinder Σ_0 to a translation invariant tangent spheres Σ_1. Their explicit form was found by Kenmotsu in [53],

$$\Sigma_b = \{(x_b(s), y_b(s)\sigma) \,:\, s \in \mathbb{R},\, \sigma \in S^1\},$$

where

$$x_b(s) = \int_0^s \frac{1 + b\sin(hr)}{\sqrt{1 + b^2 + 2b\sin(hr)}}\, dr \qquad \text{and} \qquad y_b(s) = \frac{1}{h}\sqrt{1 + b^2 + 2b\sin(hs)}.$$

It is clearly that Σ_b is invariant under the translations $z \mapsto z + x_b(2k\pi/h)e_1$, with $e_1 = (1, 0, 0)$ and $k \in \mathbb{Z}$. Moreover $\Sigma_0 = \mathbb{R} \times \frac{1}{h}S^1$, the straight cylinder of width $\frac{1}{h}$. Furthermore, with the change of variables $\cos(\frac{\theta}{2}) = \frac{h}{2}y_1(s)$, for $s \in (-\frac{\pi}{2h}, \frac{\pi}{2h})$ and $\theta = \theta(s) \in (0,\pi)$, we see that

$$x_1(s) = \frac{\sqrt{2}}{h} - \frac{2}{h}\sin(\theta) \qquad \text{and} \qquad y_1(s) = \frac{2}{h}\cos(\theta).$$

That is

$$\Sigma_1 = \left\{ \frac{2}{h}\left(-\sin(\theta), \cos(\theta)\sigma\right) + \frac{\sqrt{2} + 4k}{h}e_1 \,:\, \theta \in [0,\pi],\, \sigma \in S^1,\, k \in \mathbb{Z} \right\},$$

where $e_1 = (1, 0, 0) \in \mathbb{R}^3$. Therefore Σ_1 is a family of tangent spheres with radius $\frac{2}{h}$, centered at the points $\frac{\sqrt{2}+4k}{h}e_1$, $k \in \mathbb{Z}$.

Next, we observe that for $b \in (0, 1)$, the map x_b is increasing on $\mathbb{R}$. Hence with the change of variable $t = x_b(s)$, and setting $\varphi(t) = y_b(x_b^{-1}(t))$, we deduce that

$$\Sigma_b = \{(t, \zeta) \in \mathbb{R} \times \mathbb{R}^2 \,:\, t \in \mathbb{R},\, |\zeta| = \varphi_b(t)\}.$$

Recently, Sicbaldi and Schlenk used in [71] the Crandall-Rabinowitz bifurcation theorem (see [22]) to derive these surfaces for b close to 0. We record it here.

Theorem 2.5. *There exist $b_0, h_* > 0$ and a smooth curve $(-b_0, b_0) \ni b \mapsto \lambda(b)$ such that $\lambda(0) = 1$ and*

$$\varphi_b(t) = \frac{1}{h_*} + \frac{b}{\lambda(b)}\left\{\cos\left(\lambda(b)t\right) + v_a(\lambda(b)t)\right\},$$

where $v_b \to 0$ in $C^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ as $b \to 0$ and $\int_{-\pi}^{\pi} v_b(t)\cos(t)\, dt = 0$ for every $b \in (-b_0, b_0)$.

[1] http://www.mathcurve.com/courbes2d/delaunay/delaunay.shtml

Remark 2.6. *We note that the family of surfaces* $(\Sigma_b)_{b>1}$ *are the immersed constant mean curvature surfaces known as the nodoids.*

Expression (2.3) was first derived by M. Sturm, in an appended note in [25], and characterizes the Delauney surfaces variationally, as the extremals of surfaces of rotation having fixed volume while maximizing lateral area. A gereralization of Delauney surfaces in higher dimension is due to Hsiang and Yu, [48].

3. Constant Nonlocal Mean Curvature hypersurfaces

Similarly to the mean curvature, the fractional or Nonlocal Mean Curvature (NMC for short) is an extrinsic geometric quantity that is invariant under global immersion representing a surface. Let $\alpha \in (0,1)$. If Σ is a smooth oriented hypersurface in $\mathbb{R}^N$ with unit normal vector field $\mathcal{V}_\Sigma$, its nonlocal mean curvature of order α at a point $x \in \Sigma$ is defined as

$$H_\alpha(\Sigma; x) = \frac{2}{\alpha} \int_\Sigma \frac{(y-x) \cdot \mathcal{V}_\Sigma(y)}{|y-x|^{N+\alpha}} \, dV(y). \tag{3.1}$$

If Σ is of class $C^{1,\beta}$ for some $\beta > \alpha$ and we assume $\int_\Sigma (1+|y|)^{1-N-\alpha} \, dV(y) < \infty$, then the integral in (3.1) is absolutely convergent in the Lebesgue sense. The orientation is chosen here so that H_α is positive for a sphere.
We note that if Σ is of class C^2, then the normalized nonlocal mean curvature $\frac{1-\alpha}{\omega'_N} H_\alpha(\Sigma; \cdot)$ converges, as $\alpha \to 1$, locally uniformly to the classical mean curvature $H(\Sigma; \cdot)$ defined in 2.1, see [1, 24]. Here ω'_N is the measure of the $(N-2)$-dimensional unit sphere.
There is an alternative expression for $H_\alpha(\Sigma; \cdot)$ in terms of a solid integral. Suppose that $\Sigma = \partial E$ for some open set $E \subset \mathbb{R}^N$ and $\mathcal{V}_{\partial E}$ is the normal exterior to E. Then, for all $x \in \Sigma$, we have

$$H_\alpha(\partial E; x) = PV \int_{\mathbb{R}^N} \frac{\tau_E(y)}{|y-x|^{N+\alpha}} \, dy = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\tau_E(y)}{|y-x|^{N+\alpha}} \, dy, \tag{3.2}$$

where $\tau_E := 1_{\mathbb{R}^N \setminus E} - 1_E$, with 1_D denotes the characteristic function of a set $D \subset \mathbb{R}^N$. This can be derived using the divergence theorem and the fact that

$$\nabla_y \cdot (y-x)|y-x|^{-N-\alpha} = -\alpha |y-x|^{-N-\alpha}. \tag{3.3}$$

Remark 3.1. *The formula of NMC in* (3.2) *is comparable with the one of the mean curvature, when written in infinitesimal solid integral,*

$$H(\partial E; x) := C \lim_{\varepsilon \to 0} \frac{-1}{\varepsilon |B_\varepsilon(x)|} \int_{B_\varepsilon(x)} \tau_E(y) \, dy,$$

for some constant $C > 0$ *depending only on* N*. Here the orientation is chosen so that the mean curvature is positive for a sphere. In particular the mean curvature of* ∂E *is an infinitesimal average, centered at* ∂E*, of the sum of the "-1" coming from inside* E *and the "+1" from outside* E*. After all by Young's law, the mean curvature measures pressure difference across the interface of two non mixing fluids at rest. On the other hand the nonlocal mean curvature is a weighted average of the sum of all the "-1" coming from inside* E *and all the "+1" from outside* E*.*

Both forms of the NMC (3.1) and (3.2) turn out to be useful depending on the users interests. For instance, global comparison principles is easily proved using (3.2), while expression

(3.1) (without principle value integration) seems more convenient to work with when dealing with regularity of the NMC operator acting on graphs. By noticing that if $E_1 \subset E_2$ then $\tau_{E_1} - \tau_{E_2} = 21_{E_1 \setminus E_2}$ on $\mathbb{R}^N$, we can state the following result.

Lemma 3.2. *Let E_1, E_2 be two open sets of class $C^{1,\beta}$, $\beta > \alpha$, in a neighborhood of $p \in \partial E_1 \cap \partial E_2$. If $E_1 \subset E_2$, then $H_\alpha(\partial E_2; p) \leq H_\alpha(\partial E_1; p)$, with equality if and only if $E_1 = E_2$, up to set of zero Lebesgue measure.*

Local comparison principle does not hold true in general as in the classical case.

Alike the mean curvature, the nonlocal mean curvature appears in the first variation of nonlocal perimeter functional as well. The fractional perimeter of a bounded open set $E \subset \mathbb{R}^N$ is given by

$$P_\alpha(E) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|1_E(x) - 1_E(y)|^2}{|y-x|^{N+\alpha}} \, dxdy = \int_E \int_{\mathbb{R}^N \setminus E} \frac{1}{|y-x|^{N+\alpha}} \, dxdy.$$

We note that, if E is a set with finite perimeter, the normalized fractional perimeter $\frac{1-\alpha}{\omega_{N-1}} P_\alpha(E)$ converges, as $\alpha \to 1$, to $P(E)$, see [4,24]. Here ω_{N-1} is the measure of the unit ball in $\mathbb{R}^{N-1}$. Isoperimetric type inequalities with respect to this functional was investigated in [41, 43]. According to a result of Frank and Seiringer [41], it is known that balls uniquely minimize P_α among all sets with a equal volume. A quantitative stability of the fractional isoperimetric inequality is proven by Fusco, Millot and Morini, [42].
Provided E is bounded with Lipschitz boundary, using integration by parts and (3.3), we can rewrite the fractional perimeter as

$$P_\alpha(E) = \frac{1}{\alpha(1+\alpha)} \int_{\partial E} \int_{\partial E} \frac{\mathcal{V}_{\partial E}(y) \cdot (y-x) \mathcal{V}_{\partial E}(x) \cdot (x-y)}{|y-x|^{N+\alpha}} \, dV(y) dV(x). \tag{3.4}$$

Note that (3.4) provides a natural way to define a nonlocal or fractional measure of an orientable compact hypersurface.
In the theory of minimal surfaces, the Plateau's problem is to show the existence of a surface that locally minimize perimeter with a given boundary. For a nonlocal setting of Plateau's problem, in [12] the authors introduced the fractional perimeter in a reference open set Ω given by

$$\begin{aligned} P_{\alpha,\Omega}(E) &= \frac{1}{2} \iint_{\mathbb{R}^{2N} \setminus (\Omega^c)^2} \frac{|1_E(x) - 1_E(y)|^2}{|x-y|^{N+\alpha}} \, dxdy \\ &= L(E \cap \Omega, E^c \cap \Omega) + L(E \cap \Omega, E \cap \Omega^c) + L(E^c \cap \Omega, E^c \cap \Omega^c), \end{aligned}$$

where $A^c := \mathbb{R}^N \setminus A$ and the interaction functional is given by

$$L(A, B) := \int_B \int_A \frac{dxdy}{|x-y|^{N+\alpha}}.$$

We note that the interaction between $E^c \cap \Omega^c$ and $E \cap \Omega^c$ is left free. This, allows a wellposed setting of a (nonparametric) nonlocal Plateau's problem: existence of sets minimizing $P_{\alpha,\Omega}$ among all sets that coincide in Ω^c. The seminal paper [12] established the first existence and regularity results for nonlocal perimeter minimizing sets in a reference set Ω, and moreover the boundary of the minimizers have, in a viscosity sense, zero NMC in Ω. In the literature the boundary of such sets are called *s-minimal or nonlocal minimal hypersurfaces.*

The following formula for the first variation of fractional perimeter was found in [12, 38].

Theorem 3.3. *Let Ω and E be two domains of $\mathbb{R}^N$, with E of class $C^{1,\beta}$, $\beta > \alpha$. Let $\lambda \in \mathbb{R}$ and $(E_t)_{t\in\mathbb{R}}$ be a variation of E with respect to $\zeta \in C_c^\infty(\Omega;\mathbb{R}^N)$. Then the map $t \mapsto J_\alpha(t) := P_{\alpha,\Omega}(E_t) - \lambda|E_t \cap \Omega|$ is differentiable at zero. Moreover*

$$J'_\alpha(0) = (N-1)\int_{\partial E} \{H_\alpha(\partial E; p) - \lambda\}\, v(p)\, dV(p),$$

where $v(p) := \langle \zeta(p), \mathcal{V}_{\partial E}(p)\rangle$.

The paper [38] contains also the second variation of fractional perimeter.
From now on, we say that Σ is a CNMC hypersurface if it is of class $C^{1,\beta}$, for some $\beta > \alpha$, and such that $H_\alpha(\Sigma, \cdot)$ is a *nonzero* constant on Σ.
Even though we will be mainly interested in this note to CNMC hyperusfrcaes, we find it important to make a brief digression in the theory of the nonlocal minimal hypersurfaces and those with vanishing NMC but not necessarily minimizing fractional perimeter, since it is also a hot topic nowadays with challenging open questions, see Bucur and Valdinoci [6].
We first recall that besides the hyperplane, which trivially has zero nonlocal mean curvature, there are some nontrivial ones: the s-Lawson cones found by Dávila, del Pino and Wei [24]; the helicoid found by Cinti, del Pino and Dávila in [19]. Moreover, local inversion arguments have been used in [24], to derive, for α close to 1, two interesting examples of rotationally symmetric surfaces with zero nonlocal mean curvature. The first one posses the shape of the catenoid whereas the other is disconnected with two ends which are asymptotic to a cone of revolution.
Regularity theory, stability and Bernstein-type results in the nonlocal setting are also studied in the recent years. Recall that in the theory of constant mean curvature surfaces, the boundary of perimeter minimizing regions are smooth except a closed singular set of Hausdorff dimension at most $N-8$. Such regularity result, in its full generality, is still not known to be true in the nonlocal setting. The progress made in this direction so far parallel the classical regularity theory up to a *possible dimension shift.* This latter fact was discovered by Dávila, del Pino and Wei in [24] proving, for α close to 0 and $N = 7$, that there are nonlocal stable minimal surfaces (s-Lawson) cones — but their globally minimizing property is an open problem. On the other hand when α is close to 1, Caffarelli and Valdinoci [14, 15] proved the $C^{1,\gamma}$ regularity of nonlocal minimal hypersurfaces, except a set of $(N-8)$-Hausdorff dimension. This, together with the subsequent results of Barrios, Figalli, and Valdinoci [5], leads, for α close to 1, to their C^∞ regularity up to dimension $N \le 7$. We note that for $N = 2$ and any $\alpha \in (0,1)$, Savin and Valdinoci [69] established that nonlocal minimal curves are C^∞. Decisive regularity estimates have been proved in [7, 8, 12, 20, 39], see also the monograph [6] and the references there in.

We close this section by noting that nonlocal capillary problems have been investigated in [57, 59] and nonlocal mean curvature flow in [18, 50, 68].

3.1. **Expressions of the NMC of some globally parameterized hypersurfaces.** In this section, we derive some formulae of some hypersurfaces which admit global parameterizations. We will use both expression of the NMC in (2.1) and (3.1). Depending on the problems under study, each expression has its own advantages. For instance to prove regularity of the NMC operator, it is in general more convenient to consider those formulae from (2.1), whereas those from (3.1) turns out to be useful in the study of qualitative properties of the NMC acting on graphs as a quasilinear nonlocal elliptic operator.

3.1.1. *Without principle value integration.* We have the following result.

Proposition 3.4. *Let $N = n + m \geq 1$, with $n, m \in \mathbb{N}$. Let $u : \mathbb{R}^m \to (0, +\infty)$ be a function of class $C^{1,\beta}$, $\beta > \alpha$, and satisfy*

$$\int_{\mathbb{R}^m} \frac{(1 + |\nabla u(\tau)|)u^{n-1}(\tau)}{(1 + |\tau| + u(\tau))^{N-1+\alpha}} d\tau < \infty.$$

Consider the set

$$E_u := \{(s, \zeta) \in \mathbb{R}^m \times \mathbb{R}^n \, : \, |\zeta| < u(s)\}.$$

(i) *For $n \geq 2$, the NMC of Σ_u at the point $q = (s, u(s)e_1)$ is given by*

$$\begin{aligned} -\frac{\alpha}{2} H_\alpha(\Sigma_u; q) = & \int_{S^{n-1}} \int_{\mathbb{R}^m} \frac{\big\{u(s) - u(s-\tau) - \tau \cdot \nabla u(s-\tau)\big\} u^{n-1}(s-\tau)}{\{|\tau|^2 + (u(s) - u(s-\tau))^2 + u(s)u(s-\tau)|\sigma - e_1|^2\}^{(N+\alpha)/2}} d\tau d\sigma \\ & - \frac{u(s)}{2} \int_{S^{n-1}} \int_{\mathbb{R}^m} \frac{|\sigma - e_1|^2 u^{n-1}(s-\tau)}{\{|\tau|^2 + (u(s) - u(s-\tau))^2 + u(s)u(s-\tau)|\sigma - e_1|^2\}^{(N+\alpha)/2}} d\tau d\sigma, \end{aligned} \tag{3.5}$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$.

(ii) *For $n = 1$, the NMC of Σ_u at the point $q = (s, u(s))$ is given by*

$$\begin{aligned} -\frac{\alpha}{2} H(\Sigma_u; q) = & \int_{\mathbb{R}^{N-1}} \frac{u(s) - u(s-\tau) - \tau \cdot \nabla u(s-\tau)}{\{|\tau|^2 + (u(s) - u(s-\tau))^2\}^{(N+\alpha)/2}} d\tau \\ & - \int_{\mathbb{R}^{N-1}} \frac{u(s) + u(s-\tau) + \tau \cdot \nabla u(s-\tau)}{\{|\tau|^2 + (u(s) + u(s-\tau))^2\}^{(N+\alpha)/2}} d\tau. \end{aligned} \tag{3.6}$$

Moreover, all integrals above converge absolutely in the Lebesgue sense.

Expression (3.5) and (3.6) are easily derived from (3.1), by change of variables, taking into account that the unit outer normal of $\partial\Sigma_u$ at the point $q = (s, u(s)\sigma)$ is given by

$$\mathcal{V}_{\partial E_u}(q) = \frac{1}{\sqrt{1 + |\nabla u|^2(s)}}(-\nabla u(s), \sigma)$$

and the volume element is $u^{n-1}(s)\sqrt{1 + |\nabla u|^2(s)} ds d\sigma$. The proof uses similar arguments as in [10]. We note that the first term in the right hand of (3.6) provides the expression of NMC of an euclidean graph $x_N = u(x')$ without principle value integral.

3.1.2. *With principle value integration.* We consider a function $u : \mathbb{R}^{N-1} \to \mathbb{R}$ of class $C^{1,\beta}$, for some $\beta > \alpha$. We let

$$E_u = \big\{(y, t) \in \mathbb{R}^{N-1} \times \mathbb{R} \, : \, t < u(y)\big\}$$

and consider the parametrization

$$\mathbb{R}^N \to \mathbb{R}^N \qquad (y, t) \mapsto \mathcal{F}(y, t) = (y, u(y) + t)\,.$$

It is well known that this parametrization is *volume preserving*. We now compute the NMC of the graph u (denoted by $H(u)$) at the point $\mathcal{F}(x,0)=(x,u(x))\in\partial E_u$. From (3.1), making change of variables and using Fubini's theorem, we have

$$\begin{aligned}H(u)(x):=H_\alpha(\partial E_u;(x,u(x)))&=\int_{\mathbb{R}^N}\frac{\tau_{E_0}(y,t)}{|(x,u(x))-\mathcal{F}(y,t)|^{N+\alpha}}dydt\\&=\int_{\mathbb{R}^{N-1}}I(x,y)dy,\end{aligned}$$

where

$$\begin{aligned}I(x,y)&=[\int_{-\infty}^{0}-\int_{0}^{+\infty}]\left\{|x-y|^2+(t+u(y)-u(x))^2\right\}^{\frac{-(N+\alpha)}{2}}dt\\&=|x-y|^{-(N+\alpha)}[\int_{-\infty}^{0}-\int_{0}^{+\infty}]\left\{1+\left(\frac{t+u(y)-u(x)}{|x-y|}\right)^2\right\}^{\frac{-(N+\alpha)}{2}}dt.\end{aligned}$$

We then make the change of variables $s=\frac{t+u(y)-u(x)}{|x-y|}$ and define

$$p_u(x,y)=\frac{u(y)-u(x)}{|x-y|},\tag{3.7}$$

to get

$$I(x,y)=-|x-y|^{-(N-1+\alpha)}[\int_{-\infty}^{p_u(x,y)}-\int_{p_u(x,y)}^{+\infty}]\left\{1+s^2\right\}^{\frac{-(N+\alpha)}{2}}ds.$$

Lgtting

$$F(p):=\int_p^{+\infty}\frac{d\tau}{(1+\tau^2)^{\frac{(N+\alpha)}{2}}},\tag{3.8}$$

we find that

$$H(u)(x)=\int_{\mathbb{R}^{N-1}}\frac{F(p_u(x,y))-F(-p_u(x,y))}{|x-y|^{N-1+\alpha}}dy.$$

We then have the following formulae for the NMC of the hypersurface ∂E_u.

Proposition 3.5. *Let $u:\mathbb{R}^{N-1}\to\mathbb{R}$ of class $C^{1,\beta}$, $\beta>\alpha$. Consider the subgraph of u,*

$$E_u=\left\{(y,t)\in\mathbb{R}^{N-1}\times\mathbb{R}\,:\,t<u(y)\right\}.$$

At a point $q=(x,u(x))\in\partial E_u$, we have

$$\begin{aligned}H(u)(x):=H_\alpha(\partial E_u;q)&=PV\int_{\mathbb{R}^{N-1}}\frac{F(p_u(x,y))-F(-p_u(x,y))}{|x-y|^{N-1+\alpha}}dy\qquad(3.9)\\&=PV\int_{\mathbb{R}^{N-1}}\frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\,q_u(x,y)\,dy,\qquad(3.10)\end{aligned}$$

where p_u and F are given by (3.7) *and* (3.8), *respectively, and*

$$q_u(x,y)=\int_{-1}^{1}\left(1+\tau^2\frac{(u(x)-u(y))^2}{|x-y|^2}\right)^{-\frac{N+\alpha}{2}}d\tau.$$

Here, (3.10), is a consequence of the fundamental theorem of calculus, which yields

$$F(p) - F(-p) = p \int_{-1}^{1} F'(\tau p)\, d\tau.$$

As in the classical case, we may expect that the difference of the NMC operator of two graphs gives rise to a nonlocal symmetric operator allowing for local comparison principles.

Lemma 3.6. *Let Ω be an open set of $\mathbb{R}^{N-1}$ and $u, v \in C^{1,\beta}_{loc}(\Omega) \cap C(\mathbb{R}^{N-1})$. Then letting $w = u - v$, for every $x \in \Omega$, we have*

$$H(u)(x) - H(v)(x) = PV \int_{\mathbb{R}^{N-1}} \frac{w(x) - w(y)}{|x-y|^{N+\alpha}} \widetilde{q}_{u,v}(x, y)\, dy, \tag{3.11}$$

where $\widetilde{q}_{u,v}(x, y) := -2 \int_0^1 F'\left(p_v(x, y) + \rho p_{u-v}(x, y)\right) d\rho$. In particular, if

$$H(u)(x) - H(v)(x) \geq 0 \qquad \text{for every } x \in \Omega$$

and $u \geq v$ on $\mathbb{R}^{N-1} \setminus \Omega$, then $u - v$ cannot attain its global minimum in Ω unless it is constant on $\mathbb{R}^{N-1}$.

Proof. Since F' is even, by the fundamental theorem of calculus, we get

$$[F(a) - F(b)] - [F(-a) - F(-b)] = 2(a - b) \int_0^1 F'\left(b + \rho(a - b)\right) d\rho.$$

In view of (3.9), this gives (3.11) since $p_w(x, y) = -\frac{w(x)-w(y)}{|x-y|}$.
For the comparison principle, we suppose that, for some $x_0 \in \Omega$, we have $w(x_0) = \min_{x \in \mathbb{R}^{N-1}} w(x)$. We then have

$$\begin{aligned} 0 &\leq H(u)(x_0) - H(v)(x_0) \\ &= -2PV \int_{\mathbb{R}^{N-1}} \frac{w(x_0) - w(y)}{|x_0 - y|^{N-1+\alpha}} \int_0^1 F'\left(p_v(x_0, y) + \rho p_w(x_0, y)\right) d\rho\, dy \leq 0. \end{aligned}$$

Therefore, since $F' < 0$ on $\mathbb{R}$, we deduce that $w \equiv w(x_0)$ on $\mathbb{R}^{N-1}$. $\square$

In the case of generalized slabs, we have the following result, the proof is similar to one in [10].

Proposition 3.7. *Let $u : \mathbb{R}^m \to (0, +\infty)$ be a a function of class $C^{1,\beta}$ and $N = m + n$. Consider the open set*

$$E_u := \{(s, \zeta) \in \mathbb{R}^m \times \mathbb{R}^n \,:\, |\zeta| < u(s)\}.$$

Then at the point $q = (s, u(s)z)$, we have

$$H_\alpha(\partial E_u; q) = PV \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \frac{-\tau_{E_1}(\bar{s}, z)}{\left((s - \bar{s})^2 + (u(s)e_1 - u(\bar{s})z)^2\right)^{\frac{N+\alpha}{2}}} u^n(\bar{s})\, dz d\bar{s}.$$

We note that in the expressions of the NMC with PV in Proposition 3.5 and 3.7, no growth control of u at infinity is required.

3.2. **Bounded constant nonlocal mean curvature hypersurfaces.** In addition to the cylinders, spheres, which are CNMC with nozero NMC, we shall see that there many more. However, this class is reduced by a nonlocal counterpart of the Alexandrov result on the characterization of spheres as the only closed embedded CMC-hypersurfaces.

Theorem 3.8 ([11, 21]). *Suppose that E is a nonempty bounded open set (not necessarily connected) with $C^{2,\beta}$-boundary for some $\beta > \alpha$ and with the property that $H_\alpha(\partial E; \cdot)$ is constant on ∂E. Then E is a ball.*

This result was obtained at the same time and independently by Cabré, Solà-Morales, Weth and the author in [11] and by Ciraolo, Figalli, Maggi, and Novaga [21]. We mention that the paper [21] contains also stability results with respect to this rigidity theorem.
This new characterization of the sphere relies on the Alexandrov's moving planes method discussed above. Indeed, pick a unit vector $e \in \mathbb{R}^N$ together with a hyperplane P_e which is perpendicular to e and not intersecting E. Call E_e the reflection of E with respect to the plane P_e. Then slide the plane along the e-direction, toward E, until one of the two properties occurs for the first time:

1. *interior touching:* there exists $x_0 \notin P_e$, with $x_0 \in \partial E \cap \partial E_e$,
2. *non-transversal intersection:* $e \in T_{x_0}\partial E$, for some $x_0 \in \partial E$.

In both cases, we will find that $E_e = E$, and since e is arbitrary, this implies that E must be a unit ball. As mentioned earlier, in contrast to the classical case, there is no local comparison principle related to the fractional mean curvature. Moreover, while comparison principle holds for graphs (see Lemma 3.6), we dare not hope for a global parameterization of E by a graph! However, we may rely on a global comparison principles inherent to problem. The montonicity of the weight $t \mapsto K_N(t) := |t|^{-N-\alpha}$ will be crucial. Indeed, in case *1.*, we have

$$
\begin{aligned}
0 = H_\alpha(\partial E; x_0) - H_\alpha(\partial E_e; x_0) &= \frac{1}{2} PV \int_{\mathbb{R}^N} (\tau_E(y) - \tau_{E_e}(y)) K_N(x_0 - y)\, dy \\
&= \frac{1}{2} PV \int_{E \setminus E_e} (\tau_E(y) - \tau_{E_e}(y)) K_N(x_0 - y)\, dy + \frac{1}{2} PV \int_{E_e \setminus E} (\tau_E(y) - \tau_{E_e}(y)) K_N(x_0 - y)\, dy \\
&= \frac{1}{2} PV \int_{E \setminus E_e} K_N(x_0 - y)\, dy - \frac{1}{2} PV \int_{E_e \setminus E} K_N(x_0 - y)\, dy \\
&= \frac{1}{2} PV \int_{E \setminus E_e} \left(K_N(x_0 - z) - K_N(x_0 - \mathcal{R}(z))\right)\, dz.
\end{aligned}
$$

We made the change of variable, $y = \mathcal{R}_e(z)$, with $\mathcal{R}_e$ being the reflection with respect to the plane P_e. Now since $E \setminus E_e$ is contained on the side of the plane containing x_0, we find that $|x_0 - z| < |x_0 - \mathcal{R}(z)|$ for every $z \in E \setminus E_e$, together with the monotonicity of K_N imply that $E = E_e$.
Case *2.* is far more tricky and requires a formula for directional derivative of $H_\alpha(\partial E; \cdot)$, see

Proposition 3.9 below. Since $e \in T_{x_0}\partial E = T_{x_0}\partial E_e$, we have

$$\begin{aligned}
0 &= \frac{1}{2}\big(\partial_e H_\alpha(\partial E; x_0) - \partial_e H_\alpha(\partial E_e; x_0)\big) \\
&= -\frac{N+\alpha}{2} PV \int_{\mathbb{R}^N} (x_0 \cdot e - y \cdot e)\big(\tau_E(y) - \tau_{E_e}(y)\big) K_{N+2}(x_0 - y)\, dy \\
&= -\frac{N+\alpha}{2} PV \int_{\mathbb{R}^N} (x_0 \cdot e - y \cdot e)(1_{E\setminus E_e}(y) - 1_{E_e \setminus E}(y)) K_{N+2}(x_0 - y)\, dy.
\end{aligned}$$

Since the integrand does not change sign on $\mathbb{R}^N$, there must be $E = E_e$. We then conclude that in either case $E = E_e$, and since e is arbitrary, we deduce that E is a ball.
Of course to make all these argument rigorous, one should take into account that the integrals are defined in the principle value sense, see [11, 21] for more details.

The following formula for the derivatives of the nonlocal mean curvature was used above, see [11, 21].

Proposition 3.9. *If $E \subset \mathbb{R}^N$ is bounded and ∂E is of class $C^{2,\beta}$ for some $\beta > \alpha$, then H_E is of class C^1 on ∂E, and we have*

$$\partial_v H_\alpha(\partial E; x) = -(N+\alpha)\, PV \int_{\mathbb{R}^N} \tau_E(y)|x-y|^{-(N+2+\alpha)}(x-y)\cdot v\, dy$$

for $x \in \partial E$ and $v \in T_x\partial E$.

In order to reduce to a single sphere, connectedness is obviously a necessary assumption in Theorem 2.3 whereas Theorem 3.8 allow for disconnected sets E with finitely many connected components. Similar pheonomenon was also observed in the study of fractional overdetermined problems, see [33].

3.3. **Unbounded constant nonlocal mean curvature hypersurfaces.** When compactness is dropped, the cylinder provides a trivial example of surfaces with constant and nonzero local/nonlcal mean curvature. Obviously, CMC curves bounding a periodic domain in 2-dimension are parallel straight lines and hence has zero boundary mean curvature. This latter fact does not hold in the nonlocal case, since two parallel straight lines in $\mathbb{R}^2$ has positive CNMC. In fact, any slab $\{(s,\zeta) \in \mathbb{R}^n \times \mathbb{R}^m : |\zeta| = 1\}$ has a positive CNMC, as it can be easily seen from Proposition 3.4. It is therefore natural to ask if there are CNMC hypersurfaces besides the cylinder?
The lack of ODE theory in the nonlocal framework make this question not trivial to answer. Up to now, all existence results use either variational methods or perturbation methods.

The first answer to the above question, for $N = 2$, has been given by Cabré, J. Solà-Morales, Weth and the author in [11], where a continuous branch of periodic CNMC sets bifurcating from the straight band was found. This was improved and generalized in [10] to higher dimensions. We note that in [23], Dávila, del Pino, Dipierro, and Valdinoci, established variationally the existence of periodic and cylindrically symmetric hypersurfaces in $\mathbb{R}^N$ which minimize the periodic fractional perimeter under a volume constraint. More precisely, [23] establishes the existence of a 1-periodic minimizer for every given volume within the slab $\{(s,\zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} : -1/2 < s < 1/2\}$, which are CNMC hypersurfaces in the viscosity sense of [12], since it is not known if they are of class $C^{1,\beta}$ for some $\beta > \alpha$.

3.3.1. *CNMC hypersurface of revolution.* We consider hypersurfaces with constant nonlocal mean curvature of the form

$$\Sigma_u = \{(s,\zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} \,:\, |\zeta| = u(s)\},$$

where $u : \mathbb{R} \to (0,\infty)$ is a positive and even function. The following result shows the existence of a smooth branch of sets which are periodic in the variable s and have all the same constant nonlocal mean curvature; they bifurcate from a straight cylinder $\Sigma_R := \{|\zeta| = R\}$. The radius R of the straight cylinder is chosen so that the periods of the new cylinders converge to 2π as they approach the straight cylinder. We state the counterpart of Theorem 2.5.

Theorem 3.10. *Let $N \geq 2$. For every $\alpha \in (0,1)$, $\beta \in (\alpha,1)$ there exist $R, a_0 > 0$ and smooth curves $a \mapsto u_a$ and $a \mapsto \lambda(a)$, with $\lambda(0) = 1$, such that, for every $a \in (-a_0, a_0)$,*

$$\Sigma_{u_a} = \{(s,\zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} \,:\, |\zeta| = u_a(s)\}$$

is a CNMC hypersurface of class $C^{1,\beta}$, with

$$H_\alpha(\Sigma_{u_a}; q) = H_\alpha(\Sigma_R; \cdot), \qquad \text{for all } q \in \partial\Sigma_{u_a}.$$

Moreover for every $a \in (-a_0, a_0)$, we have

$$u_a(s) = R + \frac{a}{\lambda(a)} \left\{\cos\left(\lambda(a)s\right) + v_a(\lambda(a)s)\right\}, \qquad u_{-a}(s) = u_a\Big(s + \frac{\pi}{\lambda(a)}\Big),$$

where $v_a \to 0$ in $C^{1,\beta}(\mathbb{R}/2\pi\mathbb{Z})$ as $a \to 0$ and $\int_{-\pi}^{\pi} v_a(t)\cos(t)\,dt = 0$ for every $a \in (-a_0, a_0)$.

The proof of Theorem 3.10 rests on the application the Crandall-Rabinowitz bifurcation theorem [22], applied to the quasilinear type fractional elliptic equation, see Proposition 3.4,

$$H_\alpha(\Sigma_u; \cdot) - H_\alpha(\Sigma_R; \cdot) = 0 \qquad \text{in } \partial\Sigma_u.$$

To do so and to obtain a smooth branch of bifurcation parameter, we need the smoothness of $u \mapsto (s \mapsto H_\alpha(\Sigma_u; (s, u(s)e_1)))$ as map from an open subset of $C^{1,\beta}(\mathbb{R})$ taking values in $C^{0,\beta-\alpha}(\mathbb{R})$. This is a nontrivial task, and as such the use of the formula for the NMC without PV (see Proposition 3.4) become crucial.

The first question that may come to the reader's mind, is the existence of global continuous $(\Sigma_{u_a})_{a\in(0,1)}$, branches of nonlocal Delaunay hypersurfaces and the transition from unduloids to a periodic array of spheres as in the local case ($\alpha = 1$) discussed in Section 2.1. It is indeed an open question. However some key differences are expected in the embedded regime $a \in (0,1)$. Indeed, two results suggest that as the bifurcation parameter a varies from 0 to 1, the hypersurfaces Σ_{u_a} should approach an infinite compound of not-round-spheres. The first one being [23], where the authors proved that the enclosed sets of their (weak) CNMC Delauney hypersurface, as their constraint volume goes to zero, tend in measure (more precisely, in the so called Fraenkel asymmetry) to a periodic array of balls. The second one is a consequence of the work of Cabré, Weth and the author in [9], where it is proven that an array of periodic disjoint round spheres is not necessary a CNMC hypersurface. We detail a bit more on the latter result in the following section.

3.3.2. *Near-sphere lattices with CNMC.* In this section, we consider CNMC hypersurfaces given by an infinite compound of aligned round spheres, tangent or disconnected, enlightening possible limiting configuration of nonlocal unduloids Σ_{u_a} (from Theorem 3.10) as the bifurcation parameter becomes large. In a more general setting, we can look for CNMC hypersurfaces which are countable union of a certain bounded domain. To be precise, we assume $N \geq 2$ and let

$$S := S^{N-1} \subset \mathbb{R}^N$$

denote the unit centered sphere of $\mathbb{R}^N$. For $M \in \mathbb{N}$ with $1 \leq M \leq N$ we regard $\mathbb{R}^M$ as a subspace of $\mathbb{R}^N$ by identifying $x' \in \mathbb{R}^M$ with $(x', 0) \in \mathbb{R}^M \times \mathbb{R}^{N-M} = \mathbb{R}^N$. Let $\{\mathbf{a}_1; \dots; \mathbf{a}_M\}$ be a basis of $\mathbb{R}^M$. By the above identification, we then consider the M-dimensional lattice

$$\mathscr{L} = \left\{ \sum_{i=1}^{M} k_i \mathbf{a}_i \, : \, k = (k_1, \dots, k_M) \in \mathbb{Z}^M \right\} \tag{3.12}$$

as a subset of $\mathbb{R}^N$. In the case where $\{\mathbf{a}_1; \dots; \mathbf{a}_M\}$ is an orthogonal or an orthonormal basis, we say that $\mathscr{L}$ is a rectangular lattice or a square lattice, respectively. We define, for $r > 0$,

$$\mathscr{S}_0^r := S + r\mathscr{L} := \bigcup_{p \in \mathscr{L}} \Big(S + rp \Big) \ \subset \ \mathbb{R}^N.$$

For r large enough, the set $\mathscr{S}_0^r$ is the union of disjoint unit spheres centered at the lattice points in $r\mathscr{L}$. Consequently, $\mathscr{S}_0^r$ is a set of constant classical mean curvature (equal to one). In contrast, we shall see that the NMC function $H_\alpha(\mathscr{S}_0^r; \cdot)$ is in general *not* constant on $\mathscr{S}_0^r$. However nearby $\mathscr{S}_0^r$, one may expect CNMC sets for r large enough. Indeed, as $r \to \infty$, the hypersurface $\mathscr{S}_0^r$ tends to the single centered sphere S, which has CNMC. On the other hand, by invariance under translations $\mathbb{R}^N$, the linearized NMC operator about S has an N-dimensional kernel spanned by the coordinates functions $x_1, \dots, x_N$. Taking advantages on the invariance of the NMC operator by even reflections, we shall see that this program is realizable. Indeed, we consider the open set

$$\mathcal{O} := \{\varphi \in C^{1,\beta}(S) \, : \, \|\varphi\|_{L^\infty(S)} < 1, \ \varphi \text{ is even on } S\},$$

with $\beta \in (\alpha, 1)$, and the deformed sphere $S_\varphi := \{(1 + \varphi(\sigma))\sigma \, : \, \sigma \in S\}, \qquad \varphi \in \mathcal{O}$. Provided that $r > 0$ is large enough, the deformed sphere lattice

$$\mathscr{S}_\varphi^r := S_\varphi + r\mathscr{L} := \bigcup_{p \in \mathscr{L}} \Big(S_\varphi + rp \Big)$$

is a noncompact periodic hypersurface of class $C^{1,\beta}$. We have the following result.

Theorem 3.11 ([9])**.** *Let $\alpha \in (0, 1)$, $\beta \in (\alpha, 1)$, $N \geq 2$, $1 \leq M \leq N$ and $\mathscr{L}$ be an M-dimensional lattice. Then, there exist $r_0 > 0$, and a C^2-curve $(r_0, +\infty) \to \mathcal{O}$, $r \mapsto \varphi_r$ such that for every $r \in (r_0, +\infty)$, the hypersurface $S_{\varphi_r} + r\mathscr{L}$ has constant nonlocal mean curvature given by $H_\alpha(S_{\varphi_r} + r\mathscr{L}; \cdot) \equiv H_\alpha(S; \cdot)$.*
Moreover if $1 \leq M \leq N - 1$, then the functions φ_r, $r > r_0$, are non-constant on S.

As a consequence a periodic array of aligned spheres does not necessarily have constant nonlocal mean curvature. A Taylor expansion of the perturbation φ_r, as $r \to \infty$, shows how the near-spheres interact to form a CNMC hypersurface. To see this, we consider the linearized operator for the NMC operator acting on graphs on the unit sphere S, given by

$$\varphi \mapsto 2(L_\alpha \varphi - \lambda_1 \varphi),$$

where

$$L_\alpha \varphi(\theta) = PV \int_S \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \, dV(\sigma).$$

The operator L_α can be seen as a *spherical fractional Laplacian*, and the above integral is understood in the principle value sense. It has the spherical harmonics as eigenfunctions corresponding to an increasing sequence of eigenvalues $\lambda_0 = 0 < \lambda_1 < \lambda_2 < ....$
The function φ_r in Theorem 3.11 expands as

$$\varphi_r(\theta) = r^{-N-\alpha} \left(-\kappa_0 + r^{-2} \left\{ \kappa_1 \sum_{p \in \mathscr{L}_*} \frac{(\theta \cdot p)^2}{|p|^{N+\alpha+4}} - \kappa_2 \right\} + o\left(r^{-2}\right) \right) \quad \text{for } \theta \in S \text{ as } r \to +\infty,$$

where $\mathscr{L}_* := \mathscr{L} \setminus \{0\}$, for some positive constants $\kappa_0, \kappa_2, \kappa_3$, only depending on $N, \alpha, \lambda_1, \lambda_2$ and on $\mathscr{L}$. Since $\kappa_0 > 0$, the above expansion shows that, for large r, the perturbed spheres S_{φ_r} become smaller than S as the perturbation parameter r decreases. With regard to the order $r^{-N-\alpha}$, the shrinking process is uniform on S, whereas non-uniform deformations of the spheres may appear at the order $r^{-N-\alpha-2}$. In the case $M = N$, it is not known whether φ_r is not constant on S. In fact, we have the following more explicit form of φ_r in the case of square lattices. Assume that $\mathscr{L}$ is a square lattice. Then

$$\varphi_r(\theta) = r^{-N-\alpha} \left(-\kappa_0 + r^{-2} \left\{ \tilde{\kappa}_1 \sum_{j=1}^{M} \theta_j^2 - \kappa_2 \right\} + o(r^{-2}) \right) \quad \text{for } \theta \in S \text{ as } r \to +\infty,$$

where $\tilde{\kappa}_1 = \frac{\kappa_1}{M} \sum\limits_{p \in \mathscr{L}_*} \frac{1}{|p|^{N+\alpha+2}}$. In particular, if $M = N$, then

$$\varphi_r(\theta) = r^{-N-\alpha} \Big(-\kappa_0 + r^{-2} \Big(\tilde{\kappa}_1 - \kappa_2 \Big) + o(r^{-2}) \Big) \qquad \text{as } r \to \infty.$$

Hence the deformation of the lattice $S_{\varphi_r} + r\mathscr{L}$ is uniform up to the order $r^{-N-\alpha-2}$. It is conjectured in [9] that that $H_\alpha(\mathscr{S}_0^r; \cdot)$ is non-constant for any N-dimensional lattice $\mathscr{L}$, as long as $\mathscr{S}_0^r$ is a hypersurface of class $C^{1,\beta}$, $\beta \in (\alpha, 1)$.
The proof of Theorem 3.11, is based on the application of the implicit function theorem, to solve the r-parameter dependence fractional quasilinear elliptic type problem on the sphere,

$$H_\alpha(\mathscr{S}_\varphi^r; \cdot) = H_\alpha(S_\varphi; \cdot) + r^{-N-\alpha} G(r, \varphi; \cdot) = H_\alpha(S_0; \cdot).$$

The term containing G in the above identity is a lower order term. In fact the function G together with all its derivatives in the φ variables are bounded, for r large enough. On the other hand the leading quasilinear operator given by $H_\alpha(S_\varphi; \cdot)$ is the NMC operator acting on graphs on the sphere. Its expression, derived from (3.1), is explicitly given in the following

Proposition 3.12. *Let $\psi \in C^{1,\beta}(S^{N-1})$ with $\psi > 0$. Then the NMC of the hypersurface*

$$\Sigma_\psi := \left\{ \sigma\psi(\sigma) \, : \, \sigma \in S^{N-1} \right\}$$

at the point $q = \theta\psi(\theta)$ is given by

$$h_\alpha(\psi)(\theta) := \frac{\alpha}{2} H_\alpha(\Sigma_\psi; q) = -\psi(\theta) \int_S \frac{\psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla\psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma)\, \mathcal{K}(\psi, \sigma, \theta)\, dV(\sigma)$$
$$+ \int_S \frac{(\psi(\theta) - \psi(\sigma))^2}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma)\, \mathcal{K}(\psi, \sigma, \theta)\, dV(\sigma) + \frac{\psi(\theta)}{2} \int_S \frac{\psi^{N-1}(\sigma)}{|\theta - \sigma|^{N+\alpha-2}}\, \mathcal{K}(\psi, \sigma, \theta)\, dV(\sigma),$$

where $\mathcal{K}(\psi, \sigma, \theta) := \frac{1}{\left(\frac{(\psi(\theta)-\psi(\sigma))^2}{|\theta-\sigma|^2} + \psi(\sigma)\psi(\theta)\right)^{(N+\alpha)/2}}$. Moreover, all integrals above converge absolutely.

Establishing the regularity of the NMC operator h_α, appearing in Proposition 3.12, as a map from open subsets of $C^{1,\beta}(S^{N-1})$ and taking values in $C^{0,\beta-\alpha}(S^{N-1})$, turns out to be an involved task. The inconveniences in the above expression is the presence of θ in the singular kernel $|\theta - \sigma|^{-N-\alpha}$ and the fact that it involves only euclidean distance instead of geodesic distance on the sphere.

4. Serrin's overdetermined problems

We consider the problem of finding domains (not necessarily bounded) and functions $u \in C^2(\overline{\Omega})$ such that

$$-\Delta_g u = 1 \quad \text{in } \Omega, \qquad u = 0, \quad \partial_\nu u = const. \quad \text{on } \partial\Omega, \tag{4.1}$$

A domain Ω is called a *Serrin domain* if it is of class C^2 and if (4.1) admits a solution. System (4.1) was considered in eulidean space by J. Serrin in 1971 in his seminal paper [72].

Theorem 4.1 (Serrin [72], Weinberger [78])**.** *Bounded Serrin domains in Euclidean space are balls.*

Serrin's argument relies on Alexandrov's moving plane method, with refined comparison principle. The proof of Weinberger [78] uses the P-function method, Pohozaev identity and maximum principles. Serrin's result can be also derived from Alexandrov's rigidity result. Namely if (4.1) has a solution then $\partial\Omega$ has CMC and thus is a ball, see Farina and Kawohl in [37] and Choulli and Henrot [17].
The additional Neumann boundary condition in (4.1) arises in many applications as a shape optimization problem for the underlying domain Ω. For a detailed discussion of some applications e.g. in fluid dynamics and the linear theory of torsion, see [72, 74]. Moreover, as observed in [60], Serrin domains arise also in the context of Cheeger sets in a Riemannian framework. To explain this connection more precisely, we recall that the Cheeger constant of a Lipschitz subdomain $\Omega \subset \mathcal{M}$ is given by

$$h(\Omega) := \inf_{A \subset \Omega} \frac{P(A)}{|A|}.$$

Here the infimum is taken over measurable subsets $A \subset \Omega$, with finite perimeter $P(A)$ and where $|A|$ denotes the volume of A (both with respect to the metric g). If $h(\Omega)$ is uniquely attained by Ω itself, then Ω is called uniquely self-Cheeger. By means of the Weinberger's P-function method, it is shown in [60] that every bounded Serrin domain in a compact Riemannian manifold $\mathcal{M}$, with Ricci curvature bounded below by some constant, is uniquely

self-Cheeger. Cheeger constants play an important role in eigenvalue estimates on Riemannian manifolds (see [16]), whereas in the classical Euclidean case $(\mathcal{M}, g) = (\mathbb{R}^N, g_{eucl})$ these notions have applications in the denoising problem in image processing, see e.g. [56, 63].
The above Serrin's classification result was extended in [55] to subdomains Ω of the round hemis-sphere $\mathcal{M} = S^N$ or $\mathcal{M}$ is a space forms of constant negative sectional curvatures. More precisely, it is proved in [55] that any smooth Serrin domain Ω contained in a hemisphere of S^N is a geodesic ball while the geodesic ball is the only Serrin domain in space forms of constant negative sectional curvatures. It is an interesting and widely open problem to construct and classify Serrin domains.

Minlend and the author in [32], found that on any compact Riemaniann manifold, there exists a Serrin domain, which is a perturbation of a geodesic balls. On the hand the symmetry group of the ambient manifold might be used to find nontrivial Serrin domains with different geometry. Indeed, very recently, Minlend, Weth and the author in [30, 31] considered the cases $\mathcal{M} = S^N$ the unit sphere and the case $\mathcal{M} = \mathbb{R}^n \times \mathbb{R}^m/2\pi\mathbb{Z}^m$, endowed with the flat metric, proving existences of Serrin domains with nonconstant principal curvatures of the boundary.

From an analytic point of view, the question of finding Serrin domains share similar features to the one of finding CNMC hypersurfaces. Indeed, an admissible class of parametrizations φ of the unknown domain boundary is considered. Then the solvability condition is formulated as an operator equation of the form $H(\varphi) = const.$, where H is a nonlinear Dirichlet-to-Neumann operator, thus sharing same traits with a quasilinear nonlocal operators. In fact, the nonlocal character of this operator become apparent when computing its linearization along a trivial branch of the problem, see e.g. [30, 31]. Here we end up with a pseudo-differential operator of order 1 that shares many features with the square-root of the negative of the Laplace operator, see [45].

From a geometric point of view, the results in [10, 11, 30, 31] support naturally the perception that the structure of the set of Serrin domains in a manifold $(\mathcal{M}, g)$ share similarites to those of CNMC hypersurfaces, which both, up to a *dimesion shift*, share some structure to CMC hypersurfaces in $\mathcal{M}$. Indeed, as in the theory CNMC hypersurfaces, there exists in $\mathbb{R}^2$, non-straight periodic Serrin domains, see [31]. We also emphasize from [30] that there exist Serrin domains in S^2 which are not bounded by geodesic circles, whereas CMC-hypersurfaces in S^2 are obviously trivial, i.e., they are geodesic circles.
Next, we recall that Alexandrov also proved in [3] that any closed embedded CMC hypersurface contained in a hemisphere of S^N is a geodesic sphere. An explicit family of embedded CMC hypersurfaces in S^N with nonconstant principal curvatures was found by Perdomo in [64] in the case $N \geq 3$. These hypersurfaces seem somewhat related to the Serrin domains in [30], although they bound a tubular neighborhood of S^1 and not of S^{N-1}.
Overdetermined boundary value problems involving different elliptic equations has intensively studied recently. Starting with the pioneering papers of Pacard, Sicbaldi [62], Sicbaldi [73] and Hauswirth, Hélein and Pacard [46], the construction of nontrivial domains giving rise to solutions of overdetermined problems has been performed in many specific settings, see e.g. [26, 61, 65, 71]. Moreover, rigidity results for these domains were derived in [34, 35, 66, 67, 77].

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