

RAPIDLY ROTATING STARS

WALTER A. STRAUSS AND YILUN WU

ABSTRACT. A rotating star may be modeled as a continuous system of particles attracted to each other by gravity and with a given total mass and prescribed angular velocity. Mathematically this leads to the Euler-Poisson system. We prove an existence theorem for such stars that are rapidly rotating, depending continuously on the speed of rotation. This solves a problem that has been open since Lichtenstein's work in 1933. The key tool is global continuation theory, combined with a delicate limiting process. The solutions form a connected set \mathcal{K} in an appropriate function space. As the speed of rotation increases, we prove that *either the supports of the stars in \mathcal{K} become unbounded or the density somewhere within the stars becomes unbounded*. We permit any equation of state of the form $p = \rho^\gamma$, $6/5 < \gamma < 2$, so long as $\gamma \neq 4/3$. We consider two formulations, one where the angular velocity is prescribed and the other where the angular momentum per unit mass is prescribed.

1. INTRODUCTION

We consider a continuum of particles attracted to each other by gravity but subject to no other forces. Initially they are static and spherical but then they begin to rotate around a fixed axis after some perturbation and thereby flatten at the poles and expand at the equator. This is a simple model of a rotating star or planet. It can also model a rotating galaxy with its billions of stars. In this paper we permit fast rotations and look for steady states of the resulting configuration. To find a family of states with a given mass is a highly desirable property. We find a connected set of such states with constant mass.

This is a very classical problem that goes back to MacLaurin, Jacobi, Poincaré, Liapunov et al., who assumed the density of the rotating fluid to be homogeneous or almost homogeneous, which is of course physically unrealistic. See Jardetzky [10] for a nice account of the classical history of the problem. More realistic work for slow rotations was begun by Lichtenstein [13] beginning in 1918 and by Heilig [8], who approached the problem of slowly rotating stars by means of an implicit function theorem in function space. They made realistic assumptions on the density but the mass of their solutions changes as the body changes its speed of rotation. Recently Jang and Makino [9] studied the problem of slowly rotating stars using a simpler implicit function approach in the case of the power law $p = C\rho^\gamma$ and constant rotation speed. However, as in Lichtenstein and Heilig's work, their perturbation also does not keep the total mass constant and their analysis is restricted to the range $\frac{6}{5} < \gamma < \frac{3}{2}$. In [18] we also constructed slowly rotating stars. We constructed solutions with a given constant mass and permitted a general equation of state and a general rotation speed (see Formulation 4 in Section 7).

A different approach was begun in 1971 by Auchmuty and Beals [3] using a variational method with a mass constraint. The main difficulty in this approach is to prove that the minimizing solution has compact support. Their approach was generalized and extended by many authors, including Auchmuty [2], Caffarelli and Friedman [4], Friedman and Turkington [6], Li [12], Chanillo and Li [5], Luo and Smoller [14], Wu [19], and Wu [20]. The variational method has the major advantages that the rotation speed is allowed to be large and that the mass is constant. However, there is no control on the nature of the compact support of the star, it does not provide a *continuous* curve of solutions depending on the angular velocity, and the equation of state is restricted to powers satisfying $\gamma > \frac{4}{3}$. This

variational method is the only one that has previously been used to prove the existence of solutions that rotate rapidly.

In the present paper we extend the implicit function approach to construct solutions that represent stars that rotate rapidly. We construct, for the first time, a *connected* set \mathcal{K} of solutions that is *global*. Keeping the mass constant is a key to our methodology, so that there is no loss or gain of particles when the star changes its rotation speed. Furthermore, we permit (a) the full range $\frac{6}{5} < \gamma < 2, \gamma \neq \frac{4}{3}$, (b) a non-uniform angular velocity, and (c) a general equation of state $p = p(\rho)$.

Now we describe our method. We begin with the steady compressible Euler-Poisson equations (EP) for the density $\rho \geq 0$, subject to the internal forces of gravity due to the particles themselves. The speed $\omega(r)$ of rotation around the x_3 -axis is allowed to depend on $r = r(x) = \sqrt{x_1^2 + x_2^2}$. The inertial forces are entirely due to the rotation. In the region $\{x \in \mathbb{R}^3 \mid \rho(x) > 0\}$ occupied by the star, EP reduces to the equation

$$(1.1) \quad \frac{1}{|x|} * \rho + \kappa^2 \int_0^r s \omega^2(s) ds - h(\rho) = \text{constant},$$

where $\omega(r)$ is a given function, κ is a constant measuring the intensity of rotation, h is the enthalpy defined by $h'(\rho) = \frac{p'(\rho)}{\rho}$ with $h(0) = 0$, and p is the pressure. The constant of gravity is assumed to be 1. The density must vanish at the boundary of the star. See the end of this introduction for the derivation of (1.1).

So far this approach is standard. For simplicity in this introduction let us consider the standard equation of state $p(\rho) = C\rho^\gamma$. As a first attempt we take the inverse of h to reformulate the problem as

$$(1.2) \quad \rho(\cdot) = \left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 \int_0^r s \omega^2(s) ds + \alpha \right]_+^{1/(\gamma-1)}, \quad \int_{\mathbb{R}^3} \rho(x) dx = M,$$

where α is the negative of the constant that appears in (1.1), M is the given value of the mass, and $[z]_+ = \max(z, 0)$. This is reminiscent of the discussion of Auchmuty [2] and the method of Jang and Makino [9]. Auchmuty [2] found rapidly rotating solutions that unfortunately do not satisfy the physical boundary conditions but instead may have large density at the boundary of the star. What is novel in our formulation is to force the total mass M to be fixed and to introduce the constant α as a variable. The case $\gamma = \frac{4}{3}$ is excluded because in that case the constant mass condition introduces a nullspace of the linearized operator. If the mass were allowed to vary, the nullspace would be trivial so that the implicit function theorem would be applicable and $4/3$ would be permitted. In Section 7 we compare our approach (1.2) to several alternative mathematical approaches.

Nonetheless, even with this method there is still no way to guarantee that ρ has compact support because the expression inside $[\dots]_+$ could be positive for large $|x|$. We get the support to be compact by artificially forcing the parameter α to be sufficiently negative (see Lemma 3.1). Then we begin the construction of rotating star solutions in the standard way by continuation from a non-rotating solution ($\kappa = 0$). It is in this first step that we require $\frac{6}{5} < \gamma < 2, \gamma \neq \frac{4}{3}$, and we refer to [18] for some lemmas and details.

Letting κ increase, we continue the construction by applying the global implicit function theorem, which is based on the Leray-Schauder degree (see Lemma 5.1). Later on, in Theorem 5.2 we obtain the whole global connected set \mathcal{K} of solutions by allowing α to increase. The most novel and intricate part of our proof occurs here. Our main result, stated somewhat informally, is as follows. See Theorem 5.2 below for a completely precise version.

Theorem 1.1. *Let M be the mass of the non-rotating solution. Assume the pressure $p(\cdot)$ and the angular velocity $\omega(\cdot)$ satisfy (2.2)-(2.4), (2.7)-(2.9), (3.2)-(3.3). By a "solution" of the problem, we mean a triple (ρ, κ, α) , where ρ is an axisymmetric function with mass M*

that satisfies (1.1) and κ refers to the intensity of rotation speed. Then there exists a set \mathcal{K} of solutions satisfying the following three properties.

- \mathcal{K} is a connected set in the function space $C_c^1(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$.
- \mathcal{K} contains the non-rotating solution.
- either

$$\sup\{\rho(x) \mid x \in \mathbb{R}^3, (\rho, \kappa, \alpha) \in \mathcal{K}\} = \infty$$

or

$$\sup\{|x| \mid \rho(x) > 0, (\rho, \kappa, \alpha) \in \mathcal{K}\} = \infty.$$

The last statement means that either the densities become pointwise unbounded or the supports become unbounded.

There is another formulation that is popular in the astronomical literature where the angular velocity ω is replaced by the angular momentum L per unit mass. Our results in the latter formulation are entirely analogous, as we describe in Section 6.

We end this introduction by describing how EP reduces to (1.1). The compressible Euler-Poisson equations (EP) are

$$(1.3) \quad \begin{cases} \rho_t + \nabla \cdot (\rho v) = 0, \\ (\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla p = \rho \nabla U, \\ U(x, t) = \int_{\mathbb{R}^3} \frac{\rho(x', t)}{|x - x'|} dx'. \end{cases}$$

The first two equations hold where $\rho > 0$, and the last equation defines U on the entire \mathbb{R}^3 . To close the system, one prescribes an isentropic equation of state $p = p(\rho)$. To model a rotating star, one looks for a steady axisymmetric rotating solution to (1.3). That is, we assume ρ is symmetric about the x_3 -axis and $v = \kappa \omega(r)(-x_2, x_1, 0)$, where $r = r(x) = \sqrt{x_1^2 + x_2^2}$ with a prescribed function $\omega(r)$. With such specifications, the first equation in (1.3) concerning mass conservation is identically satisfied. The second equation in (1.3) concerning momentum conservation simplifies to

$$(1.4) \quad -\rho \kappa r \omega^2(r) e_r + \nabla p = \rho \nabla \left(\frac{1}{|\cdot|} * \rho \right), \quad e_r = \frac{1}{r(x)}(x_1, x_2, 0).$$

The first term in (1.4) can be written as $-\rho \nabla \left(\int_0^r \omega^2(s) s ds \right)$. Introducing the specific enthalpy h as above, (1.4) becomes

$$(1.5) \quad \nabla \left(\frac{1}{|\cdot|} * \rho + \kappa \int_0^r \omega^2(s) s ds - h(\rho) \right) = 0,$$

which is the same as (1.1).

2. PROPERTIES OF NON-ROTATING SOLUTIONS

In this section, we summarize some properties of the non-rotating radial (spherically symmetric) solutions to the semilinear elliptic equation

$$(2.1) \quad \Delta u + 4\pi h^{-1}(u_+) = 0 \text{ in } \mathbb{R}^3.$$

Such radial solutions will be the starting point of the global set of axisymmetric solutions we will construct.

We make the following assumptions on the equation of state $p(s)$:

$$(2.2) \quad p(s) \in C_{loc}^2(0, \infty), \quad p'(s) > 0.$$

There exists $\gamma \in (1, 2)$ such that

$$(2.3) \quad \lim_{s \rightarrow 0^+} s^{2-\gamma} p''(s) = c_0 > 0.$$

There exists $\gamma^* \in (\frac{6}{5}, 2)$ such that

$$(2.4) \quad \lim_{s \rightarrow \infty} s^{1-\gamma^*} p'(s) = c_1 > 0.$$

As shown in Lemma 3.1 of [18], these assumptions imply that the enthalpy h , defined by $h'(\rho) = p'(\rho)/\rho$, $h(0) = 0$, is a one-to-one map from $[0, \infty)$ to $[0, \infty)$. Its inverse h^{-1} is locally $C^{1,\beta}$ on $[0, \infty)$, with $h^{-1}(0) = (h^{-1})'(0) = 0$ and

$$(2.5) \quad \lim_{s \rightarrow \infty} \frac{h^{-1}(s)}{s} = \infty, \quad \lim_{s \rightarrow \infty} \frac{h^{-1}(s)}{s^5} = 0.$$

It follows that for all $R_0 > 0$, equation (2.1) has a positive radial (spherically symmetric) solution $u_0 \in C^2(\overline{B_{R_0}})$ with zero boundary values on $\partial B_{R_0} = \{x : |x| = R_0\}$ (see Lemma 3.2 in [18]). Thus $\rho_0 := h^{-1}(u_0)$ belongs to $C^{1,\beta}(\mathbb{R}^3)$ when extended to be zero outside B_{R_0} (see Lemma 3.3 in [18]). Radial solutions of (2.1) solve the ODE

$$(2.6) \quad u'' + \frac{2}{|x|}u' + 4\pi h^{-1}(u_+) = 0,$$

where $'$ denotes the radial derivative. We denote by $u(|x|; a)$ the solution of (2.6) satisfying $u(0; a) = a$, $u'(0; a) = 0$. (In [18], $u(|x|; a)$ is denoted by $v(r; a)$.) For $a > 0$, there are only two possibilities for the behavior of $u(|x|; a)$:

- (i) There exists a unique $R(a) > 0$ such that $u(R(a); a) = 0$.
- (ii) $u(|x|; a) > 0$ for all $|x| \geq 0$.

Let us denote by \mathcal{A} the set of all a 's such that possibility (i) holds. Note that $u_0(0) \in \mathcal{A}$. Furthermore, \mathcal{A} is an open set, as is easily seen by considering the fact that for $a_0 \in \mathcal{A}$ we have $u(R(a_0); a_0) = 0$ and $u'(R(a_0); a_0) \neq 0$. The implicit function theorem implies that $u(R(a); a) = 0$ has a solution $R(a)$ for all a sufficiently near a_0 .

Now for $a \in \mathcal{A}$, we can define the *physical mass* of the compactly supported radial solution $[u(|x|; a)]_+$ as

$$(2.7) \quad M(a) = \int_{B_{R(a)}} h^{-1}(u(|x|; a)) \, dx = \int_0^{R(a)} 4\pi h^{-1}(u(r; a)) r^2 \, dr.$$

Note that $M(a) > 0$ and $M(\cdot)$ is differentiable on $(0, \infty)$. Throughout this paper we make the following assumptions on the function $M(a)$:

$$(2.8) \quad M'(u_0(0)) \neq 0$$

and

$$(2.9) \quad M(a) \neq M(u_0(0)), \quad \forall a \in \mathcal{A}, a \neq u_0(0).$$

Assumptions (2.8) and (2.9) are used in Lemmas 4.3 and 5.1, respectively. Now we provide two examples of equations of state that satisfy both of these assumptions.

Lemma 2.1. *Suppose that either one of the following conditions holds for the equation of state $p(s)$:*

- (a) $p(s) = s^\gamma$, where $\gamma \in (\frac{6}{5}, 2)$, $\gamma \neq \frac{4}{3}$.
- (b) $p(s)$ satisfies (2.2), (2.3), (2.4), and

$$(2.10) \quad p'(s) < h(s) \leq 2p'(s) \text{ for } s > 0.$$

Then $\mathcal{A} = (0, \infty)$, and (2.8) and (2.9) are satisfied.

Proof. First, if $p(s) = s^\gamma$, then $h^{-1}(s) = \left(\frac{\gamma-1}{\gamma}s\right)^{1/(\gamma-1)}$. By the scaling symmetry of (2.6) for this function h^{-1} , we have

$$(2.11) \quad u(|x|; a) = \frac{a}{a_0} u\left(\left(\frac{a}{a_0}\right)^{(2-\gamma)/(2\gamma-2)} |x|; a_0\right).$$

Thus $\mathcal{A} = (0, \infty)$. It follows from (2.11) and (2.7) that

$$(2.12) \quad M(a) = \left(\frac{a}{a_0}\right)^{(3\gamma-4)/(2\gamma-2)} M(a_0)$$

for $a, a_0 > 0$. It is now obvious that both (2.8) and (2.9) are satisfied if $\gamma \in (\frac{6}{5}, 2)$, $\gamma \neq \frac{4}{3}$.

Secondly, suppose (b) is satisfied. The condition $h(s) \leq 2p(s)$ in (2.10) implies that $h(s) \leq 2sh'(s)$ by definition of h . Thus with $t = h(s)$ we have

$$(2.13) \quad t(h^{-1})'(t) \leq 2h^{-1}(t) \text{ for } t > 0.$$

Integration of this inequality yields

$$(2.14) \quad h^{-1}(t) \geq \frac{h^{-1}(1)}{t^2} \text{ for } 0 < t < 1.$$

Thus the integral $\int_0^1 h^{-1}(t)t^{-4} dt$ diverges. So by Theorem 1 in [15], no solution to (2.6) can stay positive for all $|x|$. This means that $\mathcal{A} = (0, \infty)$, so that the physical mass $M(a)$ is defined for all $a \in (0, \infty)$.

Now if $u(|x|; a)$ is supported on the ball of radius $R(a)$, then $\tilde{u}(|x|) = u(R(a)|x|; a)$ is supported on B_1 and satisfies

$$\tilde{u}'' + \frac{2}{|x|}\tilde{u}' + \widetilde{h^{-1}}(\tilde{u}_+) = 0$$

where $\widetilde{h^{-1}} = R^2(a)h^{-1}$ satisfies the same kind of inequality as h^{-1} . Replacing u by \tilde{u} and h^{-1} by $\widetilde{h^{-1}}$, we can therefore assume without loss of generality that $u(|x|; a)$ is supported on B_1 . Now the proof of Lemma 4.3 in [18] (without specializing the value of a) shows that $M'(a) = -u'_a(1; a)$. The subscript denotes the derivative with respect to a , while the prime denotes the derivative with respect to $|x|$. Letting $w = |x|u$ and $g(w, |x|) = 4\pi r h^{-1}(w/|x|)$, we have $u'_a(1; a) = w'_a(1; a) - w_a(1; a)$. Thus the conclusion of Lemma 4.9 in [18] implies that $u'_a(1; a) < 0$. Therefore both (2.8) and (2.9) are satisfied. \square

3. FORMULATION BY ANGULAR VELOCITY

For simplicity of notation we assume $R_0 = 1$ for the solution ρ_0 in Section 2 from now on. Let $M = \int_{B_1} \rho_0(x) dx$ and

$$(3.1) \quad j(x) = \int_0^{r(x)} s \omega^2(s) ds.$$

We will sometimes abuse notation and write $j(x)$ as $j(r(x))$. We assume that the rotation speed satisfies

$$(3.2) \quad s\omega^2(s) \in L^1(0, \infty), \quad \omega^2(s) \text{ is not compactly supported,}$$

and

$$(3.3) \quad \lim_{r(x) \rightarrow \infty} r(x) \left(\sup_x j - j(x) \right) = 0.$$

This means that $\omega(r)$ decays to zero sufficiently fast as $r \rightarrow \infty$. It does not really matter because our purpose is to construct stars that have compact support, but it is a convenient assumption that was also made in [3] for instance.

We define the operators

$$\mathcal{F}_1(\rho, \kappa, \alpha) = \rho(\cdot) - h^{-1} \left(\left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 j(\cdot) + \alpha \right]_+ \right),$$

$$\mathcal{F}_2(\rho) = \int_{\mathbb{R}^3} \rho(x) dx - M,$$

and the pair

$$\mathcal{F}(\rho, \kappa, \alpha) = (\mathcal{F}_1(\rho, \kappa, \alpha), \mathcal{F}_2(\rho)).$$

It is not hard to see that a solution to $\mathcal{F}(\rho, \kappa, \alpha) = 0$ with $\rho \in C_{loc}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ will give rise to a solution of (1.5) with mass M . Indeed, on the set where ρ is positive, one has

$$\frac{1}{|\cdot|} * \rho(x) + \kappa^2 j(x) - h(\rho(x)) + \alpha = 0,$$

which is the same as (1.5). For fixed constants $s > 3$, we define the weighted space

$$\mathcal{C}_s = \left\{ f : \mathbb{R}^3 \rightarrow \mathbb{R} \mid f \text{ is continuous, axisymmetric, even in } x_3, \text{ and } \|f\|_s < \infty \right\},$$

where

$$\|f\|_s =: \sup_{x \in \mathbb{R}^3} \langle x \rangle^s |f(x)| < \infty.$$

We also define for $N > 0$,

$$(3.4) \quad \mathcal{O}_N = \left\{ (\rho, \kappa, \alpha) \in \mathcal{C}_s \times \mathbb{R}^2 \mid \alpha + \kappa^2 \sup_x j(x) < -\frac{1}{N} \right\}.$$

We are looking for solutions of $\mathcal{F}(\rho, \kappa, \alpha) = 0$. We will find them by a continuation argument starting from the non-rotating solution, which satisfies $\mathcal{F}(\rho_0, 0, \alpha_0) = 0$. A key device in our proof is to control the supports of the stars. We begin with the following simple, but important, observation.

Lemma 3.1. *For all $(\rho, \kappa, \alpha) \in \mathcal{O}_N$, the expression $\left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 j(\cdot) + \alpha \right]_+$ is supported in the ball $\{x \in \mathbb{R}^3 : |x| \leq C_0 N \|\rho\|_s\}$, where C_0 is an absolute constant.*

Proof. First we note that $\left| \frac{1}{|\cdot|} * \rho(\cdot)(x) \right| \leq C_0 \|\rho\|_s \frac{1}{\langle x \rangle}$ because $s > 3$. Hence for $|x| > C_0 N \|\rho\|_s$,

$$\left[\frac{1}{|\cdot|} * \rho(\cdot)(x) + \kappa^2 j(x) + \alpha \right] \leq C_0 \|\rho\|_s \frac{1}{\langle x \rangle} - \frac{1}{N} < 0$$

since $(\rho, \kappa, \alpha) \in \mathcal{O}_N$. Therefore its positive part vanishes for such x . \square

4. BASIC PROPERTIES

Lemma 4.1. *\mathcal{F} maps \mathcal{O}_N into $\mathcal{C}_s \times \mathbb{R}$. It is C^1 Fréchet differentiable, with Fréchet derivative given by*

$$(4.1) \quad \frac{\partial \mathcal{F}}{\partial(\rho, \kappa, \alpha)}(\delta\rho, \delta\kappa, \delta\alpha) = \left(\delta\rho - \mathcal{L}(\delta\rho, \delta\kappa, \delta\alpha), \int_{\mathbb{R}^3} \delta\rho(x) dx \right),$$

where

$$(4.2) \quad \mathcal{L}(\delta\rho, \delta\kappa, \delta\alpha) = (h^{-1})' \left(\left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 j(\cdot) + \alpha \right]_+ \right) \left(\frac{1}{|\cdot|} * \delta\rho + 2\kappa(\delta\kappa)j + \delta\alpha \right).$$

Proof. \mathcal{F}_2 is very simple so we concentrate on \mathcal{F}_1 . We need to show that $\mathcal{F}_1 \in \mathcal{C}_s$. By Lemma 3.1, we may focus on the ball $|x| \leq C_0 N \|\rho\|_s$. Since h^{-1} is increasing, we have

$$(4.3) \quad \begin{aligned} & \sup_{x \in \mathbb{R}^3} \langle x \rangle^s h^{-1} \left(\left[\frac{1}{|\cdot|} * \rho(\cdot)(x) + \kappa^2 j(x) + \alpha \right]_+ \right) \\ & \leq \sup_{|x| \leq C_0 N \|\rho\|_s} \langle x \rangle^s h^{-1} \left(C_0 \|\rho\|_s \frac{1}{\langle x \rangle} \right) \\ & \leq \langle C_0 N \|\rho\|_s \rangle^s h^{-1} (C_0 \|\rho\|_s). \end{aligned}$$

This shows that $\mathcal{F}_1(\rho, \kappa, \alpha) \in \mathcal{C}_s$. In order to prove the Fréchet differentiability, we again use Lemma 3.1 to deduce that $\left[\frac{1}{|\cdot|} * (\rho + \delta\rho)(\cdot) + (\kappa + \delta\kappa)^2 j + \alpha + \delta\alpha \right]_+$ is supported in some fixed ball B_R for fixed $(\rho, \kappa, \alpha) \in \mathcal{O}_N$ and sufficiently small $(\delta\rho, \delta\kappa, \delta\alpha)$. Note that for $u \in \mathcal{C}_s$ supported in B_R , $\|u\|_s \leq \langle R \rangle^s \|u\|_{C^0(\overline{B_R})}$. Now we only need to recognize the obvious fact that $u \mapsto h^{-1}(u_+)$ as a mapping from $C^0(\overline{B_R})$ to itself is differentiable with derivative $(h^{-1})'(u_+)$. Equation (4.2) follows by the chain rule. The continuity of the Fréchet derivative follows in a similar way, as $u \mapsto (h^{-1})'(u_+)$ is continuous on $C^0(\overline{B_R})$. \square

Lemma 4.2. *For each $(\rho, \kappa, \alpha) \in \mathcal{O}_N$, $\frac{\partial \mathcal{F}}{\partial(\rho, \alpha)}(\rho, \kappa, \alpha)$ is a Fredholm operator on $\mathcal{C}_s \times \mathbb{R}$.*

Proof. By (4.1), we only need to show $\mathcal{L}(\cdot, 0, \cdot)$ is compact. By Lemma 3.1, $\mathcal{L}(\cdot, 0, \cdot)$ is supported in B_R with R depending only on (ρ, κ, α) . It is obvious that $\delta\rho \mapsto \frac{1}{|\cdot|} * \delta\rho(\cdot)$ is compact from \mathcal{C}_s to $C^0(\overline{B_R})$. This implies the Fredholm property. \square

Lemma 4.3. *Let $(\rho_0, 0, \alpha_0)$ be the non-rotating solution. If (2.8) is true, then the nullspace of the linear operator $\frac{\partial \mathcal{F}}{\partial(\rho, \alpha)}(\rho_0, 0, \alpha_0)$ is trivial. Therefore this operator is an isomorphism.*

Proof. From $\mathcal{F}(\rho_0, 0, \alpha_0) = 0$ we get

$$\rho_0 - h^{-1} \left(\left[\frac{1}{|\cdot|} * \rho_0 + \alpha_0 \right]_+ \right) = 0.$$

Denoting $u_0 = h(\rho_0)$ as in Section 2, we have

$$(4.4) \quad u_0 = \left[\frac{1}{|\cdot|} * \rho_0 + \alpha_0 \right]_+.$$

We also note the relation

$$(4.5) \quad \rho'_0 = (h^{-1})'(u_0) \cdot u'_0.$$

From $\frac{\partial \mathcal{F}}{\partial(\rho, \alpha)}(\rho_0, 0, \alpha_0)(\delta\rho, \delta\alpha) = 0$, we get

$$(4.6) \quad \delta\rho - (h^{-1})'(u_0) \left(\frac{1}{|\cdot|} * \delta\rho + \delta\alpha \right) = 0,$$

$$(4.7) \quad \int_{\mathbb{R}^3} \delta\rho(x) dx = 0.$$

Since ρ_0 and u_0 are supported on B_1 , (4.6) implies that $\delta\rho$ is also supported on B_1 . Define $w = \frac{1}{|\cdot|} * \delta\rho + \delta\alpha$. By (4.6), $\delta\rho$ is Hölder continuous on \mathbb{R}^3 . Thus $\Delta w = -4\pi\delta\rho$. By (4.5) and (4.6), we have

$$(4.8) \quad \Delta w = \begin{cases} -4\pi \frac{\rho'_0}{u'_0} w & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Using spherical coordinates, we may regard w as a function on $\mathbb{S}^2 \times \mathbb{R}^+$. Multiplying (4.8) by the non-radial ($l \geq 1$) spherical harmonic Y_{lm} and integrating over \mathbb{S}^2 , we can write

$$(4.9) \quad \Delta w_{lm} - \frac{l(l+1)}{|x|^2} w_{lm} = \begin{cases} -4\pi \frac{\rho'_0}{u'_0} w_{lm} & \text{for } 0 < |x| \leq 1, \\ 0 & \text{for } |x| > 1, \end{cases}$$

where $w_{lm} = \langle w, Y_{lm} \rangle_{\mathbb{S}^2}$. The same argument as in the proof of Lemma 4.5 of [18] (where w_{lm} is called φ_{lm}) will give us $w_{lm} = 0$. There is a technical point in that argument which requires $\lim_{|x| \rightarrow 0^+} \frac{w_{lm}(|x|)}{u'_0(|x|)} = 0$, or equivalently $\lim_{|x| \rightarrow 0^+} \frac{w_{lm}(|x|)}{|x|} = 0$. In fact, this is true because

$$\begin{aligned} \left| \frac{w_{lm}(|x|)}{|x|} \right| &= \left| \int_{\mathbb{S}^2} \frac{w(|x|\omega)}{|x|} \overline{Y_{lm}(\omega)} d\omega \right| = \left| \int_{\mathbb{S}^2} \frac{w(|x|\omega) - w(0)}{|x|} \overline{Y_{lm}(\omega)} d\omega \right| \\ &\leq C \sup_{|y| \leq |x|} |\nabla w(y)| \end{aligned}$$

The last quantity tends to 0 as $|x| \rightarrow 0^+$, because $w \in C^1$, and $\nabla w(0) = 0$ by the symmetry of $\delta\rho$.

We have now proven that w must be a radial function. Integrating $\Delta w = -4\pi\delta\rho$ over B_1 , using (4.7), and using the fact that $\delta\rho$ is supported on B_1 , we get $w'(1) = 0$. Thus w solves the boundary value problem

$$(4.10) \quad \Delta w + 4\pi \frac{\rho'_0}{u'_0} w = 0, \quad w'(1) = 0$$

on B_1 . By Lemma 4.3 of [18], w vanishes in B_1 . Thus $\delta\rho = 0$ on \mathbb{R}^3 . Equation (4.6) now implies $\delta\alpha = 0$. This means that the nullspace is trivial. \square

Lemma 4.4. *The nonlinear operator $(\rho, \kappa, \alpha) \mapsto h^{-1} \left(\left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 j(\cdot) + \alpha \right]_+ \right)$ is compact from \mathcal{O}_N into \mathcal{C}_s .*

Proof. Following Nirenberg [16], a continuous map f is called *compact* if $\overline{f(K)}$ is a compact set for every closed bounded set K . Now by Lemma 3.1, if (ρ, κ, α) is bounded, the support of $\left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 j(\cdot) + \alpha \right]_+$ is contained in some ball B_R . The map is obviously compact from \mathcal{O}_N to $C^0(\overline{B_R})$. Using again the trivial bound $\|u\|_{\mathcal{C}_s} \leq \langle R \rangle^s \|u\|_{C^0(\overline{B_R})}$ for $u \in \mathcal{C}_s$ supported in B_R , we obtain the compactness of this mapping into \mathcal{C}_s . \square

5. GLOBAL CONTINUATION

We now use the following form of the Global Implicit Function Theorem.

Theorem 5.1. *Let X and Z be Banach spaces and let U be an open subset of $X \times \mathbb{R}$. Let $F : U \rightarrow Z$ be a C^1 mapping in the Fréchet sense. Let $(\xi_0, \kappa_0) \in U$ such that $F(\xi_0, \kappa_0) = 0$. Assume that the linear operator $\frac{\partial F}{\partial \xi}(\xi_0, \kappa_0)$ is bijective from $X \times \mathbb{R}$ to Z . Assume that the mapping $(\xi, \kappa) \rightarrow F(\xi, \kappa) - \xi$ is compact from U to X . Let \mathcal{S} be the closure in $X \times \mathbb{R}$ of the solution set $\{(\xi, \kappa) \mid F(\xi, \kappa) = 0\}$. Let \mathcal{K} be the connected component of \mathcal{S} to which (ξ_0, κ_0) belongs. Then one of the following three alternatives is valid.*

- (i) \mathcal{K} is unbounded in $X \times \mathbb{R}$.
- (ii) $\mathcal{K} \setminus \{(\xi_0, \kappa_0)\}$ is connected.
- (iii) $\mathcal{K} \cap \partial U \neq \emptyset$.

Proof. This is a standard theorem basically due to Rabinowitz, Theorem 3.2 in [17] in the case that $U = X \times \mathbb{R}$ and under some extra structural assumption. A more general version also appears in Theorem II.6.1 of [11]; its proof is easy to generalize to permit a general open set U . The case of a general open set U also appears explicitly in [1]. \square

Lemma 5.1. *There is a connected set \mathcal{K}_N of solutions for which*

- either the solutions are unbounded in $\mathcal{C}_s \times \mathbb{R}^2$
- or they approach the boundary of \mathcal{O}_N .

Proof. We apply Theorem 5.1 with $X = Z = \mathcal{C}_s \times \mathbb{R}$, $U = \mathcal{O}_N$ and $\xi = (\rho, \alpha)$. The starting point is $\kappa_0 = 0$, $\xi_0 = (\rho_0, \alpha_0)$. The second alternative from that theorem is that it forms a “loop”, but we exclude the case of a loop as follows.

Suppose there were a loop. This means that $\mathcal{K}_N \setminus (\rho_0, 0, \alpha_0)$ is connected. Since \mathcal{K}_N is connected and the operator is even in κ , it follows that $\mathcal{K}_N \setminus (\rho_0, 0, \alpha_0)$ must contain a different point with $\kappa = 0$, say $(\rho_1, 0, \alpha_1) \neq (\rho_0, 0, \alpha_0)$. For this new point, $\kappa = 0$ means there is no rotation. Defining $U_1 = \frac{1}{|\cdot|} * \rho_1$, we have

$$\Delta U_1 = -4\pi\rho_1 = -4\pi h^{-1}([U_1 + \alpha_1]_+) := f(U_1).$$

This function f is C^1 . Of course, $\rho_1 \geq 0$ so that $U_1 > 0$ in \mathbb{R}^3 . So we can apply Theorem 4 in [7] to deduce that ρ_1 is radial (spherically symmetric). Letting $u_1 = U_1 + \alpha_1$, we get

$$(5.1) \quad u_1'' + \frac{2}{|x|} u_1' + 4\pi h^{-1}([u_1]_+) = 0, \quad u_1'(0) = 0.$$

Also by Lemma 3.1, $[u_1]_+$ is compactly supported. If $u_1(0) \neq u_0(0)$, then by (2.9) we would have $\int_{\mathbb{R}^3} \rho_1(x) dx \neq \int_{\mathbb{R}^3} \rho_0(x) dx = M$. This would violate the equation $\mathcal{F}_2 = 0$. Thus $u_1(0) = u_0(0)$. By uniqueness of solutions to the initial value problem of equation (2.6), we

infer that $u_1 = u_0$. It follows that $\rho_1 = \rho_0, \alpha_1 = \alpha_0$, which is a contradiction. So there is no loop. We deduce that either (i) or (iii) is valid; that is, either

$$\sup_{\mathcal{K}_N} (\|\rho\|_s + |\kappa| + |\alpha|) = \infty$$

or

$$\inf_{\mathcal{K}_N} \left| \kappa^2 \sup_x j(x) + \alpha + \frac{1}{N} \right| = 0.$$

In other words, we have either unboundedness or approach to the boundary. \square

Theorem 5.2. *Define the connected set $\mathcal{K} = \bigcup_{N=1}^{\infty} \mathcal{K}_N$. Uniformly along \mathcal{K} , either ρ is unbounded in L^∞ or the support of ρ is unbounded.*

Proof. Because the sets \mathcal{K}_N are nested, \mathcal{K} is also connected and one of the following statements is true:

- (a) $\sup_{\mathcal{K}} (\|\rho\|_s + |\kappa| + |\alpha|) = \infty$.
- (b) $\inf_{\mathcal{K}} |\kappa^2 \sup_x j(x) + \alpha| = 0$.

In order to prove the theorem, we argue by contradiction. Suppose that both $\sup_{\mathcal{K}} \sup_{x \in \mathbb{R}^3} \rho(x) < \infty$ and $R_* =: \sup_{\mathcal{K}} \sup\{x \in \mathbb{R}^3 \mid \rho(x) \neq 0\} < \infty$. We will first prove that (a) is true.

Suppose that (a) is false. Then (b) is true and $\sup_{\mathcal{K}} (\|\rho\|_s + |\kappa| + |\alpha|) < \infty$. Since $|x - y| \leq |x| + R_*$ for all y in the support of ρ , we have

$$\left(\frac{1}{|\cdot|} * \rho \right) (x) = \int \frac{1}{|x - y|} \rho(y) dy \geq \frac{M}{|x| + R_*}.$$

We may now write

$$(5.2) \quad \frac{1}{|\cdot|} * \rho(x) + \kappa^2 j(x) + \alpha \geq \frac{M}{|x| + R_*} - \kappa^2 (\sup_x j - j(x)) + (\kappa^2 \sup_x j(x) + \alpha).$$

Let $\kappa_0 = \sup_{\mathcal{K}} |\kappa|$. Let us consider a point x in the plane $x_3 = 0$, whence $|x| = r(x)$. By (3.3), $\sup_x j - j(x) = o\left(\frac{1}{|x|}\right)$ as $|x| \rightarrow \infty$. Thus by (5.2),

$$(5.3) \quad \frac{1}{|\cdot|} * \rho(x) + \kappa^2 j(x) + \alpha \geq \frac{M}{|x| + R_*} - o\left(\frac{\kappa_0^2}{|x|}\right) + (\kappa^2 \sup_x j(x) + \alpha).$$

Choosing $|x| > R_*$ sufficiently large, we can make the sum of the first two terms on the right side of (5.3) positive. Because of (b), there exists a solution $(\rho, \kappa, \alpha) \in \mathcal{K}$ such that the right side of (5.3) is positive. Due to $\mathcal{F}_1(\rho, \kappa, \alpha) = 0$, we have $\rho(x) > 0$. This contradicts the assumption that the support of ρ is bounded by R_* .

Thus (a) must be true. Since ρ is pointwise bounded and its support is also bounded all along \mathcal{K} , it follows that ρ is also bounded in the space \mathcal{C}_s . Because of (a), we know that $|\kappa| + |\alpha|$ must be unbounded. From the definition of \mathcal{O}_N , we know that $\alpha < 0$. In case κ were bounded, it would have to be the case that $\alpha \rightarrow -\infty$ along a sequence. Then the equation $\mathcal{F}_1 = 0$ would imply that $\rho \equiv 0$, which contradicts the mass constraint.

It follows that $\kappa_n \rightarrow \infty$ for some sequence $(\rho_n, \kappa_n, \alpha_n) \in \mathcal{K}$ with $\alpha_n < 0$. For each n , let us choose any point x_n such that $\rho_n(x_n) > 0$. By (3.2), we may choose a point y_0 such that $r(y_0) > R_*$ and $j(y_0) > j(R_*)$. Since $\rho_n(y_0) = 0$ and $\rho_n(x_n) > 0$, we have

$$0 \geq \left[\frac{1}{|\cdot|} * \rho_n(\cdot) + \kappa_n^2 j(\cdot) + \alpha_n \right] (y_0) \geq \left[\frac{1}{|\cdot|} * \rho_n(\cdot) + \kappa_n^2 j(\cdot) + \alpha_n \right] \Big|_{x_n}^{y_0}.$$

On the right side, the α_n cancels. Due to our assumption that the values of ρ_n and the supports of ρ_n are uniformly bounded, we deduce that

$$0 \geq \kappa_n^2 [j(r(y_0)) - j(r(x_n))] - C,$$

where C is a fixed constant. Thus $j(r(x_n)) \rightarrow j(r(y_0))$ since $\kappa_n \rightarrow \infty$. But $r(x_n) \leq R_* < r(y_0)$ and j is an increasing function of r , so that $j(r(x_n)) \leq j(R_*) < j(r(y_0))$. This is the desired contradiction. \square

6. FORMULATION BY ANGULAR MOMENTUM

A different formulation of the rotating star problem that is popular in the literature (see [3]) is to prescribe the angular momentum per unit mass $L(m)$ instead of the angular velocity $\omega(r)$. Under this formulation the velocity field is determined by the function $L(m)$ and the density $\rho(x)$ in the following way. One first defines the mass within a cylinder by

$$(6.1) \quad m_\rho(r) = \int_{x_1^2+x_2^2 \leq r^2} \rho(x) \, dx.$$

Then the function L is related to the angular velocity $\omega(r)$ by

$$(6.2) \quad L(m_\rho(r)) = r^4 \omega^2(r).$$

In other words, L is the square of $r|v|$, the angular momentum per unit mass. In this section we will entirely eliminate consideration of $\omega(r)$, and replace it by $L(m)$.

We make the following assumptions on the function $L(m)$:

$$(6.3) \quad L \geq 0, \quad L \in C_{loc}^{1,\delta}([0, \infty)), \quad L(0) = L'(0) = 0$$

for some $0 < \delta < 1$. The Euler–Poisson equations are reformulated as

$$(6.4) \quad \mathcal{F}(\rho, \kappa, \lambda) = (\mathcal{F}_1(\rho, \kappa, \lambda), \mathcal{F}_2(\rho)) = 0,$$

where

$$(6.5) \quad \mathcal{F}_1(\rho, \kappa, \lambda)(x) = \rho(x) - h^{-1} \left(\left[\frac{1}{|\cdot|} * \rho(x) - \kappa^2 \int_{r(x)}^\infty L(m_\rho(s)) s^{-3} \, ds + \lambda \right]_+ \right),$$

and

$$(6.6) \quad \mathcal{F}_2(\rho) = \int_{\mathbb{R}^3} \rho(x) \, dx - M.$$

Here λ plays a similar role as α did in the earlier formulation but it is not the same constant. We define \mathcal{C}_s as above, and define

$$(6.7) \quad \mathcal{O}_N^* = \left\{ (\rho, \kappa, \lambda) \in \mathcal{C}_s \times \mathbb{R}^2 \mid \lambda < -\frac{1}{N} \right\}.$$

Lemma 6.1. *The analogues of Lemmas 4.1-4.4 and 5.1 are true.*

Proof. By the same argument as in Lemma 3.1, there is a bound on the support of

$$\left[\frac{1}{|\cdot|} * \rho(x) - \kappa^2 \int_{r(x)}^\infty L(m_\rho(s)) s^{-3} \, ds + \lambda \right]_+.$$

We also obtain Lemma 4.1, with \mathcal{L} replaced by

$$(6.8) \quad \begin{aligned} \mathcal{L}(\delta\rho, \delta\kappa, \delta\alpha)(x) = & (h^{-1})' \left(\left[\frac{1}{|\cdot|} * \rho(x) - \kappa^2 \int_{r(x)}^\infty L(m_\rho(s)) s^{-3} \, ds + \lambda \right]_+ \right) \\ & \left[\frac{1}{|\cdot|} * \delta\rho(x) - \kappa^2 \int_{r(x)}^\infty L'(m_\rho(s)) m_{\delta\rho}(s) s^{-3} \, ds \right. \\ & \left. - 2\kappa(\delta\kappa) \int_{r(x)}^\infty L(m_\rho(s)) s^{-3} \, ds + \delta\lambda \right]. \end{aligned}$$

The key to justifying the Fréchet derivative is the estimate

$$\begin{aligned}
 & \left| \int_{r(x)}^{\infty} [L(m_{\rho+\delta\rho}(s)) - L(m_{\rho}(s)) - L'(m_{\rho}(s))m_{\delta\rho}(s)] s^{-3} ds \right| \\
 & \leq \int_{r(x)}^{\infty} \int_0^{m_{\delta\rho}(x)} |L'(m_{\rho}(s)+t) - L'(m_{\rho}(s))| dt s^{-3} ds \\
 (6.9) \quad & \leq \|L\|_{C^{1,\delta}([0,A])} \int_{r(x)}^{\infty} [m_{\delta\rho}(s)]^{1+\delta} s^{-3} ds.
 \end{aligned}$$

where $A = 2\|\rho\|_{L^1}$. Using the simple fact that

$$(6.10) \quad m_{\delta\rho}(r) \leq C\|\delta\rho\|_s \min(1, r^2),$$

we see that (6.9) is uniformly bounded on compact sets by a constant multiple of $\|\delta\rho\|_s^{1+\delta}$.

Lemma 4.2 and Lemma 4.3 only involve the $\kappa = 0$ case, thus they are valid without change. To prove Lemma 4.4, we must show that a subsequence of $j_n(x) = \int_{r(x)}^{\infty} L(m_{\rho_n}(s))s^{-3} ds$ converges uniformly on compact sets if $\{\rho_n\}$ is bounded in \mathcal{C}_s . In fact, using (6.10) again as above, we see that $j_n(x)$ is uniformly bounded on a finite ball B_R . To obtain the equicontinuity of $j_n(x)$, we estimate

$$\begin{aligned}
 & \int_{r(x)}^{r(y)} L(m_{\rho_n}(s))s^{-3} ds \\
 & \leq \|L\|_{C^{1,\delta}([0,C\|\rho_n\|_s])} \int_{r(x)}^{r(y)} (m_{\rho_n}(s))^{1+\delta} s^{-3} ds \\
 & \leq C\|L\|_{C^{1,\delta}([0,C\|\rho_n\|_s])} \|\rho_n\|_s^{1+\delta} \int_{r(x)}^{r(y)} s^{2\delta-1} ds \\
 & \leq C\|L\|_{C^{1,\delta}([0,C\|\rho_n\|_s])} \|\rho_n\|_s^{1+\delta} |x-y|^{\min(2\delta,1)}.
 \end{aligned}$$

We can now prove Lemma 5.1 in a similar way as before, thereby deducing that there is a connected set $\mathcal{K}^* \subset \bigcup_{N=1}^{\infty} \mathcal{O}_N^*$ of solutions to (6.4) such that at least one of the following statements is true:

- (a) $\sup_{\mathcal{K}} (\|\rho\|_s + |\kappa| + |\lambda|) = \infty$.
- (b) $\sup_{\mathcal{K}} \lambda = 0$.

□

We are now ready to prove

Theorem 6.1. *Along the connected set \mathcal{K}^* , either ρ is unbounded in L^∞ or the support of ρ is unbounded.*

Proof. Arguing by contradiction, we suppose that $\sup_{\mathcal{K}} \|\rho\|_{L^\infty} < \infty$ and $R_* =: \sup_{\mathcal{K}} \sup\{x \in \mathbb{R}^3 \mid \rho(x) \neq 0\} < \infty$.

Suppose also that (a) is false. Then (b) is true and $\sup_{\mathcal{K}} (\|\rho\|_s + |\kappa| + |\lambda|) < \infty$. We argue as in the proof of Theorem 5.2. We pick an x on the $x_3 = 0$ plane and such that $|x| > R_*$ is sufficiently large. Thereby we obtain the following estimate instead of (5.3):

$$\begin{aligned}
 & \frac{1}{|\cdot|} * \rho(x) - \kappa^2 \int_{r(x)}^{\infty} L(m_{\rho}(s))s^{-3} ds + \lambda \\
 & \geq \frac{M}{|x|+R_*} - \kappa_0^2 \int_{r(x)}^{\infty} L(M)s^{-3} ds + \lambda \\
 (6.11) \quad & \geq \frac{M}{|x|+R_*} - \frac{C\kappa_0^2 L(M)}{r(x)^2} + \lambda
 \end{aligned}$$

We have used the fact that $m_\rho(s) = M$ because $r(x) > R_*$. Then the sum of the first two terms in (6.11) is positive. We now use (b) and choose a solution along \mathcal{K}^* so that λ is sufficiently close to zero to make (6.11) positive. Hence for this solution, and this point x , we have $\rho(x) > 0$, contradicting the definition of R_* .

Thus (a) must be true. Since we assume that ρ is bounded in L^∞ and $R_* < \infty$, it follows that $\|\rho\|_s$ is also bounded. Suppose $|\kappa|$ is bounded. Then $|\lambda|$ must be unbounded. Since $\lambda < 0$ for solutions in $\bigcup_{N=1}^\infty \mathcal{O}_N^*$, it must be true that $\lambda \rightarrow -\infty$ along a sequence. However in this case the equation $\mathcal{F}_1 = 0$ would imply that $\rho \equiv 0$ for λ sufficiently negative, which contradicts the mass constraint.

It follows that $|\kappa_n| \rightarrow \infty$ and $\lambda_n \rightarrow -\infty$ along some sequence $(\rho_n, \kappa_n, \lambda_n) \in \mathcal{K}^*$. Arguing as in the proof of Theorem 5.2, we choose any point y_0 such that $r(y_0) > R_*$, and any point x_n such that $\rho_n(x_n) > 0$. So $r(x_n) < R_*$. It follows that

$$\begin{aligned} 0 &\geq \left[\frac{1}{|\cdot|} * \rho_n(\cdot) - \kappa_n^2 \int_{r(\cdot)}^\infty L(m_{\rho_n}(s)) s^{-3} ds + \lambda_n \right] (y_0) \\ &\geq \left[\frac{1}{|\cdot|} * \rho_n(\cdot) - \kappa_n^2 \int_{r(\cdot)}^\infty L(m_{\rho_n}(s)) s^{-3} ds + \lambda_n \right] \Big|_{x_n}^{y_0} \\ &\geq \kappa_n^2 \int_{r(x_n)}^{r(y_0)} L(m_{\rho_n}(s)) s^{-3} ds - C \geq \kappa_n^2 \int_{R_*}^{r(y_0)} L(M) s^{-3} ds - C \\ &\geq \frac{\kappa_n^2 L(M)}{2} \left(\frac{1}{R_*^2} - \frac{1}{r^2(y_0)} \right) - C. \end{aligned}$$

The desired contradiction follows because $|\kappa_n| \rightarrow \infty$. \square

7. COMPARISON BETWEEN DIFFERENT ANGULAR VELOCITY FORMULATIONS

The rotating star problem appears in several different formulations in the literature. Although these formulations are not equivalent, all of them produce rotating star solutions to the Euler–Poisson equations under certain circumstances. Here we provide a comparison of the formulations in the case of prescribed angular velocity $\omega(r)$. The case of prescribed angular momentum per unit mass can be discussed in a similar way. In our discussion the density function ρ is assumed to be an axisymmetric function on \mathbb{R}^3 , $\omega(r)$ is a continuous function on $[0, \infty)$, and $h(s)$ is a strictly increasing continuous function from $[0, \infty)$ onto $[0, \infty)$. The inverse of h is denoted by h^{-1} . The original Euler–Poisson equation (1.1) is made precise as follows.

Formulation 1. *Let ρ be a non-negative function in $C_{loc}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. It is called a rotating star solution under Formulation 1 if there exists a real number α such that the equation*

$$(7.1) \quad \frac{1}{|\cdot|} * \rho(x) + \int_0^{r(x)} s \omega^2(s) ds - h(\rho(x)) + \alpha = 0$$

is valid on the positivity set $\{x \in \mathbb{R}^3 \mid \rho(x) > 0\}$,

Note that $\frac{1}{|\cdot|} * \rho(x)$ is defined and continuous because $\rho \in C_{loc}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. The second formulation is basically the approach taken in this paper.

Formulation 2. *Let $\rho \in C_{loc}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. It is called a rotating star solution under Formulation 2 if there exists a real number α such that*

$$(7.2) \quad \rho(x) = h^{-1} \left(\left[\frac{1}{|\cdot|} * \rho(x) + \int_0^{r(x)} s \omega^2(s) ds + \alpha \right]_+ \right)$$

for all $x \in \mathbb{R}^3$.

The third formulation is basically the one used by Auchmuty in [2] and is closely related to the one used by Jang and Makino in [9].

Formulation 3. Let $\rho \in C(\overline{B_R})$ for some ball B_R of radius R centered at the origin. Extend it to be zero outside $\overline{B_R}$. Then ρ is called a rotating star solution under Formulation 3 if there exists a real number α such that (7.2) is true for all $x \in \overline{B_R}$.

The fourth formulation is used by the authors in [18]. The density is explicitly designed to be a mass-invariant perturbation of a non-rotating solution. An earlier precursor of this formulation was used by Lichtenstein [13] and Heilig [8]; however, their version did not keep the mass invariant.

Formulation 4. Let ρ_0 be a radial (spherically symmetric) continuous function on \mathbb{R}^3 that is positive in a ball B_{R_0} centered at the origin, vanishes in its complement, and solves the equation

$$(7.3) \quad \frac{1}{|\cdot|} * \rho_0(x) - h(\rho_0(x)) + \alpha_0 = 0$$

for some real number α_0 and all $x \in B_{R_0}$. Let $\zeta : \overline{B_{R_0}} \rightarrow \mathbb{R}$ be an axisymmetric continuous function vanishing at the origin to sufficiently high order such that

$$(7.4) \quad g_\zeta(x) = x \left(1 + \frac{\zeta(x)}{|x|^2} \right)$$

is a homeomorphism from $\overline{B_{R_0}}$ to $g_\zeta(\overline{B_{R_0}})$. Define

$$(7.5) \quad \rho_\zeta(x) = \frac{\int_{B_{R_0}} \rho_0(x) dx}{\int_{g_\zeta(B_{R_0})} \rho_0(g_\zeta^{-1}(x)) dx} \rho_0(g_\zeta^{-1}(x))$$

for $x \in g_\zeta(B_{R_0})$ and extend it to be zero elsewhere. The function ζ is said to give rise to a rotating star solution ρ_ζ if there exists a real number α such that

$$(7.6) \quad \frac{1}{|\cdot|} * \rho_\zeta(x) + \int_0^{r(x)} s\omega^2(s) ds - h(\rho_\zeta(x)) + \alpha = 0$$

for all $x \in g_\zeta(B_{R_0})$.

Note that the L^1 norm (mass) of ρ_ζ is designed to be the same as that of ρ_0 . Moreover, if one can find a ζ that gives rise to a rotating star solution, one not only obtains some solution, but in fact the solution ρ_ζ is created by a simple deformation along *radial directions* from the non-rotating one ρ_0 . Thus a solution under Formulation 4 reveals deeper structure about its relationship to a non-rotating star.

As alluded to earlier, the above formulations are not equivalent, at least as the definitions explicitly allow. We begin by stating how the other formulations imply Formulation 1.

Proposition 7.1. *The following implications hold.*

- (a) *Formulation 2 implies Formulation 1.*
- (b) *Formulation 3, together with the condition $\rho(x) = 0$ for all $x \in \partial B_R$, implies Formulation 1.*
- (c) *Formulation 4 implies Formulation 1.*

Proof. To prove (a), note that if ρ is a rotating star solution under Formulation 2, then whenever $\rho(x) > 0$, the term in the square bracket of (7.2) must also be positive. Thus in that region one can ignore the + subscript (the positive part of the square bracket), so that (7.1) follows. Assertion (b) is proven in a similar way, once it is noticed that the additional assumption $\rho(x) = 0$ on ∂B_R guarantees that $\rho \in C_{loc}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Assertion (c) is obvious. \square

We now discuss the weaknesses of each formulation compared with the original Formulation 1. The drawback of Formulation 2 is that it does not capture all the solutions to Formulation 1. The reason is that Formulation 1 does not require equality of the two sides of (7.2) when $\rho(x) = 0$, whereas Formulation 2 does. Formulation 2 requires the expression $\mathcal{U}(x)$ in square brackets to be non-positive outside the support of ρ , but Formulation 1 does not. Thus Formulation 2 misses many solutions which are valid according to Formulation 1, especially if the term involving $\omega(s)$ grows positively at infinity. In that case, a valid solution under Formulation 1 may make $\mathcal{U}(x)$ very big for large $|x|$, while the left side remains 0. In fact, in order to actually work with Formulation 2, one requires the right side of (7.2) to have enough decay near infinity, which is virtually impossible if the term involving $\omega(s)$ were to grow near infinity.

Formulation 3 misses some solutions of Formulation 1 in the same way that Formulation 2 does, although it does avoid the difficulty at infinity by restricting to an artificially chosen ball B_R . However, it is in general difficult to prove that $\rho(x)$ vanishes on the boundary of B_R . If one chooses B_R larger than the support of a non-rotating solution, one can show that sufficiently small perturbations of that non-rotating solution will remain zero on the boundary of B_R . However, as soon as one continues the solution branch to fast rotations, nonzero boundary values may appear, which would violate the physical vacuum boundary condition of a rotating star. Nor are we aware of a general mechanism that can force the support to grow gradually until it hits the boundary of B_R . In principle, the only physical solutions one can get via this approach seem to be merely the very small perturbations of a nonrotating star.

Formulation 4 has the advantage of enforcing an equation only where $\rho_\zeta(x) > 0$. It is thus closer in spirit to Formulation 1. However, we are not aware of any evidence that large deviations from a non-rotating solution will still have the structure of radial deformation that appears in Formulation 4. Formulation 4 is also significantly more complicated than the other formulations when it comes to the actual construction of the function ζ (see [18]), especially with regard to the required compactness property, the analogue of Lemma 4.4.

Like Formulation 1, Formulation 4 does not require (7.2) on the set where $\rho(x) = 0$. Thus it is not clear that Formulation 4 implies Formulation 2 or Formulation 3. However, in the following special situation, a solution to Formulation 4 does indeed solve Formulation 3. For a given ρ_0 in Formulation 4, choose the ball B_R in Formulation 3 to have a fixed radius $R > R_0$. Suppose the solution ρ_ζ is sufficiently close to the radial solution ρ_0 in the sense that $g_\zeta(\overline{B_{R_0}}) \subset B_R$, and ρ_ζ and ρ_0 are sufficiently close to each other in $C(\overline{B_R})$. Furthermore, suppose that $\omega(r)$ is a smooth function with sufficiently small $C(\overline{B_R})$ norm. Heuristically, the above conditions describe a small perturbation of the nonrotating solution ρ_0 . Finally, assume the technical condition that $r\omega(r)$ is non-decreasing. From (7.5) we see that $\rho_\zeta(x) > 0$ for $x \in g_\zeta(B_{R_0})$, and $\rho_\zeta(x) = 0$ for $x \in \overline{B_R} \setminus g_\zeta(B_{R_0})$. To prove that ρ_ζ is also a solution to Formulation 3, it remains to prove that

$$(7.7) \quad f(x) := \frac{1}{|\cdot|} * \rho_\zeta(x) + \int_0^{r(x)} s\omega^2(s) ds + \alpha \leq 0$$

for $x \in \overline{B_R} \setminus g_\zeta(B_{R_0})$. In this ‘‘annular’’ region we have $\Delta f(x) = \Delta \int_0^{r(x)} s\omega^2(s) ds = \frac{1}{r}(r^2\omega^2(r))' \geq 0$. Hence we only have to show $f(x) \leq 0$ on $g_\zeta(\partial B_{R_0}) \cup \partial B_R$. By (7.6) and the continuity of ρ_ζ , we obviously have $f(x) = h(\rho_\zeta) = 0$ for $x \in g_\zeta(\partial B_{R_0})$. It remains to prove that $f(x) \leq 0$ on ∂B_R . For this purpose note that the function $f_0(x) := \frac{1}{|\cdot|} * \rho_0(x) + \alpha_0$ is harmonic outside B_{R_0} and that $f_0 = 0$ and $f_0' < 0$ on ∂B_{R_0} . It follows that $f_0(x) < 0$ for $|x| > R_0$. Since $f(x)$ is sufficiently close to $f_0(x)$ in supremum norm by the smallness assumptions, we have $f(x) < 0$ on ∂B_R . This shows that ρ_ζ is also a solution to Formulation 3. Since the typical construction of solutions via Formulation 3 guarantees local uniqueness, this reasoning shows that the unique solution must have the structure detailed in Formulation 4.

If the rotating star problem is treated as a classical free boundary problem, then a fifth possible formulation emerges. Let us begin with Formulation 1 with a connected set $\Omega = \{\rho > 0\}$. Let $q = h(\rho)$. Taking the Laplacian of (1.1), the function q satisfies the elliptic equation

$$(7.8) \quad \Delta q = 4\pi h^{-1}(q) - \kappa^2 \Delta j$$

in Ω with j defined by (3.1), together with the pair of boundary conditions

$$(7.9) \quad q = 0 \quad \text{and} \quad \frac{1}{|\cdot|} * h^{-1}(q) + \kappa^2 j = \text{constant on } \partial\Omega.$$

Now we use a transformation of hodograph type to convert Ω to a fixed domain. Using standard spherical coordinates (s, θ, ϕ) , we exchange independent and dependent variables by defining

$$(7.10) \quad s' = 1 - q(s, \theta, \phi) \quad \text{and} \quad w(s', \theta, \phi) = s.$$

Then Ω goes into the unit ball while its boundary $\partial\Omega$ goes into the unit sphere $\partial B_1 = \{s' = 1\}$. The first boundary condition in (7.9) is automatically satisfied. The whole problem is thereby transformed into a nonlinear elliptic equation for $w(s', \theta, \phi)$ in the unit ball B_1 with a single nonlinear boundary condition. This is Formulation 5. We continue to assume axisymmetry, which means that w does not depend on ϕ . This formulation has the primary advantage that the domain is fixed. However it appears to be rather complicated to analyze because both the equation and the boundary condition are highly nonlinear and have variable coefficients. We refrain from providing the details.

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DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912