

# On fundamental solutions for multidimensional Helmholtz equation with three singular coefficients

Ergashev T.G.

Institute of Mathematics, Uzbek Academy of Sciences, Tashkent, Uzbekistan.  
ergashev.tukhtasin@gmail.com

The main result of the present paper is the construction of fundamental solutions for a class of multidimensional elliptic equations with three singular coefficients, which could be expressed in terms of a confluent hypergeometric function of four variables. In addition, the order of the singularity is determined and the properties of the found fundamental solutions that are necessary for solving boundary value problems for degenerate elliptic equations of second order are found.

**Key words:** multidimensional elliptic equation with three singular coefficients, fundamental solutions, confluent hypergeometric functions of four variables.

## 1 Introduction

It is known that fundamental solutions have an essential role in studying partial differential equations. Formulation and solving of many local and non-local boundary value problems are based on these solutions. Moreover, fundamental solutions appear as potentials, for instance, as simple-layer and double-layer potentials in the theory of potentials.

The explicit form of fundamental solutions gives a possibility to study the considered equation in detail. For example, in the works of Barros-Neto and Gelfand [1], fundamental solutions for Tricomi operator, relative to an arbitrary point in the plane were explicitly calculated. Among other results in this direction, we note a work by Itagaki [11], where 3D high-order fundamental solutions for a modified Helmholtz equation were found. The fundamental solutions can be applied to some boundary value problems [6, 12, 13, 14, 19, 20].

In the present work we find fundamental solutions for the equation

$$\sum_{i=1}^p u_{x_i x_i} + \frac{2\alpha_1}{x_1} u_{x_1} + \frac{2\alpha_2}{x_2} u_{x_2} + \frac{2\alpha_3}{x_3} u_{x_3} - \lambda^2 u = 0 \quad (1.1)$$

in the domain  $R_p^{3+} := \{(x_1, \dots, x_p) : x_1 > 0, x_2 > 0, x_3 > 0\}$ , where  $p$  is a dimension of the Euclidean space;  $p \geq 3$ ;  $\alpha_j$  are real constants and  $0 < 2\alpha_j < 1$  ( $j = 1, 2, 3$ );  $\lambda$  is real or pure imaginary constant.

Various modifications of the equation (1.1) in the two- and three-dimensional cases were considered in many papers [7, 8, 22, 23]. However, relatively few papers have been devoted to finding the fundamental solutions for multidimensional equations, we only note the works [16, 22]. In the paper [17], fundamental solutions of equation (1.1) with a single singular coefficient are found and investigated.

In this article, at first we shall introduce one confluent hypergeometric function of four variables. Furthermore, by means of the introduced hypergeometric function we construct fundamental solutions of the equation (1.1) in an explicit form. For studying the properties of the fundamental solutions, the introduced confluent hypergeometric function is expanded in products of hypergeometric functions of Gauss. With the help of the obtained expansion it is proved that the constructed fundamental solutions of equation (1.1) have a singularity of order  $1/r^{p-2}$  at  $r \rightarrow 0$ .

## 2 About one confluent hypergeometric function

In [3] (see also [21, p.74,(4b)]) a hypergeometric function of many variables of the form

$$H_{n,p}(a, b_1, \dots, b_n, c_{p+1}, \dots, c_n; d_1, \dots, d_p; x_1, x_2, \dots, x_n) = \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_p-m_{p+1}-\dots-m_n} (b_1)_{m_1} \dots (b_n)_{m_n} (c_{p+1})_{m_{p+1}} \dots (c_n)_{m_n}}{(d_1)_{m_1} \dots (d_p)_{m_p} m_1! \dots m_n!} x_1^{m_1} \dots x_n^{m_n}, \quad (2.1)$$

$0 \leq p \leq n$ , is considered, where  $(\kappa)_m := \Gamma(\kappa + m)/\Gamma(\kappa)$  is the Pochhammer symbol,  $m$  is a integer number,  $a$  is a complex number, and  $\kappa \neq 0, -1, -2, \dots$ , if the Pochhammer symbol  $(\kappa)_m$  is on the denominator.

The hypergeometric function (2.1) in four variables case has a form

$$H_{4,3}(a, b_1, b_2, b_3, b_4, c_4; d_1, d_2, d_3; x, y, z, t) = \sum_{m,n,k,l=0}^{\infty} \frac{(a)_{m+n+k-l}(b_1)_m(b_2)_n(b_3)_k(b_4)_l(c_4)_l}{(d_1)_m(d_2)_n(d_3)_k m! n! k! l!} x^m y^n z^k t^l, \quad (2.2)$$

where  $|x| + |y| + |z| < 1$ ,  $|t| < 1/(1 + |x| + |y| + |z|)$ .

From the hypergeometric function (2.2) we shall define the following confluent hypergeometric function

$$H_{4,3}^0(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = \lim_{\varepsilon \rightarrow 0} H_{4,3}\left(a, b_1, b_2, b_3, \frac{1}{\varepsilon}, \frac{1}{\varepsilon}; d_1, d_2, d_3; x, y, z, \varepsilon^2 t\right).$$

At the determination of the hypergeometric function  $H_{4,3}^0(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t)$  the equality  $\lim_{\varepsilon \rightarrow 0} (1/\varepsilon)_n \cdot \varepsilon^n = 1$  ( $n$  is a natural number) has been used and the found confluent hypergeometric function represented as

$$H_{4,3}^0(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = \sum_{m,n,k,l=0}^{\infty} \frac{(a)_{m+n+k-l}(b_1)_m(b_2)_n(b_3)_k}{(d_1)_m(d_2)_n(d_3)_k m! n! k! l!} x^m y^n z^k t^l, \quad |x| + |y| + |z| < 1. \quad (2.3)$$

Using the formula of derivation

$$\frac{\partial^{i+j+k+l}}{\partial x^i \partial y^j \partial z^k \partial t^l} H_{4,3}^0(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = \frac{(a)_{i+j+k-l}(b_1)_i(b_2)_j(b_3)_k}{(d_1)_i(d_2)_j(d_3)_k} \times \\ \times H_{4,3}^0(a+i+j+k-l, b_1+i, b_2+j, b_3+k; d_1+i, d_2+j, d_3+k; x, y, z, t)$$

it is easy to show that the confluent hypergeometric function in (2.3) satisfies the system of hypergeometric equations

$$\begin{cases} x(1-x)\omega_{xx} - xy\omega_{xy} - xz\omega_{xz} + xt\omega_{xt} + [d_1 - (a+b_1+1)x]\omega_x - b_1y\omega_y - b_1z\omega_z + b_1t\omega_t - ab_1\omega = 0 \\ y(1-y)\omega_{yy} - xy\omega_{xy} - yz\omega_{yz} + yt\omega_{yt} - b_2x\omega_x + [d_2 - (a+b_2+1)y]\omega_y - b_2z\omega_z + b_2t\omega_t - ab_2\omega = 0 \\ z(1-z)\omega_{zz} - xz\omega_{xz} - yz\omega_{yz} + zt\omega_{zt} - b_3x\omega_x - b_3y\omega_y + [d_3 - (a+b_3+1)z]\omega_z + b_3t\omega_t - ab_3\omega = 0 \\ t\omega_{tt} - x\omega_{xt} - y\omega_{yt} - z\omega_{zt} + (1-a)\omega_t + \omega = 0, \end{cases} \quad (2.4)$$

where

$$\omega(x, y, z, t) = H_{4,3}^0(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t).$$

Having substituted  $\omega(x, y, z, t) = x^\tau y^\nu z^\mu t^\delta \psi(x, y, z, t)$  in the system of hypergeometric equations (2.4), it is possible to find 8 linearly independent solutions of system (2.4), which is given in the table form

	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$
$\tau$	0	$1-d_1$	0	0	$1-d_1$	$1-d_1$	0	$1-d_1$
$\nu$	0	0	$1-d_2$	0	$1-d_2$	0	$1-d_2$	$1-d_2$
$\mu$	0	0	0	$1-d_3$	0	$1-d_3$	$1-d_3$	$1-d_3$
$\delta$	0	0	0	0	0	0	0	0

or in explicit form as follows

$$\omega_1(x, y, z, t) = H_{4,3}^0(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t), \quad (2.5)$$

$$\omega_2(x, y, z, t) = x^{1-d_1} H_{4,3}^0(a+1-d_1, b_1+1-d_1, b_2, b_3; 2-d_1, d_2, d_3; x, y, z, t), \quad (2.6)$$

$$\omega_3(x, y, z, t) = y^{1-d_2} H_{4,3}^0(a+1-d_2, b_1, b_2+1-d_2, b_3; d_1, 2-d_2, d_3; x, y, z, t), \quad (2.7)$$

$$\omega_4(x, y, z, t) = z^{1-d_3} H_{4,3}^0(a+1-d_3, b_1, b_2, b_3+1-d_3; d_1, d_2, 2-d_3; x, y, z, t), \quad (2.8)$$

$$\omega_5(x, y, z, t) = x^{1-d_1} y^{1-d_2} H_{4,3}^0(a+2-d_1-d_2, b_1+1-d_1, b_2+1-d_2, b_3; 2-d_1, 2-d_2, d_3; x, y, z, t), \quad (2.9)$$

$$\omega_6(x, y, z, t) = x^{1-d_1} z^{1-d_3} H_{4,3}^0(a+2-d_1-d_3, b_1+1-d_1, b_2, b_3+1-d_3; 2-d_1, d_2, 2-d_3; x, y, z, t), \quad (2.10)$$

$$\omega_7(x, y, z, t) = y^{1-d_2} z^{1-d_3} H_{4,3}^0(a+2-d_2-d_3, b_1, b_2+1-d_2, b_3+1-d_3; d_1, 2-d_2, 2-d_3; x, y, z, t), \quad (2.11)$$

$$\omega_8(x, y, z, t) = x^{1-d_1} y^{1-d_2} z^{1-d_3} H_{4,3}^0(a+3-d_1-d_2-d_3, b_1+1-d_1, b_2+1-d_2, b_3+1-d_3; 2-d_1, 2-d_2, 2-d_3; x, y, z, t). \quad (2.12)$$

### 3 Decomposition formulas

For a given multivariable function, it is useful to find a decomposition formula which would express the multivariable function in terms of products of several simpler hypergeometric functions involving fewer variables. For this purpose Burchnall and Chaundy [2] had given a number of expansions of double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the inverse pair of symbolic operators

$$\nabla(h) := \frac{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)}, \quad \Delta(h) := \frac{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)}{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)}, \quad (3.1)$$

where

$$\delta_1 := x_1 \frac{\partial}{\partial x_1}, \delta_2 := x_2 \frac{\partial}{\partial x_2}$$

Recently Hasanov and Srivastava [9, 10] generalized the operators  $\nabla(h)$  and  $\Delta(h)$  defined by (3.1) in the forms

$$\tilde{\nabla}_{x_1; x_2, \dots, x_m}(h) := \frac{\Gamma(h)\Gamma(\delta_1 + \dots + \delta_m + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + \dots + \delta_m + h)} \quad (3.2)$$

and

$$\tilde{\Delta}_{x_1; x_2, \dots, x_m}(h) := \frac{\Gamma(\delta_1 + h)\Gamma(\delta_2 + \dots + \delta_m + h)}{\Gamma(h)\Gamma(\delta_1 + \dots + \delta_m + h)}, \quad (3.3)$$

where

$$\delta_i := x_i \frac{\partial}{\partial x_i} \quad (i = 1, \dots, m), \quad (3.4)$$

and they obtained very interesting results. For example, in the special case when  $m = 3$  a Lauricella function in three variables is defined by (cf.[15]; see also [21, p.33, 1.4(1)])

$$F_A^{(3)}(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z) = \sum_{l, m, n=0}^{\infty} \frac{(a)_{l+m+n} (b_1)_l (b_2)_m (b_3)_n x^l y^m z^n}{(d_1)_l (d_2)_m (d_3)_n l! m! n!}$$

and the following decomposition formula holds true [9]

$$\begin{aligned} F_A^{(3)}(a; b_1, b_2, b_3; d_1, d_2, d_3; x, y, z) &= \sum_{l, m, n=0}^{\infty} \frac{(a)_{l+m+n} (b_1)_{l+m} (b_2)_{l+n} (b_3)_{m+n} x^{l+m} y^{l+n} z^{m+n}}{(d_1)_{l+m} (d_2)_{l+n} (d_3)_{m+n} l! m! n!} \\ &\cdot F(a + l + m, b_1 + l + m; d_1 + l + m; x) F(a + l + m + n, b_2 + l + n; d_2 + l + n; y) \\ &\cdot F(a + l + m + n, b_3 + m + n; d_3 + m + n; z), \end{aligned} \quad (3.5)$$

where

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

is a Gaussian hypergeometric function [3, p.56,(2)].

It should be noted that the symbolic notations (3.4) in the one-dimensional case take the form  $\delta := xd/dx$  and such a notation is used in solving problems of the operational calculus [18, p.26].

We now introduce here the other multivariable analogues of the Burchnall-Chaundy symbolic operators  $\nabla(h)$  and  $\Delta(h)$  defined by (3.1):

$$\tilde{\nabla}_{x,y}^{m,n}(h) := \frac{\Gamma(h)\Gamma(h + \delta_1 + \dots + \delta_m - \sigma_1 - \dots - \sigma_n)}{\Gamma(h + \delta_1 + \dots + \delta_m)\Gamma(h - \sigma_1 - \dots - \sigma_n)} = \sum_{s=0}^{\infty} \frac{(-\delta_1 - \dots - \delta_m)_s (\sigma_1 + \dots + \sigma_n)_s}{(h)_s s!}, \quad (3.6)$$

$$\tilde{\Delta}_{x,y}^{m,n}(h) := \frac{\Gamma(h + \delta_1 + \dots + \delta_m)\Gamma(h - \sigma_1 - \dots - \sigma_n)}{\Gamma(h)\Gamma(h + \delta_1 + \dots + \delta_m - \sigma_1 - \dots - \sigma_n)} = \sum_{s=0}^{\infty} \frac{(\delta_1 + \dots + \delta_m)_s (-\sigma_1 - \dots - \sigma_n)_s}{(1-h)_s s!}, \quad (3.7)$$

where

$$x := (x_1, \dots, x_m), y := (y_1, \dots, y_n), \quad (3.8)$$

$$\delta_i := x_i \frac{\partial}{\partial x_i}, \sigma_j := y_j \frac{\partial}{\partial y_j}, \quad i = 1, \dots, m, j = 1, \dots, n; m, n \in \mathbb{N}.$$

In addition, we consider operators which coincide with Hasanov- Srivastava's symbolic operators  $\tilde{\nabla}(h)$  and  $\tilde{\Delta}(h)$  defined by (3.2) and (3.3) as a particular case:

$$\tilde{\nabla}_{x,-}^{m,0}(h) := \tilde{\nabla}_{x_1;x_2,\dots,x_m}(h), \quad \tilde{\Delta}_{x,-}^{m,0}(h) := \tilde{\Delta}_{x_1;x_2,\dots,x_m}(h), \quad m \in \mathbb{N};$$

$$\tilde{\nabla}_{-,y}^{0,n}(h) := \tilde{\nabla}_{-y_1;-y_2,\dots,-y_n}(h), \quad \tilde{\Delta}_{-,y}^{0,n}(h) := \tilde{\Delta}_{-y_1;-y_2,\dots,-y_n}(h), \quad n \in \mathbb{N}.$$

It is obvious that

$$\tilde{\nabla}_{x,-}^{1,0}(h) = \tilde{\Delta}_{x,-}^{1,0}(h) = \tilde{\nabla}_{-,y}^{0,1}(h) = \tilde{\Delta}_{-,y}^{0,1}(h) = 1.$$

We introduce the notation:

$$D_z^s f(z) = \sum_{I(k,s)} \frac{z_1^{i_1} \dots z_k^{i_k}}{i_1! \dots i_k!} \frac{\partial^s f}{\partial z_1^{i_1} \dots \partial z_k^{i_k}},$$

where

$$z = (z_1, \dots, z_k); \quad I(k, s) = \{(i_1, \dots, i_k) : i_1 \geq 0, \dots, i_k \geq 0, i_1 + \dots + i_k = s\}.$$

**Lemma.** Let be  $f := f(x)$  and  $g := g(y)$  functions with variables  $x$  and  $y$  in (3.8). Then following equalities hold true for any  $m, n \in \mathbb{N}$ :

$$\left( -x_1 \frac{\partial}{\partial x_1} - \dots - x_m \frac{\partial}{\partial x_m} \right)_s f(x) = (-1)^s s! D_x^s f(x), \quad s \in \mathbb{N} \cup \{0\}; \quad (3.9)$$

$$\left( y_1 \frac{\partial}{\partial y_1} + \dots + y_n \frac{\partial}{\partial y_n} \right)_s g(y) = \begin{cases} g(y), & \text{if } s = 0, \\ s! \sum_{q=1}^s C_s^q C_{s-1}^{q-1} \cdot (s-q)! D_y^s g(y), & \text{if } s \in \mathbb{N}. \end{cases} \quad (3.10)$$

The lemma is proved by method of mathematical induction [5].

In the present paper we shall use two particular cases ( $m = 3$  and  $n = 1$ ) of the formulas (3.9) and (3.10):

$$\left( -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \right)_s f(x, y, z) = (-1)^s s! D_{x,y,z}^s f(x, y, z), \quad s \in \mathbb{N} \cup \{0\}; \quad (3.11)$$

$$\left( t \frac{d}{dt} \right)_s g(t) = \begin{cases} g(t), & s = 0, \\ \sum_{q=1}^s C_s^q C_{s-1}^{q-1} \cdot (s-q)! t^q g^{(q)}(t), & s \in \mathbb{N}. \end{cases} \quad (3.12)$$

Using the formulas (3.6) and (3.7), we obtain

$$H_{4,3}^0(a; b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = \tilde{\nabla}_{x,y,z,t}^{3,1}(a) F_A^{(3)}(a; b_1, b_2, b_3; d_1, d_2, d_3; x, y, z) {}_0F_1(1-a; -t), \quad (3.13)$$

where  ${}_0F_1(a; x) = \sum_{n=0}^{\infty} \frac{x^n}{(a)_n n!}$  is a generalized hypergeometric function [4, Chapter IV].

Now considering the equalities (3.6), (3.11) and (3.12) from the formula (3.13) we have

$$H_{4,3}^0(a; b_1, b_2, b_3; d_1, d_2, d_3; x_1, x_2, x_3, y) = \sum_{s=0}^{\infty} \sum_{q=0}^s \sum_{I(3,s)} A(s, q) C_s^q \frac{(-1)^{s+q} (b_1)_i (b_2)_j (b_3)_k x^i y^j z^k t^q}{(1-a)_q (d_1)_i (d_2)_j (d_3)_k i! j! k! q!} \cdot F_A^{(3)}(a+s; b_1+i, b_2+j, b_3+k; d_1+i, d_2+j, d_3+k; x, y, z) {}_0F_1(1-a+q; -t), \quad (3.14)$$

where

$$A(s, q) = \begin{cases} 1, & \text{if } s = 0 \text{ and } q = 0, \\ q/s, & \text{if } s \geq 1 \text{ and } q \geq 0, \end{cases} \quad I(3, s) = \{(i, j, k) : i \geq 0, j \geq 0, k \geq 0, i + j + k = s\}.$$

Applying the decomposition formula (3.5) to the expansion (3.14), we obtain

$$\begin{aligned} & H_{4,3}^0(a; b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) \\ &= \sum_{l,m,n,s=0}^{\infty} \sum_{q=0}^s \sum_{I(3,s)} A(s, q) s! C_s^q \frac{(-1)^{s+q} (a)_{l+m+n+s} (b_1)_{i+l+m} (b_2)_{j+l+n} (b_3)_{k+m+n} x^{i+l+m} y^{j+l+n} z^{k+m+n} t^q}{(a)_s (1-a)_q (d_1)_{i+l+m} (d_2)_{j+l+n} (d_3)_{k+m+n} i! n! j! m! k! l! q!} \\ & \cdot F(a+s+l+m, b_1+i+l+m; d_1+i+l+m; x) F(a+s+l+m+n, b_2+j+l+n; d_2+j+l+n; y) \\ & \cdot F(a+s+l+m+n, b_3+k+m+n; d_3+k+m+n; z) {}_0F_1(1-a+q; -t). \end{aligned} \quad (3.15)$$

By virtue of the formula [4, p.64,(22)]

$$F(a, b; c; x) = (1-x)^{-b} F\left(c-a, b; c; \frac{x}{x-1}\right),$$

the expansion (3.15) yields

$$\begin{aligned} & H_{4,3}^0(a; b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = (1-x)^{-b_1} (1-y)^{-b_2} (1-z)^{-b_3} \\ & \cdot \sum_{l,m,n,s=0}^{\infty} \sum_{q=0}^s \sum_{I(3,s)} A(s, q) s! C_s^q \frac{(-1)^{s+q} (a)_{l+m+n+s} (b_1)_{i+l+m} (b_2)_{j+l+n} (b_3)_{k+m+n}}{(a)_s (1-a)_q (d_1)_{i+l+m} (d_2)_{j+l+n} (d_3)_{k+m+n} i! j! k! n! m! l! q!} t^q \\ & \cdot \left(\frac{x}{1-x}\right)^{i+l+m} F\left(d_1 - a - j - k, b_1 + i + l + m; d_1 + i + l + m; \frac{x}{x-1}\right) \\ & \cdot \left(\frac{y}{1-y}\right)^{j+l+n} F\left(d_2 - a - i - k - m, b_2 + j + l + n; d_2 + j + l + n; \frac{y}{y-1}\right) \\ & \cdot \left(\frac{z}{1-z}\right)^{k+m+n} F\left(d_3 - a - i - j - l, b_3 + k + m + n; d_3 + k + m + n; \frac{z}{z-1}\right) {}_0F_1(1-a+q; -t). \end{aligned} \quad (3.16)$$

Expansion (3.16) will be used for studying properties of the fundamental solutions.

## 4 Fundamental solutions

We consider equation (1.1) in  $R_p^{3+}$ . Let  $x := (x_1, \dots, x_p)$  be any point and  $x_0 := (x_{01}, \dots, x_{0p})$  be any fixed point of  $R_p^{3+}$ . We search for a solution of (1.1) as follows:

$$u(x, x_0) = P(r)w(\sigma), \quad (4.1)$$

where

$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4), \quad r^2 = \sum_{i=1}^p (x_i - x_{0i})^2, \quad r_k^2 = (x_k + x_{0k})^2 + \sum_{i=1, i \neq k}^p (x_i - x_{0i})^2, \quad P(r) = (r^2)^{-\alpha},$$

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 - 1 + \frac{p}{2}, \quad \sigma_k = \frac{r^2 - r_k^2}{r^2} = -\frac{4x_k x_{0k}}{r^2}, \quad k = 1, 2, 3; \quad \sigma_4 = -\frac{1}{4} \lambda^2 r^2.$$

We calculate all necessary derivatives and substitute them into equation (1.1):

$$\sum_{m=1}^4 A_m \omega_{\sigma_m \sigma_m} + \sum_{m=1}^3 \sum_{n=m+1}^3 B_{m,n} \omega_{\sigma_m \sigma_n} + \sum_{m=1}^3 C_m \omega_{\sigma_m \sigma_4} + \sum_{m=1}^4 D_m \omega_{\sigma_m} + E \omega = 0, \quad (4.2)$$

where

$$\begin{aligned} A_k &= -\frac{4P(r)}{r^2} \frac{x_k}{x_{0k}} \sigma_k (1 - \sigma_k), \quad C_k = \frac{4P(r)}{r^2} \frac{x_{0k}}{x_k} \sigma_k \sigma_4 + \frac{\lambda^2}{2} P(r) \sigma_k, \\ B_{k,l} &= \frac{4P(r)}{r^2} \left( \frac{x_{0k}}{x_k} + \frac{x_{0l}}{x_l} \right) \sigma_k \sigma_l, \quad k \neq l, l = 1, 2, 3, \\ D_k &= -\frac{4P(r)}{r^2} \left\{ -\sigma_k \sum_{m=1}^3 \frac{x_{0m}}{x_m} \alpha_m + \frac{x_{0k}}{x_k} [2\alpha_k - \alpha \sigma_k] \right\}, \quad A_4 = \lambda^2 P(r) \sigma_4, \\ D_4 &= \frac{4P(r)}{r^2} \sigma_4 \sum_{m=1}^3 \frac{x_{0m}}{x_m} \alpha_m + \lambda^2 P(r) \alpha, \quad E = \frac{4\alpha P(r)}{r^2} \sum_{m=1}^3 \frac{x_{0m}}{x_m} - \lambda^2 P(r). \end{aligned}$$

Using the above given representations of coefficients we simplify equation (4.2) and obtain the following system of equations:

$$\begin{cases} \sigma_1(1-\sigma_1)\omega_{\sigma_1\sigma_1} - \sigma_1\sigma_2\omega_{\sigma_1\sigma_2} - \sigma_1\sigma_3\omega_{\sigma_1\sigma_3} + \sigma_1\sigma_4\omega_{\sigma_1\sigma_4} \\ + [2\alpha_1 - (\alpha + \alpha_1 + 1)\sigma_1]\omega_{\sigma_1} - \alpha_1\sigma_2\omega_{\sigma_2} - \alpha_1\sigma_3\omega_{\sigma_3} + \alpha_1\sigma_4\omega_{\sigma_4} - \alpha\alpha_1\omega = 0 \\ \sigma_2(1-\sigma_2)\omega_{\sigma_2\sigma_2} - \sigma_1\sigma_2\omega_{\sigma_1\sigma_2} - \sigma_2\sigma_3\omega_{\sigma_2\sigma_3} + \sigma_2\sigma_4\omega_{\sigma_2\sigma_4} \\ + [2\alpha_2 - (\alpha + \alpha_2 + 1)\sigma_2]\omega_{\sigma_2} - \alpha_2\sigma_1\omega_{\sigma_1} - \alpha_2\sigma_3\omega_{\sigma_3} + \alpha_2\sigma_4\omega_{\sigma_4} - \alpha\alpha_2\omega = 0 \\ \sigma_3(1-\sigma_3)\omega_{\sigma_3\sigma_3} - \sigma_1\sigma_3\omega_{\sigma_1\sigma_3} - \sigma_2\sigma_3\omega_{\sigma_2\sigma_3} + \sigma_3\sigma_4\omega_{\sigma_3\sigma_4} \\ + [2\alpha_3 - (\alpha + \alpha_3 + 1)\sigma_3]\omega_{\sigma_3} - \alpha_3\sigma_1\omega_{\sigma_1} - \alpha_3\sigma_2\omega_{\sigma_2} + \alpha_3\sigma_4\omega_{\sigma_4} - \alpha\alpha_3\omega = 0 \\ \sigma_4\omega_{\sigma_4\sigma_4} - \sigma_1\omega_{\sigma_1\sigma_4} - \sigma_2\omega_{\sigma_2\sigma_4} - \sigma_3\omega_{\sigma_3\sigma_4} + (1-\alpha)\omega_{\sigma_4} + \omega = 0 \end{cases} \quad (4.3)$$

Considering the solutions (2.5)-(2.12) of the system (2.4), we define the solutions  $\omega_i(\sigma), i = 1, \dots, 8$  of the system (4.3) and substituting those found solutions into the expression (4.1), we get some fundamental solutions of the equation (1.1)

$$q_1(x, x_0) = k_1 (r^2)^{-\alpha} H_{4,3}^0(\alpha, \alpha_1, \alpha_2, \alpha_3; 2\alpha_1, 2\alpha_2, 2\alpha_3; \sigma), \quad (4.4)$$

$$q_2(x, x_0) = k_2 (r^2)^{2\alpha_1 - \alpha - 1} (x_1 x_{01})^{1 - 2\alpha_1} \cdot H_{4,3}^0(1 + \alpha - 2\alpha_1, 1 - \alpha_1, \alpha_2, \alpha_3; 2 - 2\alpha_1, 2\alpha_2, 2\alpha_3; \sigma), \quad (4.5)$$

$$q_3(x, x_0) = k_3 (r^2)^{2\alpha_2 - \alpha - 1} (x_2 x_{02})^{1 - 2\alpha_2} \cdot H_{4,3}^0(1 + \alpha - 2\alpha_1, \alpha_1, 1 - \alpha_2, \alpha_3; 2\alpha_1, 2 - 2\alpha_2, 2\alpha_3; \sigma), \quad (4.6)$$

$$q_4(x, x_0) = k_4 (r^2)^{2\alpha_3 - \alpha - 1} (x_3 x_{03})^{1 - 2\alpha_3} \cdot H_{4,3}^0(1 + \alpha - 2\alpha_1, \alpha_1, \alpha_2, 1 - \alpha_3; 2\alpha_1, 2\alpha_2, 2 - 2\alpha_3; \sigma), \quad (4.7)$$

$$q_5(x, x_0) = k_5 (r^2)^{2\alpha_1 + 2\alpha_2 - \alpha - 2} (x_1 x_{01})^{1 - 2\alpha_1} (x_2 x_{02})^{1 - 2\alpha_2} \cdot H_{4,3}^0(2 + \alpha - 2\alpha_1 - 2\alpha_2, 1 - \alpha_1, 1 - \alpha_2, \alpha_3; 2 - 2\alpha_1, 2 - 2\alpha_2, 2\alpha_3; \sigma), \quad (4.8)$$

$$q_6(x, x_0) = k_6 (r^2)^{2\alpha_1 + 2\alpha_3 - \alpha - 2} (x_1 x_{01})^{1 - 2\alpha_1} (x_3 x_{03})^{1 - 2\alpha_3} \cdot H_{4,3}^0(2 + \alpha - 2\alpha_1 - 2\alpha_3, 1 - \alpha_1, \alpha_2, 1 - \alpha_3; 2 - 2\alpha_1, 2\alpha_2, 2 - 2\alpha_3; \sigma), \quad (4.9)$$

$$q_7(x, x_0) = k_7 (r^2)^{2\alpha_2 + 2\alpha_3 - \alpha - 2} (x_2 x_{02})^{1 - 2\alpha_2} (x_3 x_{03})^{1 - 2\alpha_3} \cdot H_{4,3}^0(2 + \alpha - 2\alpha_2 - 2\alpha_3, \alpha_1, 1 - \alpha_2, 1 - \alpha_3; 2\alpha_1, 2 - 2\alpha_2, 2 - 2\alpha_3; \sigma), \quad (4.10)$$

$$q_8(x, x_0) = k_8 (r^2)^{2\alpha_1 + 2\alpha_2 + 2\alpha_3 - \alpha - 3} (x_1 x_{01})^{1 - 2\alpha_1} (x_2 x_{02})^{1 - 2\alpha_2} (x_3 x_{03})^{1 - 2\alpha_3} \cdot H_{4,3}^0(3 + \alpha - 2\alpha_1 - 2\alpha_2 - 2\alpha_3, 1 - \alpha_1, 1 - \alpha_2, 1 - \alpha_3; 2 - 2\alpha_1, 2 - 2\alpha_2, 2 - 2\alpha_3; \sigma), \quad (4.11)$$

where  $k_1, \dots, k_8$  are constants which will be determined at solving boundary value problems for equation (1.1).

## 5 Singularity properties of fundamental solutions

Let us show that the found solutions (4.4)-(4.11) have a singularity. We choose a solution  $q_1(x, x_0)$ . For this aim we use the expansion (3.16) for the confluent hypergeometric function (2.3). As a result, solution (4.4) can be written as follows

$$q_1(x, x_0) = r^{2-p} r_1^{-2\alpha_1} r_2^{-2\alpha_2} r_3^{-2\alpha_3} f(r^2, r_1^2, r_2^2, r_3^2),$$

where

$$\begin{aligned} f(r^2, r_1^2, r_2^2, r_3^2) &= k_1 \sum_{l,m,n,s=0}^{\infty} \sum_{q=0}^s \sum_{I(3,s)} A(s, q) s! C_s^q \frac{(-1)^q (\alpha)_{l+m+n+s} (\alpha_1)_{i+l+m} (\alpha_2)_{j+l+n} (\alpha_3)_{k+m+n}}{(\alpha)_s (1-\alpha)_q (2\alpha_1)_{i+l+m} (2\alpha_2)_{j+l+n} (2\alpha_3)_{k+m+n} i! j! k! n! m! l! q!} \\ &\cdot \left(1 - \frac{r^2}{r_1^2}\right)^{i+l+m} \left(1 - \frac{r^2}{r_2^2}\right)^{j+l+n} \left(1 - \frac{r^2}{r_3^2}\right)^{k+m+n} \left(\frac{1}{2}\lambda r\right)^{2q} \\ &\cdot F\left(2\alpha_1 - \alpha - j - k, \alpha_1 + i + l + m; 2\alpha_1 + i + l + m; 1 - \frac{r^2}{r_1^2}\right) \\ &\cdot F\left(2\alpha_2 - \alpha - i - k - m, \alpha_2 + j + l + n; 2\alpha_2 + j + l + n; 1 - \frac{r^2}{r_2^2}\right) \\ &\cdot F\left(2\alpha_3 - \alpha - i - j - l, \alpha_3 + k + m + n; 2\alpha_3 + k + m + n; 1 - \frac{r^2}{r_3^2}\right) {}_0F_1\left(1 - \alpha + q; \frac{1}{4}\lambda^2 r^2\right), \end{aligned} \quad (5.1)$$

$$k_1 = \frac{4^{\alpha_1 + \alpha_2 + \alpha_3 - 1} \Gamma(\alpha) \Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3)}{\pi^{p/2} \Gamma(2\alpha_1) \Gamma(2\alpha_2) \Gamma(2\alpha_3)}.$$

Following the work [8] and applying several times a well-known summation formula [4, p.61,(14)]

$$F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}, \quad c \neq 0, -1, -2, \dots, c - a - b > 0,$$

it is easy to show that

$$f(0, r_{10}^2, r_{20}^2, r_{30}^2) = \frac{4^{\alpha_1 + \alpha_2 + \alpha_3 - 1}}{\pi^{p/2}} \Gamma\left(\frac{p-2}{2}\right), \quad p > 2. \quad (5.2)$$

Expressions (5.1) and (5.2) give us the possibility to conclude that the solution  $q_1(x, x_0)$  reduces to infinity of the order  $r^{2-p}$  at  $r \rightarrow 0$ . Similarly it is possible to be convinced that solutions  $q_i(x, x_0)$ ,  $i = 2, 3, \dots, 8$  also reduce to infinity of the order  $r^{2-p}$  when  $r \rightarrow 0$ .

#### References

1. Barros-Neto J.J., Gelfand I.M., Fundamental solutions for the Tricomi operator I,II,III, Duke Math.J. 98(3),1999. P.465-483; 111(3),2001.P.561-584; 128(1) 2005. P.119-140.
2. Burchnall J.L., Chaundy T.W. Expansions of Appell's double hypergeometric functions. The Quarterly Journal of Mathematics, Oxford, Ser.11,1940. P.249-270.
3. Erdelyi A. Integraldarstellungen für Produkte Whittakerscher Funktionen. Nieuw Archief voor Wiskunde. 1939, 2,20. P.1-34.
4. Erdelyi A., Magnus W., Oberhettinger F. and Tricomi F.G., Higher Transcendental Functions, Vol.I (New York, Toronto and London:McGraw-Hill Book Company), 1953.
5. Ergashev T.G. Fundamental solutions for a class of multidimensional elliptic equations with several singular coefficients. ArXiv.org:1805.03826.
6. Golberg M.A., Chen C.S. The method of fundamental solutions for potential, Helmholtz and diffusion problems, in: Golberg M.A.(Ed.), Boundary Integral Methods-Numerical and Mathematical Aspects, Comput.Mech.Publ.,1998. P.103-176.
7. Hasanov A., Fundamental solutions bi-axially symmetric Helmholtz equation. Complex Variables and Elliptic Equations. Vol. 52, No.8, 2007. pp.673-683.
8. Hasanov A., Karimov E.T, Fundamental solutions for a class of three-dimensional elliptic equations with singular coefficients. Applied Mathematic Letters, 22 (2009). pp.1828-1832.
9. Hasanov A., Srivastava H., Some decomposition formulas associated with the Lauricella function  $F_A^r$  and other multiple hypergeometric functions, Applied Mathematic Letters, 19(2) (2006), 113-121.
10. Hasanov A., Srivastava H., Decomposition Formulas Associated with the Lauricella Multivariable Hypergeometric Functions, Computers and Mathematics with Applications, 53:7 (2007), 1119-1128.
11. Itagaki M. Higher order three-dimensional fundamental solutions to the Helmholtz and the modified Helmholtz equations. Eng. Anal. Bound. Elem. 15,1995. P.289-293.
12. Karimov E.T. On a boundary problem with Neumann's condition for 3D elliptic equations. Applied Mathematics Letters. 2010,23. pp.517-522.
13. Karimov E.T. A boundary-value problems for 3-D elliptic equation with singular coefficients. Progress in Analysis and Its Applications. Proceedings of the 7th International ISAAC Congress. 2010. pp.619-625.
14. Karimov E.T., Nieto J.J. The Dirichlet problem for a 3D elliptic equation with two singular coefficients. Computers and Mathematics with Applications. 62, 2011. P.214-224.
15. Lauricella G. Sulle funzioni ipergeometriche a più variabili, Rend.Circ.Mat.Palermo. 1893, 7. pp. 111-158.
16. Mavlyaviev R.M., Construction of Fundamental Solutions to B-Elliptic Equations with Minor Terms. Russian Mathematics, 2017, Vol.61, No.6, pp.60-65. Original Russian Text published in Izvestiya Vysshikh Uchebnikh Zavedenii. Matematika, 2017, No.6. pp.70-75.
17. Mavlyaviev R.M., Garipov I.B. Fundamental solution of multidimensional axisymmetric Helmholtz equation. Complex Variables and elliptic equations. 62(3) (2017), pp.287-296.
18. Poole E.G.C. Introduction to the theory of linear differential equations. Oxford, Clarendon Press,1936. 202 p.
19. Salakhitdinov M.S., Hasanov A. A solution of the Neumann-Dirichlet boundary-value problem for generalized bi-axially symmetric Helmholtz equation. Complex Variables and Elliptic Equations. 53 (4) (2008), pp.355-364.
20. Salakhitdinov M.S., Hasanov A. To the theory of the multidimensional equation of Gellerstedt. Uzbek Math.Journal, 2007, No 3, pp. 95-109.
21. Srivastava H.M. and Karlsson P.W., Multiple Gaussian Hypergeometric Series. New York,Chichester,Brisbane and Toronto: Halsted Press, 1985. 428 p.
22. Urinov A.K., Karimov E.T. On fundamental solutions for 3D singular elliptic equations with a parameter. Applied Mathematic Letters, 24 (2011). pp.314-319.
23. Urinov A.K. On fundamental solutions for the some type of the elliptic equations with singular coefficients. Scientific Records of Ferghana State university, 1 (2006). pp.5-11.