EXPLICIT ∞-HARMONIC FUNCTIONS IN HIGH DIMENSIONS

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ABSTRACT. The aim of this work is to derive new explicit solutions to the ∞ -Laplace equation, the fundamental PDE arising in Calculus of Variations in the space L^{∞} . These solutions obey certain symmetry conditions and are derived in arbitrary dimensions, containing as particular sub-cases the already known classes two-dimensional infinity-harmonic functions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u \in C^2(\Omega)$ a continuous twice differential function. In this paper we study the existence of solutions to the PDE

(1.1)
$$\Delta_{\infty} u := \sum_{i,j=1}^{n} \mathcal{D}_{i} u \mathcal{D}_{j} u \mathcal{D}_{ij}^{2} u = 0$$

of the form

$$u(x) = \prod_{i=1}^{n} f_i(x_i),$$

where f_i are possibly non-linear for $1 \leq i \leq n$, and $x = (x_1, ..., x_n)^{\top}$, $x \in \Omega$. Solutions of this form are called *separated* ∞ -harmonic functions. In the above $D_i \equiv \frac{\partial}{\partial x_i}$ and $D_{ij}^2 \equiv \frac{\partial^2}{\partial x_i \partial x_j}$. The equation 1.1 is called ∞ -Laplacian (being a special case of the so-called more general the Aronsson equation) and it arises in Calculus of Variations in L^{∞} as the analogue of *Euler-Lagrange* equation of the functional

$$E_{\infty}(u, \mathcal{O}) := ||Du||_{L^{\infty}(\mathcal{O})}, \quad \mathcal{O} \in \Omega, \quad u \in W_{loc}^{1,\infty}(\Omega, \mathbb{R}).$$

These objects first arose in the work of G. Aronsson in the 1960s (see [A1],[A2]) and nowadays this is an active field of research for vectorial case $N \geq 2$ for $u \in W^{1,\infty}_{loc}(\Omega,\mathbb{R}^N)$ which has begun much more recently in 2010s (see e.g. [K1]). Since then, the field is developed enormously by N. Katzourakis in the series of papers ([K3]-[K11]) and also in collaboration with the author, Abugirda, Croce, Manfredi, Moser, Parini, Pisante and Pryer ([AyK], [AK], [CKP], [KM], [KMo], [KPa], [KP1] - [KP3]). A standard difficulty of 1.1 is that it is nondivergence and since in general smooth solutions do not exist, the definition of generalised solutions is an issue. To this end, the theory of viscosity solutions of Crandall-Ishii-Lions is utilised (see e.g. [K2]).

In this paper all the separated ∞ -harmonic functions are found for n=2 in polar coordinates, for n=3 in spherical coordinates and for all $n\geq 2$ in cartesian coordinates. Some of these new solutions derived herein coincide with previously known classes of solutions. For instance, the well-known G. Aronsson's solution

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 $u(x,y)=|x|^{\frac{4}{3}}-|y|^{\frac{4}{3}}$ which has a $C^{1,1/3}$ regularity, described in Remark 5. Also M.-F. Bidaut-Veron, M. Garcia Huidobro and L. Veron have found a solution ([VHV]) which is coincides with first two solutions of the theorem 1 and I.L. Freire, A. C. Faleiros have found a solutions of 1.1 in [FF], but only one non-trivial of their solutions coincides with a particular case of Theorem 3 when A=1. There may exist other additional solutions but this topic is not discussed herein.

The main results of this paper are contained in the following theorems.

Theorem 1 (Separated two-dimensional ∞ -Harmonic functions in polar coordinates). Let $u: \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a separated ∞ -harmonic function of the ∞ -Laplace equation in polar coordinates

(1.2)
$$u_r^2 u_{rr} + \frac{2}{r^2} u_r u_\theta u_{r\theta} + \frac{1}{r^4} u_\theta^2 u_{\theta\theta} - \frac{1}{r^3} u_r u_\theta^2 = 0$$

of the form $u(r,\theta)=f(r)g(\theta)$, where f,g are non-linear. Then, one of the following holds: either

(i)
$$|f(r)| = r^A$$
 and $|g(\theta)| = e^{B\theta}$, where A and B any constants, such that
$$A^2 - A + B^2 = 0$$

or

(ii)
$$|f(r)| = r^A$$
 and $|g(\theta)| = |g(\theta_0)|e^{\int_{\theta_0}^{\theta} G(t)dt}$, where $G(t)$ satisfies the following

$$t+c = \begin{cases} -\arctan\frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2 - A}}\arctan\frac{G(t)}{\sqrt{A^2 - A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan\frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}}\ln\left|\frac{G(t) - \sqrt{A-A^2}}{G(t) + \sqrt{A-A^2}}\right|, & \text{if } A^2 - A < 0 \end{cases}$$

or

(iii) $|g(\theta)| = e^{B\theta}$ and $|f(r)| = |f(r_0)|e^{\int_{r_0}^r \frac{\Phi(t)}{t}dt}$, where $\Phi(t)$ satisfies the following

$$\ln|t| + c = \begin{cases} \frac{1}{2} \ln\left|\frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2}\right| - \frac{1}{2} \frac{1}{\sqrt{B^2 - \frac{1}{4}}} \arctan\frac{\Phi(t) - \frac{1}{2}}{\sqrt{B^2 - \frac{1}{4}}}, & if \ B^2 - \frac{1}{4} > 0 \\ \frac{1}{2} \ln\left|\frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2}\right| + \frac{1}{2} \frac{1}{\Phi(t) - \frac{1}{2}}, & if \ B^2 - \frac{1}{4} = 0 \\ \frac{1}{2} \ln\left|\frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2}\right| - \frac{1}{4\sqrt{\frac{1}{4} - B^2}} \ln\left|\frac{\Phi(t) - \frac{1}{2} - \sqrt{\frac{1}{4} - B^2}}{\Phi(t) - \frac{1}{2} + \sqrt{\frac{1}{4} - B^2}}\right|, & if \ B^2 - \frac{1}{4} < 0, \end{cases}$$

where c is any constant.

Theorem 2 (Separated three-dimensional ∞ -Harmonic functions in spherical coordinates). Let $u: \Omega \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a separated ∞ -harmonic function of the ∞ -Laplace equation in spherical coordinates

$$(1.3) \quad u_r^2 u_{rr} + \frac{2}{r^2 \sin^2 \alpha} u_r u_\theta u_{r\theta} + \frac{1}{r^4 \sin^4 \alpha} u_\theta^2 u_{\theta\theta} - \frac{1}{r^3 \sin^2 \alpha} u_r u_\theta^2 + \frac{2}{r^2} u_r u_\alpha u_{r\alpha} + \frac{1}{r^4} u_\alpha^2 u_{\alpha\alpha} - \frac{1}{r^3} u_r u_\alpha^2 + \frac{2}{r^4 \sin^2 \alpha} u_\theta u_\alpha u_{\theta\alpha} - \frac{\cos \alpha}{r^4 \sin^3 \alpha} u_\alpha u_\theta^2 = 0$$

of the form $u(r, \theta, \alpha) = f(r)g(\theta)h(\alpha)$, where f, g and h are non-linear. Then, one of the following holds: either

(i) $|f(r)| = r^A$ and $|g(\theta)| = e^{B\theta}$, where A and B any constants, $h(\alpha) = h(\alpha_0)e^{\int_{\alpha_0}^{\alpha} H(t)dt}$, where H(t) satisfies the following

$$\left| A^2 - A + \frac{B^2}{\sin^2 t} + H^2(t) \right| = c(A, B, t_0) \left(A^2 + \frac{B^2}{\sin^2 t} + H^2(t) \right) e^{-2A \int_{t_0}^t \frac{1}{H(\lambda)} d\lambda}$$

or

(ii) $|f(r)| = r^A$, $|h(\alpha)| = |\sin \alpha|^A$ and $|g(\theta)| = |g(\theta_0)| e^{\int_{\theta_0}^{\theta} G(t)dt}$, where G(t) satisfies the following

$$t+c = \begin{cases} -\arctan\frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2-A}}\arctan\frac{G(t)}{\sqrt{A^2-A}}, & if \ A^2 - A > 0 \\ \frac{1}{G(t)}, & if \ A = 0 \\ -\arctan G(t), & if \ A = 1 \\ -\arctan\frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}}\ln\left|\frac{G(t)-\sqrt{A-A^2}}{G(t)+\sqrt{A-A^2}}\right|, & if \ A^2 - A < 0 \end{cases}$$

or

(iii) $|g(\theta)| = e^{B\theta}$, $|f(r)| = |f(r_0)|e^{\int_{r_0}^r \frac{\Phi(t)}{t}dt}$, where B is constant and $\Phi(t)$ satisfies the following

$$\ln|t| + c = \begin{cases}
\frac{1}{2} \ln\left| \frac{\Phi^{2}(t) + C^{2}}{\Phi^{2}(t) - \Phi(t) + C^{2}} \right| - \frac{1}{2} \frac{1}{\sqrt{C^{2} - \frac{1}{4}}} \arctan \frac{\Phi(t) - \frac{1}{2}}{\sqrt{C^{2} - \frac{1}{4}}}, & if C^{2} - \frac{1}{4} > 0 \\
\frac{1}{2} \ln\left| \frac{\Phi^{2}(t) + C^{2}}{\Phi^{2}(t) - \Phi(t) + C^{2}} \right| + \frac{1}{2} \frac{1}{\Phi(t) - \frac{1}{2}}, & if C^{2} - \frac{1}{4} = 0 \\
\frac{1}{2} \ln\left| \frac{\Phi^{2}(t) + C^{2}}{\Phi^{2}(t) - \Phi(t) + C^{2}} \right| - \frac{1}{4\sqrt{\frac{1}{4} - C^{2}}} \ln\left| \frac{\Phi(t) - \frac{1}{2} - \sqrt{\frac{1}{4} - C^{2}}}{\Phi(t) - \frac{1}{2} + \sqrt{\frac{1}{4} - C^{2}}} \right| & if C^{2} - \frac{1}{4} < 0,
\end{cases}$$

and

$$|h(\alpha)| = |h(\alpha_0)|e^{B \arcsin \frac{B \cot(\alpha)}{\sqrt{C^2 - B^2}} - C \arctan \frac{C \cot \alpha}{\sqrt{C^2 - \frac{B^2}{\sin^2 \alpha}}}$$

or

$$|h(\alpha)| = |h(\alpha_0)|e^{-B \arcsin \frac{B \cot(\alpha)}{\sqrt{C^2 - B^2}} + C \arctan \frac{C \cot \alpha}{\sqrt{C^2 - \frac{B^2}{\sin^2 \alpha}}}$$

where c is any constant.

Theorem 3 (Separated two-dimensional ∞ -Harmonic functions). Let $u:\Omega\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ be a separated ∞ -harmonic function of the ∞ -Laplace equation

(1.4)
$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$$

of the form u(x,y) = f(x)g(y), where f,g are non-linear. Then, one of the following holds: either

(i)
$$|f(x)| = |f(x_0)|e^{A(x-x_0)}$$
 and $|g(y)| = |g(y_0)|e^{\int_{y_0}^y G(t) dt}$, where $G(t)$ satisfies

$$t+c = \begin{cases} \frac{1}{G(t)}, & \text{if } A = 0\\ -\frac{1}{2A}\arctan\frac{G(t)}{A} + \frac{G(t)}{2\left(A^2 + G^2(t)\right)}, & \text{otherwise} \end{cases}$$

or

(ii)
$$|f(x)| = |f(x_0)|e^{\int_{x_0}^x F(t) dt}$$
 and $|g(y)| = |g(y_0)|e^{B(y-y_0)}$, where $F(t)$ satisfies

$$t+c = \begin{cases} \frac{1}{F(t)}, & if B = 0\\ -\frac{1}{2B} \arctan \frac{F(t)}{B} + \frac{F(t)}{2(B^2 + F^2(t))}, & otherwise. \end{cases}$$

Theorem 4 (Separated n-dimensional ∞ -Harmonic functions). Let $n \geq 2$ and $u: \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a separated ∞ -harmonic function of the ∞ -Laplace equation

(1.5)
$$\sum_{i,j=1}^{n} D_{i} u D_{j} u D_{ij}^{2} u = 0.$$

then

$$|f_i(x_i)| = |f_i(x_i^0)| e^{A_i(x_i - x_i^0)} \text{ for } 1 \le i \ne j \le n$$

and

$$|f_j(x_j)| = |f_j(x_j^0)| e^{\int_{x_j^0}^{x_j} F_j(t) dt}$$

where $F_i(t)$ satisfies

$$t + c = -\frac{1}{2\left(\sum_{i \neq j} A_i^2\right)^{1/2}} \arctan \frac{F_j(t)}{\left(\sum_{i \neq j} A_i^2\right)^{1/2}} + \frac{F_j(t)}{2\left(\sum_{i \neq j} A_i^2 + F_j^2(t)\right)}.$$

2. Proofs of main results

In this section we prove our main results. The general idea of our method, which is essentially the same for all our proofs, is to use a substitution to derive a "better" PDE. Then, we take any points from the domain which are different only in one component put them to the "new" equation and subtract the two equations from each other.

Proof of Theorem 1. For $u \neq 0$ the equation (1.2) can be written as

(2.1)
$$\frac{u_r^2}{u^2} \frac{u_{rr}}{u} + \frac{2}{r^2} \frac{u_r}{u} \frac{u_\theta}{u} \frac{u_{r\theta}}{u} + \frac{1}{r^4} \frac{u_\theta^2}{u^2} \frac{u_{\theta\theta}}{u} - \frac{1}{r^3} \frac{u_r}{u} \frac{u_\theta^2}{u^2} = 0.$$

Let $F = \frac{u_r}{u}$ and $G = \frac{u_{\theta}}{u}$, then $F_r + F^2 = \frac{u_{rr}}{u}$, $G_{\theta} + G^2 = \frac{u_{\theta\theta}}{u}$ and $\frac{1}{2}F_{\theta} + \frac{1}{2}G_r + FG = \frac{u_{r\theta}}{u}$. Note that $u(r,\theta) = f(r)g(\theta)$ hence F and G doesn't depend on θ and r respectively, since $F(r,\theta) = \frac{f'(r)}{f(r)}$ and $G(r,\theta) = \frac{g'(\theta)}{g(\theta)}$. Thus (2.1), becomes

(2.2)
$$F^{2}F_{r} + F^{4} + \frac{2}{r^{2}}F^{2}G^{2} + \frac{1}{r^{4}}G^{2}G_{\theta} + \frac{1}{r^{4}}G^{4} - \frac{1}{r^{3}}FG^{2} = 0.$$

Set $\Phi = Fr$, then $r\Phi_r - \Phi = F_r r^2$. Multiplying (2.2) by r^4 , we have

$$(2.3) \qquad (\Phi^2 + G^2)(\Phi^2 + G^2 - \Phi) + r\Phi^2\Phi_r + G^2G_\theta = 0.$$

We have the following 4 cases for the functions Φ and G:

Case (A) Φ and G are constant functions.

Case (B) Φ is constant and G is non-constant functions.

Case (C) Φ is non-constant and G is constant functions.

Case (D) Φ and G are non-constant functions.

Case (A) Let $\Phi \equiv A$ and $G \equiv B$, then (2.3) gives $A \equiv B \equiv 0$ or $A^2 - A + B^2 = 0$ which can be rewritten as $(A - \frac{1}{2})^2 + B^2 = \frac{1}{4}$ and as the consequent of substitutions $f(r) = r^A$ and $g(\theta) = e^{B\theta}$ up to a constants.

Case (B) Let $\Phi \equiv A$, then G is satisfying (2.3)

$$(2.4) (A^2 + G^2)(A^2 + G^2 - A) + G^2G_{\theta} = 0.$$

Therefore

$$(A^{2} + G^{2})(A^{2} + G^{2} - A) = -G^{2} \frac{dG}{d\theta}.$$

Consequently

(2.5)
$$\int d\theta = \int \frac{-G^2}{(A^2 + G^2)(A^2 + G^2 - A)} dG$$
$$= \int \frac{-A}{A^2 + G^2} dG - \int \frac{1 - A}{A^2 - A + G^2} dG.$$

Hence

$$t + c = \begin{cases} -\arctan\frac{G(t)}{A} + \frac{A - 1}{\sqrt{A^2 - A}}\arctan\frac{G(t)}{\sqrt{A^2 - A}}, & \text{if } A^2 - A > 0\\ \frac{1}{G(t)}, & \text{if } A = 0\\ -\arctan G(t), & \text{if } A = 1\\ -\arctan\frac{G(t)}{A} + \frac{A - 1}{2\sqrt{A - A^2}}\ln\left|\frac{G(t) - \sqrt{A - A^2}}{G(t) + \sqrt{A - A^2}}\right|, & \text{if } A^2 - A < 0. \end{cases}$$

If A is equal to 0 or 1, then G=0 is also a solution, hence $g\equiv c$ and u is a linear solution.

Case (C) Let $G \equiv B$, then Φ is satisfying (2.3)

$$(2.6) \qquad (\Phi^2 + B^2)(\Phi^2 + B^2 - \Phi) + r\Phi^2\Phi_r = 0.$$

Therefore

$$(\Phi^{2} + B^{2})(\Phi^{2} + B^{2} - \Phi) = -r\Phi^{2}\frac{d\Phi}{dr}.$$

Consequently

$$\int \frac{1}{r} dr = \int \frac{-\Phi^2}{(\Phi^2 + B^2)(\Phi^2 - \Phi + B^2)} d\Phi$$

$$= \int \frac{\Phi}{\Phi^2 + B^2} d\Phi - \int \frac{\Phi - \frac{1}{2}}{(\Phi - \frac{1}{2})^2 + B^2 - \frac{1}{4}} d\Phi - \int \frac{\frac{1}{2}}{(\Phi - \frac{1}{2})^2 + B^2 - \frac{1}{4}} d\Phi.$$

Hence

$$\ln|t| + c = \begin{cases} \frac{1}{2} \ln\left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| - \frac{1}{2} \frac{1}{\sqrt{B^2 - \frac{1}{4}}} \arctan\frac{\Phi(t) - \frac{1}{2}}{\sqrt{B^2 - \frac{1}{4}}}, & \text{if } B^2 - \frac{1}{4} > 0 \\ \frac{1}{2} \ln\left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| + \frac{1}{2} \frac{1}{\Phi(t) - \frac{1}{2}}, & \text{if } B^2 - \frac{1}{4} = 0 \\ \frac{1}{2} \ln\left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| - \frac{1}{4\sqrt{\frac{1}{4} - B^2}} \ln\left| \frac{\Phi(t) - \frac{1}{2} - \sqrt{\frac{1}{4} - B^2}}{\Phi(t) - \frac{1}{2} + \sqrt{\frac{1}{4} - B^2}} \right|, & \text{if } B^2 - \frac{1}{4} < 0. \end{cases}$$

If B=0, then $\Phi=0$ is also a solution, hence u is a constant.

Case (D) Let Φ and G are non-constant functions, then there exist $r_1 \neq r_2$ and $\theta_1 \neq \theta_2$ such that $\Phi(r_1) \neq \Phi(r_2)$ and $G(\theta_1) \neq G(\theta_2)$ satisfying (2.3). Thus (2.8)

$$(2.6)$$

$$r_1 \Phi(r_1)^2 \Phi_r(r_1) - \Phi^3(r_1) + \Phi(r_1)^4 + 2\Phi(r_1)^2 G(\theta)^2 + G(\theta)^2 G_\theta(\theta) + G(\theta)^4 - \Phi(r_1) G(\theta)^2 = 0$$
(2.9)

$$r_2\Phi(r_2)^2\Phi_r(r_2) - \Phi^3(r_2) + \Phi(r_2)^4 + 2\Phi(r_2)^2G(\theta)^2 + G(\theta)^2G_\theta(\theta) + G(\theta)^4 - \Phi(r_2)G(\theta)^2 = 0.$$

Subtracting (2.8) and (2.9) we get for any θ

(2.10)

$$G^{2}(\theta)(\Phi(r_{1}) - \Phi(r_{2}))(2(\Phi(r_{1}) + \Phi(r_{2})) - 1) = r_{2}\Phi^{2}(r_{2})\Phi_{r}(r_{2}) - \Phi(r_{2})^{3} + \Phi(r_{2})^{4} - r_{1}\Phi^{2}(r_{1})\Phi_{r}(r_{1}) + \Phi(r_{1})^{3} - \Phi(r_{1})^{4}.$$

if $2(\Phi(r_1) + \Phi(r_2)) - 1 \neq 0$, then $G^2(\theta)$ is a constant function or $G(\theta)$ is a step function because $G(\theta_1) = -G(\theta_2)$ and $G(\theta_1) \neq G(\theta_2)$, otherwise $2(\Phi(r_1) + \Phi(r_2)) - 1 = 0$ for any $r_1 \neq r_2$ such that $\Phi(r_1) \neq \Phi(r_2)$, which means $\Phi(r)$ is a step function and the image of Φ is symmetrical to $y = \frac{1}{4}$ since $\Phi(r_1) - \frac{1}{4} = -(\Phi(r_2) - \frac{1}{4})$. For both cases we have a contradiction to $C^{1,\alpha}$ regularity for ∞ -Harmonic mappings in two dimensions ([ES], [S]).

Finally integrating and substituting we complete the proof.

Remark 5 (The Arronson solution). If $A = \frac{4}{3}$ then $A^2 - A > 0$ and G(t) function satisfies

$$t+c = -\arctan\frac{3}{4}G(t) + \frac{1}{2}\arctan\frac{3}{2}G(t),$$

which can be rewritten as

$$27G^{3}(t) + 54G^{2}(t)\tan 2(t+c) + 32\tan 2(t+c) = 0.$$

Solving a third degree equation with respect to G(t), we get

$$G(t) = -\frac{4}{3} \frac{\tan^{\frac{1}{3}}(t+c) + \tan^{\frac{5}{3}}(t+c) + \tan(t+c)}{1 - \tan^{2}(t+c)}.$$

Therefore

$$\int G(t) dt = \ln \left| \frac{\left(1 - \tan^{\frac{2}{3}}(t+c)\right) \left(1 + \tan^{\frac{2}{3}}(t+c)\right)^{\frac{1}{3}}}{\left(\tan^{\frac{4}{3}}(t+c) - \tan^{\frac{2}{3}}(t+c) + 1\right)^{\frac{2}{3}}} \right|.$$

Hence

$$e^{\int_{\theta_0}^{\theta} G(t) dt} = \frac{|1 - \tan^{\frac{2}{3}}(\theta + c)| |1 + \tan^{\frac{2}{3}}(\theta + c)|^{\frac{1}{3}}}{|\tan^{\frac{4}{3}}(\theta + c) - \tan^{\frac{2}{3}}(\theta + c) + 1|^{\frac{2}{3}}} \cdot c(\theta_0)$$

$$= \frac{|1 - \tan^{\frac{4}{3}}(\theta + c)|}{|1 + \tan^2(\theta + c)|^{\frac{2}{3}}} \cdot c(\theta_0)$$

$$= |\cos^{\frac{4}{3}}(\theta + c) - \sin^{\frac{4}{3}}(\theta + c)| \cdot c(\theta_0).$$

Finally

$$|g(\theta)| = |g(\theta_0)| e^{\int_{\theta_0}^{\theta} G(t)} dt$$
$$= |g(\theta_0)| \left(\left| \cos^{\frac{4}{3}} (\theta + c) - \sin^{\frac{4}{3}} (\theta + c) \right| \right)$$
$$|f(r)| = r^{\frac{4}{3}}.$$

Thus, one of the possible solutions

$$u(r,\theta) = f(r)g(\theta)$$

$$= r^{\frac{4}{3}} \left(\cos^{\frac{4}{3}}(\theta+c) - \sin^{\frac{4}{3}}(\theta+c)\right).$$

$$u(x,y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}.$$

Remark 6 (The Aronsson solution). If $A = -\frac{1}{3}$ then $A^2 - A > 0$, similarly as in previous case we can find that $u(r,\theta) = r^{-\frac{1}{3}} \left(\cos^{\frac{4}{3}}(\frac{\theta+c}{2}) - \sin^{\frac{4}{3}}(\frac{\theta+c}{2})\right)$ is the solution of the ∞ -Laplace equation which was described in [A3] as

$$g(\theta) = \frac{\cos t}{(1 + 3\cos^2 t)^{\frac{2}{3}}}, \qquad \theta = t - 2\arctan\left(\frac{\tan t}{2}\right), \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

The key fact two solutions are identically equal is $\tan \frac{\theta}{2} = -\tan^3 \frac{t}{2}$.

Proof of Theorem 3. It is particular case of the Theorem 4, when n = 2. \square **Proof of Theorem 4.** For $u \neq 0$ equation (1.5) can be written as

(2.11)
$$\sum_{i,j=1}^{n} \frac{\mathbf{D}_{i}u}{u} \frac{\mathbf{D}_{j}u}{u} \frac{\mathbf{D}_{ij}^{2}u}{u} = 0.$$

Let $F_i = \frac{D_i u}{u}$ then $D_i F_i + F_i^2 = \frac{D_{ii} u}{u}$ and $F_i F_j = \frac{D_{ij} u}{u}$. Thus (2.11), becomes

(2.12)
$$\left(\sum_{i=1}^{n} F_i(x_i)^2\right)^2 + \sum_{i=1}^{n} F_i(x_i)^2 D_i F_i(x_i) = 0.$$

Since $u(x) = \prod_{i=1}^n f_i(x_i)$, then F_i depends only on x_i , consequently $D_i F_i(x_i) = F_i'(x_i)$. Set $x^1, x^2 \in \Omega$ such that $x^1 = (x_1, x_2, ..., x_j^1, ..., x_n)$ and $x^2 = (x_1, x_2, ..., x_j^2, ..., x_n)$, where $x_j^1 \neq x_j^2$ in (2.12) and subtract two equations. We find

$$\left(F_j^2(x_j^1) - F_j^2(x_j^2)\right) \left(2\sum_{i \neq j}^n F_i^2(x_i) + 2F_j^2(x_j^1) + 2F_j^2(x_j^1)\right) + F_j^2 F_j'(x_j^1) - F_j^2 F_j'(x_j^2) = 0,$$

assuming $F_j^2(x_j^1) \neq F_j^2(x_j^2)$, we have

$$(2.13) 2\sum_{i\neq j}^{n} F_i^2(x_i) = -\frac{F_j^2 F_j'(x_j^1) - F_j^2 F_j'(x_j^2)}{F_j^2(x_j^1) - F_j^2(x_j^2)} - 2F_j^2(x_j^1) - 2F_j^2(x_j^1).$$

LHS of (2.13) does not depend on x_i^1 and x_i^2 so

$$\sum_{i \neq j}^{n} F_i^2(x_i) \equiv c$$

for all x_i . Hence $F_i(x_i) = A_i$, where A_i is a constant for all $i \neq j$. Thus (2.12) gives

$$\left(\sum_{i=1}^{n} A_i^2 + F_j(x_j)^2\right)^2 + F_j(x_j)^2 F_j'(x_j) = 0,$$

consequently

$$dx_j = -\frac{F_j^2}{\left(\sum_{i \neq j}^n A_i^2 + F_j^2\right)^2} dF_j,$$

hence

$$x_j + c = -\frac{1}{2\sqrt{\sum_{i \neq j} A_i^2}} \arctan \frac{F_j(x_j)}{\sqrt{\sum_{i \neq j} A_i^2}} + \frac{F_j(x_j)}{2\left(\sum_{i \neq j} A_i^2 + F_j^2(x_j)\right)}, \text{ if } \sum_{i \neq j}^n A_i^2 \neq 0.$$

Otherwise

$$F_j^4(x_j) + F^2(x_j)F'(x_j) = 0,$$

so

(2.14)
$$F^2(x_j) + F'_j(x_j) = 0$$
, since we assume $F^2_j(x_j^1) \neq F^2_j(x_j^2)$.

Solving (2.14) we get $F_j(x_j) = -\frac{1}{x_j+c}$. Hence $f_i(x_i) = x_i + c_i$ for all $i \neq j$ and $f_j(x_j) = \frac{c_j}{|x_j+c|}$, where c_i are constants for all $1 \leq i \leq n$.

If there is no j such that $F_j^2(x_j^1) \neq F_j^2(x_j^2)$ then $F_j^2(x_j) \equiv c_j$ for all $1 \leq j \leq n$ and (2.12) gives that all $c_j = 0$ for all $1 \leq j \leq n$. So $f_i(x_i) = C_i$, where C_i is constant for all i.

Proof of Theorem 2. For $u \neq 0$ we can rewrite (1.3) as

(2.15)

$$\begin{split} &\frac{u_r^2}{u^2}\frac{u_{rr}}{u} + \frac{2}{r^2\sin^2\alpha}\frac{u_r}{u}\frac{u_\theta}{u}\frac{u_{r\theta}}{u} + \frac{1}{r^4\sin^4\alpha}\frac{u_\theta^2}{u^2}\frac{u_{\theta\theta}}{u} - \frac{1}{r^3\sin^2\alpha}\frac{u_r}{u}\frac{u_\theta^2}{u^2} + \frac{2}{r^2}\frac{u_r}{u}\frac{u_\alpha}{u}\frac{u_{r\alpha}}{u} + \\ &+ \frac{1}{r^4}\frac{u_\alpha^2}{u^2}\frac{u_{\alpha\alpha}}{u} - \frac{1}{r^3}\frac{u_r}{u}\frac{u_\alpha^2}{u^2} + \frac{2}{r^4\sin^2\alpha}\frac{u_\theta}{u}\frac{u_\alpha}{u}\frac{u_{\theta\alpha}}{u} - \frac{\cos\alpha}{r^4\sin^3\alpha}\frac{u_\alpha}{u}\frac{u_\theta^2}{u^2} = 0. \end{split}$$

Let

$$F = \frac{u_r}{u}, \ G = \frac{u_\theta}{u} \ \text{and} \ H = \frac{u_\alpha}{u}$$

Then, we have

$$F' + F^2 = \frac{u_{rr}}{u}, \ G' + G^2 = \frac{u_{\theta\theta}}{u} \ \text{and} \ H' + H^2 = \frac{u_{\alpha\alpha}}{u}.$$

Also,

$$\frac{u_{r\theta}}{u} = F'G', \quad \frac{u_{r\alpha}}{u} = F'H' \quad \text{and} \quad \frac{u_{\theta\alpha}}{u} = G'H'.$$

Note that F, G and H depend on only r, θ and α respectively, since $\frac{u_r}{u} = \frac{f'}{f}$, $\frac{u_{\theta}}{u} = \frac{g'}{q}$ and $\frac{u_{\alpha}}{u} = \frac{h'}{h}$. Thus (2.14) becomes

$$\begin{split} F^2(F'+F^2) + \frac{2}{r^2 \sin^2 \alpha} F^2 G^2 + \frac{1}{r^4 \sin^4 \alpha} G^2(G^2+G') - \frac{1}{r^3 \sin^2 \alpha} F G^2 + \\ + \frac{2}{r^2} F^2 H^2 + \frac{1}{r^4} H^2 (H^2+H') - \frac{1}{r^3} F H^2 + \frac{2}{r^4 \sin^2 \alpha} G^2 H^2 - \frac{\cos \alpha}{r^4 \sin^3 \alpha} H G^2 = 0, \end{split}$$

which is equivalent to

$$(2.16) \qquad \left(F^2 + \frac{1}{r^2 \sin^2 \alpha} G^2 + \frac{1}{r^2} H^2\right)^2 + F^2 F' + \frac{1}{r^4 \sin^4 \alpha} G^2 G' + \frac{1}{r^4} H^2 H' - \frac{\cos \alpha}{r^4 \sin^3 \alpha} H G^2 - \frac{1}{r^3 \sin^2 \alpha} F G^2 - \frac{1}{r^3} F H^2 = 0.$$

Let $\Phi = rF$. Then, $r\Phi' - \Phi = r^2F'$. Multiplying (2.16) by r^4 we have

$$(2.17) \quad \frac{\left(\Phi(r)^{2} + \frac{1}{\sin^{2}\alpha}G(\theta)^{2} + H(\alpha)^{2}\right)\left(\Phi(r)^{2} - \Phi(r) + \frac{1}{\sin^{2}\alpha}G(\theta)^{2} + H(\alpha)^{2}\right) - \frac{\cos\alpha}{\sin^{3}\alpha}H(\alpha)G(\theta)^{2} + r\Phi(r)^{2}\Phi'(r) + \frac{1}{\sin^{4}\alpha}G(\theta)^{2}G'(\theta) + H(\alpha)^{2}H'(\alpha) = 0.$$

Setting (r_1, θ, α) , $(r_2, \theta, \alpha) \in \Omega$ such that $r_1 \neq r_2$ in (2.17) and subtracting two equation we get

$$\left(2\Phi^{2}(r_{1}) - \Phi(r_{1}) - 2\Phi^{2}(r_{2}) + \Phi(r_{2})\right) \left(H^{2}(\alpha) + \frac{1}{\sin^{2}\alpha}G^{2}(\theta)\right) + \Phi^{4}(r_{1}) - \Phi^{4}(r_{2}) - \Phi^{3}(r_{1}) + \Phi^{3}(r_{2}) + r_{1}\Phi^{2}(r_{1})\Phi'(r_{1}) - r_{2}\Phi^{2}(r_{2})\Phi'(r_{2}) = 0.$$

Case 1. If
$$2\Phi^2(r) - \Phi(r) \not\equiv c$$
, then $H^2(\alpha) + \frac{1}{\sin^2 \alpha} G^2(\theta) = C^2$, hence $G(\theta) \equiv B$, $H(\alpha) = \pm \sqrt{C^2 - \frac{B^2}{\sin^2 \alpha}}$ and (2.17) becomes (2.18)
$$\left(\Phi^2(r) + C^2\right) \left(\Phi^2(r) - \Phi(r) + C^2\right) + r\Phi^2(r)\Phi'(r) = H(\alpha)B^2 \frac{\cos \alpha}{\sin^3 \alpha} - H^2(\alpha)H'(\alpha).$$

LHS in (2.18) depends on r only and RHS in (2.18) depends on α only, so we have the system

(2.19)
$$\begin{cases} \left(\Phi^{2}(r) + C^{2}\right) \left(\Phi^{2}(r) - \Phi(r) + C^{2}\right) + r\Phi^{2}(r)\Phi'(r) = c_{1} \\ H(\alpha)B^{2} \frac{\cos\alpha}{\sin^{3}\alpha} - H^{2}(\alpha)H'(\alpha) = c_{1} \\ H(\alpha) = \pm \sqrt{C^{2} - \frac{B^{2}}{\sin^{2}\alpha}} \\ G(\theta) \equiv B. \end{cases}$$

Solving (2.19) we get $c_1=0$, $G(\theta)\equiv B$, $H(\alpha)=\pm\sqrt{C^2-\frac{B^2}{\sin^2\alpha}}$ and from the previous result (2.6) Φ satisfies the following

$$\ln|t|+c = \begin{cases} \frac{1}{2} \ln\left|\frac{\Phi^2(t)+C^2}{\Phi^2(t)-\Phi(t)+C^2}\right| - \frac{1}{2} \frac{1}{\sqrt{C^2-\frac{1}{4}}} \arctan\frac{\Phi(t)-\frac{1}{2}}{\sqrt{C^2-\frac{1}{4}}}, & \text{if } C^2-\frac{1}{4}>0\\ \frac{1}{2} \ln\left|\frac{\Phi^2(t)+C^2}{\Phi^2(t)-\Phi(t)+C^2}\right| + \frac{1}{2} \frac{1}{\Phi(t)-\frac{1}{2}}, & \text{if } C^2-\frac{1}{4}=0\\ \frac{1}{2} \ln\left|\frac{\Phi^2(t)+C^2}{\Phi^2(t)-\Phi(t)+C^2}\right| - \frac{1}{4\sqrt{\frac{1}{4}-C^2}} \ln\left|\frac{\Phi(t)-\frac{1}{2}-\sqrt{\frac{1}{4}-C^2}}{\Phi(t)-\frac{1}{2}+\sqrt{\frac{1}{4}-C^2}}\right|, & \text{if } C^2-\frac{1}{4}<0. \end{cases}$$

If B=0, then C=0, $\Phi=0$ is also a solution of (2.19).

Case 2. If $2\Phi^2(r) - \Phi(r) \equiv c$, then $\Phi(r) \equiv A$. Setting $(r, \theta_1, \alpha), (r, \theta_2, \alpha) \in \Omega$ such that $\theta_1 \neq \theta_2$ in (2.17) and subtracting two equations, we get

$$\left(G^{2}(\theta_{1}) - G^{2}(\theta_{2})\right) \left(2H^{2}(\alpha) - \cot(\alpha)H(\alpha) + 2A^{2} - A\right) \frac{1}{\sin^{2}\alpha} + \frac{1}{\sin^{4}\alpha} \left(G^{2}(\theta_{1})G'(\theta_{1}) - G^{2}(\theta_{2})G(\theta_{2}) + G^{4}(\theta_{1}) - G^{4}(\theta_{2})\right) = 0.$$

Case 2a. If $G^2(\theta) \not\equiv c$, then $\left(2H^2(\alpha) - \cot(\alpha)H(\alpha) + 2A^2 - A\right)\sin^2\alpha \equiv C$ and (2.17) becomes

$$\begin{split} \frac{G^4(\theta)}{\sin^4\alpha} + \frac{G^2(\theta)}{\sin^2\alpha} \Big(2H^2(\alpha) - \cot(\alpha)H(\alpha) + 2\Phi(r)^2 - \Phi(r) \Big) + \frac{G^2(\theta)G'(\theta)}{\sin^4\alpha} = \\ \Phi^3(r) - \Phi^4(r) - 2H^2(\alpha)\Phi^2(\theta) + \Phi(r)H^2(\alpha) - H^4(\alpha) - H^2(\alpha)H'(\alpha), \end{split}$$

thus

(2.20)
$$G^{4}(\theta) + G^{2}(\theta)C + G^{2}(\theta)G'(\theta) = \sin^{4}\alpha \Big(\Phi^{3}(r) - \Phi^{4}(r) - 2H^{2}(\alpha)\Phi^{2}(\theta) + \Phi(r)H^{2}(\alpha) - H^{4}(\alpha) - H^{2}(\alpha)H'(\alpha)\Big).$$

The LHS of (2.20) depends on θ only and the RHS depends on r and α only, hence we have a system

(2.21)

$$\begin{cases} G^4(\theta) + G^2(\theta)C + G^2(\theta)G'(\theta) = c_1\\ \sin^4 \alpha \Big(\Phi^3(r) - \Phi^4(r) - 2H^2(\alpha)\Phi^2(\theta) + \Phi(r)H^2(\alpha) - H^4(\alpha) - H^2(\alpha)H'(\alpha) \Big) = c_1\\ \Phi(r) \equiv A\\ \Big(2H^2(\alpha) - \cot(\alpha)H(\alpha) + 2A^2 - A \Big) \sin^2 \alpha \equiv C. \end{cases}$$

Solving (2.21) we get $C = 2A^2 - A$, $c_1 = -A^4 + A^3$, $\Phi(r) = A$, $H(\alpha) = A \cot(\alpha)$ and $G(\theta)$ satisfies the following

$$(G^2(\theta) + A^2)(G^2(\theta) + A^2 - A) + G^2(\theta)G'(\theta) = 0,$$

which is equivalent from the previous result (2.4) to

$$t+c = \begin{cases} -\arctan\frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2 - A}}\arctan\frac{G(t)}{\sqrt{A^2 - A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan\frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}}\ln\left|\frac{G(t) - \sqrt{A-A^2}}{G(t) + \sqrt{A-A^2}}\right|, & \text{if } A^2 - A < 0. \end{cases}$$

If A is equal to 0 or 1, then G = 0 is also a solution of (2.21).

Case 2b. If $G^2(\theta) \equiv c$, then $G(\theta) = B$. Thus (2.17) gives (2.22)

$$\left(A^2 - A + \frac{B^2}{\sin^2 \alpha} + H^2(\alpha)\right) \left(A^2 + \frac{B^2}{\sin^2 \alpha} + H^2(\alpha)\right) - \frac{B^2 \cos \alpha}{\sin^3 \alpha} H(\alpha) + H^2(\alpha)H'(\alpha) = 0.$$

Let $\Psi(\alpha) = \frac{B^2}{\sin^2 \alpha} + H^2(\alpha)$, then (2.22) becomes

$$(A^2 - A + \Psi)(A^2 + \Psi) + H(\alpha)\frac{1}{2}\Psi'(\alpha) = 0,$$

which gives us

$$1 + \frac{H}{2A} \left(\ln \left| \frac{A^2 - A + \Psi}{A^2 + \Psi} \right| \right)_{\alpha} = 0,$$

consequently

$$|A^{2} - A + \Psi(t)| = c(A, \Psi(t_{0}))(A^{2} + \Psi(t))e^{-2A\int_{t_{0}}^{t} \frac{1}{H(\lambda)}d\lambda},$$

so that

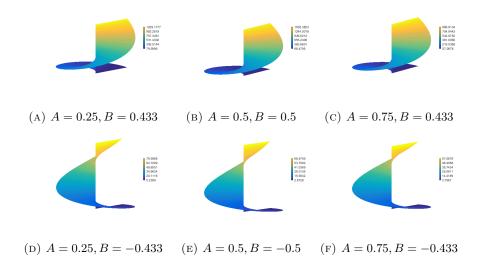
$$\left| A^2 - A + \frac{B^2}{\sin^2 t} + H^2(t) \right| = c(A, B, H(t_0)) \left(A^2 + \frac{B^2}{\sin^2 t} + H^2(t) \right) e^{-2A \int_{t_0}^t \frac{1}{H(\lambda)} d\lambda}.$$

Finally integrating and substituting we complete the proof.

3. Numerical approximations of ∞ -harmonic functions

In this section we illustrate the ∞ -Harmonic functions derived earlier, depending on the parameter(s). The results illustrate that we may have a family of solutions depending on the 2π -interval even if the parameter(s) is/are fixed. For example: the solution on Figure 3h is a combination of those in Figure 3i and Figure 3j when θ belongs to 1st and 2nd 2π - interval of the domain respectively.

FIGURE 1. The approximation to u of the Theorem 1 i, depending on the parameters A and B.

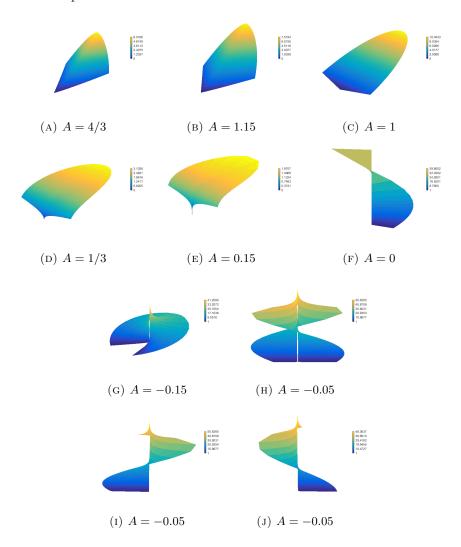


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FIGURE 2. The approximation to u of the Theorem 1 ii, depending on the parameter A.



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FIGURE 3. The approximation to u of the Theorem 1 iii, depending on the parameter B.

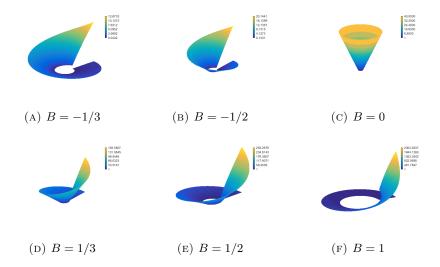
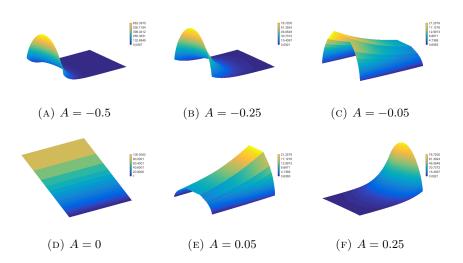


FIGURE 4. The approximation to u of the Theorem 3 i, depending on the parameter A.



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