

Multiplicity of solutions to an elliptic problem with singularity and measure data

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Abstract

In this paper, we prove the existence of multiple nontrivial solutions of the following equation.

$$\begin{aligned} -\Delta_p u &= \frac{\lambda}{u^\gamma} + g(u) + \mu \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ u &> 0 \text{ in } \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 3$, $1 < p - 1 < q$, $\lambda > 0$, $\gamma > 0$, g satisfies certain conditions, $\mu \geq 0$ is a bounded Radon measure.

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1 Introduction

Elliptic equations with singularity has gained a huge attention owing to its richness both from the theoretical and application point of view. Early traces of research pertaining to problems involving singularity can be found in [1], where the authors have addressed the following problem.

$$\begin{aligned} -\Delta u &= \frac{f(x)}{u^\gamma} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a strictly convex, bounded domain in \mathbb{R}^N with C^2 boundary. The existence of a unique solution was guaranteed iff $0 < \gamma < 3$. The authors in [1], has also shown the existence

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of a solution in $C^1(\bar{\Omega})$, for $0 < \gamma < 1$. Haitao [2] studied the perturbed singular problem

$$\begin{aligned} -\Delta u &= \frac{\lambda}{u^\gamma} + u^p \ \& \ u > 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \quad (1.2)$$

and guaranteed the existence of two weak solutions for $\lambda < \Lambda$, no solution for $\lambda > \Lambda$ and atleast one solution for $0 < \gamma < 1 < p \leq \frac{N+2}{N-2}$ and some $\Lambda > 0$. A further generalization to this problem can be found in [5], where the existence of two solutions were shown for some $0 < \gamma < 1 < p - 1 < q \leq p^* - 1$. An important problem involving singularity in the literature can be found in the work due to Crandall et al [4], where the authors have addressed the problem.

$$\begin{aligned} -\Delta u &= f(u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \quad (1.3)$$

where f is a function with singularity near 0. The authors in [4], have shown the existence of a unique classical solution in $C^2(\Omega) \cap C(\bar{\Omega})$. Another noteworthy work is due to Giacomoni and Sreenadh [3], where the authors have investigated the following quasilinear and singular problem.

$$\begin{aligned} -\Delta_p u &= \frac{\lambda}{u^\delta} + u^q \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ u &> 0 \text{ in } \Omega, \end{aligned} \quad (1.4)$$

where Ω is a bounded domain in \mathbb{R}^N with smooth boundary, $1 < p - 1 < q$ and $\lambda, \delta > 0$. The authors have shown the existence of weak solutions for small $\lambda > 0$ in $W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ if and only if $\delta < 2 + \frac{1}{p-1}$. Further they have investigated the radial symmetry case, i.e. for $\Omega = B_R(0)$, where they have proved the global multiplicity of solutions in $C(\bar{\Omega})$ with $\delta > 0$, $1 < p - 1 < q$, by using shooting method. Readers interested in ‘singularity involving problem’ can refer to [7, 8, 9, 10] and of late Panda et al. [6], who have investigated a problem involving singularity and a measure. Motivated by the work due to [20], which stemmed out from the work due to [14], by generalizing their result for the p -Laplacian, we will study the following problem.

$$(P) \quad \begin{cases} -\Delta_p u = \frac{\lambda}{u^\gamma} + g(u) + \mu \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \quad (1.5)$$

where Ω is a strictly convex, bounded domain in \mathbb{R}^N with C^2 boundary, $N > 2$, $1 < p < N$, $\Delta_p u = \operatorname{div}\{|\nabla u|^{p-2} \nabla u\}$, $\lambda > 0$, $\gamma > 0$ and μ is a bounded Radon measure. The function g obeys certain growth conditions, i.e there exists some constants $C > 0$ such that,

$$C^{-1}t^{1+q} \leq tg(t) \leq Ct^{1+q},$$

where $p - 1 < q < \frac{N(p-1)}{N-p}$.

2 Notations and Definitions

We will use the notations due to [30], to denote $W_0^{k,p}(\Omega)$, to be the space obtained by considering the closure of $C_c^\infty(\Omega)$ in the Sobolev space $W^{k,p}(\Omega)$ and $W_{loc}^{k,p}(\Omega)$ to be the local Sobolev space, which consists of functions u such that for any compact $K \subset \Omega$, $u \in W^{k,p}(K)$. The Hölder Space is denoted by $C^{k,\beta}(\bar{\Omega})$ with $0 < \beta \leq 1$ (again a notation borrowed from [30]), which consists of all functions $u \in C^k(\bar{\Omega})$ such that the norm

$$\sum_{|\alpha| \leq k} \sup |D^\alpha u| + \sup_{x \neq y} \left\{ \frac{|D^k u(x) - D^k u(y)|}{|x - y|^\beta} \right\} < \infty.$$

We will use the truncation functions for fixed $k > 0$,

$$T_k(t) = \max\{-k, \min\{k, t\}\} \text{ and } G_k(t) = (|t| - k)^+ \text{sign}(t)$$

with $t \in \mathbb{R}$. Observe that $T_k(t) + G_k(t) = t$ for any $t \in \mathbb{R}$ and $k > 0$.

We denote $\mathbb{M}(\Omega)$ as the space of all finite Radon measures on Ω . For every $\mu \in \mathbb{M}(\Omega)$, we define

$$\|\mu\|_{\mathbb{M}(\Omega)} = \int_{\Omega} d|\mu|.$$

We will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ (or weak $L^q(\Omega)$) defined for every $0 < q < \infty$, as the space of all measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that the corresponding distribution functions satisfy an estimate of the form

$$m(\{x \in \Omega : |f(x)| > t\}) \leq \frac{C}{t^q} \quad t > 0, \quad C < \infty.$$

Indeed, for bounded domain Ω we have $\mathcal{M}^q \subset \mathcal{M}^{\bar{q}}$ if $q \geq \bar{q}$, for some fixed positive \bar{q} . Further, the following continuous embeddings holds

$$L^q(\Omega) \hookrightarrow \mathcal{M}^q(\Omega) \hookrightarrow L^{q-\epsilon}(\Omega), \quad (2.6)$$

for every $1 < q < \infty$ and $0 < \epsilon < q - 1$. We will use this embedding result to show the existence of solutions. We now give the definition of convergence in the measure space.

Definition 2.1. *Let (μ_n) be the sequence of measurable functions in $\mathbb{M}(\Omega)$. We say (μ_n) converges to $\mu \in \mathbb{M}(\Omega)$ in the sense of measure [31] i.e. $\mu_n \rightharpoonup \mu$ in $\mathbb{M}(\Omega)$, if*

$$\int_{\Omega} f d\mu_n \rightarrow \int_{\Omega} f d\mu, \quad \forall f \in C_0(\Omega).$$

In order to show the existence of solutions to the problem (1.5), we will consider the following sequence of problems (P_n) .

$$(P_n) \quad \begin{cases} -\Delta_p u = \frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \quad (2.7)$$

whose solution will be denoted by u_n . The weak formulation to (2.7) is defined as

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \phi dx = \lambda \int_{\Omega} \frac{\phi}{(u_n + \frac{1}{n})^\gamma} + \int_{\Omega} g(u_n) \phi dx + \int_{\Omega} \mu_n \phi dx, \forall \phi \in C_0^1(\bar{\Omega}) \quad (2.8)$$

where, (μ_n) is a sequence of smooth non-negative functions bounded in $L^1(\Omega)$ and converging weakly to μ in the sense of Definition 2.1. We now give the definition of weak solution to the problem (P) in (1.5).

Definition 2.2. *We say a function $u \in W_{loc}^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a weak solution to the problem (1.5) if $\frac{\phi}{u^\gamma} \in L^1(\Omega)$ and it satisfies*

$$\int_{\Omega} |\nabla u|^{p-2} \cdot \nabla u \cdot \nabla \phi dx = \lambda \int_{\Omega} \frac{\phi}{u^\gamma} dx + \int_{\Omega} g(u) \phi dx + \int_{\Omega} \phi d\mu \quad (2.9)$$

for every $\phi \in W_0^{1,p}(\Omega')$ with $\Omega' \subset\subset \Omega$.

In the subsequent section, we will prove a few lemmas which will be required to prove our main result in Section 4. Note that the solution will be named as u_n in multiple places for different problems.

3 Important Lemmas

In this section we will prove a few important lemmas, Lemma (3.1) - (3.7), which are the main tools needed to prove the main result of existence of solution to the problem (1.5).

Lemma 3.1. *The problem*

$$\begin{aligned} -\Delta_p u &= \frac{\lambda}{(u + \frac{1}{n})^\gamma} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \quad (3.10)$$

possesses a nonnegative weak solution in $W_{loc}^{1,p}(\Omega) \cap L^\infty(\Omega)$ for each $n \in \mathbb{N}$.

Proof. The idea of the proof is to apply Schauder's fixed point argument. For a fixed $n \in \mathbb{N}$ and a fixed $v \in L^p(\Omega)$, we define the map $J_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$, as follows,

$$J_\lambda(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} \frac{u}{(|v| + \frac{1}{n})^\gamma} dx.$$

It is easy to see that, J_λ is continuous, coercive and strictly convex in $W_0^{1,p}(\Omega)$. Therefore, the existence of a unique minimizer $w \in W_0^{1,p}(\Omega)$ corresponding to a $v \in L^p(\Omega)$ is certain.

We define, $H : L^p(\Omega) \rightarrow L^p(\Omega)$ by

$$H(v) = (-\Delta_p)^{-1} \left[\frac{\lambda}{(|v| + \frac{1}{n})^\gamma} \right] := w.$$

On choosing w as a test function from $W_0^{1,p}(\Omega)$ in the weak formulation of (3.10), we have

$$\begin{aligned} \int_{\Omega} |\nabla w|^p &= \int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla w = \int_{\Omega} \frac{\lambda}{(|v| + \frac{1}{n})^\gamma} w \\ &\leq \lambda n^\gamma \int_{\Omega} |w|. \end{aligned}$$

Hence, by using the Poincaré inequality and the Hölder's inequality on the left and right hand side respectively, we get

$$\|w\|_p \leq C(n, \gamma, \lambda). \quad (3.11)$$

Let us consider a sequence (v_k) that converges to v in $L^p(\Omega)$. By using the dominated convergence theorem, we have

$$\left\| \frac{\lambda}{(|v_k| + \frac{1}{n})^\gamma} - \frac{\lambda}{(|v| + \frac{1}{n})^\gamma} \right\|_{L^p(\Omega)} \rightarrow 0.$$

Thus, the convergence of $w_k = H(v_k)$ to $w = H(v)$ in $L^p(\Omega)$ can be followed from the uniqueness of the weak solution. Hence, the continuity of H over $L^p(\Omega)$ is followed. By the estimate in equation (3.11) and by the Rellich-Kondrochov theorem, we get that $H(L^p(\Omega))$ is relatively compact in $L^p(\Omega)$. We now can apply the Schauder's fixed point theorem to guarantee the existence of a fixed point say w . By the regularity theorem of Lieberman [17], we have $u_n \in C^1(\bar{\Omega}), \forall n \in \mathbb{N}$. Using the strong maximum principle [27], we have $w > 0$ in Ω and this concludes the proof. \square

Lemma 3.2. *The sequence (u_n) is increasing w.r.t n and for every $K \subset\subset \Omega$, there exists C_K (only depends on K) such that $u_n \geq C_K > 0$, a.e. in K with $\|u_n\|_\infty \leq R\lambda^{\frac{1}{\gamma+p-1}}, \forall n \in \mathbb{N}$, R is independent of n .*

Proof. Consider a sequence of problems

$$\begin{aligned} -\Delta_p u &= \frac{\lambda}{(u + \frac{1}{n})^\gamma} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \quad (3.12)$$

For each n , let u_n be the solution to the problem (3.12). Consider,

$$\int_{\Omega} (|\nabla u_n|^{p-2} \cdot \nabla u_n - |\nabla u_{n+1}|^{p-2} \cdot \nabla u_{n+1}) \cdot \nabla \phi \, dx = \lambda \int_{\Omega} \left((u_n + \frac{1}{n})^{-\gamma} - (u_{n+1} + \frac{1}{n+1})^{-\gamma} \right) \phi \, dx.$$

We choose, the test function $\phi = (u_n - u_{n+1})^+$ to obtain,

$$\begin{aligned} \int_{\Omega} (|\nabla u_n|^{p-2} \cdot \nabla u_n - |\nabla u_{n+1}|^{p-2} \cdot \nabla u_{n+1}) \cdot \nabla (u_n - u_{n+1})^+ \, dx \\ \leq \lambda \int_{\Omega} \left((u_n + \frac{1}{n})^{-\gamma} - (u_{n+1} + \frac{1}{n+1})^{-\gamma} \right) (u_n - u_{n+1})^+ \, dx. \end{aligned}$$

Using the inequalities from [26], we get for $p \geq 2$,

$$\begin{aligned} \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_{n+1}|^{p-2} \nabla u_{n+1}) \cdot \nabla (u_n - u_{n+1})^+ dx &\geq C_p \|\nabla (u_n - u_{n+1})^+\|^p \\ &\geq 0 \end{aligned}$$

and for $1 < p < 2$,

$$\begin{aligned} \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_{n+1}|^{p-2} \nabla u_{n+1}) \cdot \nabla (u_n - u_{n+1})^+ dx &\geq C_p \frac{\|u_n - u_{n+1}\|^2}{(\|u_n\| + \|u_{n+1}\|)^{2-p}} \\ &\geq 0. \end{aligned}$$

Therefore, we have

$$0 \leq \lambda \int_{\Omega} \left\{ \left(u_n + \frac{1}{n+1} \right)^{-\gamma} - \left(u_{n+1} + \frac{1}{n+1} \right)^{-\gamma} \right\} (u_n - u_{n+1})^+ dx \leq 0.$$

Hence, we get $\|\nabla (u_n - u_{n+1})^+\| = 0$. This implies u_n is monotonically increasing w.r.t n . Now, using the Strong Maximum principle [15], we get $u_1 > 0$ in Ω , where u_1 is the solution of the problem (3.12) with $n = 1$. Since, u_n is monotonically increasing with respect to n , we have $u_n > u_1$ in Ω and hence we conclude that $u_n > C_K > 0$, for every $K \subset\subset \Omega$ with C_K being independent of n .

Claim: (u_n) is uniformly bounded in Ω .

Case 1: When $\lambda = 1$. Define, $M(k) = \{x \in \Omega : u_n > k\}$ and

$$S_k(u_n) = \begin{cases} u_n - k; & \text{if } u_n > k \\ 0; & \text{if } u_n \leq k. \end{cases}$$

We choose, $S_k(u_n)$ as the test function in the weak formulation of (3.12) to get,

$$\begin{aligned} \int_{M(k)} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_n &= \int_{M(k)} |\nabla u_n|^p \\ &= \int_{M(k)} \frac{u_n - k}{\left(u_n + \frac{1}{n}\right)^\gamma} \\ &< \int_{M(k)} \frac{u_n - k}{u_n^\gamma} \\ &\leq \|u_n - k\|_{L^p(M(k))} |M(k)|^{\frac{1}{p'}} \\ &\leq C \|\nabla u_n\|_{L^p(M(k))} |M(k)|^{\frac{1}{p'}}, \quad (\text{by the Poincaré inequality}). \end{aligned}$$

By using the Sobolev embedding theorem, we get

$$\|u_n\|_{L^{p^*}(M(k))}^{p-1} < \frac{C}{S^{p-1}} |M(k)|^{\frac{1}{p'}}, \quad \text{where } p^* = \frac{Np}{N-p} \text{ (Sobolev conjugate of } p).$$

It is easy to see that, for $1 < k < l$, $M(l) \subset M(k)$. Hence,

$$|M(l)| \leq \left\{ \frac{C}{S^{p-1}} \right\}^{\frac{p^*}{p-1}} \frac{1}{(l-k)^{p^*}} |M(k)|^{\frac{p^*}{p}}.$$

By the Lemma 4.1 of [11], we can guarantee the existence of a $T > 0$ independent of n such that $|M(T)| = 0$. Therefore, $\|u_n\|_\infty \leq T$.

Case 2: Suppose v is such that

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \phi < \lambda \int_{\Omega} \frac{\phi}{v^\gamma} \quad \forall \lambda \in W_0^{1,p}(\Omega), \phi \geq 0. \quad (3.13)$$

Let $\lambda > 0$. Choose $v = (\frac{1}{\lambda})^{\frac{1}{\gamma+p-1}} w$. We can see that v satisfies

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \phi < \int_{\Omega} \frac{\phi}{v^\gamma}, \quad \forall \phi \in W_0^{1,p}(\Omega), \phi > 0.$$

Therefore, using the result from *Case 1*, for $\lambda = 1$, we have $\|v\|_\infty \leq T$, which implies that $\|u_n\|_\infty \leq R\lambda^{\frac{1}{\gamma+p-1}}$. Hence, (u_n) is uniformly bounded in Ω . Finally, on using a result due to Lieberman [17], helps us to conclude that $u_n \in C^1(\Omega)$, $\forall n \in \mathbb{N}$. \square

Lemma 3.3. *Every bounded nontrivial solution v of the problem $-\Delta_p u = g(u) + \mu_n$ in Ω , is uniformly bounded below in $L^\infty(\Omega)$, i.e. $\|v\|_\infty > \delta$, for some $\delta > 0$.*

We instead first prove the following lemma.

Lemma 3.4. *Every bounded nontrivial solution u of the problem $-\Delta_p u = g(u)$ in Ω , is uniformly bounded below in $L^\infty(\Omega)$, i.e. $\|u\|_\infty > \delta$, for some $\delta > 0$.*

Proof. Let us consider a sequence of nontrivial solutions (u_m) such that $\|u_m\|_\infty \rightarrow 0$ as $m \rightarrow \infty$. Then we can define $w_m(x) = u_m(x) \|u_m\|_\infty^{-1}$. Clearly, $\|w_m\|_\infty = 1$. As u_m satisfies $-\Delta_p u = g(u)$, we have

$$\begin{aligned} \Delta_p w_m &= \Delta_p (u_m(x) \|u_m\|_\infty^{-1}) \\ &= \nabla (|\nabla (u_m(x) \|u_m\|_\infty^{-1})|^{p-2} \nabla (u_m(x) \|u_m\|_\infty^{-1})) \\ &= \Delta_p u_m \|u_m\|_\infty^{1-p} \\ &= g(u_m) \|u_m\|_\infty^{1-p} \\ &\leq C u_m^q \|u_m\|_\infty^{1-p} \\ &\leq C w_m^q \|u_m\|_\infty^{1-p+q} \\ &= f_m. \end{aligned}$$

Now for very large m , these f_m 's are uniformly bounded in $L^\infty(\Omega)$. So, $\|w_m\|_{C^{1,\beta}(\bar{\Omega})} \leq M$ for some $\beta \in (0, 1)$, by regularity results in [18], where M is independent of m . Hence, by the Ascoli-Arzelà theorem, the sequence (w_m) converges uniformly to w in $C_0^1(\Omega)$. This implies $w = 0$. But with the consideration of the Lemma 1.1 of [19], we have a unique solution w in $C_0^1(\Omega)$, which contradicts the fact that $\|w_m\|_\infty = 1$. Hence, there exists $\delta > 0$ such that $\|u\|_\infty > \delta$. \square

Proof of Lemma 3.3. Since $\mu_n \geq 0$, then the solutions of the problem in Lemma 3.3 are supersolutions of the problem in Lemma 3.4. Therefore, if v and u are solutions of the problem in Lemma 3.3 and Lemma 3.4 respectively, then $\|v\|_\infty \geq \|u\|_\infty > \delta > 0$, for some $\delta > 0$.

Lemma 3.5. *There exists a $\bar{\lambda} > 0$ such that the following problem*

$$\begin{aligned} -\Delta_p u &= \frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ u &> 0 \text{ in } \Omega \end{aligned} \tag{3.14}$$

does not have any weak solution $u \in W_0^{1,p}(\Omega)$ for $\lambda \geq \bar{\lambda}$.

Proof. Let λ_1 be the first eigenvalue of the operator $-\Delta_p$ and its corresponding eigenfunction $\phi_1 \geq 0$ be such that

$$\begin{aligned} -\Delta_p \phi_1 &= \lambda_1 \phi_1^{p-1} \text{ in } \Omega, \\ \phi_1 &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Its weak formulation with the test function $\phi = \phi_1$ is given by

$$\int_{\Omega} |\nabla \phi_1|^p = \lambda_1 \int_{\Omega} \phi_1^p.$$

Let u_n be the weak solution of (2.7), then by the strong maximum principle [15], we get $\frac{\phi_1^p}{u_n^{p-1}} \in W_0^{1,p}(\Omega)$. On applying the Picone's Identity (Theorem 2.1 in [20]), we have

$$\begin{aligned} &\int_{\Omega} |\nabla \phi_1|^p dx - \int_{\Omega} \nabla \left(\frac{\phi_1^p}{u_n^{p-1}} \right) |\nabla u_n|^{p-2} \nabla u_n dx \geq 0 \\ \Rightarrow &\int_{\Omega} \lambda_1 \phi_1^p - \frac{\phi_1^p}{u_n^{p-1}} \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} - g(u_n) \frac{\phi_1^p}{u_n^{p-1}} - \mu_n \frac{\phi_1^p}{u_n^{p-1}} dx \geq 0 \\ \Rightarrow &\int_{\Omega} \left(\lambda_1 u_n^{p-1} - \lambda \left(u_n + \frac{1}{n} \right)^{-\gamma} - g(u_n) - \mu_n \right) \phi_1^p dx \geq 0. \end{aligned}$$

Consider $\bar{\lambda}$ defined as $\bar{\lambda} = \max_{x \in \Omega} \frac{\lambda_1 u_n^{p-1} - g(u_n) - \mu_n}{(u_n + 1)^{-\gamma}}$. Now for every $\epsilon > 0$, there exists a $\delta > 0$ such that $v^q < \epsilon v^{p-1}, \forall v \in [0, \delta]$. Therefore, $\bar{\lambda} > 0$ for some ϵ and for $\lambda \geq \bar{\lambda}$, we have

$$\begin{aligned} \lambda &\geq \max_{x \in \Omega} \frac{\lambda_1 u_n^{p-1} - g(u) - \mu_n}{(u + 1)^{-\gamma}} \\ &\geq \frac{\lambda_1 u_n^{p-1} - g(u) - \mu_n}{(u + \frac{1}{n})^{-\gamma}} \\ &\Rightarrow \left(\lambda_1 u_n^{p-1} - \lambda \left(u + \frac{1}{n} \right)^{-\gamma} - g(u) - \mu_n \right) < 0 \end{aligned} \tag{3.15}$$

which is a contradiction to our assumption. Hence, for $\lambda \geq \bar{\lambda}$, the problem (2.7) does not possess any solution $u \in W_0^{1,p}(\Omega)$. \square

Lemma 3.6. *Let Ω be a strictly convex domain and u_n be a solution of problem (2.7). Then there exists $M > 0$, which does not depend on n , such that $\|u_n\|_\infty \leq M$.*

Proof. We divide the proof of this lemma into six steps.

Step 1 (Uniform Höpf Lemma). Our aim is to show that $\frac{\partial u_n}{\partial \hat{n}}(x) < c < 0$ for any $n \in \mathbb{N}$, where c is some constant which is independent of n but depends on x . \hat{n} is the unit outward normal to the boundary $\partial\Omega$ at the point x .

Now Ω satisfies the interior ball condition as it has a C^2 boundary, i.e. for some $x_0 \in \partial\Omega$, there exists a $B_r(y) \subset \Omega$ such that $\partial B_r(y) \cap \partial\Omega = \{x_0\}$. Let us define $v : B_r(y) \rightarrow \mathbb{R}$ given by

$$v(x) = [2^{\frac{N-p}{p-1}} - 1]^{-1} r^{\frac{N-p}{p-1}} |x - y|^{\frac{p-N}{p-1}} - [2^{\frac{N-p}{p-1}} - 1]^{-1}.$$

We observe that,

- (i) $v(x) = 1$ on $\partial B_{\frac{r}{2}}(y)$ and $v(x) = 0$ on $\partial B_r(y)$, and
- (ii) if $x \in B_r(y) \setminus B_{\frac{r}{2}}(y)$ with $|\nabla v(x)| > c > 0$ for some constant c independent of n .

Therefore, we have $0 < v(x) < 1$. Let us define $m = \inf\{u_n(x) | x \in \partial B_{\frac{r}{2}}(y)\}$. By using the Lemma 3.2, we can conclude that $m > 0$ and is independent of n . on choosing $w = mv$, we see that w satisfies

$$\begin{aligned} -\Delta_p w &= 0 \text{ in } B_r(y) - \overline{B_{\frac{r}{2}}(y)}, \\ w &= m \text{ if } x \in \partial B_{\frac{r}{2}}(y), \\ w &= 0 \text{ if } x \in \partial B_r(y). \end{aligned}$$

We have $u_n \geq w$ on the boundary of $B_r(y) - \overline{B_{\frac{r}{2}}(y)}$ and $-\Delta_p w \leq -\Delta_p u_n$ in Ω . Hence, by the weak comparison principle, we have $u_n \geq w$ in $B_r(y) - \overline{B_{\frac{r}{2}}(y)}$. Since, $u_n(x_0) = w(x_0) = 0$, then from the properties of v in (i) and (ii) above, we obtain

$$\begin{aligned} \frac{\partial u_n}{\partial \hat{n}}(x_0) &= \lim_{t \rightarrow 0} \frac{u_n(x_0 - t\hat{n})}{t} \leq \lim_{t \rightarrow 0} \frac{w(x_0 - t\hat{n})}{t} \\ &= \frac{\partial w}{\partial \hat{n}}(x_0) = m \frac{\partial v}{\partial \hat{n}} < -c < 0, \text{ where } c > 0 \text{ is independent of } n. \end{aligned}$$

Step 2 (Existence of a neighbourhood of the boundary which does not contain any critical points of u_n). Let us denote $C(u_n) = \{x \in \Omega : \nabla u_n(x) = 0\}$, as the set of critical points of u_n . From Step 1, we have $\frac{\partial u_n}{\partial \hat{n}} < 0$ on the boundary. Hence, $\text{dist}(\partial\Omega, C(u_n)) = b_n > 0$, $\forall n \in \mathbb{N}$ as $\partial\Omega$ and $C(u_n)$ are compact subsets in Ω .

Claim: There exists $\epsilon > 0$, independent of n , such that $b_n > \epsilon > 0$. In other words there exists a neighbourhood $\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \epsilon\}$, such that $C(u_n) \cap \Omega_\epsilon = \emptyset$.

Proof. We prove this by a contrapositive argument. Let there does not exist any such $\epsilon > 0$ such that $C(u_n) \cap \Omega_\epsilon \neq \emptyset$. Then there exists $x_n \in C(u_n)$ such that $\text{dist}(x_n, \partial\Omega) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, upto a subsequence $x_{n_k} \rightarrow x_0$ and $x_0 \in \partial\Omega$. But from Step 1, we obtain $\frac{\partial u_n}{\partial \hat{n}}(x_0) < c < 0$. Hence, there exists $l > 0$ such that $|\nabla u_n(x)| > \frac{c}{2}$ for $x \in B_l(x_0) \cap \Omega$, where c is independent of n . This implies that $B_l(x_0) \cap C(u_n) = \emptyset$. This is a contradiction, since we can find $x_{n_0} \in B_l(x_0) \cap \Omega$ such that $\nabla u_{n_0}(x_{n_0}) = 0$. Hence the claim.

Step 3 (Monotonicity of u_n). Let $e \in \mathbb{S}^{N-1}, \delta \in \mathbb{R}$, then for a fixed $n \in \mathbb{N}$, we define the following

- (i) The hyperplane $\mathbb{L}_{\delta,e} = \{x \in \mathbb{R}^N : x \cdot e = \delta\}$ and $\sigma_{\delta,e} = \{x \in \mathbb{R}^N : x \cdot e < \delta\}$.
- (ii) \hat{x} be the reflection of x with respect to the hyperplane $\mathbb{L}_{\delta,e}$ i.e. $\hat{x} = x + 2(\delta - x \cdot e)e$.
- (iii) $a(e) = \inf_{x \in \Omega} \{x \cdot e\}$ and the reflected cap of $\sigma_{\delta,e}$ with respect to $\mathbb{L}_{\delta,e}$ for any $\delta > a(e)$ denoted as $\hat{\sigma}_{\delta,e}$.
- (iv) $\hat{\sigma}_{\delta,e}$ is not internally tangent to $\partial\Omega$ at some point $p \notin \mathbb{L}_{\delta,e}$.
- (v) $\hat{n}(x)$ be the unit inward normal to $\partial\Omega$ at x , then $\hat{n}(x) \cdot e \neq 0, \forall x \in \partial\Omega \cap \mathbb{L}_{\delta,e}$.
- (vi) $\xi(e) = \{\mu_0 > a(e) : \forall \delta \in (a(e), \mu_0), 4 \text{ and } 5 \text{ holds}\}$. and $\bar{\xi}(e) = \sup\{\xi(e)\}$.

If Ω is strictly convex, then the map $e \mapsto \bar{\xi}(e)$ is continuous by Proposition 2 of [21]. Let us denote $v_n(x) = u_n(\hat{x})$. Considering the strict convexity of Ω and the property (4), we see that $\hat{\sigma}_{\delta,e}$ is contained in Ω for any $\delta \leq \delta_1$ where δ_1 only depends on Ω . Since, Δ_p is invariant under reflection and both u_n and v_n satisfy equation (2.7) hence both the functions take the same value on the hyperplane $\mathbb{L}_{\delta,e}$. Let us define $\delta_0 = \min(\delta_1, \epsilon)$. Also for $x \in \partial\Omega \cap \partial\sigma_{\delta,e}$, we have $u_n(x) = 0$ and $v_n(x) = u_n(\hat{x}) > 0$ as $\hat{x} \in \Omega$. Therefore,

$$\begin{aligned} -\Delta_p u_n + \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} + g(u_n) + \mu_n &= -\Delta_p v_n + \frac{\lambda}{(v_n + \frac{1}{n})^\gamma} + g(v_n) + \mu_n \text{ in } \sigma_{\delta,e} \\ u_n &\leq v_n \text{ on } \partial\sigma_{\delta,e} \cap \partial\Omega. \end{aligned}$$

Then $u_n \leq v_n$ in $\sigma_{\delta,e}$ for any $\delta \in (a(e), \delta_0)$, by the comparison principle [22]. Hence, u_n is nondecreasing for all $x \in \sigma_{\delta_0,e}$ along the e -direction.

Step 4 (Existence of a measurable proper subset of Ω of nonzero measure on which u is nondecreasing). For a fixed $x_0 \in \partial\Omega$, let $e = e(x_0)$ be the unit outward normal to $\partial\Omega$ at x_0 . Then by the results in Step 3, we conclude that u_n is nondecreasing in the direction of e for all $x \in \sigma_{\delta,e}$ and $a(e) < \delta < \delta_0$. For any $\theta \in \mathbb{S}^{N-1}$ in a small neighbourhood of e , the reflection of $\sigma_{\delta,\theta}$ w.r.t. $\mathbb{L}_{\delta,\theta}$ is a member of Ω , since the domain is strictly convex and hence the sequence u_n will be nondecreasing in the θ direction. Fix $\delta = \frac{\delta_0}{2}$. Since Ω is strictly convex, there exists a neighbourhood $\Theta \in \mathbb{S}^{N-1}$ such that $\sigma_{\frac{\delta_0}{2},e} \subset \sigma_{\delta_0,\theta}$ for all $\theta \in \Theta$. Thus, we can conclude that u_n is nondecreasing in every direction for $\theta \in \Theta$ and for any x with $x \cdot e < \frac{\delta_0}{2}$.

Consider

$$\sigma_0 = \left\{ x \in \Omega : \frac{\delta_0}{8} < x \cdot e < \frac{3\delta_0}{8} \right\}.$$

Obviously, $\sigma_0 \subset \sigma_{\frac{\delta_0}{2},e}$ and u_n is nondecreasing in every direction $\theta \in \Theta$ and $x \in \sigma_0$. Choose $\epsilon = \frac{\delta_0}{8}$ and fix a point $x \in \Omega_\epsilon$. Let x_0 be the projection of the point x onto $\partial\Omega$. We define $\mathbb{I}_x \subset \sigma_0$ to be the truncated cone having vertex at $x_0 - \epsilon e$ and an opening angle $\frac{\theta}{2}$. Then \mathbb{I}_x satisfies the following properties.

- (i) $|\mathbb{I}_x| > k$ for some k , where k depends only on Ω and ϵ ,

(ii) $u_n(x) \leq u_n(y)$ for all $y \in \mathbb{I}_x$ and $n \in \mathbb{N}$.

Then, we have $u_n(x) \leq u_n(x_0 - \epsilon e) \leq u_n(y)$, for all $y \in \mathbb{I}_x$.

Step 5 (A boundary ‘estimate’). Let us consider the first eigenfunction ϕ_1 of the p -Laplacian eigenvalue problem over Ω . Using the Picone’s identity on ϕ_1 , u_n and then applying the strong maximum principle [15], we have $\frac{\phi_1^p}{u_n^{p-1}} \in W_0^{1,p}(\Omega)$. Denote $f_n(u_n) = \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} + \mu_n$. Then, we have

$$\begin{aligned} \int_{\Omega} \frac{[f_n(u_n) + g(u_n)]\phi_1^p}{u_n^{p-1}} &= \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \left(\frac{\phi_1^p}{u_n^{p-1}} \right) \\ &\leq \int_{\Omega} |\nabla \phi_1|^p dx \\ &\leq C(\Omega). \end{aligned} \tag{3.16}$$

Let $\phi_1(z) \geq \xi > 0$ for all $z \in \Omega - \Omega_{\frac{\xi}{2}}$. Hence, from (3.16), we have

$$\xi^p \int_{\Omega - \Omega_{\frac{\xi}{2}}} \frac{[f_n(u_n) + g(u_n)]}{u_n^{p-1}} \leq C(\Omega).$$

This implies

$$\int_{\mathbb{I}_x} \frac{[f_n(u_n) + g(u_n)]}{u_n^{p-1}} \leq \frac{C(\Omega)}{\xi^p}.$$

Now since,

$$\int_{\mathbb{I}_x} \frac{[f_n(u_n) + g(u_n)]}{u_n^{p-1}} \geq \int_{\mathbb{I}_x} g(u_n) u_n^{1-p}(z) dz \geq u_n^{q-p+1}(x) |\mathbb{I}_x| \tag{3.17}$$

we have

$$u_n^{q-p+1}(x) \leq \frac{C_1(\Omega)}{\xi^p},$$

for some constant $C_1 > 0$, i.e. $u_n(x) \leq C'$, for all $x \in \Omega_\epsilon$ and for all $n \in \mathbb{N}$.

Step 6 (Blow-up analysis). We will show that for every open set, $K \subset\subset \Omega$, there exists $C_K > 0$ such that $\|u_n\|_\infty < C_K$, for every solution u_n of (2.7). We will prove it by contrapositive argument. Suppose, there exist a sequence (u_n) of positive solutions of the problem (2.7) and a sequence of points $(Z_n) \subset \Omega$ such that $M_n = u_n(Z_n) = \max\{u_n(x) : x \in \bar{K}\} \rightarrow \infty$ as $n \rightarrow \infty$. Using the boundary estimates one can assume that $Z_n \rightarrow x_0$ as $n \rightarrow \infty$, where $x_0 \in \bar{K}$. Let $dist(\bar{K}, \partial\Omega) = 2d$ and $\Omega_d = \{x \in \Omega : dist(x, \Omega) < d\}$.

Let R_n be the sequence of positive real numbers with $R_n^{\frac{p}{q-p+1}} M_n = 1$. Observe that $M_n \rightarrow \infty$ iff $R_n \rightarrow 0$ as $n \rightarrow \infty$. Define, $w_n : B_{\frac{d}{R_n}}(0) \rightarrow \mathbb{R}$ such that

$$w_n(y) = R_n^{\frac{p}{q-p+1}} u_n(Z_n + R_n y).$$

Now u_n has a maximum at Z_n , hence we have $\|w_n\|_\infty = w_n(0) = 1$. Since $R_n \rightarrow 0$ there exists n_0 such that $B_R(0) \subset B_{\frac{d}{R_n}}(0)$ for fixed $R > 0$. Again, we have that w_n satisfies the following

$$\nabla w_n(y) = R_n^{\frac{p}{q-p+1}+1} \nabla u_n(Z_n + R_n y)$$

and

$$-\Delta_p w_n(y) = R_n^{\frac{pq}{q-p+1}} [\lambda f_n(u_n(Z_n + R_n y)) + R_n^{\frac{-pq}{q-p+1}} w_n^q(Z_n + R_n y) + R_n^{\frac{-pq}{q-p+1}} \mu_n(Z_n + R_n y)].$$

From Lemma (3.1) and Lemma (3.3), for any $y \in B_R(0)$, we have $Z_n + R_n y \in \bar{\Omega}_d \subset \Omega$ and

$$R_n^{\frac{pq}{q-p+1}} [\lambda f_n(u_n(Z_n + R_n y)) + R_n^{\frac{-pq}{q-p+1}} w_n^q(Z_n + R_n y) + R_n^{\frac{-pq}{q-p+1}} \mu_n(Z_n + R_n y)] \leq C(\bar{\Omega}_d), \quad (3.18)$$

for every $n \geq n_0$. Let us fix a ball B such that $\bar{B} \subset B_{\frac{d}{R_n}}(0)$, $\forall n \geq n_0$. Then by the interior estimates of Lieberman [17] and Tolksdorf [18], we have the existence of a constant $C = C(N, p, B) > 0$ and $\beta = \beta(N, p, B) \in (0, 1)$ such that

$$w_n \in C^{1,\beta}(\bar{B}) \text{ and } \|w_n\|_{1,\beta} \leq C.$$

Using the Arzela-Ascoli theorem, we guarantee the existence of a function $w \in C^1(\bar{B})$ such that there exists a convergent subsequence $w_n \rightarrow w$ in $C^1(\bar{B})$. On passing the limit $n \rightarrow \infty$, we have

$$\int_B |\nabla w|^{p-2} \nabla w \cdot \nabla \phi \geq C \int_B w^q \phi, \quad \forall \phi \in C_c^\infty(B), \quad w \in C^1(\bar{B}), \quad w \geq 0 \text{ on } \bar{B},$$

where the constant is obtained from the growth condition over g and the condition in (3.18). Also, we have $\|w\|_\infty = 1$. Hence, by using the strong maximum principle [15], we have $w(x) > 0$, $\forall x \in B$. Now for a sequence of balls with increasing radius, the Cantor diagonal subsequence converges to $w \in C^1(\mathbb{R}^N)$, on every compact subsets of \mathbb{R}^N and satisfy the following

$$\int_{\mathbb{R}^N} |\nabla w|^{p-2} \nabla w \cdot \nabla \phi \geq C \int_{\mathbb{R}^N} w^q \phi; \quad \forall \phi \in C_c^\infty(\mathbb{R}^N), \quad w \in C^1(\mathbb{R}^N), \quad w > 0 \text{ on } \mathbb{R}^N.$$

This contradicts the Theorem 4.9. □

Lemma 3.7. *For a strictly convex domain Ω , there exists $\bar{\lambda} > 0$ such that for $0 < \lambda < \bar{\lambda}$ and $\gamma > 0$ atleast two solutions (say u_n, v_n) exist for the problem (2.7) in $W_{loc}^{1,p}(\Omega)$.*

Proof. We define $\bar{J}_\lambda : C(\bar{\Omega}) \rightarrow C(\bar{\Omega})$ by

$$\bar{J}_\lambda(u) = (-\Delta_p)^{-1} \left(\frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n \right), \quad \lambda \geq 0.$$

Now equation (2.7) can be written as $u = \bar{J}_\lambda(u)$. The map \bar{J}_λ is compact since, we know $(-\Delta_p)^{-1}$ is a compact operator on $C(\bar{\Omega})$. So, we assume the map \bar{J}_λ is also compact. For $0 < \lambda < \bar{\lambda}$, we have (u_n) as solutions to the problem (2.7) and $\|u_n\|_\infty \leq M$, using Lemma 3.5 and Lemma 3.6. Let us define, $S_1 = \{u \in C(\bar{\Omega}) : u \geq 0 \text{ in } \Omega\}$, $\bar{J}_0 : S_1 \rightarrow S_1$ by $\bar{J}_0(u) = (-\Delta_p)^{-1}(g(u) + \mu_n)$ and $G : \bar{B}_R \times [0, \infty) \rightarrow S_1$ such that $G(u, \lambda) = \bar{J}_\lambda$.

Claim 1. There exists a supersolution to the problem (2.7).

Proof. Let us define, $N(r) = \frac{1}{3} \left(\left(\frac{r}{R}\right)^{\gamma+p-1} - Cr^{\gamma+q} \right)$, for $r \in [0, \infty)$ where R is the bound used

in Lemma 3.2 and $C > 0$ is the constant used in the growth condition of g and $\eta = \max_{0 \leq r \leq \beta_0} N(r)$,

where $\beta_0 = \frac{1}{2}(2q - 2p + 3)^{\frac{1}{p-q-1}} R^{\frac{\gamma+p-1}{p-q-1}}$.

Observe that, $N(r) > 0$ for $r \in (0, \beta_1)$, where $\beta_1 \in (0, \min(\gamma, \beta_0))$. Now applying the intermediate value property of continuous functions, we get that there exists a $\beta_2 \in (0, \beta_1)$ such that $N(\beta_2) = \lambda_0$.

Denote $\lambda^* = \left(\frac{\beta_2}{R}\right)^{\gamma+p-1}$. So,

$$\begin{aligned}\lambda_0 &= N(\beta_2) = \frac{1}{2} (\lambda^* - C\beta_2^{\gamma+q}) \\ \lambda^* &> \lambda_0 + \beta_2^{\gamma+q} = \lambda_0 + C[R(\lambda^*)^{\frac{1}{\gamma+p-1}}]^{\gamma+q}.\end{aligned}$$

Let u_{n,λ^*} satisfy (2.7). Then for $n \geq n_0$, we have

$$\begin{aligned}\lambda^* &> \lambda_0 + C(\|u_{n,\lambda^*}\|_\infty)^q \left(\|u_{n,\lambda^*}\| + \frac{1}{n} \right)^\gamma \\ &> \lambda + C(u_{n,\lambda^*})^q \left(u_{n,\lambda^*} + \frac{1}{n} \right)^\gamma, \text{ for } \lambda \leq \lambda_0\end{aligned}\tag{3.19}$$

$$> \lambda + g(u_{n,\lambda^*}) \left(u_{n,\lambda^*} + \frac{1}{n} \right)^\gamma.\tag{3.20}$$

Hence,

$$-\Delta_p u_{n,\lambda^*} = \frac{\lambda^*}{(u_{n,\lambda^*} + \frac{1}{n})^\gamma} + \mu_n > \frac{\lambda}{(u_{n,\lambda^*} + \frac{1}{n})^\gamma} + \mu_n + g(u_{n,\lambda^*}), \text{ for } \lambda \leq \lambda^* \text{ and } n \geq n_0.$$

Therefore, $u_{n,\lambda^*} \in C^{1,\alpha}(\bar{\Omega})$ is a positive supersolution for some $\alpha > 0$ and u_{n,λ^*} is a supersolution of

$$\begin{aligned}-\Delta_p u &= \frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n, \\ u &= 0 \text{ on } \partial\Omega,\end{aligned}\tag{3.21}$$

with $\|u_{n,\lambda^*}\|_\infty \leq \beta_2$.

Claim 2. Problem (2.7) possesses a unique solution.

To prove the Claim 2, we define,

$$f_n(x, r) = \frac{\lambda(r + \frac{1}{n})^{-\gamma} + g(r)}{r^{p-1}}, \text{ for } r \in [0, \infty).$$

Now the derivative of f_n w.r.t r is given by

$$\begin{aligned}f'_n(x, r) &= \frac{1}{r^p} \left[\frac{\lambda\{(1-p-\gamma)r + \frac{1-p}{n}\}}{(r + \frac{1}{n})^{1+\gamma}} \right] + \frac{rg'(r) - g(r)(p-1)}{r^p} \\ &< \frac{1}{r^p} \left[\frac{\lambda[(1-p-\gamma)r + \frac{1-p}{n}]}{(r + \frac{1}{n})^{1+\gamma}} \right] + (q-p+1)r^{q-p}.\end{aligned}$$

As the function $r^q(r + \frac{1}{n})^{1+\gamma}$ is convex, so there exists a unique $C_n > 0$, which is increasing with respect to λ such that

$$\lambda \left[(p + \gamma - 1)C_n + \frac{p-1}{n} \right] > (q - p + 1)C_n^q \left(C_n + \frac{1}{n} \right)^{1+\gamma}.$$

Now for $r \leq C_n$, we have

$$(q - p + 1)r^q \left(r + \frac{1}{n} \right)^{1+\gamma} \leq \lambda \left[(p + \gamma - 1)r + \frac{p-1}{n} \right].$$

Hence, $f'_n(x, r) < 0$. Consider

$$F_n(x, r) = \frac{\lambda \left(r + \frac{1}{n} \right)^{-\gamma} + g(r) + \mu_n}{r^{p-1}}, \text{ for } r \in [0, \infty).$$

Clearly, $F'_n(x, r) = f'_n(x, r) - \frac{\mu_n(p-1)}{r^p} < 0$. Therefore, F_n is decreasing and using the result of Díaz-Saá [25], we guarantee that the problem (2.7) has unique solution and $\|u_n\|_\infty \leq C_n$. Thus, we have $\beta_2 \leq \delta_0$. So,

$$\frac{q - p + 1}{\gamma + p - 1} \beta_2^{\gamma+q} < \lambda_0, \text{ for } \gamma > 1.$$

Choose

$$\lambda_m = \frac{\{(q - p + 1)(\beta_2 + \epsilon)^q - \mu_m(p - 1)\}(\beta_2 + \epsilon + \frac{1}{m})^{1+\gamma}}{(p + \gamma - 1)(\beta_2 + \epsilon) + \frac{p-1}{m}} < \lambda_0,$$

then for all $n \geq m$, we have $C_n(\lambda_0) \geq C_n(\lambda_n) = \beta_2 + \epsilon$. So, $\|u_n\|_\infty \leq \beta_2 + \epsilon$.

We can see that using Lemma 3.3, Lemma 3.5 and Lemma 3.6, \bar{J}_0 and G satisfy all the conditions of Lemma (4.10) taken from [24] for some $0 < r < \beta_2 < R$. Since $\beta_2 < \alpha$, $(I - \bar{J}_0)(u)$ has no solution on ∂B_r . Now considering Lemma 3.5 and using Lemma 4.8 of [16], we can obtain a continuum $A_n \subset A = \{(\lambda, u) \in [0, \bar{\lambda}] \times C(\bar{\Omega}) : u - \bar{J}_\lambda(u) = 0\}$ such that

$$A_n \cap (\{0\} \times B_r) \neq \phi, \quad A_n \cap (\{0\} \times (B_R - B_r)) \neq \phi. \quad (3.22)$$

Next, we define $F : [0, \lambda_0] \rightarrow C_0^{1,\alpha}(\bar{\Omega})$ a continuous map such that $F(\lambda) = u_{n,\lambda^*}$. Using Lemma 4.7, we conclude that there exists $u_n \in A_n^{\lambda_0} = \{u \in C(\bar{\Omega}) : (\lambda_0, u) \in A_n\}$ such that $0 < u_n < u_{n,\lambda^*}$. We have $\|u_{n,\lambda^*}\|_\infty \leq \beta_2$ and hence $\|u_n\|_\infty \leq \|u_{n,\lambda^*}\|_\infty \leq \beta_2$.

We have $A_n \cap (\{0\} \times (B_R - B_r)) \neq \phi$ by equation (3.22). Hence, for $n \geq \max(n_0, m)$, there exists v_n such that $\|v_n\|_\infty \geq \beta_2 + \epsilon$. For $\lambda = \lambda_0$ we have at least two solutions u_n and v_n to the problem (2.7). As $\lambda_0 < \bar{\lambda}$ is arbitrary, it concludes the proof. \square

Theorem 3.8. *Given $\gamma > 0$ there exists $\bar{\lambda} > 0$ such that the problem (1.5) admits atleast two solutions u, v in $W_{loc}^{1,p}(\Omega)$, provided Ω is strictly convex with $1 < p < N$, $p-1 < q < \frac{p(N-1)}{N-p} - 1$ and for $0 < \lambda < \bar{\lambda}$.*

Proof. From the above Lemma 3.7, we can conclude the existence of atleast two solutions u_n and v_n of the problem (2.7). Also for a suitable choice of $c > 0$, $\underline{u} = (c\phi_1 + n^{\frac{1+p-\gamma}{p}})^{\frac{p}{\gamma+p-1}} - \frac{1}{n}$ will be a weak subsolution to the problem (3.10) for $\lambda = \lambda_0$.

Again, using $\frac{\lambda_0}{(r+\frac{1}{n})^\gamma} \leq \frac{\lambda_0}{(r+\frac{1}{n})^\gamma} + g(r) + \mu_n$ for all $r \geq 0$ we can conclude that each solutions of the problem (2.7) with $\lambda = \lambda_0$ is a weak supersolution of (3.10). Now by the strong comparison principle [27], we have

$$\bar{u} \leq u_{n,\lambda_0} \leq u_n \leq \beta_2, \quad \bar{u} \leq u_{n,\lambda_0} \leq v_n \text{ and } \|v_n\|_\infty \geq \beta_2 + \epsilon. \quad (3.23)$$

Let us take $z_n = u_n$ or v_n , then from (3.23) and the Lemma 3.6 we have,

$$\bar{u} \leq z_n \leq M,$$

where M is independent of n . By using the strong comparison principle [27] and Lemma 3.2, we have

$$\forall K \subset\subset \Omega, \exists C_K \text{ such that } z_n \geq C_K > 0 \text{ in } K, \forall n \in \mathbb{N}. \quad (3.24)$$

Claim. (z_n) is bounded in $W_{loc}^{1,p}(\Omega)$. *Proof.* Consider $z_n\phi^p$ as a test function in the equation (2.7) for $\phi \in C_0^1(\Omega)$, then we get

$$\int_{\Omega} |\nabla z_n|^p \phi^p = -p \int_{\Omega} \phi^{p-1} z_n |\nabla z_n|^{p-2} \nabla \phi \cdot \nabla z_n + \int_{\Omega} \frac{\lambda_0 z_n \phi^p}{(z_n + \frac{1}{n})^\gamma} + \int_{\Omega} z_n g(z_n) \phi^p + \int_{\Omega} z_n \mu_n$$

By using the modified Young's inequality we have, $\int_{\Omega} |\nabla z_n|^p \phi^p \leq C_\phi \forall n \in \mathbb{N}$, where C_ϕ is a constant depending only on ϕ . Hence, $z_n \in W_{loc}^{1,p}(\Omega)$ and there exists $z \in W_{loc}^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $z_n \rightarrow z$ a.e upto a subsequence and $z_n \rightarrow z$ weakly in $W^{1,p}(K)$ for all $K \subset\subset \Omega$. From the Theorem 4.4 of [13], $\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \phi$ converges to $\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi$.

Again, by using dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{\lambda_0 \phi}{(z_n + \frac{1}{n})^\gamma} + \phi g(z_n) \right) dx = \lambda_0 \int_{\Omega} \frac{\phi}{z^\gamma} dx + \int_{\Omega} \phi g(z) dx$$

Since, $\|u_n\|_\infty \leq \beta_2$, $\|v_n\|_\infty \geq \beta_2 + \epsilon > \beta_2$ and $u_n \rightarrow u, v_n \rightarrow v$, we have the existence of two distinct solutions u and v . \square

We will now prove the existence result of the problem (1.5).

4 Existence result

4.1 The case of $\gamma < 1$.

Let us consider the problem in (2.7) for the case of $\gamma < 1$.

Lemma 4.1. *Let u_n be a solution of (2.7) with $\gamma < 1$. Then (u_n) is bounded in $W_0^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.*

Proof. We will prove the boundedness of (∇u_n) in the Marcinkiewicz space $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$. For this, let us take $\varphi = T_k(u_n)$ as a test function in the weak formulation (2.8) and we have

$$\int_{\Omega} |\nabla T_k(u_n)|^p = \int_{\Omega} \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} T_k(u_n) + \int_{\Omega} g(u_n) T_k(u_n) + \int_{\Omega} T_k(u_n) \mu_n. \quad (4.25)$$

Observe

$$\frac{T_k(u_n)}{(u_n + \frac{1}{n})^\gamma} \leq \frac{u_n}{(u_n + \frac{1}{n})^\gamma} = \frac{u_n^\gamma}{(u_n + \frac{1}{n})^\gamma u_n^{\gamma-1}} \leq u_n^{1-\gamma}$$

and

$$\int_{\Omega} T_k(u_n) \mu_n \leq k \|\mu_n\|_{L^1(\Omega)} \leq Ck.$$

Therefore, we have,

$$\int_{\Omega} |\nabla T_k(u_n)|^p \leq Ck. \quad (4.26)$$

Now consider the following set inclusion

$$\begin{aligned} \{|\nabla u_n| \geq t\} &= \{|\nabla u_n| \geq t, u_n < k\} \cup \{|\nabla u_n| \geq t, u_n \geq k\} \\ &\subset \{|\nabla u_n| \geq t, u_n < k\} \cup \{u_n \geq k\} \subset \Omega. \end{aligned}$$

With the help of the subadditivity property of Lebesgue measure m we have,

$$m(\{|\nabla u_n| \geq t\}) \leq m(\{|\nabla u_n| \geq t, u_n < k\}) + m(\{u_n \geq k\}). \quad (4.27)$$

By the Sobolev inequality, we have

$$\frac{1}{\lambda_1} \left(\int_{\Omega} |T_k(u_n)|^{p^*} \right)^{\frac{p}{p^*}} \leq \int_{\Omega} |\nabla T_k(u_n)|^p \leq Ck \quad (4.28)$$

where λ_1 is the first eigenvalue of the p -Laplacian operator. Now, on restricting the left hand side of the integral (4.28) on $I = \{x \in \Omega : u_n \geq k\}$, such that $T_k(u_n) = k$, we obtain

$$\begin{aligned} k^p m(\{u_n \geq k\})^{\frac{p}{p^*}} &\leq Ck \\ \Rightarrow m(\{u_n \geq k\}) &\leq \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \quad \forall k \geq 1. \end{aligned}$$

Hence, (u_n) is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$.

Similarly on restricting (4.28) on $I' = \{|\nabla u_n| \geq t, u_n < k\}$, we have

$$m(\{|\nabla u_n| \geq t, u_n < k\}) \leq \frac{1}{t^p} \int_{\Omega} |\nabla T_k(u_n)|^p \leq \frac{Ck}{t^p}, \quad \forall k > 1.$$

Now (4.27) becomes

$$m(\{|\nabla u_n| \geq t\}) \leq m(\{|\nabla u_n| \geq t, u_n < k\}) + m(\{u_n \geq k\}) \leq \frac{Ck}{t^p} + \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \quad \forall k > 1.$$

Let us choose, $k = t^{\frac{N-p}{N-1}}$ and hence we get

$$m(\{|\nabla u_n| \geq t\}) \leq \frac{C}{t^{\frac{N(p-1)}{N-1}}}, \quad \forall t \geq 1.$$

We have proved that (∇u_n) is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$. This implies by property (2.6) that (u_n) is bounded in $W_0^{1,r}(\Omega)$, for every $r < \frac{N(p-1)}{N-1}$. \square

Theorem 4.2. *Let $\gamma < 1$. Then there exists a weak solution u of (1.5) in $W_0^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.*

Proof. Lemma 4.1, implies that there exists u such that a subsequence of u_n converges weakly to u in $W_0^{1,r}(\Omega)$, for every $r < \frac{N(p-1)}{N-1}$. This implies that for φ in $C_c^1(\Omega)$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \nabla u_n \cdot \nabla \varphi = \int_{\Omega} \nabla u \cdot \nabla \varphi.$$

Also due to the compact embeddings we can assume that u_n converges to u both strongly in $L^1(\Omega)$ and a.e. in Ω . Thus, taking φ in $C_c^1(\Omega)$, we get,

$$\begin{aligned} 0 &\leq \left| \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} \varphi \right| \\ &\leq C\lambda \|\varphi\|_{L^\infty(\Omega)} \end{aligned}$$

This is sufficient to apply the dominated convergence theorem to obtain

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} \varphi = \int_{\Omega} \frac{\lambda}{u^\gamma} \varphi.$$

Further, since (u_n) is bounded in $W_0^{1,r}(\Omega)$, we have by the compact embedding that $u_n \rightarrow u$ in $L^r(\Omega)$. By the same standard argument, there exists a subsequence that converge to u uniformly except on a set of arbitrarily small Lebesgue measure. Since, by the hypothesis g is continuous, the limit $n \rightarrow \infty$ can be passed on. On applying a similar argument as in step 4 of the Theorem 3.2 in [7], we have a.e. convergence of the ∇u_n towards ∇u that follows in a standard way by proving that $\nabla T_k(u_n)$ goes to $\nabla T_k(u)$, in $L_{loc}^r(\Omega)$ for $r < p$, for every $k > 0$. Finally, we can pass the limit $n \rightarrow \infty$ in the last term of (2.8) involving μ_n . This concludes the proof of the result as it is easy to pass to the limit in (2.8). Therefore, we obtain a weak solution of (1.5) in $W_0^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$. \square

4.2 The case of $\gamma \geq 1$.

Due to the strong singularity we can hold some local estimates on u_n in the Sobolev space. We shall give global estimates on $T_k^{\frac{\gamma+p-1}{2}}(u_n)$ in $W_0^{1,2}(\Omega)$ with the aim of giving sense, at least in a weak sense, to the boundary values of u .

Lemma 4.3. *Let u_n be a solution of (2.7) with $\gamma \geq 1$. Then $T_k^{\frac{\gamma+p-1}{p}}(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every fixed $k > 0$.*

Proof. Consider $\varphi = T_k^\gamma(u_n)$ as a test function in (2.8). We have

$$\begin{aligned} \gamma \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla T_k(u_n) T_k^{\gamma-1}(u_n) \\ = \int_{\Omega} \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} T_k^\gamma(u_n) + \int_{\Omega} g(u_n) T_k^\gamma(u_n) + \int_{\Omega} T_k^\gamma(u_n) \mu_n. \end{aligned} \quad (4.29)$$

We can estimate the term on the left hand side of (4.29) as,

$$\gamma \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla T_k(u_n) T_k^{\gamma-1}(u_n) = \gamma \int_{\Omega} |\nabla T_k^{\frac{\gamma+p-1}{p}}(u_n)|^p. \quad (4.30)$$

As $\frac{T_k^\gamma(u_n)}{(u_n + \frac{1}{n})^\gamma} \leq \frac{u_n^\gamma}{(u_n + \frac{1}{n})^\gamma} \leq 1$, the term on the right hand side of (4.29) can be estimated as,

$$\begin{aligned} \int_{\Omega} \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} T_k^\gamma(u_n) + \int_{\Omega} g(u_n) T_k^\gamma(u_n) + \int_{\Omega} T_k^\gamma(u_n) \mu_n \\ \leq C \lambda k^\gamma + C \int_{\Omega} u_n^q T_k^\gamma(u_n) + k^\gamma \int_{\Omega} \mu_n \\ \leq C \lambda k^\gamma + C M k^\gamma + k^\gamma \int_{\Omega} \mu_n \\ \leq C(k, \gamma) k^\gamma. \end{aligned} \quad (4.31)$$

On combining the previous inequalities (4.30) and (4.31) we get

$$\int_{\Omega} |\nabla T_k^{\frac{\gamma+p-1}{p}}(u_n)|^p \leq C k^\gamma \quad (4.32)$$

then, $\left(T_k^{\frac{\gamma+p-1}{p}}(u_n) \right)$ is bounded in $W_0^{1,p}(\Omega)$ for every fixed $k > 0$. \square

Now, so as to pass to the limit $n \rightarrow \infty$ in the weak formulation (2.8), we require to prove some local estimates on u_n . We first prove the following.

Lemma 4.4. *Let u_n be a solution of (2.7) with $\gamma \geq 1$. Then (u_n) is bounded in $W_{loc}^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.*

Proof. We prove the theorem in two steps.

Step 1. We claim that $(G_1(u_n))$ is bounded in $W_0^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.

We can see that $G_1(u_n) = 0$ when $0 \leq u_n \leq 1$, $G_1(u_n) = u_n - 1$, otherwise i.e when $u_n > 1$. So $\nabla G_1(u_n) = \nabla u_n$ for $u_n > 1$.

Now, we need to show that $(\nabla G_1(u_n))$ is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$, where $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$ is the Marcinkiewicz space. Then we have

$$\begin{aligned} \{|\nabla u_n| > t, u_n > 1\} &= \{|\nabla u_n| > t, 1 < u_n \leq k+1\} \cup \{|\nabla u_n| > t, u_n > k+1\} \\ &\subset \{|\nabla u_n| > t, 1 < u_n \leq k+1\} \cup \{u_n > k+1\} \subset \Omega. \end{aligned}$$

Hence,

$$m(\{|\nabla u_n| > t, u_n > 1\}) \leq m(\{|\nabla u_n| > t, 1 < u_n \leq k+1\}) + m(\{u_n > k+1\}). \quad (4.33)$$

In order to estimate (4.33) we take $\varphi = T_k(G_1(u_n))$, for $k > 1$, as a test function in (2.7). We observe that $\nabla T_k(G_1(u_n)) = \nabla u_n$ only when $1 < u_n \leq k+1$, otherwise is zero, and $T_k(G_1(u_n)) = 0$ on $\{u_n \leq 1\}$, we have

$$\begin{aligned} \int_{\Omega} |\nabla T_k(G_1(u_n))|^p &= \int_{\Omega} \frac{\lambda}{(u + \frac{1}{n})^\gamma} T_k(G_1(u_n)) + \int_{\Omega} g(u_n) T_k(G_1(u_n)) + \int_{\Omega} T_k(G_1(u_n)) \mu_n \\ &\leq C\lambda k + Ck \int_{\Omega} u_n^q + k \int_{\Omega} \mu_n \\ &\leq Ck \end{aligned}$$

and by restricting the above integral on $I_1 = \{1 < u_n \leq k+1\}$, we get

$$\begin{aligned} \int_{\{1 < u_n \leq k+1\}} |\nabla T_k(G_1(u_n))|^p &= \int_{\{1 < u_n \leq k+1\}} |\nabla u_n|^p \\ &\geq \int_{\{|\nabla u_n| > t, 1 < u_n \leq k+1\}} |\nabla u_n|^p \\ &\geq t^p m(\{|\nabla u_n| > t, 1 < u_n \leq k+1\}) \end{aligned}$$

so that,

$$m(\{|\nabla u_n| > t, 1 < u_n \leq k+1\}) \leq \frac{Ck}{t^p} \quad \forall k \geq 1.$$

According to (4.32) in the proof of Lemma 3.2, one can see that

$$\int_{\Omega} |\nabla T_k^{\frac{\gamma+p-1}{p}}(u_n)|^p \leq Ck^\gamma \quad \text{for any } k > 1.$$

Therefore, from the Sobolev inequality

$$\frac{1}{\lambda_1} \left(\int_{\Omega} |T_k^{\frac{\gamma+p-1}{p}}(u_n)|^{p^*} \right)^{\frac{p}{p^*}} \leq \int_{\Omega} |\nabla T_k^{\frac{\gamma+p-1}{p}}(u_n)|^p \leq Ck^\gamma,$$

where, λ_1 is the first eigenvalue of the p -Laplacian operator. Now, if we restrict the integral on the left hand side on $I_2 = \{u_n > k+1\}_{x \in \Omega}$, on which $T_k(u_n) = k$, we then obtain

$$k^{\gamma+p-1} m(\{u_n > k+1\})^{\frac{p}{p^*}} \leq Ck^\gamma,$$

so that

$$m(\{u_n > k+1\}) \leq \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \quad \forall k \geq 1.$$

So, (u_n) is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$, i.e. $(G_1(u_n))$ is also bounded in $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$.
Now (4.33) becomes

$$\begin{aligned} m(\{|\nabla u_n| > t, u_n > 1\}) &\leq m(\{|\nabla u_n| > t, 1 < u_n \leq k+1\}) + m(\{u_n > k+1\}) \\ &\leq \frac{Ck}{t^p} + \frac{C}{k^{\frac{N(p-1)}{N-p}}}, \forall k > 1. \end{aligned}$$

We then choose, $k = t^{\frac{N-p}{N-1}}$ and we get

$$m(\{|\nabla u_n| > t, u_n > 1\}) \leq \frac{C}{t^{\frac{N(p-1)}{N-1}}} \quad \forall t \geq 1.$$

We just proved that $(\nabla u_n) = (\nabla G_1(u_n))$ is bounded in $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$. This implies by property (2.6) that $(G_1(u_n))$ is bounded in $W_0^{1,r}$ for every $r < \frac{N(p-1)}{N-1}$.

Step 2. We claim that $T_1(u_n)$ is bounded in $W_{loc}^{1,r}(\Omega)$.

We have to examine the behavior of u_n for small values of u_n for each n . We want to show that for every $K \subset\subset \Omega$,

$$\int_K |\nabla T_1(u_n)|^p \leq C. \quad (4.34)$$

We have already proved that $u_n \geq C_K > 0$ on K in Lemma 3.2. We will use $\varphi = T_1^\gamma(u_n)$ as a test function in (2.8) to get

$$\begin{aligned} \gamma \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla T_k(u_n) T_k^{\gamma-1}(u_n) \\ = \int_{\Omega} \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} T_k^\gamma(u_n) + \int_{\Omega} g(u_n) T_k^\gamma(u_n) + \int_{\Omega} T_k^\gamma(u_n) \mu_n \\ \leq C. \end{aligned} \quad (4.35)$$

Now observe that

$$\begin{aligned} \gamma \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla T_1(u_n) T_1^{\gamma-1}(u_n) &\geq \int_K |\nabla T_1(u_n)|^p T_1^{\gamma-1}(u_n) \\ &\geq C_K^{\gamma-1} \int_K |\nabla T_1(u_n)|^p. \end{aligned} \quad (4.36)$$

On combining (4.35) and (4.36) we get (4.34). We completed the proof as $u_n = T_1(u_n) + G_1(u_n)$. Hence, (u_n) is bounded in $W_{loc}^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$. \square

Now, we can finally state and prove the existence result for $\gamma \geq 1$.

Theorem 4.5. *Let $\gamma \geq 1$. Then there exists a weak solution u of (1.5) in $W_{loc}^{1,r}(\Omega)$ for every $r < \frac{N(p-1)}{N-1}$.*

Proof. The proof of this theorem is a straightforward application of the Theorem 4.2 and using the results in Lemma 4.3 and Lemma 4.4. \square

Some Important results

Define, $X = \{u \in C_0^{1,\alpha}(\bar{\Omega}) : u(x) \geq 0 \text{ in } \bar{\Omega}\}$ and let ξ is a unit outward normal at $\partial\Omega$, then define $X_0 = \{u \in C_0^{1,\alpha}(\bar{\Omega}) : u(x) > 0 \text{ and } \frac{\partial u}{\partial \xi}(x) < 0, \forall x \in \partial\Omega\}$. Clearly X_0 is the interior of X .

Lemma 4.6. *If $u_1, u \in C_0^{1,\alpha}(\bar{\Omega})$ with $u_1 \neq u$ and*

$$-\Delta_p u_1 > \frac{\lambda}{(u_1 + \frac{1}{n})^\gamma} + g(u_1) + \mu_n,$$

$$-\Delta_p u = \frac{\lambda}{(u + \frac{1}{n})^\gamma} + g(u) + \mu_n,$$

then $(u_1 - u) \notin \partial X$.

Proof. We prove this Lemma by contradiction. Suppose $(u_1 - u) \in \partial X$. Then $u_1(x) \geq u(x)$. By Strong maximum principle [27], we can obtain $(u_1 - u) \in X_0$. But $X_0 \cap \partial X = \emptyset$, for which we get a contradiction. Therefore, $u_1 - u$ does not belong to ∂X . \square

Lemma 4.7. *Assume I is an interval in \mathbb{R} and $A = I \times C_0^{1,\alpha}(\bar{\Omega})$ is a connected set of solutions of (2.7). Define $F : I \rightarrow C_0^{1,\alpha}(\bar{\Omega})$ is continuous such that $F(\lambda)$ is a supersolution to the problem (2.7).*

If $u_1 \leq F(\lambda_1)$ in Ω , $u_1 \neq F(\lambda_1)$ for some $(\lambda_1, u_1) \in A$, then $u < F(\lambda)$ in Ω , $\forall (\lambda, u) \in A$.

Proof. Let $Z : A \rightarrow C_0^{1,\alpha}(\bar{\Omega})$ is a continuous map such that $Z(\lambda, u) = F(\lambda) - u$. A is connected, so by continuity $Z(A)$ is connected in $C_0^{1,\alpha}(\bar{\Omega})$.

Using Lemma 4.6, $F(\lambda_1) - u_1 = Z(\lambda_1, u_1) \notin \partial X$. Hence, $Z(\lambda_1, u_1) \in X_0$. So, $Z(A) \subset X_0$, as $Z(A)$ is connected.

Therefore, $F(\lambda) - u > 0$, which implies $F(\lambda) > u$, $\forall (\lambda, u) \in A$. Hence, we get our required result. \square

Lemma 4.8. *[Ambrosetti-Arcoya [16]]. Given X be a real Banach space with $U \subset X$ be open, bounded set. Let $a, b \in \mathbb{R}$ such that the equation $u - T(\lambda, u) = 0$ has no solution on ∂U for all $\lambda \in [a, b]$ and that $u - T(\lambda, u) = 0$ has no solution in \bar{U} for $\lambda = b$. Also let $U_1 \subset U$ be open such that $u - T(\lambda, u) = 0$ has no solution in ∂U_1 for $\lambda = a$ and $\deg(I - K_a, U_1, 0) \neq 0$.*

Then there exists a continuum C in $\Sigma = \{(\lambda, u) \in [a, b] \times X : u - T(\lambda, u) = 0\}$ such that

$$C \cap (\{a\} \times U_1) \neq \emptyset \text{ and } C \cap (\{a\} \times (U - U_1)) \neq \emptyset.$$

Theorem 4.9. *[Mitidieri-Pohozaev [28]]. If $p - 1 < q < \frac{N(p-1)}{N-p}$, $p < N$ and $C > 0$, then the problem*

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \geq C \int_{\mathbb{R}^n} u^q \phi; \quad \phi \in C_c^\infty(\mathbb{R}^n)$$

does not have any positive solution in $C^1(\mathbb{R}^n)$.

Theorem 4.10. [De Figueiredo et al. [24]]. Let C be a cone in a Banach space X and $\phi : C \rightarrow C$ be a compact map such that $\phi(0) = 0$. Assume that there exists $0 < r < R$ such that

1. $x \neq t\phi(x)$ for $0 \leq t \leq 1$ and $\|x\| = r$
2. a compact homotopy $F : \bar{B}_R \times [0, \infty) \rightarrow C$ such that $F(x, 0) = \phi(x)$ for $\|x\| = R$, $F(x, t) \neq x$ for $\|x\| = R$ and $0 \leq t < \infty$ and $F(x, t) = x$ has no solution for $x \in \bar{B}_R$ for $t \geq t_0$.

Then if, $U = \{x \in C : r < \|x\| < R\}$ and $B_\rho = \{x \in C : \|x\| < \rho\}$ we have $\deg(I - \phi, B_R, 0) = 0$, $\deg(I - \phi, B_r, 0) = 1$ and $\deg(I - \phi, U, 0) = -1$.

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