An operator algebra associated with a pair of intersecting manifolds

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Abstract

Given a pair of smooth transversally intersecting manifolds in some ambient manifold, we construct an operator algebra generated by pseudodifferential operators and the (co)boundary operators associated with the submanifolds. We show that this algebra has 18 types of additive generators. Then we define the symbols of the operators in this algebra and obtain the composition formula.

1 Introduction

Let X_0 be a closed smooth manifold and X_1, X_2 be its submanifolds of arbitrary dimensions with a nonempty intersection. We consider a natural class of boundary value problems associated with this geometry. Namely, we consider operators with boundary conditions posed on X_1 and X_2. These problems were studied, e.g., in the works [1–3] (see also [4,5]), where the Fredholm property for some problems of such type was obtained, index formulas were proved (these formulas involve contributions of X_1 and X_2 considered as strata of the manifold with singularities $X_1 \cup X_2$). Later these results and methods were applied to study nonlocal problems with boundary conditions on a smooth submanifold (see [6,7]).

In the present paper, we study several algebraic aspects of this theory. More precisely, we consider an operator algebra multiplicatively generated by pseudodifferential operators (ψDOs) on the ambient manifold and on the submanifolds, and the (co)boundary operators associated with the submanifolds (by a boundary operator we mean the restriction operator to a submanifold, and by a coboundary operator we mean its dual, that is, an operator which extends functions on a submanifold to distributions on the ambient manifold). We show that this algebra has 18 types of additive generators, and general elements in this algebra can be written as the following 3×3 matrices

$$\mathcal{D} = \begin{pmatrix} D_0 + G_1 + G_2 + M_0 & C_1 + C_1' & C_2 + C_2' \\ B_1 + B_1' & D_1 + M_1 & T_{12} \\ B_2 + B_2' & T_{21} & D_2 + M_2 \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{H}', \qquad (1.1)$$

where $\mathcal{H}, \mathcal{H}'$ stand for direct sums of Sobolev spaces on X_0, X_1, X_2 (of some orders for each of the manifolds). The entries in (1.1) are of the following types:

- D_0, D_1, D_2 are ψDOs on X_0, X_1, X_2, respectively;
- B_1, B_2 and C_1, C_2 are boundary and coboundary operators (see [8]), localized at X_1, X_2;
- G_1, G_2 are Green operators (see, e.g., [9–11]), localized at X_1, X_2;
- M_0, M_1, M_2 are Mellin operators (see, e.g., [5]), localized at $X_1 \cap X_2$;
- T_{12}, T_{21} are translators (see [2]), localized at $X_1 \cap X_2$;
- B_1', B_2' and C_1', C_2' are boundary and coboundary operators, localized at the intersection $X_1 \cap X_2$.

We note that while these operators were considered in the literature individually, our approach allows one to study them from a unified point of view, and this considerably simplifies the theory. Moreover, our classification of operators (1.1) is carried out in terms of the strata on which these operators are localized at. We also define symbols of these operators and establish the composition formula.

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2 Statement of the problem

Let $X_0 = \mathbb{R}^n$ with the standard coordinates $x_1, \ldots, x_n$. We consider two coordinate subspaces $X_1, X_2 \subset X_0$ such that $\dim X_k < \dim X_0$, $k = 1, 2$, and $\dim(X_1 \cap X_2) > 0$. We will take appropriate components of the coordinates $x_1, \ldots, x_n$ on the ambient space as coordinates on X_k, $k = 1, 2$.

For $k = 1, 2$ denote $n_k = \operatorname{codim}_{X_k}(X_1 \cap X_2)$, $\nu_k = \operatorname{codim}_{X_0} X_k$. Also denote $\nu_3 = \operatorname{codim}_{X_0}(X_1 \cap X_2)$.

We associate the following operators with the triple (X_0, X_1, X_2):

1. Pseudodifferential operators (ψDOs) on X_0, X_1, X_2:

$$A_k \colon H^s(X_k) \longrightarrow H^{s-m}(X_k), \quad k = 0, 1, 2. \tag{2.1}$$

   Hereinafter we consider only ψDOs (2.1) with compactly supported Schwartz kernels.

2. Elementary boundary operators corresponding to X_k, $k = 1, 2$:

$$i^k \colon H^s(X_0) \longrightarrow H^{s-\nu_k/2}(X_k), \quad u(y, z) \longmapsto u(y, 0), \quad s - \nu_k/2 > 0, \tag{2.2}$$

   where (y, z) are the coordinates on X_0 such that $X_k = \{y = 0\}$.

3. Elementary coboundary operators corresponding to X_k, $k = 1, 2$:

$$i_k \colon H^{-s+\nu_k/2}(X_k) \longrightarrow H^{-s}(X_0), \quad u(z) \longmapsto u(z) \otimes \delta(y), \quad s - \nu_k/2 > 0, \tag{2.3}$$

   where $\delta(y)$ stands for the Dirac delta function, and the coordinates (y, z) are chosen as above.

Below we always assume that all operators act between Sobolev spaces of some fixed orders (we will denote these spaces by $H(Z)$, where $Z \subset X_0$ is a manifold, omitting the order from the notation). Moreover, for compositions $D_1 D_2$ we assume that the domain of D_1 is equal to the range of D_2, that is, the corresponding Sobolev spaces are the same.

Now, for any $k, l \in \{0, 1, 2\}$, consider the linear space $\mathrm{Mor}_{k,l}$ which consists of operators $H(X_l) \to H(X_k)$ of some fixed order and is multiplicatively generated by the operators (2.1), (2.2), and (2.3). More precisely, an element of $\mathrm{Mor}_{k,l}$ is a finite sum of operators of the form

$$\mathcal{D}_{kl} = D_{k,i_1} D_{i_1,i_2} \ldots D_{i_N,l} \colon H(X_l) \longrightarrow H(X_k), \tag{2.4}$$

where $D_{\alpha,\beta} \colon H(X_\beta) \to H(X_\alpha)$ is a composition of ψDOs and the elementary boundary operator (when $X_\alpha \subset X_\beta$) or the elementary coboundary operator (when $X_\beta \subset X_\alpha$). Now denote by Mor the direct sum

$$\mathrm{Mor} = \bigoplus_{k,l=0,1,2} \mathrm{Mor}_{k,l}\,.$$

Elements of this space are called *morphisms* (cf. [1]).

Further, for simplicity we consider only morphisms of order zero in the spaces

$$\mathcal{H} = \bigoplus_{k=0}^{2} H^{s_k}(X_k) \tag{2.5}$$

for some fixed $s_k \in \mathbb{R}$. By construction such morphisms form an algebra with respect to the operator composition (we denote this algebra also by Mor).

It is an interesting problem to study operators (morphisms) from the algebra Mor. More precisely, it is necessary to examine the structure of these operators, define their symbols and the notion of ellipticity, establish the Fredholm property of elliptic operators (the finiteness theorem) and obtain the index formula.

In this paper, we carry out a classification of morphisms, define their symbols and determine the composition formula (the symbol homomorphism). Fredholm property and index theorem will be studied elsewhere.

3 Classification of morphisms

The union $X_1 \cup X_2$ is a stratified manifold with singularities in X_0. Let us classify the elements of Mor by means of the strata they are localized at. First, we introduce the corresponding concept.

Note that the space $\mathcal{H}$ (see (2.5)) is a $C_c^\infty(X_0)$-module with respect to the multiplication by functions in $C_c^\infty(X_0)$ and their restrictions to X_1 and X_2.

Definition 3.1. A morphism $\mathcal{D} \colon \mathcal{H} \to \mathcal{H}$ is *localized* at a submanifold $Z \subset X_0$ if compositions $\varphi\mathcal{D}$ or $\mathcal{D}\varphi$ are operators of order $\leq \operatorname{ord} \mathcal{D}$ for any $\varphi \in C_c^\infty(X_0 \setminus Z)$.

One can easily show that the elementary boundary and coboundary operators are localized at the submanifolds they are associated with. This implies the following

Lemma 3.2. *The composition* (2.4) *is localized at the intersection*

$$X_k \cap X_{i_1} \cap \cdots \cap X_{i_N} \cap X_l. \tag{3.1}$$

Lemma 3.2 allows us to classify compositions (2.4) into four classes, according to the intersection (3.1): this intersection is equal to one of the submanifolds X_0, X_1, X_2, and $X_1 \cap X_2$. Let us examine the form of the operators from each of the four classes.

From now on, we represent the morphisms from Mor as 3×3 matrix operators acting in $\mathcal{H} \to \mathcal{H}$.

1. *The intersection* (3.1) *is equal to* X_0. Clearly, this is the case if and only if

$$X_k = X_{i_1} = \cdots = X_{i_N} = X_l = X_0.$$

Thus, we have a ψDO on X_0. The corresponding matrix operator is

$$\begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{H}.$$

2. *The intersection* (3.1) *is equal to* X_1. This is the case if and only if all the submanifolds in (3.1) are equal to either X_0 or X_1. Thus, we deal with a composition of ψDOs on X_0 and X_1 with (at least one) the elementary boundary or coboundary operators i_1, i^1. The corresponding matrix operator is

$$\begin{pmatrix} G_1 & C_1 & 0 \\ B_1 & D_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{H}.$$

Here D_1 is always a ψDO on X_1 because the composition $i^1 A i_1$, where A is a ψDO on X_0, is a ψDO on X_1 (see [8, 12]). Next, B_1 is equal to

$$A_1\, i^1\, A_0, \tag{3.2}$$

where A_1, A_0 are ψDOs on X_1, X_0, respectively. Note that "long" compositions

$$A_1\, i^1\, A_0\, i_1\, A_1'\, i^1\, A_0',$$

where A_1', A_0' are ψDOs on X_1, X_0, respectively, can always be represented as in (3.2), since, as noted above, $i^1 A_0 i_1$ is a ψDO on X_1. The operator B_1 is a *boundary operator* localized at X_1. In a dual manner, C_1 is

$$A_0\, i_1\, A_1.$$

It is a *coboundary operator* localized at X_1. Finally, we have

$$G_1 = A_0\, i_1\, A_1\, i^1\, A_0',$$

where A_0, A_0' are ψDOs on X_0, and A_1 is a ψDO on X_1. This operator is called *Green operator* localized at X_1.

3. *The intersection* (3.1) *is equal to* X_2. This case is analogous to the previous one. One obtains matrix operators

$$\begin{pmatrix} G_2 & 0 & C_2 \\ 0 & 0 & 0 \\ B_2 & 0 & D_2 \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{H},$$

where D_2 is a ψDO on X_2; B_2 and C_2 are boundary and coboundary operators localized at X_2, respectively; G_2 is a Green operator localized at X_2.

4. *The intersection* (3.1) *is equal to* $X_1 \cap X_2$. In this case, one obtains matrix operators

$$\begin{pmatrix} M_0 & C_1' & C_2' \\ B_1' & M_1 & T_{12} \\ B_2' & T_{21} & M_2 \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{H}.$$

Each term in this matrix is a composition with at least one (co)boundary operator for X_1 and at least one (co)boundary operator for X_2. The operators M_0, M_1, M_2 are called *Mellin operators* localized at $X_1 \cap X_2$; the operators B_1', B_2' and C_1', C_2' are called boundary and coboundary operators localized at $X_1 \cap X_2$; the operators T_{12}, T_{21} are called *translators* between X_1 and X_2 (see [2]).

The operators of the above four classes constitute the set of additive generators of the algebra Mor. Therefore, a general morphism is of the form

$$\mathcal{D} = \begin{pmatrix} D_0 + G_1 + G_2 + M_0 & C_1 + C_1' & C_2 + C_2' \\ B_1 + B_1' & D_1 + M_1 & T_{12} \\ B_2 + B_2' & T_{21} & D_2 + M_2 \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{H}. \qquad (3.3)$$

It follows that the algebra Mor has 18 types of additive generators.

4 Symbols of morphisms

4.1 Symbols of ψDOs and (co)boundary operators

In this section, we define symbols of general morphisms in Mor. First, we define symbols of the generators on various submanifolds.

1. *Symbols for ψDOs.* Let A be a ψDO on X_0 and Z be a stratum in X_0. Denote by (y, z) the coordinates on X_0 such that $Z = \{y = 0\}$. Denote by (η, ζ) the corresponding coordinates in the fibers of the bundle T^*X_0. The principal symbol of A is denoted by $A(y, z, \eta, \zeta)$. Now, the *symbol* $\sigma_Z(A)$ of A on Z is defined as the following operator-function

$$\sigma_Z(A)(z, \zeta) = A\left(0, z, -i\frac{\partial}{\partial y}, \zeta\right) : H(\mathbb{R}^k_y) \longrightarrow H(\mathbb{R}^k_y), \quad (z, \zeta) \in T^*_0 Z. \qquad (4.1)$$

Note that (4.1) is obtained by freezing the coefficients of A at a point on Z and applying the Fourier transform with respect to the tangent variables.

The symbols of ψDOs on X_1, X_2, $X_1 \cap X_2$ are defined similarly.

2. *Symbols of elementary boundary operators.* The *symbol* $\sigma_{X_1}(i^1)$ of the operator i^1 on the stratum X_1 is the operator-function

$$\sigma_{X_1}(i^1)(z, \zeta) : H(\mathbb{R}^{\nu_1}_y) \longrightarrow \mathbb{C}, \quad u(y) \longmapsto u(0), \quad (z, \zeta) \in T^*_0 X_1,$$

where (z, y) are the coordinates on X_0 such that $X_1 = \{y = 0\}$. To define the symbol of i^1 on the stratum $X_1 \cap X_2$, we choose the local coordinates (x, y, z) on

X_0 such that $X_1 = \{y = 0\}$, and $X_1 \cap X_2 = \{(x, y) = (0, 0)\}$. Then the *symbol* $\sigma_{X_1 \cap X_2}(i^1)$ is the operator-function

$$\sigma_{X_1 \cap X_2}(i^1)(z, \zeta)\colon H(\mathbb{R}^{\nu_3}_{x,y}) \longrightarrow H(\mathbb{R}^{n_1}_x), \quad u(x, y) \longmapsto u(x, 0),$$

where $(z, \zeta) \in T^*_0(X_1 \cap X_2)$.

The symbols $\sigma_{X_2}(i^2)$ and $\sigma_{X_1 \cap X_2}(i^2)$ are defined along the same lines.

3. *Symbols of coboundary operators.* The symbol of the operator i_1 is defined in a dual manner to that of i^1. More precisely, we set

$$\sigma_{X_1}(i_1)(z, \zeta)\colon \mathbb{C} \longrightarrow H(\mathbb{R}^{\nu_1}_y), \quad q \longmapsto q\,\delta(y), \quad (z, \zeta) \in T^*_0 X_1,$$

and

$$\sigma_{X_1 \cap X_2}(i^1)(z, \zeta)\colon H(\mathbb{R}^{n_1}_x) \longrightarrow H(\mathbb{R}^{\nu_3}_{x,y}), \quad u(x) \longmapsto u(x) \otimes \delta(y),$$

where $(z, \zeta) \in T^*_0(X_1 \cap X_2)$.

The symbols $\sigma_{X_2}(i^2)$ and $\sigma_{X_1 \cap X_2}(i^2)$ of i^2 are defined similarly.

4.2 Symbols of general morphisms

Let Z be any of the strata X_0, X_1, X_2, $X_1 \cap X_2$.

Definition 4.1. The composition

$$\sigma_Z(\mathcal{D}_{kl}) = \sigma_Z(D_{k,i_1})\,\sigma_Z(D_{i_1,i_2})\,\ldots\,\sigma_Z(D_{i_N,l}), \tag{4.2}$$

is called the *symbol* $\sigma_Z(\mathcal{D}_{kl})$ of the morphism (2.4) on Z. Note that all the terms σ_Z on the right hand side were defined above.

Therefore, the following symbols are defined for a general morphism (3.3).

1. The symbol on X_0 is equal to the symbol of the ψDO component D_0:

$$\sigma_{X_0}(\mathcal{D})(z, \zeta) = \sigma(D_0)(z, \zeta)\colon \mathbb{C} \longrightarrow \mathbb{C}, \quad (z, \zeta) \in T^*_0 X_0.$$

2. The symbol on X_1 is the operator-function

$$\sigma_{X_1}(\mathcal{D})(z, \zeta) = \begin{pmatrix} \sigma_{X_1}(D_0 + G_1) & \sigma_{X_1}(C_1) \\ \sigma_{X_1}(B_1) & \sigma(D_1) \end{pmatrix}(z, \zeta), \quad (z, \zeta) \in T^*_0 X_1$$

ranging in operators acting in the spaces

$$\sigma_{X_1}(\mathcal{D})(z, \zeta)\colon \begin{matrix} H(\mathbb{R}^{\nu_1}) \\ \oplus \\ \mathbb{C} \end{matrix} \longrightarrow \begin{matrix} H(\mathbb{R}^{\nu_1}) \\ \oplus \\ \mathbb{C} \end{matrix}.$$

3. The symbol on X_2 is the operator-function

$$\sigma_{X_2}(\mathcal{D})(z, \zeta) = \begin{pmatrix} \sigma_{X_2}(D_0 + G_2) & \sigma_{X_2}(C_2) \\ \sigma_{X_2}(B_2) & \sigma(D_2) \end{pmatrix}(z, \zeta), \quad (z, \zeta) \in T^*_0 X_2,$$

with values in operators acting in the spaces

$$\sigma_{X_2}(\mathcal{D})(z, \zeta)\colon \begin{matrix} H(\mathbb{R}^{\nu_2}) \\ \oplus \\ \mathbb{C} \end{matrix} \longrightarrow \begin{matrix} H(\mathbb{R}^{\nu_2}) \\ \oplus \\ \mathbb{C} \end{matrix}.$$

4. Finally, the symbol on the intersection $Z = X_1 \cap X_2$ is the operator-function

$$\sigma_Z(\mathcal{D})(z,\zeta) = \\ = \begin{pmatrix} \sigma_Z(D_0+G_0+G_1+M_0) & \sigma_Z(C_1+C_1') & \sigma_Z(C_2+C_2') \\ \sigma_Z(B_1+B_1') & \sigma_Z(D_1+M_1) & \sigma_Z(T_{12}) \\ \sigma_Z(B_2+B_2') & \sigma_Z(T_{21}) & \sigma_Z(D_2+M_2) \end{pmatrix}(z,\zeta),$$

where $(z,\zeta) \in T_0^*Z$, with values in operators acting in the spaces

$$\sigma_Z(\mathcal{D})(z,\zeta)\colon \begin{matrix} H(\mathbb{R}^{\nu_3}) \\ \oplus \\ H(\mathbb{R}^{n_1}) \\ \oplus \\ H(\mathbb{R}^{n_2}) \end{matrix} \longrightarrow \begin{matrix} H(\mathbb{R}^{\nu_3}) \\ \oplus \\ H(\mathbb{R}^{n_1}) \\ \oplus \\ H(\mathbb{R}^{n_2}) \end{matrix}.$$

4.3 The composition formula

In this section we show that symbols of morphisms are well defined and establish the corresponding composition formula. This is the main result of the present paper.

Theorem 4.2. *For any morphism $\mathcal{D} \in \mathrm{Mor}$, its symbol $\sigma_Z(\mathcal{D})$ (see Definition 4.1) on any stratum Z does not depend on the choice of representation of $\mathcal{D}$ in terms of generators* (2.1), (2.2), *and* (2.3) *of the algebra* Mor. *Moreover, for any two morphisms $\mathcal{D}_1, \mathcal{D}_1 \in \mathrm{Mor}$ the following composition formula holds*

$$\sigma_Z(\mathcal{D}_1\mathcal{D}_2) = \sigma_Z(\mathcal{D}_1)\,\sigma_Z(\mathcal{D}_2). \tag{4.3}$$

Proof. Note that (4.3) readily follows from Definition 4.1. Thus, we only have to prove that the symbol is well-defined (that is, it does not depend on the choice of representation of a morphism in terms of generators).

1. Define the following order reduction operators

$$(\Lambda_0,\Lambda_1,\Lambda_2)\colon \bigoplus_{k=0,1,2} H^{s_k}(X_k) \longrightarrow \bigoplus_{k=0,1,2} L^2(X_k),$$

where Λ_k is an elliptic ψDO on X_k of order s_k. Now, we can reduce any morphism $\mathcal{D} \in \mathrm{Mor}$ to an operator acting in L^2-spaces by multiplying it by appropriate powers of order reduction operators. Then it suffices to prove the theorem for the resulting operators; the general case easily follows.

Thus, we assume that $\mathcal{D} \in \mathrm{Mor}$ acts in the spaces

$$\mathcal{D}\colon \bigoplus_{k=0,1,2} L^2(X_k) \longrightarrow \bigoplus_{k=0,1,2} L^2(X_k). \tag{4.4}$$

2. Now, we show that symbols of the morphism (4.4) are well defined. We note that our approach is based on the ideas described in [13] (for a smooth manifold without boundary) and in [14] (for boundary value problems).

First, we introduce an auxiliary operator family. Namely, consider the space $\mathbb{R}^{k+\nu}$ with coordinates (z,y). Given a point $(z_0,\zeta_0) \in T_0^*\mathbb{R}^k$, define the operator family (cf. [14])

$$R_{\lambda,z,y}\colon L^2(\mathbb{R}^{k+\nu}_{z,y}) \longrightarrow L^2(\mathbb{R}^{k+\nu}_{z,y}), \quad \lambda > 0, \tag{4.5}$$

where

$$R_{\lambda,z,y}\colon u(z,y) \longmapsto \lambda^{k/4+\nu/2}\, e^{i\lambda z\zeta_0}\, u\big(\lambda^{1/2}(z-z_0), \lambda y\big).$$

A straightforward computation shows that the operators (4.5) are unitary, and for any $u \in C_c^\infty(\mathbb{R}^{k+\nu})$ the sequence of functions $R_{\lambda,z,y}u$ tends to 0 weakly in $L^2(\mathbb{R}^{k+\nu}_{z,y})$ as $\lambda \to \infty$.

Let us return to the morphism (4.4). Consider its component

$$\mathcal{D}_{kl}\colon L^2(X_l) \longrightarrow L^2(X_k)$$

and its symbol $\sigma_Z(\mathcal{D}_{kl})$ on some stratum $Z \subset X_1 \cup X_2$. Choose the coordinates $(z,y) \in \mathbb{R}^{n+\nu} = X_k$, $(z,y') \in \mathbb{R}^{n+\nu'} = X_l$, such that Z is defined by the equations $Z = \{(z,0)\}$ in X_k and in X_l. Here ν stands for the codimension of Z in X_k, and ν' is the codimension of Z in X_l.

The following lemma implies that the symbol $\sigma_Z(D_{kl})$ is well defined.

Lemma 4.3. *For any two functions $u \in C_c^\infty(\mathbb{R}^n_z)$, $v \in C_c^\infty(\mathbb{R}^{\nu'}_{y'})$, and any point $(z_0,\zeta_0) \in T_0^*Z$ the following equality holds*

$$\lim_{\lambda\to\infty} \big\| R^{-1}_{\lambda,z,y}\, \mathcal{D}_{kl}\, R_{\lambda,z,y'}(u\otimes v) - u \otimes \big[\sigma_Z(\mathcal{D}_{kl})(z_0,\zeta_0)\big] v \big\|_{L^2(X_k)} = 0. \tag{4.6}$$

Proof. By linearity and multiplicativity of the expression under the norm sign in (4.6), it suffices to prove that the limit is equal to zero in the following three special cases:

1. $k = l$, and $\mathcal{D}_{kk}$ is a ψDO of order zero;
2. $l = 0$, $k > 0$, and
$$\mathcal{D}_{k0} = \Lambda_k\, i^k\, \Lambda_0\colon L^2(X_0) \longrightarrow L^2(X_k),$$
where Λ_0 and Λ_k are order reduction operators on X_0 and X_k respectively.
3. $k = 0$, $l > 0$, and
$$\mathcal{D}_{0l} = \Lambda_0\, i_l\, \Lambda_l\colon L^2(X_l) \longrightarrow L^2(X_0),$$
where Λ_0 and Λ_l are order reduction operators on X_0 and X_l respectively.

For brevity, we omit the details of the corresponding verification and refer the reader to [14] where a similar calculation is carried out. □

Now, it follows from Lemma 4.3 that the symbol of (4.4) is well defined. In turn, this implies that symbols of general morphisms are well defined.

The proof of Theorem 4.2 is now complete. □

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