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LONG-RANGE SCATTERING MATRIX FOR SCHRÖDINGER-TYPE OPERATORS

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Abstract. We show that the scattering matrix for a class of Schrödinger-type operators with long-range perturbations is a Fourier integral operator with the phase function which is the generating function of the modified classical scattering map.

1. Introduction

In this paper, we consider Schrödinger type-operator:

$$H = H_0 + V$$

on $L^2(\mathbb{R}^d)$, $d \geq 1$. The unperturbed operator H_0 has the form: $H_0 = p_0(D_x)$ on $\mathbb{R}^d$, where $D_x = -i\partial_x$ and p_0 is a real-valued smooth function. We suppose:

Assumption A. Let $m > 0$. $p_0 \in S^m$, i.e., for any multi-index $\alpha \in \mathbb{Z}_+^d$, there is $C_\alpha > 0$ such that

$$\left|\partial_\xi^\alpha p_0(\xi)\right| \leq C_\alpha \langle\xi\rangle^{m-|\alpha|}, \quad \xi \in \mathbb{R}^d.$$

Moreover, we suppose p_0 is elliptic, i.e., there is $c_0, c_1 > 0$ such that

$$\left|p_0(\xi)\right| \geq c_0\langle\xi\rangle^m - c_1, \quad \xi \in \mathbb{R}^d.$$

The perturbation term V is a symmetric pseudodifferential operator, and satisfies the following assumption. We let

$$g = \frac{dx^2}{\langle x\rangle^2} + d\xi^2$$

be our standard metric on $T^*\mathbb{R}^d$. Then we use the Hörmander $S(m,g)$-class notation with respect this metric, i.e., for a weight function $m(x,\xi)$, $a \in S(m,g)$ if for any $\alpha, \beta \in \mathbb{Z}_+^d$, there is $C_\alpha\beta > 0$ such that

$$\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha\beta} m(x,\xi)\langle x\rangle^{-|\alpha|}, \quad x,\xi \in \mathbb{R}^d.$$

Assumption B. Let m be as in Assumption A. There is $\mu \in (0,1)$ such that $V \in S(\langle x\rangle^{-\mu}\langle\xi\rangle^m, g)$, and V is real-valued.

We denote the Weyl quantization of V by the same symbol: $V = V^W(x, D_x)$, and V is a symmetric operator on $L^2(\mathbb{R}^d)$.

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Remark 1.1. For $\mu > 1$, the perturbation is short-range, and the scattering theory is much simpler (see [12]). Here we also exclude the case $\mu = 1$. This case is especially important, because the (mollified) Coulomb potential satisfies this assumption. Though such potential satisfies the condition with any $\mu < 1$, more precise results should hold, and we address this in a separate paper [14].

We denote the symbol of H by p_1:

$$p_1(x,\xi) = p_0(\xi) + V(x,\xi), \quad x,\xi \in \mathbb{R}^d.$$

Since V decays as $|x| \to \infty$, we have ellipticity of p_1 if $|x|$ is sufficiently large. However, since our perturbations include the metric perturbation, we need to assume global ellipticity:

Assumption C. There is $c_2, c_3 > 0$ such that

$$|p_1(x,\xi)| \geq c_2\langle\xi\rangle^m, \quad \text{if } |\xi| \geq c_3, x,\xi \in \mathbb{R}^d.$$

Under these assumptions, it is well-known that H is self-adjoint on $H^m(\mathbb{R}^d)$. We write the unique self-adjoint extension by H as well as the pseudodifferential operator.

We now fix an energy interval $I = [E_0, E_1] \Subset \mathbb{R}$, and we consider the scattering on I. We note, by Assumption B,

$$\Omega_I^0 = \bigl\{\xi \in \mathbb{R}^d \bigm| p_0(\xi) \in I\bigr\} \subset \mathbb{R}^d$$

is bounded. We assume the following non-degenerate condition on the interval I:

Assumption D. For $x \in \Omega_I^0$, $\partial_\xi p_0(\xi) \neq 0$.

Under these assumptions, we can apply the Mourre theory with the conjugate operator $A = \frac{1}{2}(x\cdot\partial_\xi p_0(D_x) + \partial_\xi p_0(D_x)\cdot x)$, and we learn the spectrum of H on I is absolutely continuous possibly except for finite number of eigenvalues (see, e.g., [1], [12]).

Following Isozaki-Kitada [8, 9], Dereziński-Gérard [4] and Robert [16], we construct time-independent modifiers $J_\pm$ in our setting in Section 3 (which depends on I). Using these we can define modified wave operators:

$$W_\pm^I = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} J_\pm e^{-itH_0} E_I(H_0)$$

Then the existence of these limits are proved by the same method as in the papers by Isozaki-Kitada [8], and $W_\pm$ are partial isometries on Ran $[E_I(H_0)]$. Moreover, the asymptotic completeness is also proved by the standard method:

$$\text{Ran}\,[W_\pm^I E_I(H_0)] = E_I(H)\mathcal{H}_c(H),$$

where $\mathcal{H}_c(H)$ is the continuous spectral subspace with respect to H. The scattering operator S^I is defined by

$$S^I = (W_+^I)^* W_-^I,$$

and it is an isometry on Ran $[E_I(H_0)]$. It is well-known that S^I commutes with the free Hamiltonian: $S^I H_0 = H_0 S^I$.

We then introduce the scattering matrix. We employ the formulation in Nakamura [12]. For $\lambda \in I$, we set the energy surface Σ_λ by

$$\Sigma_\lambda = p_0^{-1}(\{\lambda\}) = \bigl\{\xi \in \mathbb{R}^d \bigm| p_0(\xi) = \lambda\bigr\}.$$

We note Σ_λ is a smooth submanifold for $\lambda \in I$ since $\partial_\xi p_0(\xi) \neq 0$ on $p_0^{-1}(I)$. We let a measure m_λ on Σ_λ defined by $m_\lambda(\xi) = |\partial_x p_0(\xi)|^{-1} dS_\xi$ so that $m_\lambda \wedge dp_0(\xi) = d\xi$, where

dS_ξ is the surface density (measure) on Σ_λ. Let $T(\lambda)$ be the trace operator from $H^s_{\mathrm{loc}}(\mathbb{R}^d_\xi)$ ($s > 1/2$) to $L^2(\Sigma_\lambda)$ defined by

$$T(\lambda) \;:\; f \mapsto f\big|_{\Sigma_\lambda} \in L^2(\Sigma_\lambda), \quad f \in H^s_{\mathrm{loc}}(\mathbb{R}^d).$$

Then

$$T(\cdot) \;:\; f \mapsto (T(\lambda)f) \in \int_I^\oplus L^2(\Sigma_\lambda, m_\lambda)d\lambda$$

is extended to a surjective partial isometry with the initial space $L^2(p_0^{-1}(I))$. In particular, $T(\cdot)\mathcal{F}$ is a spectral representation of H_0 on Ran $[E_I(H_0)]$. Then $\mathcal{F}S^I\mathcal{F}^*$ is decomposed on this spectral representation space, and we have

$$T(\cdot)\mathcal{F}S^I\mathcal{F}^*T(\cdot)^* = \int_I^\oplus S(\lambda)d\lambda,$$

with $S(\lambda) \in \mathcal{B}(L^2(\Sigma_\lambda, m_\lambda))$. $S(\lambda)$ is the scattering matrix, and it is easy to show $S(\lambda)$ is unitary for (at least) almost all $\lambda \in I$.

Theorem 1.1. *Let $\lambda \in I \backslash \sigma_{\mathrm{p}}(H)$. Then there are $\psi(y,\eta) \in S^1_{1,0}$ on $T^*\Sigma_\lambda$ and $a(y,\eta) \in S^0_{1,0}$ such that*

$$S(\lambda)\varphi(\eta) = (2\pi)^{-(d-1)} \iint e^{-i\psi(y,\eta)+iy\cdot\zeta}\tilde{\Theta}(y,\eta)a(y,\eta)\varphi(\zeta)d\zeta dy$$

for $\varphi \in C_0^\infty(\Sigma_\lambda)$ in a local coordinate of Σ_λ, where

$$\tilde{\Theta}(y,\eta) = \left|\det(\partial_y\partial_\eta\psi(y,\eta))\right|^{1/2}.$$

Moreover, $\psi(y,\eta) - y\cdot\eta \in S^{1-\mu}_{1,0}$ and the principal symbol of $a(y,\eta)$ is 1, i.e., $a(y,\eta) - 1 \in S^{-1}_{1,0}$. Here we have used the standard Kohn-Nirenberg symbol notation $S^m_{\rho,\delta}$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$.

Remark 1.2. Even though $S(\lambda)$ is not a pseudodifferential operator in general, it has pseudo-local property since $\partial_y\psi(y,\eta) - \eta = O(\langle y\rangle^{-\mu})$ as $|y| \to \infty$. Hence it is sufficient to consider such operators in a local coordinate, as well as pseudodifferential operators. This class of Fourier integral operators is somewhat different form the standard Hörmander-type Fourier integral operators, where the phase functions are supposed to be homogeneous of order one with respect to the conjugate variables (y in our setting). We note the calculus of Asada-Fujiwara [3] still applies to our class of operators.

Remark 1.3. $\psi(y,\eta)$ is the generating function of classical scattering map, and we discuss the construction in detail later in this paper. The factor $\tilde{\Theta}(y,\eta)$ corresponds to the modification of the volume form, and it makes the operator approximately unitary.

Remark 1.4. In principle, we can compute $\psi(y,\eta)$ explicitly in terms of classical mechanics. For many examples, at least if $\mu > 1/2$, we can compute the asymptotic behavior as $|y| \to \infty$ (see [19]). We can consider $\exp(-i\psi(-D_\eta,\eta))$ as a good approximation of the scattering matrix, and hence we expect the spectral properties of $S(\lambda)$ is decided by the behavior. See Nakamura [14] for the case $\mu = 1$.

The long-range scattering theory for Schrödinger operators has a long history, and there is substantial literature on this subject, especially for two-body case. We refer textbooks Reed-Simon [15] §11.9, Dereziński-Gérard [4], Yafaev [20] , [21] and references therein. Long-range scattering for discrete Schrödinger operators has also been studied by several authors recently ([11], [18]), and in this paper we consider relatively large class of operators, of which the method is easily applied to discrete Schrödinger operators as well. The literature on the scattering matrix for long-range scattering has been relatively few. The off-diagonal smoothness of the scattering matrix was proved by Isozaki-Kitada [9], and also studied by Yafaev (see [20] and references therein). The Fourier integral operator representation of the scattering matrix was studied by Yafaev in the case of $\mu > 1/2$ using the Dollard-type approximate solutions to the eikonal equation ([19]). In this paper we employ explicit construction of the solutions to eikonal equation with precise control of the classical trajectories and ideas from interaction pictures.

Our argument relies heavily on the formulation of long-range scattering in terms of time-independent modifiers by Isozaki and Kitada ([7, 8, 9, 10]. See also alternative constructions by Robert [16], Dereziński-Gérard §4.15, Yafaev [20]). In this paper, we give relatively detailed analysis of the classical mechanics, partly because the system we consider is more general than the Schrödinger (or Newton) equation, but also because the settings and constructions are somewhat different, and the construction itself is important to understand the meaning of the representation. The other source of the proof of Theorem 1.1 is a recent result by the author on the short-range scattering matrix [12], and we modify its argument to apply to the long-range case. Another recent result on microlocal resolvent estimates [13] is also crucial in the proof (for our generalized system).

The paper is constructed as folows: In Section 2, we prepare global-in-time estimates for the solutions to Hamilton equations with nontrapping dynamical system, and construct solutions to Hamilton-Jacobi equations and eikonal equations, using the idea of interaction picture. In Section 3, we construct the time-independent modifiers following the idea of Isozaki and Kitada. In Section 4, we give the proof of Theorem 1.1, using the idea of [12]. We use microlocal analysis extensively in Section 3 and Section 4, and we refer Hörmander [6], Sogge [17] and Asada-Fujiwara [3].

2. Preparation on classical mechanics

2.1. **Classical mechanics with space cut-off.** We introduce a constant $R > 0$, and we set

$$V_R(x,\xi) = \chi_1(|x|/R)V(x,\xi),$$

where $\chi_1 \in C^\infty(\mathbb{R})$ is a smooth cut off function such that $\chi_1(s) = 0$ if $s \leq 1$; $\chi_1(s) = 1$ if $s \geq 2$. We then set

$$p(x,\xi) = p_0(x,\xi) + V_R(x,\xi).$$

We fix R later in this subsection. We now consider the classical mechanics generated by $p(x,\xi)$. Namely, we consider solutions to the Hamilton equation:

$$\frac{d}{dt}x(t) = \frac{\partial p}{\partial \xi}(x(t),\xi(t)), \quad \frac{d}{dt}\xi(t) = -\frac{\partial p}{\partial x}(x(t),\xi(t))$$

with the initial condition: $x(0) = x_0$, $\xi(0) = \xi_0$. We denote the solution to the equation by

$$\exp tH_p(x_0,\xi_0) = (x(x_0,\xi_0;t),\xi(x_0,\xi_0;t)) \in \mathbb{R}^{2d}, \quad x_0,\xi_0 \in \mathbb{R}^d, t \in \mathbb{R}.$$

We now recall Assumption D, and we consider a trajectories with the energy λ in a neighborhood of $I = [E_0, E_1]$. We choose $\varepsilon_0 > 0$ so that there is $c_4 > 0$ such that

$$|\partial_\xi p_0(\xi)| \geq c_4, \quad \text{for } \xi \in \Omega^0_{I_6},$$

where

$$I_k = [E_0 - k\varepsilon_0, E_1 + k\varepsilon_0], \quad k = 0,1,2,\dots 6.$$

$\partial_\xi p_0(\xi)$ is the free velocity, and we denote it as

$$v(\xi) = \partial_\xi p_0(\xi).$$

We denote the Poisson bracket of $a, b \in C^\infty(\mathbb{R}^{2d})$ by

$$\{a, b\} = \sum_{j=1}^{d} \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j} - \frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j},$$

and we write the unit matrix on $\mathbb{C}^d$ by E. We also denote

$$\Omega_J = \big\{(x,\xi)\ \big|\ p(x,\xi) \in J\big\}, \quad J \subset \mathbb{R}.$$

Lemma 2.1. *There is $R_0 > 0$ such that if $R \geq R_0$, then there is $c_5 > 0$ such that*

$$\frac{d^2}{dt^2}|x(t)|^2 = \{\{|x|^2, p\}, p\}(x,\xi) \geq c_5, \quad x, \xi \in \mathbb{R}^d,$$

if $(x_0, \xi_0) \in \Omega_{I_5}$. Moreover, for each x_0, $t \in \mathbb{R}$, $\xi_0 \mapsto \xi(x_0, \xi_0; t)$ is a diffeomorphism, and

$$\det\left[\frac{\partial \xi}{\partial \xi_0}(x_0, \xi_0; t)\right] \geq 1/2$$

for any $x_0, \xi_0 \in \mathbb{R}^d$ such that $(x_0, \xi_0) \in \Omega_{I_5}$, $t \in \mathbb{R}$.

Proof. This is a variation of the so-called *classical Mourre estimate*, and we only sketch the idea. By Assumption D, we learn that if R_0 is sufficiently large,

$$|V(x,\xi)| \leq \frac{1}{2}|p_0(\xi)| + c_1 \quad \text{for } |x| \geq R_0, \xi \in \mathbb{R}^d.$$

Then we have, provided $R \geq R_0$,

$$|p_0(\xi)| \leq 2|p(x,\xi)| + 2c_1 = 2|p(x_0,\xi_0)| + 2c_1, \quad \text{for } (x_0,\xi_0) \in \Omega_{I_5},$$

and hence $|\xi| \leq M$ with some $M > 0$, uniformly on Ω_{I_6}. We choose R so large that

$$|V_R(x,\xi)| \leq \varepsilon_0 \quad \text{for } (x_0,\xi_0) \in \Omega_{I_5}$$

holds. Then, if $(x,\xi) \in \Omega_{I_5}$ then $p_0(\xi) \in I_6$ and hence $|v(\xi)| \geq c_4$.

Now, by the direct computations, we learn

$$\begin{aligned}\{\{|x|^2, p\}, p\} &= 2|v(\xi)|^2 + O(\langle \xi \rangle^{2m-2} \langle x \rangle^{-\mu} \chi_{\{|x| \geq R\}}) \\ &= 2|v(\xi)|^2 + O(R^{-\mu}), \quad \text{on } \Omega_{I_5},\end{aligned}$$

and hence

$$\{\{|x|^2, p\}, p\} \geq c_4^2 > 0, \quad \text{for } (x_0,\xi_0) \in \Omega_{I_5},$$

provided R is chosen sufficiently large. This also implies, in particular, for any solution to the Hamilton equation, there is $t_0 \in \mathbb{R}$ such that

$$|x(t)| \geq (|x(t_0)|^2 + c_4(t-t_0)^2/2)^{1/2} \geq (c_4/2)^{1/2}|t-t_0|, \quad t \in \mathbb{R}.$$

Hence we have

$$\left|\frac{d}{dt}\xi(t)\right| \leq \left|\frac{\partial V_R}{\partial x}(x(t),\xi(t))\right| \leq C(\langle t-t_0 \rangle + R)^{-\mu-1},$$

and this implies

$$|\xi(t) - \xi(s)| \leq C\int_s^t (\langle u - t_0 \rangle + R)^{-\mu-1} du \leq C' R^{-\mu}, \quad -\infty < s < t < \infty.$$

Similarly we can show

$$\left\| \frac{\partial \xi(t)}{\partial \xi_0} - E \right\|_{\mathbb{C}^d \to \mathbb{C}^d} \leq CR^{-\mu}, \tag{2.1}$$

and we conclude the last assertion by choosing R large enough. □

In the following, we suppose R is large enough that the argument of the above proof is valid.

2.2. **Solution to the Hamilton-Jacobi equation.** We consider the Hamilton-Jacobi equation in ξ-space:

$$\frac{\partial}{\partial t}\phi(t,\xi) = p\left(\frac{\partial \phi}{\partial \xi}(t,\xi), \xi\right), \quad \xi \in \Omega^0_{I_4}, t \in \mathbb{R},$$

with the initial condition

$$\phi(0,\xi) = 0, \quad \xi \in \mathbb{R}^d.$$

We write

$$\Lambda_t : \eta \mapsto \xi(0,\eta;t),$$

Then, by Lemma 2.1, Λ_t is locally diffeomorphic, and diffeomorphism from $\Omega^0_{I_5}$ into $\Omega^0_{I_6}$, and the range contains $\Omega^0_{I_4}$ (note (2.1)). By the standard theory of Hamilton-Jacobi equation (see, e.g., Evans [5] Chapter 3, Arnold [2] §47), the solution is constructed as follows: We set

$$u(t,\eta) = \int_0^t \big\{ p(x(0,\eta;s), \xi(0,\eta;s)) - x(0,\eta;s) \cdot \partial_x V_R(x(0,\eta;s), \xi(0,\eta;s)) \big\} ds.$$

If we set

$$\phi(t,\xi) = u(t, \Lambda_t^{-1}(\xi)), \quad \xi \in \Omega^0_{I_4},$$

then ϕ is the solution to the Hamilton-Jacobi equation. We will show $\phi(t,\xi)$ satisfies suitable symbol properties in the following. For simplicity, in this subsection we write

$$x(t) = x(t,\eta) = x(0,\eta;t), \quad \xi(t) = \xi(t,\eta) = \xi(0,\eta;t).$$

At first we recall that there are $c > 0$ such that

$$c^{-1}|t| \leq |x(t,\eta)| \leq c|t|, \quad t \in \mathbb{R}$$

for any $\eta \in \Omega^0_{I_5}$ by Lemma 2.1. Here we may suppose $V_R = 0$ in a neighborhood of 0.

Lemma 2.2. *For any $\alpha \in \mathbb{Z}^d_+$, there is $C_\alpha > 0$ such that*

$$\big|\partial_\eta^\alpha x(t,\eta)\big| \leq C_\alpha \langle t \rangle, \quad \big|\partial_\eta^\alpha \xi(t,\eta)\big| \leq C_\alpha,$$

uniformly in $t \in \mathbb{R}$.

Proof. For $\alpha = 0$, the claim is obvious. By differentiating the Hamilton equation, we have

$$\partial_t(\partial_\eta x) = (\partial_\xi \partial_x p)\partial_\eta x + (\partial_\xi \partial_\xi p)\partial_\eta \xi,$$
$$\partial_t(\partial_\eta \xi) = -(\partial_x \partial_x p)\partial_\eta x - (\partial_x \partial_\xi p)\partial_\eta \xi.$$

We note

$$\partial_\xi \partial_\xi p(x(t),\xi(t)) = \partial_\xi \partial_\xi p_0(x(t),\xi(t)) + \partial_\xi \partial_\xi V_R(x(t),\xi(t)) = O(1),$$
$$\partial_\xi \partial_x p(x(t),\xi(t)) = \partial_\xi \partial_x V(x(t),\xi(t)) = O(\langle t \rangle^{-1-\mu}),$$
$$\partial_x \partial_x p(x(t),\xi(t)) = \partial_x \partial_x V(x(t),\xi(t)) = O(\langle t \rangle^{-2-\mu}).$$

Using these, we learn

$$|\partial_t(\partial_\eta x)| \le C\langle t\rangle^{-1-\mu}|\partial_\eta x| + C|\partial_\eta \xi|,$$
$$|\partial_t(\partial_\eta \xi)| \le C\langle t\rangle^{-2-\mu}|\partial_\eta x| + C\langle t\rangle^{-1-\mu}|\partial_\eta \xi|,$$

and these imply

$$\partial_t(\langle t\rangle^{-1-\mu/2}|\partial_\eta x|) \le C\langle t\rangle^{-1-\mu}(\langle t\rangle^{-1-\mu/2}|\partial_\eta x|) + C\langle t\rangle^{-1-\mu/2}|\partial_\eta \xi|,$$
$$\partial_t|\partial_\eta \xi| \le C\langle t\rangle^{-1-\mu/2}(\langle t\rangle^{-1-\mu/2}|\partial_\eta x|) + C\langle t\rangle^{-1-\mu}|\partial_\eta \xi|.$$

Here ∂_t should be considered in distribution sense, and we have used the fact: $\partial_t\langle t\rangle^{-1-\mu/2} \le 0$. Combining them, we have

$$\partial_t(\langle t\rangle^{-1-\mu/2}|\partial_\eta x| + |\partial_\eta \xi|) \le 2C\langle t\rangle^{-1-\mu/2}(\langle t\rangle^{-1-\mu/2}|\partial_\eta x| + |\partial_\eta \xi|),$$

with $(\langle t\rangle^{-1-\mu/2}|\partial_\eta x| + |\partial_\eta \xi|)|_{t=0} = 1$. Then by the Gronwall's inequality, we learn

$$\langle t\rangle^{-1-\mu/2}|\partial_\eta x| + |\partial_\eta \xi| \le C' < \infty$$

for all $t \in \mathbb{R}$, since $\langle t\rangle^{-1-\mu/2}$ is integrable in t. This implies $\partial_\eta x = O(\langle t\rangle^{1+\mu/2})$ and $\partial_\eta \xi = O(1)$. Substituting these to the above equations again to learn $\partial_t(\partial_\eta x) = O(1)$ and hence $\partial_\eta x = O(\langle t\rangle)$.

For higher derivatives, we use induction in $|\alpha|$. Suppose the claim holds for $|\alpha| \le N-1$, and suppose $|\alpha| = N \ge 2$ We note

$$\begin{aligned}\partial_t(\partial_\eta^\alpha x) &= \partial_\eta^\alpha(\partial_\xi p(x(t,\eta),\xi(t,\eta))) \\ &= (\partial_\xi\partial_x p)\partial_\eta^\alpha x + (\partial_\xi\partial_\xi p)\partial_\eta^\alpha \xi \\ &\quad + \sum_* c_*(\partial_\xi\partial_x^\beta\partial_\xi^\gamma p)\prod_{i=1}^d\biggl(\prod_{j=1}^{\beta_i}(\partial_\eta^{\tilde\beta(i,j)}x_i)\prod_{k=1}^{\gamma_i}(\partial_\eta^{\tilde\gamma(i,k)}\xi_i)\biggr),\end{aligned}$$

where the last sum is taken over $\beta,\gamma,\tilde\beta(i,j),\tilde\gamma(i,k) \in \mathbb{Z}_+^d$ such that $|\beta+\gamma| \ge 2$; $\tilde\beta(i,j) \ne 0$ for $i = 1,\dots,d$, $j = 1,\dots,\beta_i$; $\tilde\gamma(i,k) \ne 0$ for $i = 1,\dots,d$, $k = 1,\dots,\gamma_i$; and

$$\sum_{i=1}^d\biggl(\sum_{j=1}^{\beta_i}\tilde\beta(i,j) + \sum_{k=1}^{\gamma_i}\tilde\gamma(i,k)\biggr) = \alpha.$$

Here c_* denote some universal constants depending only on the indices. By the fact $\partial_\xi\partial_x^\beta\partial_\xi^\gamma p = O(\langle t\rangle^{-|\beta|})$ and the induction hypothesis, we learn that the last term is $O(1)$. Then by the same argument as above, we have

$$|\partial_t(\partial_\eta^\alpha x)| \le C\langle t\rangle^{-1-\mu}|\partial_\eta^\alpha x| + C|\partial_\eta^\alpha \xi| + C,$$

and similarly

$$|\partial_t(\partial_\eta^\alpha \xi)| \le C\langle t\rangle^{-2-\mu}|\partial_\eta^\alpha x| + C\langle t\rangle^{-1-\mu}|\partial_\eta^\alpha \xi| + C\langle t\rangle^{-1-\mu}.$$

Combining them, we learn

$$\partial_t(\langle t\rangle^{-1-\mu/2}|\partial_\eta^\alpha x| + |\partial_\eta^\alpha \xi| + 1) \le 2C\langle t\rangle^{-1-\mu/2}(\langle t\rangle^{-1-\mu/2}|\partial_\eta^\alpha x| + |\partial_\eta^\alpha \xi| + 1).$$

Again by Gronwall's inequality, we have $\langle t\rangle^{-1-\mu/2}|\partial_\eta^\alpha x| + |\partial_\eta^\alpha \xi| + 1 \le C$, and hence $\partial_\eta^\alpha x = O(\langle t\rangle^{1+\mu/2})$, $\partial_\eta^\alpha \xi = O(1)$. We substitute these to the above inequality again to conclude $\partial_\eta^\alpha x = O(\langle t\rangle)$. □

Lemma 2.3. *For any $\alpha\in\mathbb{Z}_+^d$, there is $C_\alpha>0$ such that*

$$\big|\partial_\xi^\alpha\phi(t,\xi)\big|\le C_\alpha\langle t\rangle,\quad t\in\mathbb{R},$$

uniformly in $\xi\in\Omega^0_{I_4}$.

Proof. By direct computation using the definition of $u(t,\eta)$ and the previous lemma, we learn

$$\partial_\eta^\alpha u(t,\eta)=O(\langle t\rangle),\quad t\in\mathbb{R},$$

for any $\alpha\in\mathbb{Z}_+^d$, uniformly in $\eta\in\Omega^0_{I_5}$. We note, by Lemma 2.1 and the previous lemma, we learn

$$\partial_\xi^\alpha\Lambda_t^{-1}(\xi)=O(1),\quad t\in\mathbb{R},$$

also uniformly on the range of $\Lambda_t(\Omega^0_{I_5})$. Combining these, we learn

$$\partial_\xi^\alpha\phi(t,\xi)=\partial_\xi^\alpha(u\circ\Lambda_t^{-1})(\xi)=O(\langle t\rangle),\quad t\in\mathbb{R},$$

uniformly in $\xi\in\Omega^0_{I_4}\subset\Lambda_t(\Omega^0_{I_5})$. □

Lemma 2.4. *For any $\alpha\in\mathbb{Z}_+^d$, there is $C_\alpha>0$ such that*

$$\big|\partial_\xi^\alpha(\phi(t,\xi)-tp_0(\xi))\big|\le C_\alpha\langle t\rangle^{1-\mu},\quad t\in\mathbb{R},\xi\in\Omega^0_{I_4}.$$

Proof. We recall, by the construction of the solution to the Hamilton-Jacobi equation, $\partial_\xi\phi(t,\xi)=x(t,\Lambda_t^{-1}(\xi))$, and hence

$$\big|\partial_\xi\phi(t,\xi)\big|=\big|x(t,\Lambda_t^{-1}(\xi))\big|\ge c|t|,\quad t\in\mathbb{R},$$

with some $c>0$, uniformly in ξ. On the other hand, by the Hamilton-Jacobi equation, we have

$$\begin{aligned}\phi(t,\xi)&=\int_0^t p(\partial_\xi\phi(s,\xi),\xi)ds=\int_0^t(p_0(\xi)+V_R(\partial_\xi\phi(s,\xi),\xi)ds\\&=tp_0(\xi)+\int_0^t V_R(\partial_\xi\phi(s,\xi),\xi)ds,\end{aligned}$$

and hence

$$\begin{aligned}\big|\phi(t,\xi)-tp_0(\xi)\big|&\le\pm\int_0^t|V_R(\partial_\xi\phi(s,\xi),\xi)|ds\\&\le\pm\int_0^t C\langle s\rangle^{-\mu}ds\le C\langle t\rangle^{1-\mu},\quad\text{for }\ \pm t\ge0.\end{aligned}$$

For derivatives: $\partial_\xi^\alpha(\phi(t,\xi)-tp_0(\xi))$, we differentiate the above equality, and we obtain the result using Lemma 2.2. □

2.3. **Out-going/in-coming conditions.** Throughout this section, we always suppose Assumptions A–D, and consider classical trajectories with $p(x_0,\xi_0)\in I_5\Subset\mathbb{R}$ as in Lemma 2.1.

Let $\beta\in(-1,1]$, and consider the condition

$$\pm\cos(x_0,v(\xi_0))=\pm\frac{x_0}{|x_0|}\cdot\frac{v(\xi_0)}{|v(\xi_0)|}\ge\beta. \tag{2.2}$$

Lemma 2.5. *Let $\beta>-1$ be fixed. Then there are $c>0$ and $L\ge0$ such that*

$$|x(x_0,\xi_0;t)|\ge c(\langle x_0\rangle+\langle t\rangle),\quad\pm t\ge0,$$

respectively, provided $|x_0|\ge L$, and the condition (2.2) is satisfied.

Proof. We may assume $\beta < 0$ without loss of generality, and we consider the case $t \geq 0$ only. The other case can be considered in the same way. We note, by the Hamilton equation,

$$\frac{d}{dt}|x(t)|^2 = 2x \cdot \partial_\xi p(x,\xi) = 2x \cdot v(\xi) + 2x \cdot \partial_\xi V_R(x,\xi).$$

By this equality, we easily observe that if $\cos(x_0, v(\xi_0)) \geq |\beta|$, then $|x(t)|^2 \geq c\langle t\rangle^2$ for $t \geq 0$, provided $|x_0|$ is sufficiently large. Thus in the following we may assume

$$|\cos(x_0, v(\xi_0))| \leq |\beta| < 1. \tag{2.3}$$

Since $v(\xi)$ and $\partial_\xi V(x,\xi)$ are bounded on Ω_{I_5}, we have

$$\frac{d}{dt}|x(t)|^2\Big|_{t=0} \geq -C_1|x_0|$$

with a constant C_1. Then, by Lemma 2.1, we learn

$$\begin{aligned} |x(t)|^2 &\geq |x_0|^2 - C_1|x_0|t + \frac{c_5}{2}t^2 \\ &= |x_0|^2 + (c_5/4)t^2 + ((c_5/4)t - C_1|x_0|)t \end{aligned}$$

for $t \geq 0$. Hence, we have

$$|x(t)|^2 \geq |x_0|^2 + (c_4/4)t^2, \quad \text{if } t \geq (4C_1/c_5)|x_0|.$$

Thus it suffices to consider the estimate for $t \in [0, C_2|x_0|]$, where $C_2 = 4C_1/c_5$.

We now consider the *impact paramter*:

$$y(t) = x(t) - (x(t) \cdot \hat{v}(\xi(t)))\hat{v}(\xi(t)), \quad t \in \mathbb{R},$$

where $\hat{v}(\xi) = |v(\xi)|^{-1}v(\xi)$. We have

$$\frac{d}{dt}y(t) = \partial_\xi V_R - (\partial_\xi V_R \cdot \hat{v})\hat{v} - \left(x \cdot \frac{d\hat{v}}{dt}\right)\hat{v} - (x \cdot \hat{v})\frac{d\hat{v}}{dt}.$$

We note

$$\frac{d\hat{v}}{dt} = \frac{d}{dt}\left(\frac{v}{|v|}\right) = \frac{1}{|v|}\frac{dv}{dt} - \left(v \cdot \frac{dv}{dt}\right)\frac{v}{|v|^3}$$

and

$$\frac{dv}{dt} = (\partial_\xi \partial_\xi p_0)\frac{d\xi}{dt} = -(\partial_\xi \partial_\xi p_0)(\partial_x V_R) = O(\langle x(t)\rangle^{-1-\mu}).$$

We also note $|v| \geq c_4$ by Assumption D. Thus we learn

$$\left|\frac{d}{dt}y(t)\right| \leq C_3\langle x(t)\rangle^{-\mu} \leq C_3\langle y(t)\rangle^{-\mu}$$

with some constant $C_3 > 0$. We solve this differential inequality as follows: at first, we note

$$\frac{d}{dt}\langle y(t)\rangle \geq -\left|\frac{d}{dt}y(t)\right| \geq -C_3\langle y(t)\rangle^{-\mu},$$

and hence

$$\langle y\rangle^\mu \frac{d}{dt}\langle y\rangle = \frac{1}{\mu+1}\frac{d}{dt}\big(\langle y\rangle^{\mu+1}\big) \geq -C_3.$$

Thus we have

$$\langle y(t)\rangle^{\mu+1} \geq \langle y(0)\rangle^{\mu+1} - (\mu+1)C_3 t, \quad t \geq 0.$$

On the other hand, by the assumption (2.2) and (2.3), we have

$$|y(0)|^2 = |x_0|^2 - |x_0 \cdot \hat{v}(\xi_0)|^2 \geq (1-\beta^2)|x_0|^2,$$

since $y(t)$ and $\hat{v}(\xi)$ are perpendicular. Combining them, we have

$$\begin{aligned}\langle y(t)\rangle^{\mu+1} &\geq (1-\beta^2)^{(\mu+1)/2}|x_0|^{\mu+1} - (\mu+1)C_3 t\\ &\geq (1-\beta^2)^{(\mu+1)/2}|x_0|^{\mu+1}\big(1 - C_4|x_0|^{-\mu}\big)\end{aligned}$$

if $t \in [0, C_2|x_0|]$, where $C_4 = (1-\beta^2)^{-(\mu+1)/2}(\mu+1)C_3C_2$. Now if we take $R > 0$ so that $1 - C_4R^{-\mu} > 1/2$, then

$$\langle x(t)\rangle \geq \langle y(t)\rangle \geq (1-\beta^2)^{1/2}2^{-1/(\mu+1)}|x_0|, \quad t \in [0, C_2|x_0|].$$

This completes the proof. □

We note that if $|x_0| \leq L$, then by Lemma 2.1 and its proof, we have

$$|x(x_0, \xi_0; t)| \geq c\langle t\rangle - C, \quad t \in \mathbb{R},$$

with some $c > 0$ and $C > 0$, uniformly. Thus the above estimates are always valid for sufficiently large $|t|$, and hence, under the above out-going/in-coming condition (2.2), we have

$$|x(t)| \geq c\langle x_0; t\rangle, \quad \pm t \geq T,$$

with some constant $c > 0$ and sufficiently large $T > 0$, uniformly in (x_0, ξ_0), where we denote

$$\langle x; t\rangle = (1 + |x|^2 + t^2)^{1/2}.$$

We use the following notation:

$$\begin{aligned}\Omega_{J,\pm}(\beta) &= \big\{(x,\xi) \bigm| p(x,\xi) \in J, \pm\cos(x, v(\xi)) > \beta\big\},\\ \Omega^0_{J,\pm}(\beta) &= \big\{(x,\xi) \bigm| x \in \mathbb{R}^d, p_0(\xi) \in J, \pm\cos(x, v(\xi)) > \beta\big\}\end{aligned}$$

for $J \subset \mathbb{R}$ and $\beta \in (-1, 1]$.

Lemma 2.6. (i) *There is a constant $C > 0$ such that*

$$\begin{aligned}&|\partial_{x_0}x(t)| \leq C(1 + \langle x_0\rangle^{-1-\mu}|t|), \quad |\partial_{x_0}\xi(t)| \leq C\langle x_0\rangle^{-1-\mu},\\ &|\partial_{\xi_0}x(t)| \leq C|t|, \quad |\partial_{\xi_0}\xi(t)| \leq C,\end{aligned}$$

uniformly for $(x_0, \xi_0) \in \Omega_{I_5,\pm}(\beta)$. Moreover,

$$|\partial_{\xi_0}\xi(t) - \mathrm{E}| \leq C\langle x_0\rangle^{-\mu}.$$

(ii) *For $\alpha, \beta \in \mathbb{Z}^d_+$, $|\alpha + \beta| \geq 2$, there is $C_{\alpha\beta} > 0$ such that*

$$|\partial^\alpha_{x_0}\partial^\beta_{\xi_0}x(t)| \leq C_{\alpha\beta}\langle x_0\rangle^{-|\alpha|}|t|, \quad |\partial^\alpha_{x_0}\partial^\beta_{\xi_0}\xi(t)| \leq C_{\alpha\beta}\langle x_0\rangle^{-|\alpha|-\mu},$$

uniformly for $\pm t \geq 0$, $(x_0, \xi_0) \in \Omega_{I_5,\pm}(\beta)$. If, moreover, $\alpha \neq 0$, then

$$|\partial^\alpha_{x_0}\partial^\beta_{\xi_0}x(t)| \leq C_{\alpha\beta}\langle x_0\rangle^{-|\alpha|-\mu}|t|.$$

Proof. We use an argument similar to the proof of Lemma 2.2, but with carefully controlling the dependence of the constants on x_0.

(i) We first note, by the Hamilton equation, we have

$$\begin{cases}\partial_t(\partial_{x_0}x) = (\partial_\xi\partial_x p)\partial_{x_0}x + (\partial_\xi\partial_\xi p)\partial_{x_0}\xi,\\ \partial_t(\partial_{x_0}\xi) = -(\partial_x\partial_x p)\partial_{x_0}x - (\partial_x\partial_\xi p)\partial_{x_0}\xi,\\ (\partial_{x_0}x)(0) = \mathrm{E}, \quad (\partial_{x_0}\xi)(0) = 0.\end{cases} \tag{2.4}$$

We recall

$$|\partial_\xi\partial_\xi p| = O(1), \quad |\partial_x\partial_\xi p| = O(\langle x_0; t\rangle)^{-1-\mu}), \quad |\partial_x\partial_x p| = O(\langle x_0; t\rangle^{-2-\mu}),$$

under our conditions. Thus we have

$$\partial_t|\partial_{x_0}x| \le C\langle x_0;t\rangle^{-1-\mu}|\partial_{x_0}x| + C|\partial_{x_0}\xi|, \tag{2.5}$$

$$\partial_t|\partial_{x_0}\xi| \le C\langle x_0;t\rangle^{-2-\mu}|\partial_{x_0}x| + C\langle x_0;t\rangle^{-1-\mu}|\partial_{x_0}\xi|. \tag{2.6}$$

Hence we have

$$\partial_t\big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}x| + |\partial_{x_0}\xi|\big) \le C\langle x_0;t\rangle^{-1-\mu/2}\big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}x| + |\partial_{x_0}\xi|\big).$$

Then by Gronwall's inequality, we learn

$$\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}x| + |\partial_{x_0}\xi| \le C\big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}x| + |\partial_{x_0}\xi|\big)_{t=0} \le C'\langle x_0\rangle^{-1-\mu/2}$$

uniformly, and hence

$$|\partial_{x_0}x(t)| \le C\langle x_0\rangle^{-1-\mu/2}\langle x_0;t\rangle^{1+\mu/2}, \quad |\partial_{x_0}\xi(t)| \le C\langle x_0\rangle^{-1-\mu/2}.$$

Substituting these to (2.5) and (2.6), and we have

$$\partial_t|\partial_{x_0}x| \le C\langle x_0\rangle^{-1-\mu/2}\langle x_0;t\rangle^{-\mu/2} + C\langle x_0\rangle^{-1-\mu/2} \le C\langle x_0\rangle^{-1-\mu/2},$$
$$\partial_t|\partial_{x_0}\xi| \le C\langle x_0\rangle^{-1-\mu/2}\langle x_0;t\rangle^{-1-\mu/2},$$

and hence (using $\int_0^\infty\langle x_0;t\rangle^{-1-\mu/2}dt \le C\langle x_0\rangle^{-\mu/2}$),

$$|\partial_{x_0}x| \le C(1+\langle x_0\rangle^{-1-\mu/2}|t|), \quad |\partial_{x_0}\xi| \le C\langle x_0\rangle^{-1-\mu}.$$

Iterating this procedure once more, we obtain

$$|\partial_{x_0}x| \le C(1+\langle x_0\rangle^{-1-\mu}|t|)$$

Similarly, we have

$$\begin{cases} \partial_t(\partial_{\xi_0}x) = (\partial_\xi\partial_x p)\partial_{\xi_0}x + (\partial_\xi\partial_\xi p)\partial_{\xi_0}\xi, \\ \partial_t(\partial_{\xi_0}\xi) = -(\partial_x\partial_x p)\partial_{\xi_0}x - (\partial_x\partial_\xi p)\partial_{\xi_0}\xi, \\ (\partial_{\xi_0}x)(0) = 0, \quad (\partial_{\xi_0}\xi)(0) = \mathrm{E}, \end{cases}$$

and hence

$$\partial_t|\partial_{\xi_0}x| \le C\langle x_0;t\rangle^{-1-\mu}|\partial_{\xi_0}x| + C|\partial_{\xi_0}\xi|,$$
$$\partial_t|\partial_{\xi_0}\xi| \le C\langle x_0;t\rangle^{-2-\mu}|\partial_{\xi_0}x| + C\langle x_0;t\rangle^{-1-\mu}|\partial_{\xi_0}\xi|.$$

We again obtain

$$\partial_t\big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{\xi_0}x| + |\partial_{\xi_0}\xi|\big) \le C\langle x_0;t\rangle^{-1-\mu/2}\big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{\xi_0}x| + |\partial_{\xi_0}\xi|\big),$$

and by Gronwall's inequality, we obtain

$$\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}x| + |\partial_{x_0}\xi| \le C\big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{\xi_0}x| + |\partial_{\xi_0}\xi|\big)_{t=0}.$$

This implies

$$|\partial_{\xi_0}x| \le C\langle x_0;t\rangle^{1+\mu/2}, \quad |\partial_{\xi_0}\xi| \le C.$$

Substituting these to the above inequality for $\partial_{\xi_0}x$ again to learn

$$\partial_t|\partial_{\xi_0}x| \le C\langle x_0;t\rangle^{-\mu/2} + C \le C',$$

and hence $|\partial_{\xi_0}x| = O(|t|)$. Also, by substituting these to the inequality for $\partial_{x_0}\xi$, we learn

$$\partial_t|\partial_{\xi_0}\xi| \le C\langle x_0;t\rangle^{-2-\mu}\langle t\rangle + C\langle x_0;t\rangle^{-1-\mu} \le C'\langle x_0;t\rangle^{-1-\mu},$$

and hence $|\partial_{\xi_0}\xi - \mathrm{E}| = O(\langle x_0\rangle^{-\mu})$.

(ii) We prove the claim by induction in $|\alpha+\beta|$. We use a weaker induction hypothesis:

(C_k): For $\alpha,\beta\in\mathbb{Z}_+^d$, $|\alpha+\beta|\le k$, there is $C_{\alpha\beta}>0$ such that

$$\bigl|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x(t)\bigr|\le C_{\alpha\beta}\langle x_0\rangle^{-|\alpha|}\langle x_0;t\rangle,\quad \bigl|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi(t)\bigr|\le C_{\alpha\beta}\langle x_0\rangle^{-|\alpha|},$$

uniformly for $t\ge 0$, $(x_0,\xi_0)\in\Omega_{I_5}$ and $\cos(x_0,v(\xi_0))\ge\beta$.

We note (C_0) is easy to show, and (C_1) is already proved in (i) of the lemma. Suppose (C_k) holds for $k<N$, and suppose $|\alpha+\beta|=N\ge 2$. By the Hamilton equation and the Leibniz rule, we have

$$\begin{aligned}\partial_t(\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x) &= (\partial_\xi\partial_x p)\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x+(\partial_\xi\partial_\xi p)\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi\\ &\quad +\sum_* c_*(\partial_\xi\partial_x^\gamma\partial_\xi^\delta p)\prod_{i=1}^d\Bigl(\prod_{j=1}^{\gamma_i}(\partial_{x_0}^{a(i,j)}\partial_{\xi_0}^{b(i,j)}x_i)\prod_{k=1}^{\delta_i}(\partial_{x_0}^{\tilde a(i,k)}\partial_{\xi_0}^{\tilde b(i,k)}\xi_i)\Bigr),\end{aligned}$$

where the last sum is taken over $\gamma,\delta,a(i,j),\tilde a(i,j),b(i,k),\tilde b(i,k)\in\mathbb{Z}_+^d$ such that $|\gamma+\delta|\ge 2$; $a(i,j)+b(i,j)\ne 0$ for $i=1,\dots,d$, $j=1,\dots,\gamma_i$; $\tilde a(i,k)+\tilde b(i,k)\ne 0$ for $i=1,\dots,d$, $k=1,\dots,\delta_i$; and

$$\sum_{i=1}^d\Bigl(\sum_{j=1}^{\gamma_i}a(i,j)+\sum_{k=1}^{\delta_i}\tilde a(i,k)\Bigr)=\alpha,\quad \sum_{i=1}^d\Bigl(\sum_{j=1}^{\gamma_i}b(i,j)+\sum_{k=1}^{\delta_i}\tilde b(i,k)\Bigr)=\beta.$$

c_* denotes suitable universal constant for each index. We denote the last term by R_1. Similarly, we have

$$\begin{aligned}\partial_t(\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi) &= (\partial_x\partial_x p)\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x+(\partial_x\partial_\xi p)\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi\\ &\quad +\sum_* c_*(\partial_x\partial_x^\gamma\partial_\xi^\delta p)\prod_{i=1}^d\Bigl(\prod_{j=1}^{\gamma_i}(\partial_{x_0}^{a(i,j)}\partial_{\xi_0}^{b(i,j)}x_i)\prod_{k=1}^{\delta_i}(\partial_{x_0}^{\tilde a(i,k)}\partial_{\xi_0}^{\tilde b(i,k)}\xi_i)\Bigr).\end{aligned}$$

We denote the last term by R_2.

We now recall

$$\bigl|\partial_x^\gamma\partial_\xi^\delta p(x(t),\xi(t))\bigr|\le C\langle x_0;t\rangle^{-|\gamma|},$$

with some constant $C>0$. Combining this with the induction hypothesis, we learn

$$\begin{aligned}&\Bigl|(\partial_\xi\partial_x^\gamma\partial_\xi^\delta p)\prod_{i=1}^d\Bigl(\prod_{j=1}^{\gamma_i}(\partial_{x_0}^{a(i,j)}\partial_{\xi_0}^{b(i,j)}x_i)\prod_{k=1}^{\delta_i}(\partial_{x_0}^{\tilde a(i,k)}\partial_{\xi_0}^{\tilde b(i,k)}\xi_i)\Bigr)\Bigr|\\ &\quad\le C\langle x_0;t\rangle^{-|\gamma|}\langle x_0;t\rangle^{|\gamma|}\langle x_0\rangle^{-|\alpha|}=C\langle x_0\rangle^{-|\alpha|},\end{aligned}$$

and hence

$$|R_1|\le C\langle x_0\rangle^{-|\alpha|}.$$

Similarly, we have

$$|R_2|\le C\langle x_0;t\rangle^{-1-\mu}\langle x_0\rangle^{-|\alpha|}.$$

Thus we have

$$\partial_t|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|\le C\langle x_0;t\rangle^{-1-\mu}|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|+C|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|+C\langle x_0\rangle^{-|\alpha|}, \tag{2.7}$$

$$\begin{aligned}\partial_t|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi| &\le C\langle x_0;t\rangle^{-2-\mu}|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|+C\langle x_0;t\rangle^{-1-\mu}|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|\\ &\quad +C\langle x_0;t\rangle^{-1-\mu}\langle x_0\rangle^{-|\alpha|}.\end{aligned} \tag{2.8}$$

Combining these, we obtain

$$\begin{aligned}&\partial_t\Big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|+|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|\Big)\\&\quad\leq C\langle x_0;t\rangle^{-1-\mu/2}\Big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|+|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|\Big)+C\langle x_0;t\rangle^{-1-\mu/2}\langle x_0\rangle^{-|\alpha|}\end{aligned}$$

and

$$\Big(\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|+|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|\Big)\Big|_{t=0}=0$$

since $|\alpha+\beta|\geq 2$. Then by Gronwall's inequality we have

$$\langle x_0;t\rangle^{-1-\mu/2}|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|+|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|\leq C\langle x_0\rangle^{-|\alpha|-\mu/2},$$

and hence

$$|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|\leq C\langle x_0;t\rangle^{1+\mu/2}\langle x_0\rangle^{-|\alpha|},\quad |\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|\leq C\langle x_0\rangle^{-|\alpha|-\mu/2}\leq C\langle x_0\rangle^{-|\alpha|}.$$

Now we substitute these to (2.7) to learn

$$\partial_t|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|\leq C\langle x_0;t\rangle^{-\mu/2}\langle x_0\rangle^{-|\alpha|}+2C\langle x_0\rangle^{-|\alpha|}\leq 3C\langle x_0\rangle^{-|\alpha|}.$$

Integrating this, we conclude

$$|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|\leq C|t|\langle x_0\rangle^{-|\alpha|}\leq C\langle x_0;t\rangle\langle x_0\rangle^{-|\alpha|}. \tag{2.9}$$

In particular, we have proved the induction step (C_k) with $k=N$, and thus (C_k) holds for all $k\geq 0$. We substitute (C_N) to (2.8), and we learn

$$\partial_t|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|\leq C\langle x_0;t\rangle^{-1-\mu}\langle x_0\rangle^{-|\alpha|},$$

and then by integrating in t, we have

$$|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|\leq C\langle x_0\rangle^{-|\alpha|-\mu}. \tag{2.10}$$

Now we suppose $\alpha\neq 0$, and consider each term in R_1 more carefully:

$$r_*=(\partial_\xi\partial_x^\gamma\partial_\xi^\delta p)\prod_{i=1}^d\bigg(\prod_{j=1}^{\gamma_i}(\partial_{x_0}^{a(i,j)}\partial_{\xi_0}^{b(i,j)}x_i)\prod_{k=1}^{\delta_i}(\partial_{x_0}^{\tilde a(i,k)}\partial_{\xi_0}^{\tilde b(i,k)}\xi_i)\bigg).$$

If $\gamma=0$, then r_* contains derivatives of ξ in x_0, and thus we learn $r_*=O(\langle x_0\rangle^{-|\alpha|-\mu})$, by virtue of (2.10) and (i) of the lemma. If $\gamma\neq 0$, then $\partial_\xi\partial_x^\gamma\partial_\xi^\delta p=O(\langle x_0;t\rangle^{-|\gamma|-\mu})$, and we also improve the estimate to obtain $r_*=O(\langle x_0\rangle^{-|\alpha|-\mu})$. Thus, if $\alpha\neq 0$, we have $R_1=O(\langle x_0\rangle^{-|\alpha|-\mu})$, and we obtain, instead of (2.7),

$$\partial_t|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|\leq C\langle x_0;t\rangle^{-1-\mu}|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|+C|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi|+C\langle x_0\rangle^{-|\alpha|-\mu}.$$

Now we substitute (2.9) and (2.10) to this inequality to learn

$$\partial_t|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|\leq C\langle x_0;t\rangle^{-\mu}\langle x_0\rangle^{-|\alpha|}+C\langle x_0\rangle^{-|\alpha|-\mu}\leq 2C\langle x_0\rangle^{-|\alpha|-\mu}.$$

Integrating this in t, we conclude $|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x|=O(\langle x_0\rangle^{-|\alpha|-\mu}|t|)$. □

2.4. **Classical mechanics in the interaction picture.** We now consider the evolution of

$$y(t)=x(x_0,\xi_0;t)-\partial_\xi\phi(t,\xi(x_0,\xi_0;t)),\quad \xi(t)=\xi(x_0,\xi_0;t),$$

where $\phi(t,x)$ is the solution to the Hamilton-Jacobi equation constructed in Subsection 2.2. The classical Hamiltonian for the evolution is given by

$$\begin{aligned}q(t,y,\xi)&=p(y+\partial_\xi\phi(t,\xi),\xi)-p(\partial_\xi\phi(t,\xi),\xi)\\&=V_R(y+\partial_\xi\phi(t,\xi),\xi)-V_R(\partial_\xi\phi(t,\xi),\xi).\end{aligned}$$

For the completeness, we verify that $q(t,y,\xi)$ generate the evolution:

Lemma 2.7. *Let $y(t)$, $\xi(t)$, $q(t,y,\xi)$ as above. Then*

$$\begin{aligned}&\frac{d}{dt}y(t)=\partial_\xi q(t,y(t),\xi(t)),\quad \frac{d}{dt}\xi(t)=-\partial_y q(t,y(t),\xi(t)),\\&y(0)=x_0,\quad \xi(0)=\xi_0.\end{aligned}$$

Proof. For $j=1,\dots,d$, we compute

$$\begin{aligned}\frac{d}{dt}y_j(t)&=\frac{d}{dt}x_j(t)-(\partial_{\xi_j}\partial_t\phi)(t,\xi(t))-\sum_k(\partial_{\xi_k}\partial_{\xi_j}\phi)(t,\xi(t))\frac{d}{dt}\xi_k(t)\\&=(\partial_{\xi_j}p)(y+\partial_\xi\phi,\xi)+\sum_k(\partial_{\xi_j}\partial_{\xi_k}\phi)(t,\xi(t))(\partial_{x_k}p)(y+\partial_\xi\phi,\xi)-\partial_{\xi_j}(p(\partial_\xi\phi(t,\xi),\xi))\\&=\partial_{\xi_j}\big(p(y+\partial_\xi\phi(t,\xi),\xi)\big)-\partial_{\xi_j}(p(\partial_\xi\phi(t,\xi),\xi))\\&=\partial_{\xi_j}q(t,y,\xi).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}\xi_j(t)&=-(\partial_{x_j}p)(y+\partial_\xi\phi(t,\xi),\xi)\\&=-\partial_y\big(p(y+\partial_\xi\phi(t,\xi),\xi)\big)=-\partial_y q(t,y,\xi).\end{aligned}$$

The initial condition is obvious from the definition. □

The existence of the classical long-range scattering is well-known, but we write it down for the completeness, and also for the later reference.

Lemma 2.8. *For each $(x_0,\xi_0)\in\Omega_{I_5}$, the limits*

$$x_\pm=\lim_{t\to\pm\infty}y(t),\quad \xi_\pm=\lim_{t\to\pm\infty}\xi(t)$$

exist, and $\xi_\pm\in\Omega^0_{I_4}$.

Proof. We recall, by Lemma 2.1, we have

$$|x(t)|\geq c|t|-c',\quad t\in\mathbb{R},$$

with some constants $c,c'>0$, and $\xi(t)$ is uniformly bounded. Now we observe

$$|\partial_t\xi(t)|\leq|\partial_x V_R(x(t),\xi(t))|\leq C\langle t\rangle^{-1-\mu},\quad t\in\mathbb{R},$$

and hence the limit

$$\lim_{t\to\pm\infty}\xi(t)=\xi_0-\int_0^{\pm\infty}\partial_x V_R(x(t),\xi(t))dt$$

exist. This also implies

$$|\xi(t)-\xi_\pm|\leq C\langle t\rangle^{-\mu},\quad \pm t>0.$$

We note

$$\begin{aligned}\frac{d}{dt}y(t) &= \partial_\xi q(t;y,\xi)\\ &= (\partial_\xi V_R)(y+\partial_\xi\phi(t,\xi),\xi) - (\partial_\xi V_R)(\partial_\xi\phi(t,\xi),\xi)\\ &\quad + (\partial_\xi\partial_\xi\phi(t,\xi))\big(\partial_x V_R(y+\partial_\xi\phi(t,\xi)) - \partial_x V_R(\partial_\xi\phi(t,\xi))\big).\end{aligned}$$

We also note $y(t)+\partial_\xi\phi(t,\xi)=x(t)$, and hence each term can be bounded using Lemma 2.1, and we learn $\frac{d}{dt}y(t)=O(\langle t\rangle^{-\mu})$. Integrating this in t, we have $y(t)=O(\langle t\rangle^{1-\mu})$. In particular, by Lemma 2.4 we have

$$\big|sy(t)+\partial_\xi\phi(t,\xi(t))\big| \geq c\langle t\rangle, \quad |t|\gg 0,\ c>0,$$

uniformly for $s\in[0,1]$.

Now by using

$$\begin{aligned}\frac{d}{dt}y_j(t) &= \frac{\partial}{\partial\xi_j}\Big(V_R(y+\partial_\xi\phi(t,\xi),\xi) - V_R(\partial_\xi\phi(t,\xi),\xi)\Big)\\ &= \sum_k\int_0^1 y_k(\partial_{x_k}\partial_{\xi_j}V_R)(sy+\partial_\xi\phi(t,\xi),\xi)ds\\ &\quad + \sum_{k,\ell}\int_0^1 y_k(\partial_{\xi_j}\partial_{\xi_\ell}\phi)(t,\xi)(\partial_{x_k}\partial_{x_\ell}V_R)(sy+\partial_\xi\phi(t,\xi),\xi)ds,\end{aligned} \tag{2.11}$$

we obtain

$$\frac{d}{dt}|y(t)| \leq C|y(t)|\langle t\rangle^{-1-\mu}, \quad t\in\mathbb{R}.$$

By Gronwall's inequality, we learn $|y(t)|$ is uniformly bounded. By substituting this boundedness to the above equation (2.11) again, we have $\frac{d}{dt}y(t)=O(\langle t\rangle^{-\mu-1})$, and hence $\frac{d}{dt}y(t)$ is integrable. This implies the convergence of $y(t)$ as $t\to\pm\infty$. □

Now we consider uniform estimates for out-going/in-coming initial conditions. We first prepare a preliminary lemma.

Lemma 2.9. *Let $\beta\in(-1,1]$, and suppose $(x_0,\xi_0)\in\Omega_{I_5,\pm}(\beta)$. Then*

(i) *There is $C_0>0$ such that*

$$|\xi(x_0,\xi_0;t)-\xi_0| \leq C_0\langle x_0\rangle^{-\mu}, \quad \pm t\geq 0,$$

uniformly in (x_0,ξ_0). In particular, $|\xi_\pm-\xi_0|=O(\langle x_0\rangle^{-\mu})$.

(ii) *There is $C_1>0$ such that*

$$|y(t)| \leq C_1\langle x_0\rangle, \quad \pm t>0.$$

Proof. We modify the proof of Lemma 2.8. By Lemma 2.5, we have

$$|\partial_t\xi(t)| = |(\partial_x V_R)(x(t),\xi(t))| \leq C\langle x_0;t\rangle^{-1-\mu}, \quad t\in\mathbb{R},(x_0,\xi_0)\in\Omega_{I_5}.$$

By integrating this inequality in t, we conclude (i).

In order to prove the claim (ii), we note that the statement is obvious for x_0 in a bounded set by virtue of Lemma 2.8. Hence, it suffices to consider it for $|x_0|\gg 0$. We also consider the case "+" only, since the other case is handled similarly. By the same argument as in the proof of Lemma 2.8, we have

$$|y(t)-x_0| = |x(t)-x_0-\partial_\xi\phi(t,\xi(t))| \leq C_2\langle t\rangle^{1-\mu}, \quad t\geq 0,$$

uniformly in $(x_0,\xi_0)\in\Omega_{I_5,\pm}(\beta)$. We recall that there is $c>0$ such that

$$|\partial_\xi\phi(t,\xi)|\geq c_0|t|,\quad t\in\mathbb{R}.$$

We choose $T_0>0$ so large that

$$C_2\langle T_0\rangle^{1-\mu}\leq (c_0/4)T_0.$$

For the moment, we suppose $|x_0|\geq (c_0/4)T_0$ so that $t\geq (4/c_0)|x_0|$ implies $t\geq T_0$. Thus, if $t\geq (4/c_0)|x_0|$, then

$$\begin{aligned}|sy(t)+\partial_\xi\phi(t,\xi)| &\geq |\partial_\xi\phi(t,\xi)|-|y(t)|\\ &\geq c_0|t|-(|x_0|+C_2\langle t\rangle^{1-\mu})\geq (c_0/2)|t|,\end{aligned}$$

uniformly in $s\in[0,1]$. Hence, by (2.11), we learn

$$\frac{d}{dt}|y(t)|\leq C|y(t)|\langle t\rangle^{-1-\mu},\quad t\geq (4/c_0)|x_0|. \tag{2.12}$$

On the other hand, since $\partial_\xi q(t,y,\xi)$ is bounded on $\tilde\Omega_{I_5}(t)=\{(x+\partial_\xi\phi(t,\xi),\xi)\mid (x,\xi)\in\Omega_{I_5}\}$, we learn

$$|y(t)|\leq |x_0|+C|t|,\quad t\in\mathbb{R}.$$

In particular,

$$|y(t)|\leq C'|x_0|,\quad t\in[0,(4/c_0)|x_0|].$$

We now use Gronwall's inequality for (2.12) with the initial condition at $t=(4/c_0)|x_0|$ to conclude $|y(t)|\leq C|x_0|$ for all $t\geq 0$. $\square$

We now consider the derivatives of the evolution $(y(t),\xi(t))$, i.e., for $\alpha,\beta\in\mathbb{Z}_+^d$, we study the properties of $\partial_{x_0}^\alpha\partial_{\xi_0}^\beta y(t)$ and $\partial_{x_0}^\alpha\partial_{\xi_0}^\beta\xi(t)$. We first prepare properties of $q(t,y,\xi)$ along the classical trajectories.

Lemma 2.10. *Let x_0, ξ_0, $y(t)$, $\xi(t)$ as in Lemma 2.9. Then*

(i) *For any $\beta\in\mathbb{Z}_+^d$, there is $C_\beta>0$ such that*

$$\big|(\partial_\xi^\beta q)(t;y(t),\xi(t))\big|\leq C_\beta\langle x_0\rangle\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu},\quad \pm t\geq 0.$$

(ii) *For any $\alpha,\beta\in\mathbb{Z}_+^d$ with $\alpha\neq 0$, there is $C_{\alpha\beta}>0$ such that*

$$\big|(\partial_y^\alpha\partial_\xi^\beta q)(t;y(t),\xi(t))\big|\leq C_{\alpha\beta}\langle x_0;t\rangle^{-\mu-|\alpha|},\quad \pm t\geq 0.$$

Proof. By direct computations, we have

$$\begin{aligned}\partial_\xi^\beta q(t,y,\xi) &= \sum_* c_*\Big((\partial_y^\gamma\partial_\xi^\delta V_R)(y+\partial_\xi\phi(t,\xi),\xi)-(\partial_y^\gamma\partial_\xi^\delta V_R)(\partial_\xi\phi(t,\xi),\xi)\Big)\times\\ &\quad\times\prod_{i=1}^d\prod_{j=1}^{\gamma_i}(\partial_\xi^{\tilde\beta(i,j)}\phi)(t,\xi),\end{aligned} \tag{2.13}$$

where the indices runs over $\gamma,\delta,\tilde\beta(i,j)\in\mathbb{Z}_+^d$, $i=1,\dots,d$, $j=1,\dots,\gamma_i$, such that $\tilde\beta(i,j)\neq 0$ for $i=1,\dots,d$, $j=1,\dots,\delta_i$, and

$$\sum_{i=1}^d\sum_{j=1}^{\gamma_i}\tilde\beta(i,j)+\delta=\beta,$$

and c_* denote suitable universal constants.

If $\alpha \neq 0$, then we also have

$$\partial_y^\alpha \partial_\xi^\beta q(t,y,\xi) = \sum_* c_* (\partial_y^{\alpha+\gamma}\partial_\xi^\delta V_R)(y+\partial_\xi\phi(t,\xi),\xi) \prod_{i=1}^d \prod_{j=1}^{\gamma_i} (\partial_\xi^{\tilde\beta(i,j)}\phi)(t,\xi),$$

with the same set of indices. If $\alpha \neq 0$, then the claim (ii) follows easily from the above expression combined with Lemmas 2.3, 2.5. It remains to show (i). We first consider the case $\alpha = \beta = 0$, i.e., bounds on $q(t,y,\xi)$ itself. We consider the "+" case only.

By Lemma 2.4, there is $c_0 > 0$ such that $|\partial_\xi\phi(t,\xi)| \geq c_0|t|$, uniformly in t and $\xi_0 \in \Omega^0_{I_3}$. Also, let C_1 as in the previous lemma, i.e., $|y(t)| \leq C_1|x_0|$ for any $t \in \mathbb{R}$. If $t \geq M|x_0|$ with $M = 2C_1/c_0$, then

$$\big|sy(t) + \partial_\xi\phi(t,\xi)\big| \geq |\partial_\xi\phi(t,x)| - |y(t)| \geq c_0|t| - C_1|x_0| \geq (c_0/2)|t|,$$

uniformly for $s \in [0,1]$. Combining this with

$$q(t,y,\xi) = \int_0^1 y(t)\cdot(\partial_x V_R)(sy(t)+\partial_\xi\phi(t,\xi),\xi)ds,$$

we learn

$$|q(t,y(t),\xi(t))| \leq C|x_0|\langle t\rangle^{-1-\mu} \leq C'|x_0|\langle x_0;t\rangle^{-1-\mu},$$

if $t > M|x_0|$. In the last inequality, we have used $|x_0|^2 + |t|^2 \leq (1+M^2)|t|^2$ and hence $\langle t\rangle^{-1} \leq (1+M^2)^{1/2}\langle x_0;t\rangle^{-1}$. On the other hand, if $0 \leq t \leq M|x_0|$, then we have

$$\begin{aligned} |q(t,y(t),\xi(t))| &\leq C\langle x(t)\rangle^{-\mu} + C\langle t\rangle^{-\mu} \\ &\leq C\langle x_0;t\rangle\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu} \leq C'\langle x_0\rangle\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu}. \end{aligned}$$

Combining these, we conclude

$$|q(t,y(t),\xi(t))| \leq C\langle x_0\rangle\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu}, \quad t \geq 0.$$

Estimates for $\partial_\xi^\alpha q(t,y,\xi)$ is similar, though more terms are involved. Actually, for $t \geq M|x_0|$, we have, using (2.13),

$$|\partial_\xi^\beta q| \leq C|x_0| \sum_{j=1}^{|\beta|} \langle t\rangle^{-1-\mu-j}\langle t\rangle^j \leq C|x_0|\langle t\rangle^{-1-\mu}$$

as before, where j corresponds to $|\gamma|$ in (2.13). For $t \in [0, M|x_0|]$, we similarly have

$$\begin{aligned} |\partial_\xi^\beta q| &\leq C\sum_{j=1}^{|\beta|} \big\{\langle x(t)\rangle^{-\mu-j} + \langle t\rangle^{-\mu-j}\big\}\langle t\rangle^j \\ &\leq C(\langle x_0;t\rangle^{-\mu} + \langle t\rangle^{-\mu}) \leq C'\langle x_0\rangle\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu} \end{aligned}$$

as above. □

In the following, the next combined estimates are sometimes useful.

Corollary 2.11. *Let x_0, ξ_0, $y(t)$, $\xi(t)$ as in Lemma 2.9. Then for any $\alpha, \beta \in \mathbb{Z}^d_+$, there is $C_{\alpha>\beta}0$ such that*

$$\big|(\partial_y^\alpha\partial_\xi^\beta q)(t;y(t),\xi(t))\big| \leq C_{\alpha\beta}\langle x_0\rangle^{1-|\alpha|}\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu}, \quad \pm t \geq 0.$$

Proof. If $\alpha = 0$, the claim is the same as (i) in Lemma 2.10. If $\alpha \neq 0$, then

$$\langle x_0;t\rangle^{-\mu-|\alpha|} = \langle x_0;t\rangle^{1-|\alpha|}\langle x_0;t\rangle^{-1}\langle x_0;t\rangle^{-\mu} \leq \langle x_0\rangle^{1-|\alpha|}\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu},$$

since $1-|\alpha| \leq 0$, $\mu > 0$. Thus the claim follows from (ii) of Lemma 2.10. □

We also use the following elementary estimate repeatedly.

Lemma 2.12. *Let $0 < \mu < 1$. Then there is $C > 0$ such that*

$$\int_0^\infty \langle a;t\rangle^{-1}\langle t\rangle^{-\mu}dt \le C\langle a\rangle^{-\mu}, \quad a \ge 0.$$

Proof. Suppose $a \ge 1$. Then

$$\begin{aligned}\int_0^\infty \langle a;t\rangle^{-1}\langle t\rangle^{-\mu}dt &\le \sqrt{2}\int_0^\infty (a+t)^{-1}t^{-\mu}dt \\ &= \sqrt{2}\int_0^\infty (1+s)^{-1}(as)^{-\mu}ds \\ &= ca^{-\mu} \le 2^{\mu/2}c\langle a\rangle^{-\mu}\end{aligned}$$

with a constant $c > 0$. If $0 < a \le 1$, then

$$\int_0^\infty \langle a;t\rangle^{-1}\langle t\rangle^{-\mu}dt \le \int_0^\infty \langle t\rangle^{-1-\mu}dt \le C \le \sqrt{2}C\langle a\rangle^{-\mu}.$$

□

Lemma 2.13. (i) *There is a constant $C > 0$ such that*

$$|\partial_{x_0}y(t)| \le C, \quad |\partial_{x_0}\xi(t)| \le C\langle x_0\rangle^{-1-\mu},$$
$$|\partial_{\xi_0}y(t)| \le C\langle x_0\rangle^{1-\mu}, \quad |\partial_{\xi_0}\xi(t)| \le C,$$

uniformly for $\pm t \ge 0$, $(x_0,\xi_0) \in \Omega_{I_5,\pm}(\beta)$. Moreover,

$$|\partial_{x_0}y(t) - \mathrm{E}| \le C\langle x_0\rangle^{-\mu}, \quad |\partial_{\xi_0}\xi(t) - \mathrm{E}| \le C\langle x_0\rangle^{-\mu}.$$

(ii) *For $\alpha,\beta \in \mathbb{Z}_+^d$, $|\alpha+\beta| \ge 2$, there is $C_{\alpha\beta} > 0$ such that*

$$|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta y(t)| \le C_{\alpha\beta}\langle x_0\rangle^{1-|\alpha|-\mu}, \quad |\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi(t)| \le C_{\alpha\beta}\langle x_0\rangle^{-|\alpha|-\mu},$$

uniformly for $\pm t \ge 0$, $(x_0,\xi_0) \in \Omega_{I_5,\pm}(\beta)$.

Proof. We recall that estimates for $\xi(t)$ have already proved in Lemma 2.6. As before, we consider the "+"-case only.

(i) By the Hamilton equation, we have

$$\partial_t(\partial_z y) = (\partial_y\partial_\xi q)\partial_z y + (\partial_\xi\partial_\xi q)\partial_z\xi,$$

where $z = x_0$ or ξ_0. By Lemma 2.10 and Lemma 2.6, we learn

$$\begin{aligned}\partial_t|\partial_{x_0}y| &\le C\langle x_0;t\rangle^{-1-\mu}|\partial_{x_0}y| + C\langle x_0\rangle\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu}\langle x_0\rangle^{-1-\mu} \\ &\le C\langle t\rangle^{-1-\mu}|\partial_{x_0}y| + C\langle t\rangle^{-1-\mu}\langle x_0\rangle^{-\mu}\end{aligned} \tag{2.14}$$

with $\partial_{x_0}y(0) = \mathrm{E}$. Then by Gronwall's inequality, we have $|\partial_{x_0}y(t)| \le C$ uniformly in $t \ge 0$. Substituting this to (2.14) again to learn

$$\partial_t|\partial_{x_0}y(t)| \le C\langle x_0;t\rangle^{-1-\mu} + C\langle x_0\rangle^{-\mu}\langle t\rangle^{-1-\mu},$$

and this implies $|\partial_{x_0}y(t) - \mathrm{E}| \le C\langle x_0\rangle^{-\mu}$, thanks to Lemma 2.12.

Similarly, we have

$$\begin{aligned}\partial_t|\partial_{\xi_0}y| &\le C\langle x_0;t\rangle^{-1-\mu}|\partial_{\xi_0}y| + C\langle x_0\rangle\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu} \\ &\le C\langle t\rangle^{-1-\mu}|\partial_{\xi_0}y| + C\langle x_0\rangle\langle x_0;t\rangle^{-1}\langle t\rangle^{-\mu}\end{aligned}$$

with $\partial_{x_0} y(0) = 0$. Then by Gronwall's inequality and Lemma 2.12, we learn

$$|\partial_{\xi_0} y(t)| \leq C\langle x_0\rangle^{1-\mu}, \quad t \geq 0.$$

(ii) We prove the claim by induction in $k = |\alpha+\beta|$. We use the induction hypothesis:

(D_k) : For $\alpha, \beta \in \mathbb{Z}_+^d$, $|\alpha+\beta| \leq k$, there is $C_{\alpha\beta} > 0$ such that

$$\left|\partial_{x_0}^\alpha \partial_{\xi_0}^\beta y(t)\right| \leq C_{\alpha\beta}\langle x_0\rangle^{1-|\alpha|}, \quad t \geq 0.$$

uniformly for $(x_0, \xi_0) \in \Omega_{I_5}$ such that $\cos(x_0, v(\xi_0)) \geq \beta$.

We note (D_0) is an immediate consequence of Lemma 2.9, and (D_1) is proved in (i). We also recall

$$\left|\partial_{x_0}^\alpha \partial_{\xi_0}^\beta \xi(t)\right| \leq C'_{\alpha\beta}\langle x_0\rangle^{-|\alpha|}$$

with some constant $C'_{\alpha\beta} > 0$ for all $\alpha, \beta \in \mathbb{Z}_+^d$ (including $\alpha = 0$).

Suppose (D_{N-1}) holds, and let $|\alpha+\beta| = N$. As in the proof of Lemma 2.6, we use the Leibniz formula:

$$\begin{aligned}\partial_t(\partial_{x_0}^\alpha \partial_{\xi_0}^\beta y) &= (\partial_\xi\partial_y q)\partial_{x_0}^\alpha \partial_{\xi_0}^\beta y + (\partial_\xi\partial_\xi q)\partial_{x_0}^\alpha \partial_{\xi_0}^\beta \xi \\ &\quad + \sum_* c_*(\partial_\xi \partial_y^\gamma \partial_\xi^\delta q) \prod_{i=1}^d \left(\prod_{j=1}^{\gamma_i} (\partial_{x_0}^{a(i,j)} \partial_{\xi_0}^{b(i,j)} y_i) \prod_{k=1}^{\delta_i} (\partial_{x_0}^{\tilde a(i,k)} \partial_{\xi_0}^{\tilde b(i,k)} \xi_i) \right),\end{aligned}$$

where the last sum is taken over $\gamma, \delta, a(i,j), \tilde a(i,j), b(i,k), \tilde b(i,k) \in \mathbb{Z}_+^d$ such that $|\gamma+\delta| \geq 2$; $a(i,j)+b(i,j) \neq 0$ for $i = 1, \dots, d$, $j = 1, \dots, \gamma_i$; $\tilde a(i,k)+\tilde b(i,k) \neq 0$ for $i = 1, \dots, d$, $k = 1, \dots, \delta_i$; and

$$\sum_{i=1}^d \left(\sum_{j=1}^{\gamma_i} a(i,j) + \sum_{k=1}^{\delta_i} \tilde a(i,k) \right) = \alpha, \quad \sum_{i=1}^d \left(\sum_{j=1}^{\gamma_i} b(i,j) + \sum_{k=1}^{\delta_i} \tilde b(i,k) \right) = \beta.$$

c_* denotes suitable universal constant for each index. We denote the last term by R_3. By the induction hypothesis and Corollary 2.11, we learn

$$|R_3| \leq C\langle x_0\rangle^{1-|\alpha|}\langle x_0; t\rangle^{-1}\langle t\rangle^{-\mu}.$$

Hence we have

$$\begin{aligned}\partial_t\left|\partial_{x_0}^\alpha \partial_{\xi_0}^\beta y\right| &\leq C\langle x_0; t\rangle^{-1-\mu}\left|\partial_{x_0}^\alpha \partial_{\xi_0}^\beta y\right| + C\langle x_0\rangle\langle x_0; t\rangle^{-1}\langle t\rangle^{-\mu}\langle x_0\rangle^{-|\alpha|} \\ &\quad + C\langle x_0\rangle^{1-|\alpha|}\langle x_0; t\rangle^{-1}\langle t\rangle^{-\mu} \\ &\leq C\langle t\rangle^{-1-\mu}\left|\partial_{x_0}^\alpha \partial_{\xi_0}^\beta y\right| + C'\langle x_0\rangle^{1-|\alpha|}\langle x_0; t\rangle^{-1}\langle t\rangle^{-\mu}\end{aligned}$$

with $\partial_{x_0}^\alpha \partial_{\xi_0}^\beta y(0) = 0$. Thus by Gronwall's inequality and Lemma 2.12 again, we have

$$|\partial_{x_0}^\alpha \partial_{\xi_0}^\beta y(t)| \leq C\langle x_0\rangle^{1-|\alpha|-\mu},$$

which proves the induction step (D_N), and also completes the proof. □

Corollary 2.14. *Under the assumptions of Lemma 2.9,*

$$|y(x_0, \xi_0, t) - x_0| \leq C|x_0|^{1-\mu}$$

uniformly in $\pm t \geq 0$, $(x_0, \xi_0) \in \Omega_{I_5,\pm}(\beta)$.

Proof. We note $y(0,\xi_0,t)=0$ for all $t\in\mathbb{R}$ by the definition. Hence,

$$\begin{aligned}|y(x_0,\xi_0,t)-x_0| &= \left|\int_0^1 \frac{d}{ds}(y(sx_0,\xi_0,t)-sx_0)ds\right| \\ &= \left|\int_0^1 x_0\cdot\big\{(\partial_{x_0}y)(sx_0,\xi_0,t)-\mathrm{E}\big\}ds\right| \\ &\le C|x_0|\int_0^1 \langle sx_0\rangle^{-\mu}ds \le C|x_0|\int_0^1 |sx_0|^{-\mu}ds \\ &= C|x_0|^{1-\mu}\int_0^1 s^{-\mu}ds = C'|x_0|^{1-\mu}.\end{aligned}$$

□

2.5. **Solutions to Hamilton-Jacobi equation in the interaction picture.** In this subsection, we construct a solution to the Hamilton-Jacobi equation in the interaction picture:

$$\partial_t\psi(t,x_0,\xi)=q(t,\partial_\xi\psi(t,x_0,\xi),\xi),\quad \psi(0,x_0,\xi)=x_0\cdot\xi,$$

and study its properties. We write,

$$\Lambda_t^{x_0}\ :\ \xi_0\mapsto\xi(t,x_0,\xi_0)\in\mathbb{R}^d$$

for $(x_0,\xi_0)\in\Omega_{I_5}$ and $t\in\mathbb{R}$. By Lemma 2.1, $\Lambda_t^{x_0}$ is a diffeomorphism from $\Omega_{I_4}^0$ into a subset of $\Omega_{I_5}^0$, which contains $\Omega_{I_3}^0$ for each $t\in\mathbb{R}$, $x_0\in\mathbb{R}^d$, and the inverse has uniformly bounded Jacobian matrix on $\Omega_{I_3}^0$.

We set

$$\varphi(t,x_0,\xi_0)=\int_0^t\big\{q(s,y(s),\xi(s))-y(s)\cdot\partial_y q(s,y(s),\xi(s))\big\}ds+x_0\cdot\xi_0,$$

where

$$y(t)=y(x_0,\xi_0;t),\quad \xi(t)=\xi(x_0,\xi_0;t),\quad t\in\mathbb{R}.$$

Then by the standard theory of the Hamilton-Jacobi equation,

$$\psi(t,x_0,\xi)=\varphi(t,x_0,(\Lambda_t^{x_0})^{-1}(\xi))$$

satisfies the above Hamilton-Jacobi equation and the initial condition. We also recall that $\psi(t,x,\xi)$ is the generating function of the evolution (see, e.g., [2] §47),

$$w_t\ :\ (x_0,\xi_0)\mapsto(y(t),\xi(t)),$$

namely, we have

$$w_t\ :\ \begin{pmatrix}x\\ \partial_x\psi(t,x,\xi)\end{pmatrix}\mapsto\begin{pmatrix}\partial_\xi\psi(t,x,\xi)\\ \xi\end{pmatrix}.$$

Then, the conservation of the energy is expressed as

$$p(x,\partial_x\psi(t,x,\xi))=p(\partial_\xi\psi(t,x,\xi)+\partial_\xi\phi(t,\xi),\xi), \tag{2.15}$$

provided $(x_0,\partial_x\psi(t,x,\xi))\in\Omega_{I_4}$.

We show $\psi(t,x,\xi)$ is a good symbol on $\Omega_{I_3,\pm}(\beta)$ for $\pm t\ge 0$, respectively.

Lemma 2.15. *For $\alpha,\beta\in\mathbb{Z}_+^d$, there is $C_{\alpha\beta}>0$ such that*

$$\big|\partial_{x_0}^\alpha\partial_\xi^\beta\big((\Lambda_t^{x_0})^{-1}(\xi)\big)\big|\le C_{\alpha\beta}\langle x_0\rangle^{-|\alpha|},$$

uniformly in x_0, ξ, $\pm t \geq 0$, *provided* $\xi = \Lambda_t^{x_0}(\xi_0)$ *with* $(x_0, \xi_0) \in \Omega_{I_5,\pm}(\beta)$. *Moreover,*

$$\left|\partial_{x_0}^\alpha \partial_\xi^\beta \big((\Lambda_t^{x_0})^{-1}(\xi) - \xi\big)\right| \leq C_{\alpha\beta} \langle x_0 \rangle^{-|\alpha|-\mu}$$

under the same conditions.

Proof. By Lemma 2.6 and the definition of $\Lambda_t^{x_0}$, we learn

$$\partial_\xi^\alpha \partial_{\xi_0}^\beta (\Lambda_t^{x_0}(\xi)) = O(\langle x_0 \rangle^{-|\alpha|}),$$

and

$$\partial_{x_0}^\alpha \partial_\xi^\beta (\Lambda_t^{x_0}(\xi) - \xi) = O(\langle x_0 \rangle^{-|\alpha|-\mu}).$$

(Note we did not prove the estimate in Lemma 2.6 for $\alpha = \beta = 0$, but this is easily shown as well.) By Lemma 2.1, $(\partial\xi/\partial\xi_0)^{-1}$ is uniformly bounded, and hence by the standard formulas of the derivatives of inverse map, we obtain the claim. □

In particular, we learn

$$\left|\partial_{x_0}^\alpha \partial_\xi^\beta \big(x_0 \cdot (\Lambda_t^{x_0})^{-1}(\xi) - x_0 \cdot \xi\big)\right| \leq C'_{\alpha\beta} \langle x_0 \rangle^{1-\mu-|\alpha|}.$$

Lemma 2.16. *For* $\alpha, \beta \in \mathbb{Z}_+^d$, *there is* $C_{\alpha\beta} > 0$ *such that*

$$\left|\partial_{x_0}^\alpha \partial_{\xi_0}^\beta \big(\varphi(t, x_0, \xi_0) - x_0 \cdot \xi_0\big)\right| \leq C_{\alpha\beta} \langle x_0 \rangle^{1-\mu-|\alpha|},$$

uniformly for $(x_0, \xi_0) \in \Omega_{I_5,\pm}(\beta)$, $\pm t \geq 0$, *respectively.*

Proof. We write

$$\ell(t, y, \xi) = q(t, y, \xi) - y \cdot \partial_y q(t, y, \xi).$$

Then by Lemma 2.10 (or by Corollary 2.11), we learn, for any $\alpha, \beta \in \mathbb{Z}_+^d$,

$$\left|(\partial_y^\alpha \partial_\xi^\beta \ell)(t, y(t), \xi(t))\right| \leq C \langle x_0 \rangle^{1-|\alpha|} \langle x_0; t \rangle^{-1} \langle t \rangle^{-\mu}.$$

Combining these with Lemma 2.13, we have, for any $\alpha, \beta \in \mathbb{Z}_+^d$,

$$\left|\partial_{x_0}^\alpha \partial_{\xi_0}^\beta (\ell(t, y(t), \xi(t)))\right| \leq C \langle x_0 \rangle^{1-|\alpha|} \langle x_0; t \rangle^{-1} \langle t \rangle^{-\mu}$$

with some $C > 0$. Integrating this in t, and using Lemma 2.12, we obtain the claim. □

Combining these two lemmas, we obtain the following estimate:

Lemma 2.17. *For* $\alpha, \beta \in \mathbb{Z}_+^d$, *there is* $C_{\alpha\beta} > 0$ *such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta \big(\psi(t, x, \xi) - x \cdot \xi\big)\right| \leq C_{\alpha\beta} \langle x \rangle^{1-\mu-|\alpha|},$$

uniformly in x, ξ, $\pm t \geq 0$, *provided* $(x, \xi_0) \in \Omega_{I_5,\pm}(\beta)$ *with* $\xi = \Lambda_t^x(\xi_0)$.

We address the conditions on the domain in the above lemma later. We here note that the condition is satisfied if $(x, \xi) \in \Omega_{I_3,\pm}(\beta')$ with $\beta' > \beta$ and $|x| \gg 0$.

Remark 2.1. In the above results, we concentrate on the properties of functions on $\Omega_{I_5,\pm}(\beta)$. These functions are defined globally (provided $(x, \xi) \in \Omega_{I_5}$), and they are smooth. The same analysis can be easily carried out locally in (x, ξ) in a neighborhood of any arbitrarily fixed point in Ω_{I_2}. Actually, by Lemma 2.1, we can show $\partial_x^\alpha \partial_\xi^\beta \psi(t; x, \xi)$ is uniformly bounded by $O(\langle x \rangle)$, but it does not satisfy the above properties globally.

2.6. **Classical wave maps and their generating functions.** We have already seen in Subsection 2.4 that $\lim_{t\to\pm\infty}(y(t),\xi(t))$ exist. We denote them by

$$w_\pm : (x_0,\xi_0) \mapsto (x_\pm,\xi_\pm) = \lim_{t\to\pm\infty}(y(t),\xi(t))$$

and we call them *classical (inverse) wave maps*. By the results (and the proof) in Subsection 2.4, we can easily show:

Lemma 2.18. *$x_\pm$, $\xi_\pm$ are smooth functions of $(x_0,\xi_0)\in\Omega_{I_5}$, and for any $\alpha,\beta\in\mathbb{Z}_+^d$,*

$$\lim_{t\to\pm\infty}\partial_{x_0}^\alpha\partial_{\xi_0}^\beta y(t) = \partial_{x_0}^\alpha\partial_{\xi_0}^\beta x_\pm, \quad \lim_{t\to\pm\infty}\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi(t) = \partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi_\pm.$$

Moreover, for $(x_0,\xi_0)\in\Omega_{I_5,\pm}(\beta)$,

$$\begin{aligned} \big|\partial_{x_0}x_\pm - \mathrm{E}\big| &\le C\langle x_0\rangle^{-\mu}, \quad \big|\partial_{\xi_0}x_\pm\big| \le C\langle x_0\rangle^{-1-\mu}, \\ \big|\partial_{x_0}\xi_\pm\big| &\le C\langle x_0\rangle^{-1-\mu}, \quad \big|\partial_{\xi_0}\xi_\pm - \mathrm{E}\big| \le C\langle x_0\rangle^{-\mu}, \end{aligned}$$

and

$$\big|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta x_\pm\big| \le C_{\alpha\beta}\langle x_0\rangle^{1-|\alpha|-\mu}, \quad \big|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi_\pm\big| \le C_{\alpha\beta}\langle x_0\rangle^{-|\alpha|-\mu}$$

if $|\alpha+\beta|\ge 2$. The convergence is uniform in the following sense:

$$\lim_{t\to\pm\infty}\sup_{(x_0,\xi_0)\in\Omega_{I_2,\pm}(\beta)}\langle x_0\rangle^{-1+|\alpha|}\big|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta y(t) - \partial_{x_0}^\alpha\partial_{\xi_0}^\beta x_\pm\big| = 0,$$

$$\lim_{t\to\pm\infty}\sup_{(x_0,\xi_0)\in\Omega_{I_2,\pm}(\beta)}\langle x_0\rangle^{|\alpha|}\big|\partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi(t) - \partial_{x_0}^\alpha\partial_{\xi_0}^\beta \xi_\pm\big| = 0,$$

We also have the limit of the generating function of w_t:

$$\psi_\pm(x,\xi) = \lim_{t\to\pm\infty}\psi(t,x,\xi).$$

Lemma 2.19. *$\psi_\pm(x,\xi)$ is smooth functions of $(x,\xi)\in\Omega_{I_4}$ and and for any $\alpha,\beta\in\mathbb{Z}_+^d$,*

$$\lim_{t\to\pm\infty}\partial_x^\alpha\partial_\xi^\beta\psi(t,x,\xi) = \partial_x^\alpha\partial_\xi^\beta\psi_\pm(x,\xi).$$

Moreover,

$$\big|\partial_x^\alpha\partial_\xi^\beta\big(\psi_\pm(x,\xi) - x\cdot\xi\big)\big| \le C_{\alpha\beta}\langle x\rangle^{1-\mu-|\alpha|}, \quad (x,\xi)\in\tilde\Omega_{I_5,\pm}(\beta),$$

where

$$\tilde\Omega_{J,\pm}(\beta) = \big\{(x,\xi)\;\big|\;(x,\partial_x\psi_\pm(x,\xi))\in\Omega_{J,\pm}(\beta)\big\}.$$

The convergence is uniform in the sense:

$$\lim_{t\to\pm\infty}\sup_{(x,\xi)\in\tilde\Omega_{I_5,\pm}(\beta)}\langle x\rangle^{-1+\mu+|\alpha|}\big|\partial_x^\alpha\partial_\xi^\beta\psi(t,x,\xi) - \partial_x^\alpha\partial_\xi^\beta\psi_\pm(x,\xi)\big| = 0.$$

We also note that $\psi_\pm$ is the generating function of $w_\pm$, i.e.,

$$w_\pm \;:\; \begin{pmatrix} x \\ \partial_x\psi_\pm(x,\xi)\end{pmatrix} \mapsto \begin{pmatrix}\partial_\xi\psi_\pm(x,\xi) \\ \xi\end{pmatrix}.$$

The energy conservation is

$$\begin{aligned} p(x,\partial_\xi\psi_\pm(x,\xi)) &= \lim_{t\to\pm\infty} p(y(t)+\partial_\xi\phi(t,\xi),\xi) \\ &= \lim_{t\to\pm\infty}\big(p_0(\xi) + V_R(y(t)+\partial_\xi\phi(t,\xi),\xi)\big), \end{aligned}$$

but since $y(t)$ is uniformly bounded and $|\partial_\xi \phi(t,\xi)| \to \infty$, we learn that the right hand side converges to $p_0(\xi)$. Thus $\psi_\pm(x,\xi)$ are solutions to the eikonal equation:

$$p(x, \partial_x \psi_\pm(x,\xi)) = p_0(\xi), \quad (x,\xi) \in \tilde{\Omega}_{I_5,\pm}(\beta). \tag{2.16}$$

Finally, we consider the definition domain of the generating function $\psi_\pm(x,\xi)$, i.e., $\tilde{\Omega}_{I_5,\pm}(\beta)$.

Lemma 2.20. *Let $\beta' > \beta$. Then there is $L > 0$ such that*

$$\Omega^0_{I_4,\pm}(\beta', L) \subset \tilde{\Omega}_{I_5,\pm}(\beta),$$

where

$$\Omega^0_{J,\pm}(\gamma, L) = \big\{(x,\xi) \bigm| p_0(\xi) \in J, |x| \ge L, \pm\cos(x, v(\xi)) > \gamma\big\}$$

for $J \subset \mathbb{R}$, $L > 0$ and $\gamma > -1$.

Proof. For a given $(x,\xi) \in \Omega^0_{I_4,\pm}(\beta', L)$, it suffices to find $(x,\xi_0) \in \tilde{\Omega}_{I_5,\pm}(\beta)$ such that $\xi = \xi_\pm(x,\xi_0)$. Thus we find a inverse map of $\xi_0 \mapsto \xi = \xi_\pm(x,\xi_0)$ for such (x,ξ). We construct the inverse map by, for example, the contraction mapping. For a fixed (x,ξ), we set

$$F_\pm(\eta) = \xi - (\xi_\pm(x,\eta) - \eta), \quad \eta \in \Omega_{I_5},$$

and then $F_\pm(\eta) = \eta$ if and only if $\xi = \xi_\pm(x,\xi)$. We note, by the construction of $\xi_\pm$ and Lemma 2.18, we have

$$|\xi_\pm(x,\eta) - \eta| \le C\langle x\rangle^{-\mu}, \quad |\partial_\eta \xi_\pm(x,\eta) - \mathrm{E}| \le C\langle x\rangle^{-\mu},$$

uniformly for $(x,\eta) \in \Omega_{I_5,\pm}(\beta')$. Thus, if $|x| \ge L$ is sufficiently large, $F_\pm$ is a contraction map in a small ball with the center at ξ_0 which is contained in $\Omega^0_{I_5}$ and $\{\eta \,|\, \pm\cos(x,\eta) > \beta\}$. Thus we can apply the fixed point theorem to conclude the existence of the fixed point. This implies the assertion. □

The above lemma implies we can apply the result of Lemma 2.19 for $(x,\xi) \in \Omega^0_{I_4,\pm}$. We note that the phase function $\psi_\pm$ is well-defined on $\mathbb{R}^d \times \Omega^0_{I_4}$, though they do not enjoy the decay properties in Lemma 2.19 globally:

Lemma 2.21. *$\psi_\pm(x,\xi)$ is well-defined for $x \in \mathbb{R}^d$, $\xi \in \Omega^0_{I_5}$, namely,*

$$\mathbb{R}^d \times \Omega^0_{I_4} \subset \tilde{\Omega}_{I_5,\pm} = \big\{(x,\xi) \bigm| (x, \partial\psi_\pm(x,\xi)) \in \Omega_{I_5}\big\}.$$

Proof. We consider the "+" case only. It suffices to show that for $x \in \mathbb{R}^d$, $\xi \in \Omega^0_{I_4}$ there is ξ_0 such that $\xi_+(x,\xi_0) = \xi$. We recall, by Lemma 2.1 and the condition on V_R, for each $x \in \mathbb{R}^d$ the map

$$\xi_0 \mapsto \xi_+(x_0,\xi_0) = \Lambda^{x_0}_+(x_0) = \lim_{t\to\infty} \Lambda^{x_0}_t(\xi_0)$$

is diffeomorphism from $\{\xi \,|\, p(x_0,\xi) \in I_5\}$ into $\Omega^0_{I_6}$, and the range covers $\Omega^0_{I_4}$. Hence the inverse map is well-defined on $\Omega^0_{I_4}$, and hence $x_0 = (\Lambda^{x_0}_+)^{-1}(\xi)$ satisfies the required property. We note that by the eikonal equation (2.16), we learn $p(x_0,\xi_0) = p_0(\xi) \in I_4$, and hence $(x_0,\xi_0) \in \Omega_{I_4}$. □

3. Time-independent modifiers

We construct the so-called Isozaki-Kitada modifiers, or time-independent modifiers, $J_\pm$, using solutions of eikonal equations $\psi_\pm(x,\xi)$ constructed in Section 2. We suppose $J_\pm$ has the form

$$J_\pm f(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i\psi_\pm(x,\xi)} b_\pm(x,\xi)\hat{f}(\xi)d\xi$$

for $f\in\mathcal{S}(\mathbb{R}^d)$, and the symbols $b_\pm(x,\xi)$ are elements of $S(1, dx^2/\langle x\rangle^2+d\xi^2)$, and supported in $\tilde\Omega_{I_4,\pm}(\beta)$ with some $-1<\beta\le 1$. Our construction is analogous to the one in Dereziński-Gérard [4] §4.15 and Robert [16], though the setting is more general. We construct $b_\pm$ in the rest of this section. We mostly consider the "+"-case. The other case can be handled similarly.

We suppose $a^\pm$ has the form

$$b_\pm(x,\xi) = \Theta_\pm(1 + a_1^\pm + a_2^\pm + \cdots),$$

where $\Theta_\pm \in S(1,g)$, $a_j^\pm \in S(\langle x\rangle^{-\mu-j}, g)$, $j = 1, 2, \dots,$ on $\tilde\Omega_{I_4,\pm}(\beta)$, where we denote $g = dx^2/\langle x\rangle^2 + d\xi^2$. We construct these symbols so that

$$HJ_\pm - J_\pm H_0 \sim 0$$

asymptotically as $|x|\to\infty$ in $\tilde\Omega_{I_4,\pm}(\beta)$. At first we prepare a formula to compute $HJ_\pm$:

Lemma 3.1. *Suppose $b_\pm \in S(\langle x\rangle^\nu, g)$, $\nu\in\mathbb{R}$, and supported in $\tilde\Omega_{I_4,\pm}(\beta)$ with some $\beta > -1$. Then*

$$\begin{aligned} &e^{-i\psi_\pm(x,\xi)} H\big[e^{i\psi_\pm(\cdot,\xi)} b_\pm(\cdot,\xi)\big] = p(x,\partial_x\psi_\pm(x,\xi))b_\pm(x,\xi)\\ &- \frac{i}{2}\partial_x\cdot\big((\partial_\xi p)(x,\partial_x\psi_\pm(x,\xi))\big)b_\pm(x,\xi) - i(\partial_\xi p)(x,\partial_x\psi_\pm(x,\xi))\cdot\partial_x b_\pm(x,\xi)\\ &+ r_\pm(x,\xi),\end{aligned}$$

with $r_\pm \in S(\langle x\rangle^{-2+\nu-\mu}, g)$. Moreover, $r_\pm$ are supported essentially in $\tilde\Omega_{I_4,\pm}(\beta)$, i.e., they decay rapidly in x, away from $\tilde\Omega_{I_4,\pm}(\beta)$.

Proof. We compute

$$\begin{aligned} e^{-i\psi_\pm(x,\xi)} H[e^{i\psi_\pm(\cdot,\xi)} b_\pm(\cdot,\xi)] &= e^{-i\psi_\pm(x,\xi)} p^W(x,D_x)[e^{i\psi_\pm(\cdot,\xi)} b_\pm(\cdot,\xi)]\\ &= (2\pi)^{-d}\iint e^{-i\psi_\pm(x,\xi)+i(x-y)\cdot\eta+i\psi_\pm(y,\xi)} p(\tfrac{x+y}{2},\eta) b_\pm(y,\xi)dyd\eta\\ &= (2\pi)^{-d}\iint e^{i(x-y)\cdot(\eta-\int_0^1\partial_x\psi_\pm(tx+(1-t)y,\xi)dt)} p(\tfrac{x+y}{2},\eta) b_\pm(y,\xi)dyd\eta\\ &= (2\pi)^{-d}\iint e^{i(x-y)\cdot\eta} p(\tfrac{x+y}{2},\eta+\Phi_\pm(x,y,\xi)) b_\pm(y,\xi)dyd\eta,\end{aligned}$$

where

$$\begin{aligned}\Phi_\pm(x,y,\xi) &= \int_0^1 \partial_x\psi_\pm(tx+(1-t)y,\xi)dt\\ &= \int_{-1/2}^{1/2}\partial_x\psi_\pm(\tfrac{x+y}{2}+t(x-y),\xi)dt.\end{aligned}$$

We note $\Phi_\pm(x,y,\xi)$ are even functions in $x-y$, and hence $(\partial_x-\partial_y)\Phi_\pm(x,y,\xi)=0$. Moreover, we have

$$\Phi_\pm(x,y,\xi)-\xi=\int_{-1/2}^{1/2}(\partial_x\psi_\pm(\tfrac{x+y}{2}+t(x-y),\xi)-\xi)dt$$

and hence, by Lemma 2.17,

$$\left|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma(\Phi_\pm(x,y,\xi)-\xi)\right|\le C_{\alpha\beta\gamma}\langle x+y\rangle^{-\mu-|\alpha|-|\beta|},\quad \text{if } \left|\frac{x}{|x|}-\frac{y}{|y|}\right|<\delta\ll 1. \tag{3.1}$$

We now use Taylor expansion in η to learn

$$\begin{aligned}p(\tfrac{x+y}{2},\eta+\Phi(x,y,\xi))&=p(\tfrac{x+y}{2},\Phi(x,y,\xi))+\eta\cdot(\partial_\xi p)(\tfrac{x+y}{2},\Phi(x,y,\xi))\\&\quad+\frac12\int_0^1\sum_{j,k}\eta_j\eta_k(\partial_{\xi_j}\partial_{\xi_k}p)(\tfrac{x+y}{2},t\eta+\Phi(x,y,\xi))dt.\end{aligned}$$

We substitute this to the above equation to obtain

$$e^{-i\psi_\pm(x,\xi)}H[e^{i\psi_\pm(\cdot,\xi)}b_\pm(\cdot,\xi)]=I_1+I_2+I_3,$$

where

$$\begin{aligned}\mathrm{I}_1&=(2\pi)^{-d}\iint e^{i(x-y)\cdot\eta}p(\tfrac{x+y}{2},\Phi(x,y,\xi))b_\pm(y,\xi)dyd\eta,\\ \mathrm{I}_2&=(2\pi)^{-d}\iint e^{i(x-y)\cdot\eta}\eta\cdot(\partial_\xi p)(\tfrac{x+y}{2},\Phi(x,y,\xi))b_\pm(y,\xi)dyd\eta,\\ \mathrm{I}_3&=(2\pi)^{-d}\iint\int_0^1\sum_{j,k}e^{i(x-y)\cdot\eta}\eta_j\eta_k(\partial_{\xi_j}\partial_{\xi_k}p)(\tfrac{x+y}{2},t\eta+\Phi(x,y,\xi))\times\\&\qquad\times b_\pm(y,\xi)dtdyd\eta.\end{aligned}$$

By oscillatory integrations, we have

$$\begin{aligned}\mathrm{I}_1&=p(x,\partial_x\psi_\pm(x,\xi))b_\pm(x,\xi),\\ \mathrm{I}_2&=-(2\pi)^{-d}\iint e^{i(x-y)\cdot\eta}i\partial_y\cdot\big\{(\partial_\xi p)(\tfrac{x+y}{2},\Phi(x,y,\xi))b_\pm(y,\xi)\big\}dyd\eta\\&=-\frac{i}{2}\partial_x\cdot\big\{(\partial_\xi p)(x,\partial_x\psi_\pm(x,\xi))\big\}b_\pm(x,\xi)-i(\partial_\xi p)(x,\partial_x\psi_\pm(x,\xi))\cdot\partial_x b_\pm(x,\xi).\end{aligned}$$

By virtue of (3.1), and using integration by parts, we also have $r_\pm=\mathrm{I}_3\in S(\langle x\rangle^{-2+\nu-\mu},g)$. It is easy to observe $r_\pm$ are essentially supported in $\tilde\Omega_{I_2,\pm}(\beta)$. $\square$

We now compute the 0-th order term $\Theta_\pm(x,\xi)$ in the above setting. This factor is actually the well-known volume factor in the WKB analysis.

Lemma 3.2. *Let* $\Theta_\pm(x,\xi)=\Big(\det\Big(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\Big)\Big)^{1/2}$, *then* $\Theta_\pm$ *satisfies*

$$\frac12\partial_x\cdot\big\{(\partial_\xi p)(x,\partial_x\psi_\pm(x,\xi))\big\}\Theta_\pm(x,\xi)+(\partial_\xi p)(x,\partial_x\psi_\pm(x,\xi))\cdot\partial_x\Theta_\pm(x,\xi)=0.$$

Moreover, $\Theta_\pm-1\in S(\langle x\rangle^{-\mu},g)$ *on* $\tilde\Omega_{I_4,\pm}(\beta)$.

Proof. By differentiating the eikonal equation (2.16) in ξ_j, we learn

$$\sum_{k=1}^d\partial_{\xi_j}\partial_{x_k}\psi_\pm(x,\xi)\partial_{\xi_k}p(x,\partial_x\psi_\pm(x,\xi))-\partial_{\xi_j}p_0(\xi)=0.$$

Then we differentiate this in x_i:

$$\sum_{k=1}^{d} \partial_{\xi_k} p\, \partial_{x_k}(\partial_{x_i}\partial_{\xi_j}\psi_\pm) + \sum_{k=1}^{d} \partial_{\xi_j}\partial_{x_k}\psi_\pm\, \partial_{x_i}(\partial_{\xi_k} p(x,\partial_x\psi_\pm)) = 0.$$

We write this in matrix form to obtain

$$\sum_{k=1}^{d} (\partial_{\xi_k} p) \frac{\partial}{\partial x_k}\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right) + \left[\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial \xi}(x,\partial_x\psi_\pm)\right)\right]\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right) = 0.$$

Since $\partial_x\partial_\xi\psi_\pm$ is invertible (as a matrix), we have

$$\sum_{k=1}^{d} (\partial_{\xi_k} p) \left[\frac{\partial}{\partial x_k}\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right)\right]\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right)^{-1} + \left[\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial \xi}(x,\partial_x\psi_\pm)\right)\right] = 0.$$

Then we take the trace:

$$\sum_{k=1}^{d} (\partial_{\xi_k} p) \mathrm{Tr}\left[\left[\frac{\partial}{\partial x_k}\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right)\right]\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right)^{-1}\right] + \sum_{k=1}^{d}\frac{\partial}{\partial x_k}\left(\frac{\partial p}{\partial \xi_k}(x,\partial_x\psi_\pm)\right) = 0.$$

On the other hand, by the derivative formula for the determinant, we learn

$$\begin{aligned}\frac{\partial}{\partial x_k}\Theta_\pm(x,\xi) &= \frac{1}{2}\frac{\partial}{\partial x_k}\left[\det\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right)\right]\left(\det\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right)\right)^{-1/2}\\ &= \frac{1}{2}\mathrm{Tr}\left[\left[\frac{\partial}{\partial x_k}\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right)\right]\left(\frac{\partial^2\psi_\pm}{\partial x\partial\xi}\right)^{-1}\right]\Theta_\pm(x,\xi),\end{aligned}$$

and hence $\Theta_\pm$ satisfies

$$\partial_\xi p(x,\partial_x\psi_\pm)\cdot\partial_x\Theta_\pm + \frac{1}{2}\partial_x\cdot((\partial_\xi p)(x,\partial_x\psi_\pm))\Theta_\pm = 0.$$

The last claim follows from the observation: $(\partial_x\partial_\xi\psi_\pm - E)\in S(\langle x\rangle^{-\mu},g)$ on $\Omega_{I_4,\pm}(\beta)$. $\square$

Now we construct symbols of $J_\pm$. By Lemmas 3.1, 3.2, we learn, at least formally,

$$\begin{aligned}&e^{-i\psi_\pm(x,\xi)}H[e^{i\psi_\pm(\cdot,\xi)}\Theta_\pm(\cdot,\xi)] = p_0(\xi)\Theta_\pm(x,\xi)\\ &\quad -\frac{i}{2}\partial_x\cdot((\partial_\xi p)(x,\partial_\xi\psi_\pm(x,\xi)))\Theta_\pm(x,\xi) - i(\partial_\xi p)(x,\partial_x\psi_\pm(x,\xi))\cdot\partial_x\Theta_\pm(x,\xi)\\ &\quad + r_\pm(x,\xi)\\ &= p_0(\xi)\Theta_\pm(x,\xi) + r_{0,\pm}(x,\xi)\Theta_\pm(x,\xi)\end{aligned}$$

where $r_{0,\pm}\in S(\langle x\rangle^{-2-\mu},g)$ on $\Omega_{I_4,\pm}(\beta)$ with any $\beta>-1$. We note $\Theta_\pm$ do not satisfy the support property of Lemma 3.1, but we will introduce cutoff functions, and the following computations are readily justified. By the same computation, for $b_\pm\in S(\langle x\rangle^{-\nu},g)$, we have

$$\begin{aligned}&e^{-i\psi_\pm(x,\xi)}H[e^{i\psi_\pm(\cdot,\xi)}\Theta_\pm(\cdot,\xi)b_\pm(\cdot,\xi)] = p_0(\xi)\Theta_\pm(x,\xi)b_\pm(x,\xi)\\ &\quad - i(\partial_\xi p)(x,\partial_x\psi_\pm(x,\xi))\cdot(\partial_x b_\pm(x,\xi))\Theta_\pm(x,\xi) + \tilde r_{0,\pm}(x,\xi)\Theta_\pm(x,\xi)\end{aligned}$$

where $\tilde r_{0,\pm}\in S(\langle x\rangle^{-2-\mu-\nu},g)$ on $\Omega_{I_4,\pm}(\beta)$. Thus, if we set

$$a_1^\pm(x,\xi) = i\int_0^{\pm\infty} r_{0,\pm}(\exp tH_p(x,\partial_x\psi_\pm(x,\xi)))dt,$$

then they solve the equations:

$$(\partial_\xi p)(x, \partial_x \psi_\pm(x,\xi)) \cdot \partial_x a_1^\pm(x,\xi) = -ir_{0,\pm}(x,\xi),$$

and hence

$$\begin{aligned} &e^{-i\psi_\pm(x,\xi)} H[e^{i\psi_\pm(\cdot,\xi)}\Theta_\pm(\cdot,\xi)(1+a_1^\pm(\cdot,\xi))] \\ &= p_0(\xi)\Theta_\pm(x,\xi)(1+a_1^\pm(x,\xi)) + r_{1,\pm}(x,\xi)\Theta_\pm(x,\xi), \end{aligned}$$

where $r_{1,\pm} \in S(\langle x\rangle^{-3-\mu}, g)$ on $O_{I_4,\pm}(\beta)$. Moreover, $a_i^\pm$ satisfy the boundary condition: $a_1^\pm(x,\xi) \to 0$ as $|x| \to \infty$ in $\Omega_{I_4,\pm}(\beta)$. We note that if we set

$$(z(t), \zeta(t)) = \exp tH_p(x, \partial_x\psi_\pm(x,\xi)),$$

then $(z(t), \zeta(t))$ is the solution to the Hamilton equation with the boundary conditions $\zeta(t) \to \xi$ as $t \to \pm\infty$, and $z(0) = x$. Thus, by using Lemmas 2.17 and 2.6, we can show $a_1^\pm \in S(\langle x\rangle^{-1-\mu}, g)$ on $\Omega_{I_4,\pm}(\beta)$.

We iterate this procedure to construct $a_j^\pm(x,\xi)$, $j = 2, 3, \dots$. Namely, we set $r_{k,\pm}$ so that

$$\begin{aligned} &r_{k,\pm}(x,\xi)\Theta_\pm(x,\xi) \\ &\quad = e^{-i\psi_\pm} H[e^{i\psi_\pm}\Theta_\pm(1+a_1^\pm+\cdots+a_k^\pm)] - p_0(\xi)\Theta_\pm(1+a_1^\pm+\cdots+a_k^\pm) \\ &\quad \in S(\langle x\rangle^{-k-2-\mu}, g) \text{ on } \Omega_{I_4,\pm}(\beta). \end{aligned}$$

Then we solve the equation

$$(\partial_\xi p)(x, \partial_x\psi_\pm(x,\xi)) \cdot \partial_x a_{k+1}^\pm(x,\xi) = -ir_{k,\pm}(x,\xi),$$

with the boundary condition: $a_k^\pm(x,\xi) \to 0$ as $|x| \to \infty$ in $\Omega_{I_4,\pm}(\beta)$. The solutions are given by

$$a_{k+1}^\pm(x,\xi) = i\int_0^{\pm\infty} r_k^\pm(\exp tH_p(x, \partial_x\psi_\pm(x,\xi)))dt,$$

and we can show $a_{k+1}^\pm \in S(\langle x\rangle^{-k-1-\mu}, g)$ on $\Omega_{I_4,\pm}(\beta)$ with any $\beta > -1$.

We define $a^\pm(x,\xi)$ as an asymptotic sum of $1 + a_1^\pm + \cdots$, i.e., $a^\pm \in S(1, g)$ on $\Omega_{I_4,\pm}(\beta)$ such that for any $N \geq 1$,

$$a^\pm(x,\xi) - \left(1 + \sum_{j=1}^N a_j^\pm(x,\xi)\right) \in S(\langle x\rangle^{-N-2-\mu}, g) \text{ on } \Omega_{I_4,\pm}(\beta),$$

with arbitrary $\beta > -1$.

Then we introduce a cut-off to these symbols. Let $R_0 \gg 0$ and $-1 < \beta_{\pm,1} < \beta_{\pm,2} < 1$. We choose smooth functions $\chi_1(x)$, $\chi_2(\lambda)$ and $\chi_{3,\pm}(\sigma)$ such that

$$\chi_1(x) = \chi_1(|x|) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ 1 & \text{if } |x| \leq 2, \end{cases}$$

$$\chi_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in I_3, \\ 0 & \text{if } \lambda \notin I_4, \end{cases}$$

$$\chi_{3,\pm}(\sigma) = \begin{cases} 0 & \text{if } \sigma \leq \beta_{\pm,1}, \\ 1 & \text{if } \sigma \geq \beta_{\pm,2}, \end{cases}$$

and $0 \leq \chi_1(x), \chi_2(\lambda), \chi_{3,\pm}(\sigma) \leq 1$. We then set

$$\chi_\pm(x,\xi) = \chi_1(x/R_0)\chi_2(p_0(\xi))\chi_{3,\pm}(\pm\cos(x, v(\partial_x\psi_\pm(x,\xi)))).$$

We can now define our time-independent modifiers by

$$J_\pm f(x) = (2\pi)^{-d/2} \int e^{i\psi_\pm(x,\xi)} \Theta_\pm(x,\xi) \chi_\pm(x,\xi) a^\pm(x,\xi) \hat{f}(\xi) d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. On the support of the cut-off functions $\chi_\pm(x,\xi)$, the above formal computations can be readily justified, and we can show the following properties of $J_\pm$. We define interaction operators $G_\pm$ by

$$G_\pm = HJ_\pm - J_\pm H_0,$$

which are bounded operators on $L^2(\mathbb{R}^d)$.

Lemma 3.3. *There are symbols* $g_\pm(x,\xi) \in S(\langle x\rangle^{-1}, g)$ *such that*

$$G_\pm f(x) = (2\pi)^{-d/2} \int e^{i\psi_\pm(x,\xi)} \Theta_\pm(x,\xi) g_\pm(x,\xi) \hat{f}(\xi) d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. *Moreover,* $g_\pm$ *are essentially supported in* $\tilde{\Omega}_{I_4,\pm}(\beta_{\pm,1}) \setminus \tilde{\Omega}_{I_3,\pm}(\beta_{\pm,2})$, *i.e., for any* $\alpha, \beta \in \mathbb{Z}_+^d$ *and* N, *there is* $C_{\alpha\beta N} > 0$ *such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta g_\pm(x,\xi)\right| \le C_{\alpha\beta N} \langle x\rangle^{-N}, \quad (x,\xi) \notin \tilde{\Omega}_{I_4,\pm}(\beta_{\pm,1}) \setminus \tilde{\Omega}_{I_3,\pm}(\beta_{\pm,2}).$$

The principal symbols of $g_\pm(x,\xi)$ *are given by* $-i(\partial_\xi p)(x, \partial_\xi \psi_\pm(x,\xi)) \cdot \partial_x \chi_\pm(x,\xi)$, *i.e,*

$$g_\pm(x,\xi) - \left[-i(\partial_\xi p)(x, \partial_x \psi_\pm(x,\xi)) \cdot \partial_x \chi_\pm(x,\xi)\right] \in S(\langle x\rangle^{-2}, g).$$

4. Wave operators, scattering operators, and scattering matrix

We follows the argument of [12], and we mainly explain the necessary modifications. In the construction of $J_\pm$ in the last section, we choose $\beta_{\pm,i}$, $i = 1, 2$, such that

$$-1 < \beta_{+,1} = \beta_{-,1} < \beta_{+,2} = \beta_{-,2} < 0,$$

and fix them. We denote $\beta_i = \beta_{\pm,i}$, $i = 1, 2$. Using these modifiers $J_\pm$, we can now define wave operators with time-independent modifiers (or Isozaki-Kitada modifiers).

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} J_\pm e^{-itH_0}.$$

Then the existence of these limits are proved by the same method as in the papers by Isozaki-Kitada [8] or Robert [16], and $W_\pm$ are partial isometries on Ran $[E_{I_3}(H_0)]$. Moreover, the asymptotic completeness is also proved by the standard method:

$$\text{Ran}\,[W_\pm E_{I_3}(H_0)] = E_{I_3}(H)\mathcal{H}_c(H),$$

where $\mathcal{H}_c(H)$ is the continuous spectral subspace with respect to H. The scattering operator S (with essentially a smooth energy cut-off $\chi_2(H_0)$) is defined by

$$S = (W_+)^* W_-,$$

and it is an isometry on Ran $[E_{I_3}(H_0)]$. It is well-known that S commutes with the free Hamiltonian: $SH_0 = H_0 S$.

We recall a representation formula for the scattering matrix:

$$S(\lambda) = -2\pi i T(\lambda) J_+^* G_- T(\lambda)^* + 2\pi i T(\lambda) G_+^* (H - \lambda - i0)^{-1} G_- T(\lambda)^* \tag{4.1}$$

for $\lambda \in I$, which is due to Isozaki-Kitada [9] and Yafaev [19]. We give a proof of the formula in Appendix A for the completeness. The second term in the right hand side is a smoothing operator by virtue of the microlocal resolvent estimate of Isozaki-Kitada type [7, 10]. The resolvent estimate under our setting is proved in Nakamura [13]. Thus it remains to compute the first term as a Fourier integral operator.

We consider the oscillatory integral:

$$\mathcal{F}J_+^* G_- f(\xi) = (2\pi)^{-d} \iiint e^{-i\psi_+(x,\xi)+i\psi_-(x,\eta)-iy\cdot\eta}\Theta_+(x,\xi)\Theta_-(x,\eta)\times \\ \times \overline{a_+(x,\xi)}g_-(x,\eta)f(y)dyd\eta dx,$$

and we compute the integration in (x,η) using the stationary phase method. The stationary phase points are given by

$$\begin{aligned} &\partial_x(-\psi_+(x,\xi)+\psi_-(x,\eta)) = 0, \quad \text{i.e., } \partial_x\psi_+(x,\xi) = \partial_x\psi_-(x,\eta), \\ &\partial_\eta(\psi_-(x,\eta) - y\cdot\eta) = 0, \quad \text{i.e., } \partial_\eta\psi_-(x,\eta) = y. \end{aligned} \tag{4.2}$$

Thus these stationary points correspond to the map

$$\begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} \partial_\eta\psi_-(x,\eta) \\ \eta \end{pmatrix} \underset{w_-}{\longleftarrow} \begin{pmatrix} x \\ \partial_x\psi_-(x,\eta) \end{pmatrix} = \begin{pmatrix} x \\ \partial_x\psi_+(x,\xi) \end{pmatrix} \underset{w_+}{\longmapsto} \begin{pmatrix} \partial_\xi\psi_+(x,\xi) \\ \xi \end{pmatrix}.$$

These classical wave maps $w_\pm$ are local diffeomorphism, and the composition is also. For fixed (y,ξ), with $p_0(\xi)\in I$, we write the stationary phase points by

$$x = x(y,\xi), \quad \eta = \eta(y,\xi),$$

and we set

$$\psi(y,\xi) = \psi_+(x(y,\xi),\xi) - \psi_-(x(y,\xi),\eta(y,\xi)) + y\cdot\eta(y,\xi)$$

be the stationary phase. We can show by the construction of $\psi_\pm$ that $\psi(x,\xi) - x\cdot\xi \in S(\langle x\rangle^{1-\mu}, g)$ on $\big\{(x,\xi) \bigm| \beta_1 < \cos(x, v(\xi)) < -\beta_1\big\}$. Then, as is expected, $\psi(y,\xi)$ is the generating function of the classical scattering map : $w_+\circ w_-^{-1}$, i.e.,

$$\partial_y\psi(y,\xi) = \eta(y,\xi), \quad \partial_\xi\psi(y,\xi) = \partial_\xi\psi_+(x(y,\xi),\xi).$$

In fact, we have

$$\begin{aligned} \partial_y\psi(y,\xi) &= (\partial_y x)\partial_x\psi_+(x,\xi) - (\partial_y x)\partial_x\psi_-(x,\eta) \\ &\quad - (\partial_y\eta)\partial_\eta\psi_-(x,\eta) + (\partial_y\eta)y + \eta \\ &= (\partial_y x)(\partial_x\psi_+(x,\xi) - \partial_x\psi_-(x,\eta)) \\ &\quad - (\partial_y\eta)(\partial_\eta\psi_-(x,\eta) - y) + \eta \;\; = \eta \end{aligned}$$

by the stationary phase equations. Similarly we have

$$\begin{aligned} \partial_\xi\psi(y,\xi) &= \partial_\xi\psi_+(x,\xi) + (\partial_\xi x)(\partial_x\psi_+(x,\xi) - \partial_x\psi_-(x,\eta)) \\ &\quad - (\partial_\xi\eta)(\partial_\eta\psi_-(x,\eta) - y) \\ &= \partial_\xi\psi_+(x,\xi). \end{aligned}$$

In order to apply the stationary phase method, we need to compute the Hessian at the stationary phase points:

Lemma 4.1. *Let* $\mathrm{Hess}(y,\xi)$ *be the Hessian of* $-\psi_+(x,\xi)+\psi_-(x,\eta)-y\cdot\eta$ *with respect to* (x,η) *at the stationary points. Then*

$$\mathrm{Hess} = (-1)^d \det(\partial_x\partial_\xi\psi_-(x,\eta))\det(\partial_x\partial_\xi\psi_+(x,\xi))\det(\partial_y\partial_\xi\psi(y,\xi))^{-1}.$$

Proof. We compute

$$\begin{aligned} \mathrm{Hess} &= \det((\partial_x,\partial_\eta)^2(-\psi_+(x,\xi)+\psi_-(x,\eta)-y\cdot\eta))\big|_{x=x(y,\xi),\eta=\eta(y,\xi)} \\ &= \det\begin{pmatrix} -\partial_x\partial_x\psi_+(x,\xi)+\partial_x\partial_x\psi_-(x,\eta) & \partial_x\partial_\eta\psi_-(x,\eta) \\ \partial_\eta\partial_x\psi_-(x,\eta) & \partial_\eta\partial_\eta\psi_-(x,\eta) \end{pmatrix}\Bigg|_{x=x(y,\xi),\eta=\eta(y,\xi)}. \end{aligned}$$

It is easy to see

$$\begin{pmatrix} -\partial_x\partial_x\psi_+ + \partial_x\partial_x\psi_- & \partial_x\partial_\eta\psi_- \\ \partial_\eta\partial_x\psi_- & \partial_\eta\partial_\eta\psi_- \end{pmatrix} \begin{pmatrix} E & 0 \\ (\partial_x\partial_\eta\psi_-)^{-1}(\partial_x\partial_x\psi_+ - \partial_x\partial_x\psi_-) & E \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \partial_x\partial_\eta\psi_- \\ \partial_\eta\partial_x\psi_- + (\partial_\eta\partial_\eta\psi_-)(\partial_x\partial_\eta\psi_-)^{-1}(\partial_x\partial_x\psi_+ - \partial_x\partial_x\psi_-) & \partial_\eta\partial_\eta\psi_- \end{pmatrix},$$

and hence

$$\mathrm{Hess} = (-1)^d \det(\partial_x\partial_\eta\psi_-)\det\big(\partial_\eta\partial_x\psi_- + (\partial_\eta\partial_\eta\psi_-)(\partial_x\partial_\eta\psi_-)^{-1}(\partial_x\partial_x\psi_+ - \partial_x\partial_x\psi_-)\big).$$

Now we differentiate the stationary phase equation (4.2) in y to learn

$$(\partial_y x)\partial_x\partial_x\psi_+ = (\partial_y x)\partial_x\partial_x\psi_- + (\partial_y\eta)\partial_\eta\partial_x\psi_-, \tag{4.3}$$

$$(\partial_y x)\partial_x\partial_\eta\psi_- + (\partial_y\eta)\partial_\eta\partial_\eta\psi_- = E. \tag{4.4}$$

From (4.3), we have

$$\partial_y\eta = (\partial_y x)(\partial_x\partial_x\psi_+ - \partial_x\partial_x\psi_-)(\partial_\eta\partial_x\psi_-)^{-1}.$$

Substituting this to (4.4), we have

$$(\partial_y x)(\partial_x\partial_\eta\psi_- + (\partial_x\partial_x\psi_+ - \partial_x\partial_x\psi_-)(\partial_\eta\partial_x\psi_-)^{-1}\partial_\eta\partial_\eta\psi_-) = E,$$

and hence

$$(\partial_y x)^{-1} = \partial_x\partial_\eta\psi_- + (\partial_x\partial_x\psi_+ - \partial_x\partial_x\psi_-)(\partial_\eta\partial_x\psi_-)^{-1}\partial_\eta\partial_\eta\psi_-,$$

or

$${}^t(\partial_y x)^{-1} = \partial_\eta\partial_x\psi_- + (\partial_\eta\partial_\eta\psi_-)(\partial_x\partial_\eta\psi_-)^{-1}(\partial_x\partial_x\psi_+ - \partial_x\partial_x\psi_-).$$

Substituting this to the above formula on the Hessian, we learn

$$\mathrm{Hess} = \det(\partial_x\partial_\eta\psi_-(x,\eta))\cdot\det(\partial_y x(y,\xi))^{-1},$$

where $x = x(y,\xi)$, $\eta = \eta(y,\xi)$. If we set

$$z(y,\xi) = (\partial_\xi\psi_+)(x(y,\xi),\xi),$$

then, since ψ is the generating function of $w_+\circ w_-^{-1}$, we learn

$$\partial_y\partial_\xi\psi(y,\xi) = \partial_y z = (\partial_y x)\cdot(\partial_x\partial_\xi\psi_+)(x,\xi).$$

Combining these, we conclude the assertion. □

Now we denote $x(y,\xi)$ be the stationary point as above, and denote the corresponding momentum at $t=0$ by

$$\zeta(y,\xi) = \partial_x\psi_-(x(y,\xi),\eta(y,\xi)) = \partial_x\psi_+(x(y,\xi),\xi). \tag{4.5}$$

We also denote

$$\Theta(y,\xi) = \left|\det\Big(\tfrac{\partial^2\psi}{\partial y\partial\xi}(y,\xi)\Big)\right|^{1/2}.$$

Then using the stationary phase method and the standard oscillatory integral calculation, we have the following expression of $J_+^* G_-$ (see, e.g., Asada-Fujiwara [3] Section 3).

Lemma 4.2. *There is $Z(x,\xi)\in S(\langle x\rangle^{-1},g)$ such that*

$$\mathcal{F}J_+^*G_-f(\xi) = (2\pi)^{-d/2}\int e^{-i\psi(y,\xi)}\Theta(y,\xi)Z(y,\xi)f(y)dy.$$

Moreover, Z is essentially supported in

$$\Omega = \big\{(y,\xi)\ \big|\ p_0(\xi)\in I, \cos(x(y,\xi), v(\zeta(y,\xi)))\in[-\beta_2,-\beta_1]\big\},$$

i.e., for any $\alpha, \beta \in \mathbb{Z}_+^d$ *and* $N \geq 0$,

$$\left|\partial_y^\alpha \partial_\xi^\beta Z(y,\xi)\right| \leq C_{\alpha\beta N} \langle y\rangle^{-N}, \quad \textit{for } (y,\xi) \notin \Omega.$$

The principal symbol of $Z(y,\xi)$ *is given by*

$$Z_0(y,\xi) = \overline{a_+(x(y,\xi),\xi)} g_-(x(y,\xi), \partial_y\psi(y,\xi)),$$

i.e., $Z - Z_0 \in S(\langle x\rangle^{-2}, g)$.

In order to compute $T(\lambda)J_+^* G_- T(\lambda)^*$, we note the following basic property of the generating function $\psi(y,\xi)$, which essentially says $\psi(y,\xi)$ restricted to Σ_λ defines a canonical map on $T^*\Sigma_\lambda$. We recall that by the energy conservation, we have

$$p_0(\partial_y\psi(y,\xi)) = p_0(\xi), \quad \xi \in p_0^{-1}(I).$$

Lemma 4.3. *For* $p_0(\xi) = \lambda \in I$,

$$\psi(y + tv(\partial_y\psi(y,\xi)),\xi) = \psi(y,\xi), \quad t \in \mathbb{R}.$$

Proof. We choose a local coordinate near Σ_λ such that $p_0(\xi) = \lambda + \xi_1$ and hence

$$\Sigma_\lambda = \left\{(0,\xi') \,\middle|\, \xi' \in \mathbb{R}^{d-1}\right\}, \quad v(\xi) = \partial_\xi p_0(\xi) = (1,0,\dots,0)$$

in the neighborhood. We may suppose ξ and $\partial_y\psi(y,\xi)$ are contained in the neighborhood, and hence $v(\xi) = v(\partial_y\psi(y,\xi)) = (1,0,\dots,0)$. We note, since $\partial_y\psi(y,\xi) \in \Sigma_\lambda$, $\partial_{y_1}\psi(y,\xi) = 0$ in this coordinate. Thus we have

$$\begin{aligned}\partial_t\psi(y + tv(\partial_y\psi(y,\xi)),\xi) &= v(\partial_y\psi(y,\xi)) \cdot \partial_y\psi(y + tv(\partial_y\psi(y,\xi)),\xi)\\ &= \partial_{y_1}\psi(y + t(1,0,\dots,0),\xi) = 0.\end{aligned}$$

This implies the assertion. □

In the following, we consider Fourier integral operators defined on Σ_λ, and here we introduce several notations. We usually work in a local coordinate in Σ_λ, and since we are interested in the behavior of operators/symbols for large $|x|$, and hence we may suppose ξ, $\partial_y\psi(y,\xi)$, $\partial_x\psi_\pm(x,\xi)$, etc., are in the same local coordinate patch. For $\xi \in \Sigma_\lambda$, we identify the cotangent space at ξ: $T_\xi^*\Sigma_\lambda$ with $v(\xi)^\perp$, i.e., the orthogonal subspace of the normal vector $v(\xi) = \partial_\xi p_0(\xi)$, as usual. We employ the standard metric on $T_\xi^*\Sigma_\lambda$. For $a(x,\xi) \in C^\infty(T^*\Sigma_\lambda)$, we write $a \in S(m(x,\xi),\tilde{g})$ if for any multi-indices $\alpha, \beta \in \mathbb{Z}_+^{d-1}$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|} m(x,\xi), \quad x \in \mathbb{R}^{d-1}, \xi \in \Sigma_\lambda,$$

in the local coordinate. We note it is not always natural to consider $x \in T_\xi^*\Sigma_\lambda$ in the above expression, since we consider Fourier integral operators, and hence x may be better to be considered as an element in another cotangent space. In our case, here we consider in a local coordinate patch, and the condition is well-defined without ambiguities. By virtue of Lemma 4.3, we may define

$$\tilde{\psi}(y,\xi) = \psi(y,\xi), \quad \tilde{\Theta}(y,\xi) = \Theta(y,\xi)$$

on $T^*\Sigma_\lambda$ using the local coordinate, where y should be considered as an element of $T_\eta^*\Sigma_\lambda$ with $\eta = \partial_y\psi(y,\xi)$.

We compute the operator $T(\lambda)J_+^*G_-T(\lambda)^*$ using the local coordinate in the above proof. Then, as well as in the proof of Lemma 5.4 of [12], for $f\in C_0^\infty(\Sigma_\lambda)$ supported in the neighborhood, we have

$$\begin{aligned}
&T(\lambda)J_+^*G_-T(\lambda)^*f(\xi')\\
&= c_d\iint e^{-i\psi(y,(0,\xi'))+iy\cdot(0,\eta')}\Theta(y,(0,\xi'))Z(y,(0,\xi'))f(\eta')d\eta' dy\\
&= c_{d-1}\iint\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\psi((t,y'),(0,\xi'))+iy'\cdot\eta'}\times\right. \qquad (4.6)\\
&\qquad \left.\times\,\Theta((t,y'),(0,\xi'))Z((t,y'),(0,\xi'))dt\right)f(\eta')d\eta' dy'\\
&= c_{d-1}\iint\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\tilde\psi(y',\xi')+iy'\cdot\eta'}\tilde\Theta(y',\xi')Z((t,y'),(0,\xi'))dt\right)f(\eta')d\eta' dy'\\
&= c_{d-1}\iint\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}Z((t,y'),(0,\xi'))dt\right)e^{-i\tilde\psi(y',\xi')+iy'\cdot\eta'}\tilde\Theta(y',\xi')f(\eta')d\eta' dy', \qquad (4.7)
\end{aligned}$$

where $c_\ell=(2\pi)^{-\ell}$. Thus we formally observe that $T(\lambda)J_+^*G_-T(\lambda)^*$ is a Fourier integral operator on Σ_λ with the phase function $\tilde\psi(y',\xi')$. In other words, we have

$$\begin{aligned}
T(\lambda)J_+^*G_-T(\lambda)^*f(\xi) &= c_{d-1}\iint\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}Z(y+tv(\partial_y\psi(y,\xi)),\xi)dt\right)\times\\
&\qquad\times e^{-i\tilde\psi(y,\xi)+iy\cdot\eta}\tilde\Theta(y,\xi)f(\eta)d\eta dy
\end{aligned} \qquad (4.8)$$

on $T^*\Sigma_\lambda$. It remains to compute the symbol and thus justify the computation.

Lemma 4.4.

$$\int_{-\infty}^{\infty}Z(y+tv(\partial_y\psi(y,\xi)),\xi)dt=i+R(y,\xi)$$

on $T^*\Sigma_\lambda$, where $R\in S(\langle x\rangle^{-1},\tilde g)$.

Proof. We fix (y_0,ξ_0), and let

$$z_0=x(y_0,\xi_0),\quad \zeta_0=\zeta(y_0,\xi_0)$$

be the stationary phase points as in (4.5). We also write $\eta_0=\partial_y\psi(y_0,\xi_0)$. Then by the construction, we observe

$$w_-(z_0,\zeta_0)=(y_0,\eta_0),\quad\text{or equivalently,}\quad (z_0,\zeta_0)=w_-^{-1}(y_0,\eta_0).$$

We note

$$\exp tH_{p_0}(y_0,\eta_0)=(y_0+tv(\eta_0),\eta_0),$$

and combining this with the intertwining property:

$$\exp tH_p\circ w_-^{-1}=w_-^{-1}\circ\exp tH_{p_0},$$

we learn

$$w_-^{-1}(y_0+tv(\eta_0),\eta_0)=\exp tH_p(z_0,\zeta_0),$$

and hence

$$(x(y_0+tv(\eta_0),\xi_0),\zeta(y_0+tv(\eta_0),\xi_0)=\exp tH_p(z_0,\zeta_0).$$

We denote $(z(t),\zeta(t))=\exp tH_p(z_0,\zeta_0)$ as in the last section. We note, by Lemmas 3.3 and 4.2, the principal symbol of $Z(y,\xi)$ is given by

$$Z_{00}(y,\xi)=-i(\partial_\xi p)(x,\partial_x\psi_-(x,\eta))\cdot\partial_x\chi_-(x,\eta).$$

with $x = x(y,\xi)$ and $\eta = \partial_y \psi(y,\xi)$. Hence we have

$$Z_{00}(y_0 + tv(\eta_0), \xi_0) = -i(\partial_\xi p)(z(t), \zeta(t)) \cdot (\partial_x \chi_-)(z(t), \eta_0), \quad t \in \mathbb{R}.$$

By the Hamilton equation, we note $(\partial_\xi p)(z(t),\zeta(t)) = \frac{d}{dt} z(t)$, and hence

$$Z_{00}(y_0 + tv(\eta_0), \xi_0) = -i\frac{d}{dt}(\chi_-(z(t), \eta_0)).$$

Since $\lim_{t\to\infty} \chi_-(z(t),\eta_0) = 0$ and $\lim_{t\to-\infty} \chi_-(z(t),\eta_0) = 1$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} Z_{00}(y_0 + tv(\eta_0), \xi_0)dt &= -i\int_{-\infty}^{\infty} \frac{d}{dt}\chi_-(z(t),\eta_0)dt \\ &= -i\Big(\lim_{t\to\infty} \chi_-(z(t),\eta_0) - \lim_{t\to-\infty} \chi_-(z(t),\eta_0)\Big) = i.\end{aligned}$$

Now it remains to estimate the contribution from the lower order term: $R(y,\xi) = Z(y,\xi) - Z_{00}(y,\xi) \in S(\langle x\rangle^{-2}, g)$.

As usual, we identify $T^*_\xi \Sigma_\lambda$ with $v(\xi)^\perp$, the orthogonal subspace of the normal vector $v(\xi)$ at $x \in \Sigma_\lambda$. Then $y_0 \perp \eta_0$ and hence

$$|y_0 + t\eta_0| = (|y_0|^2 + t^2|v(\eta_0)|^2)^{1/2} \geq c_0(|y_0| + t|v(\eta_0)|),$$

where $c_0 = 1/\sqrt{2}$. Thus we have

$$|R(y_0,\xi_0)| \leq C\int_{-\infty}^{\infty} (1 + |y_0| + t|v(\eta_0)|)^{-2}dt \leq C'\langle y_0\rangle^{-1},$$

where $\xi_0 \in \Sigma_\lambda$, $y_0 \in T^*_{\eta_0}\Sigma_\lambda$. Similarly, we can show, for any $\alpha, \beta \in \mathbb{Z}_+^{d-1}$,

$$\left|\partial_y^\alpha \partial_\xi^\beta R(y,\xi)\right| \leq C_{\alpha\beta}\langle y\rangle^{-1-|\alpha|},$$

which completes the proof. □

Thus we learn, combining the lemma with (4.8),

$$T(\lambda)J_+^* G_- T(\lambda)^* f(\xi) = \frac{c_{d-1}}{2\pi}\iint (i + R(y,\eta))e^{-i\tilde\psi(y,\xi)+iy'\cdot\eta}\tilde\Theta(y,\xi)f(\eta)d\eta dy$$

with $R \in S(\langle x\rangle^{-1}, \tilde g)$. Substituting this to the representation formula, (4.1), we obtain

$$S(\lambda)f(\xi) = c_{d-1}\iint (1 - iR(y,\eta))e^{-i\tilde\psi(y,\xi)+iy'\cdot\eta}\tilde\Theta(y,\xi)f(\eta)d\eta dy.$$

This complete the proof of Theorem 1.1. □

Appendix A. Representation formula of the scattering matrix

In this appendix, we sketch the proof of (4.1). We suppose $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that $\hat f, \hat g \in C_0^\infty(p_0^{-1}(I))$, and we write $f(\lambda) = T(\lambda)f$, $g(\lambda) = T(\lambda)g$, $\lambda \in I$. We first note, by the standard Cook-Kuroda method, we have

$$W_\pm^I f = J_\pm f + i\int_0^{\pm\infty} e^{itH}G_\pm e^{-itH_0} f dt. \tag{A.1}$$

We also note, by the construction of $J_\pm$,

$$\left\|J_\pm e^{-itH_0} f\right\| \to 0, \quad \text{as } t \to \mp\infty,$$

and hence

$$\begin{aligned} W^I_\pm f &= \lim_{t\to\pm\infty}\Big(e^{itH}J_\pm e^{-itH_0}f - e^{-itH}J_\pm e^{itH_0}f\Big)\\ &= \pm i\int_{-\infty}^{\infty} e^{itH}G_\pm e^{-itH_0}f\,dt. \end{aligned} \tag{A.2}$$

Using (A.2), we compute

$$\begin{aligned} \big\langle f, S^I g\big\rangle &= \big\langle W^I_+ f, W^I_- g\big\rangle\\ &= -i\int_{-\infty}^{\infty}\big\langle W^I_+ f, e^{itH}G_- e^{-itH_0}g\big\rangle dt\\ &= -i\int_{-\infty}^{\infty}\big\langle W^I_+ e^{-itH_0} f, G_- e^{-itH_0}g\big\rangle dt. \end{aligned}$$

In the last line, we have used the intertwining property. Then we substitute (A.1) to learn

$$\begin{aligned} \big\langle f, S^I g\big\rangle = &-i\int_{-\infty}^{\infty}\big\langle J_+ e^{-itH_0} f, G_- e^{-itH_0}g\big\rangle dt\\ &-\int_0^\infty\int_{-\infty}^{\infty}\big\langle e^{isH}G_+ e^{-i(s+t)H_0} f, G_- e^{-itH_0}g\big\rangle dt\,ds. \end{aligned}$$

Now we use the spectral representation:

$$f=\int_I T(\lambda)^* f(\lambda)d\lambda,\quad g=\int_I T(\sigma)^* g(\sigma)d\sigma$$

to obtain (at least formally)

$$\begin{aligned} &\big\langle f, S^I g\big\rangle = -i\int_{-\infty}^{\infty}dt\int_I d\lambda\int_I d\sigma\big\langle J_+ e^{-itH_0}T(\lambda)^*f(\lambda), G_- e^{-itH_0}T(\sigma)^*g(\sigma)\big\rangle\\ &-\int_0^\infty ds\int_{-\infty}^{\infty}dt\int_I d\lambda\int_I d\sigma\big\langle e^{isH}G_+ e^{-i(s+t)H_0}T(\lambda)^*f(\lambda), G_- e^{-itH_0}T(\sigma)^*g(\sigma)\big\rangle\\ &= -i\int_{-\infty}^{\infty}dt\int_I d\lambda\int_I d\sigma e^{it(\lambda-\sigma)}\big\langle J_+T(\lambda)^*f(\lambda), G_-T(\sigma)^*g(\sigma)\big\rangle\\ &-\int_0^\infty ds\int_{-\infty}^{\infty}dt\int_I d\lambda\int_I d\sigma e^{it(\lambda-\sigma)}\big\langle e^{is(H-\lambda)}G_+T(\lambda)^*f(\lambda), G_-T(\sigma)^*g(\sigma)\big\rangle. \end{aligned}$$

Here we note that

$$\mathrm{WF}(\mathcal{F}T^*(\lambda)f(\lambda))\subset\big\{(\xi,x)\bigm|\xi\in\Sigma_\lambda, x\perp\Sigma_\lambda=\mathbb{R}v(\xi)\big\},$$

and the essential support of the amplitudes of $G_\pm$ are disjoint from it. Hence

$$G_+T(\lambda)^*f(\lambda), G_-(\sigma)T(\sigma)^*g(\sigma)\in\mathcal{S}(\mathbb{R}^d),$$

and these integrants are well-defined, smooth in the parameters. Thus we can change the order of integration, and using the formula

$$\int_{-\infty}^{\infty}e^{it(\lambda-\sigma)}dt=\delta(\lambda-\sigma)$$

in the distribution sense, we learn

$$\begin{aligned}\big\langle f, S^I g\big\rangle &= -2\pi i \int_I d\lambda \big\langle T(\lambda)^* f(\lambda), J_+^* G_- T(\lambda)^* g(\lambda)\big\rangle \\ &\quad - 2\pi \int_0^\infty ds \int_I d\lambda \big\langle T(\lambda)^* f(\lambda), G_+^* e^{-is(H-\lambda)} G_- T(\lambda)^* g(\lambda)\big\rangle.\end{aligned}$$

By the microlocal resolvent estimate [12], we learn that

$$\int_0^\infty G_+^* e^{-is(H-\lambda)} G_- ds = -iG_+^*(H-\lambda-i0)^{-1}G_-$$

makes sense, and we conclude

$$\big\langle f, S^I g\big\rangle = -2\pi i \int_I d\lambda \big\langle T(\lambda)^* f(\lambda), \big(J_+^* G_- - G_+^*(H-\lambda-i0)^{-1}G_-\big)T(\lambda)^* g(\lambda)\big\rangle.$$

This implies (4.1). □

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