

**COUPLED GRAVITATIONAL AND ELECTROMAGNETIC
PERTURBATIONS OF REISSNER-NORDSTRÖM SPACETIME
IN A POLARIZED SETTING
II - Combined estimates for the system of wave equations**

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Abstract

In a previous paper on coupled gravitational and electromagnetic perturbations of Reissner-Nordström spacetime in a polarized setting ([2]), we derived a system of wave equations for two independent quantities, one related to the Weyl curvature and one related to the Ricci curvature of the perturbed spacetime. We analyze here the system of coupled wave equations, deriving combined energy-Morawetz and r^p -estimates for the system, in the case of small charge.

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Introduction

The final state conjecture in General Relativity for charged black holes states that the Kerr-Newman spacetime is stable under small perturbation of initial data as solutions to the Einstein-Maxwell equation:

$$\text{Ric}(g)_{\mu\nu} = T(F)_{\mu\nu} := 2F_{\mu\lambda}F^\lambda{}_\nu - \frac{1}{2}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \tag{1}$$

where F is a 2-form satisfying Maxwell's equations

$$\nabla_{[\alpha}F_{\beta\gamma]} = 0, \quad \nabla^\alpha F_{\alpha\beta} = 0. \tag{2}$$

In the case of charged black holes, gravitational and electromagnetic perturbations have to be considered together, and they are coupled one another. One step towards the proof of the non-linear stability of Kerr-Newman spacetime is the linear stability of the spherically symmetric charged

black holes, namely the Reissner-Nordström solution, corresponding to Kerr-Newman spacetime with zero angular momentum. For a brief review of previous results on stability of the Einstein and the Einstein-Maxwell's equation see the introduction in [2], and references therein.

In the present paper, we continue the project started in [2] in the context of stability of Reissner-Nordström spacetime under polarized perturbations. In [2], we derived a coupled system of wave equations for two independent quantities \mathfrak{q} and $\mathfrak{q}^{\mathbf{F}}$, related to the Weyl curvature and to the Ricci curvature of the perturbed spacetime respectively. We recall here the main definitions of Section 6.1 of [2]. The main quantities, related to the Weyl curvature and to the Ricci curvature of the polarized electrovacuum spacetime, form a hierarchy of five quantities in the following way:

$$\begin{aligned}\psi_0 &= r^2 \underline{\kappa}^2 \alpha, \\ \psi_1 &= \underline{P}(\psi_0), \\ \psi_2 &= \underline{P}(\psi_1) = \underline{P}(\underline{P}(\psi_0)) =: \mathfrak{q}, \\ \psi_3 &= r^2 \underline{\kappa} \mathfrak{f}, \\ \psi_4 &= \underline{P}(\psi_3) =: \mathfrak{q}^{\mathbf{F}}\end{aligned}\tag{3}$$

where the operator \underline{P} is a rescaled null derivative in the ingoing direction e_3 defined as

$$\underline{P}(f) = r \underline{\kappa}^{-1} e_3 f + \frac{1}{2} r f.\tag{4}$$

The quantities $\psi_2 = \mathfrak{q}$ and $\psi_4 = \mathfrak{q}^{\mathbf{F}}$ are of particular importance because they verify Regge-Wheeler type equations which are coupled one another. Notice that ψ_0 and ψ_1 are lower order terms with respect to \mathfrak{q} , in the sense that they can be obtained by \mathfrak{q} integrating along the e_3 direction. Similarly, ψ_3 is a lower order term with respect to $\mathfrak{q}^{\mathbf{F}}$. We recall the main theorem in [2], summarizing the system of coupled wave equations.

Theorem 0.1 (Theorem 6.12 in [2]). *Let $(\mathbf{M}, \mathbf{g}, \mathbf{Z})$ be an axially symmetric polarized spacetime solution of the Einstein-Maxwell equation, which is a $O(\epsilon)$ -perturbation of Reissner-Nordström spacetime. Denote its quasi-local charge by e . Then there exist $O(\epsilon^2)$ -invariant quantities \mathfrak{q} and $\mathfrak{q}^{\mathbf{F}}$ related to the Weyl curvature and to the Ricci curvature respectively that verify the following coupled system of wave equations, modulo $O(\epsilon^2)$ terms,*

$$\begin{cases} \left(\square_2 + \kappa \underline{\kappa} - 10 {}^{(F)}\rho^2 \right) \mathfrak{q} = e \left(\frac{2}{r} \underline{\Delta}_2 \mathfrak{q}^{\mathbf{F}} - \frac{2}{r^2} Q \mathfrak{q}^{\mathbf{F}} - \frac{2}{r^2} \kappa \underline{\kappa} P \mathfrak{q}^{\mathbf{F}} + \frac{1}{r} (5 \kappa \underline{\kappa} + 8 \rho + 4 {}^{(F)}\rho^2) \mathfrak{q}^{\mathbf{F}} \right) + e(l.o.t.)_1, \\ \left(\square_2 + \kappa \underline{\kappa} + 3 \rho \right) \mathfrak{q}^{\mathbf{F}} = e \left(-\frac{2}{r^3} \mathfrak{q} \right) + e^2(l.o.t.)_2 \end{cases}\tag{5}$$

where Q is a rescaled null derivatives in the outgoing direction defined as

$$Q(f) = r \underline{\kappa} e_4 f + \frac{1}{2} r \kappa \underline{\kappa} f\tag{6}$$

and $(l.o.t.)_1$ and $(l.o.t.)_2$ are lower order terms with respect to \mathfrak{q} and $\mathfrak{q}^{\mathbf{F}}$, explicitly,

$$\begin{aligned}(l.o.t.)_1 &= -6 \rho \psi_3 + e \left(-\frac{4}{r^3} \psi_1 - \frac{2}{r^2} \psi_0 \right) + e^2 \left(-\frac{20}{r^4} \psi_3 \right), \\ (l.o.t.)_2 &= \frac{4}{r^3} \psi_3\end{aligned}$$

According to Theorem 0.1, we can write the system (5) in the following concise form:

$$\begin{cases} (\square_2 - V_1)\mathbf{q} = e C_1[\mathbf{q}^{\mathbf{F}}] + e L_1[\mathbf{q}^{\mathbf{F}}] + e^2 L_1[\mathbf{q}] + N_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}], \\ (\square_2 - V_2)\mathbf{q}^{\mathbf{F}} = e C_2[\mathbf{q}] + e^2 L_2[\mathbf{q}^{\mathbf{F}}] + N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] \end{cases} \quad (7)$$

where

$$V_1 = -\kappa_{\underline{\kappa}} + 10^{(F)}\rho^2, \quad (8)$$

$$V_2 = -\kappa_{\underline{\kappa}} - 3\rho, \quad (9)$$

$$C_1[\mathbf{q}^{\mathbf{F}}] = \frac{2}{r}\mathcal{A}_2\mathbf{q}^{\mathbf{F}} - \frac{2}{r^2}Q\mathbf{q}^{\mathbf{F}} - \frac{2}{r^2}\kappa_{\underline{\kappa}}P\mathbf{q}^{\mathbf{F}} + \frac{1}{r}(5\kappa_{\underline{\kappa}} + 8\rho + 4^{(F)}\rho^2)\mathbf{q}^{\mathbf{F}}, \quad (10)$$

$$C_2[\mathbf{q}] = -\frac{2}{r^3}\mathbf{q}, \quad (11)$$

$$L_1[\mathbf{q}] = -\frac{4}{r^3}\psi_1 - \frac{2}{r^2}\psi_0, \quad (12)$$

$$L_1[\mathbf{q}^{\mathbf{F}}] = -6\rho\psi_3 - e^2\frac{20}{r^4}\psi_3, \quad (13)$$

$$L_2[\mathbf{q}^{\mathbf{F}}] = \frac{4}{r^3}\psi_3 \quad (14)$$

The terms C_s and L_s are the linear terms of the equations, all multiplied by the charge of the spacetime, and the terms N_1 and N_2 are the quadratic, i.e. $O(\epsilon^2)$, error terms. In particular:

- The terms C_1 and C_2 are the terms representing the coupling between the Weyl curvature and the Ricci curvature. In the wave equation for \mathbf{q} , the coupling term $C_1 = C_1[\mathbf{q}^{\mathbf{F}}]$ is an expression in terms of $\mathbf{q}^{\mathbf{F}}$ and in the wave equation for $\mathbf{q}^{\mathbf{F}}$, the coupling term $C_2 = C_2[\mathbf{q}]$ is an expression in terms of \mathbf{q} .
- The terms L_1 and L_2 collect the lower order terms: in particular $L_1[\mathbf{q}]$ are lower order terms with respect to \mathbf{q} , i.e. ψ_0 and ψ_1 , while $L_1[\mathbf{q}^{\mathbf{F}}]$ and $L_2[\mathbf{q}^{\mathbf{F}}]$ are lower terms with respect to $\mathbf{q}^{\mathbf{F}}$, i.e. ψ_3 . The index 1 or 2 denotes if they appear in the first or in the second equation.

The aim of this paper is to prove the following theorem.

Theorem 0.2. *Let $(\mathbf{M}, \mathbf{g}, \mathbf{Z})$ be an axially symmetric polarized spacetime solution of the Einstein-Maxwell equation, which is a $O(\epsilon)$ -perturbation of Reissner-Nordström spacetime, and let $\psi_0, \psi_1, \mathbf{q}, \psi_3, \mathbf{q}^{\mathbf{F}}$ be its curvature terms as defined in (3). Then energy boundedness and weighted r^p -estimates hold as in Theorem 2.1.*

As remarked in Appendix A of [2], the system (5) is also valid for non-polarized electrovacuum perturbations of Reissner-Nordström spacetime. Therefore, Theorem 0.2 is a fundamental step in the proof of linear stability of Reissner-Nordström spacetime.

The paper is organized as follows. In Section 1, we define the main weighted energies and bulks used in the estimates. The energy boundedness and weighted r^p -estimates for the system are obtained in Section 2. The strategy to obtain the combined estimates is the following:

1. **Separated estimates** Consider the two equations of the system separately as

$$(\square_2 - V_1)\mathbf{q} = \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] \quad (15)$$

and

$$\left(\square_2 - V_2\right)q^{\mathbf{F}} = \mathbf{M}_2[q, q^{\mathbf{F}}] \quad (16)$$

where the linear terms of $\mathbf{M}_1[q, q^{\mathbf{F}}]$ and $\mathbf{M}_2[q, q^{\mathbf{F}}]$ are $O(e)$. More precisely

$$\begin{aligned} \mathbf{M}_1[q, q^{\mathbf{F}}] &= e C_1[q^{\mathbf{F}}] + e L_1[q^{\mathbf{F}}] + e^2 L_1[q] + N_1[q, q^{\mathbf{F}}], \\ \mathbf{M}_2[q, q^{\mathbf{F}}] &= e C_2[q] + e^2 L_2[q^{\mathbf{F}}] + N_2[q, q^{\mathbf{F}}]. \end{aligned}$$

We apply to both equations separately the standard procedures used to derive energy-Morawetz estimates to equation of the forms $\square_{\mathbf{g}}\Psi = V\Psi + \mathbf{M}[\Psi]$. Using the Morawetz vector field as multiplier and the r^p -method, we derive separated Morawetz estimates and r^p -weighted estimates, (43)-(44), as long as higher derivative estimates, (49)-(50), for the equations (15) and (16). We will summarize the results we need in Section 2.1, omitting the complete proofs for the sake of the clarity of the derivation of the combined estimates. The proofs will appear in [3].

Notice that these separated estimates will contain on the right hand side terms involving \mathbf{M}_1 and \mathbf{M}_2 that at this stage are not controlled. In particular \mathbf{M}_1 and \mathbf{M}_2 contain both the coupling terms C and the lower order terms L . We will not deal with the non-linear terms N_1 and N_2 in this paper.

2. **Absorption of the coupling terms** The coupling term $C_1[q^{\mathbf{F}}]$ in the first equation involves up to second derivative of $q^{\mathbf{F}}$. On the other hand, the coupling term $C_2[q]$ in the second equation contains no derivative of q . In order to take into account the difference in the presence of derivatives, we consider the 0th-order estimate for equation (15) and the 1st-order estimate for equation (16) and we add them together. This operation will create a combined estimate, where the Morawetz bulks on the left hand side of each equations will absorb the coupling term on the right hand side of the other equation, as in Proposition 2.5. In the trapping region, this absorption is delicate because of the degeneracy of the bulk norms. Nevertheless, the special structure of the coupling terms $C_1[q^{\mathbf{F}}]$ and $C_2[q]$ implies a cancellation of problematic terms in the trapping region, as in Proposition 2.6. This is done in Section 2.
3. **Absorption of the lower order terms** To absorb the lower order terms $L_1[q]$, $L_1[q^{\mathbf{F}}]$ and $L_2[q^{\mathbf{F}}]$ in the combined estimate, we derive transport estimates for ψ_0 , ψ_1 and ψ_3 . We make use of the differential relation (3) to get non-degenerate energy estimates in Proposition 2.8. Using these estimates, we will be able to control the norms involving the lower terms, as done in Proposition 2.9. This is done in Section 2.3.

Summing the separated estimates and absorbing the coupling terms and the lower order terms on the right hand side we obtain a combined estimate for the system as in Theorem 2.1.

The presence of the lower order terms in the equations is treated as for the Teukolsky equation in Kerr spacetime, as previously done in [6] and [1]. On the other hand, the absorption of the coupling terms is proper of the electrovacuum case, with coupled and independent gravitational and electromagnetic perturbations.

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1 Preliminaries and definition of main weighted quantities

We consider an electrovacuum spacetime \mathcal{M} which is a $O(\epsilon)$ -perturbation of Reissner-Nordström spacetime, as defined in Definition 4.1 of [2]. Suppose that the original Reissner-Nordström spacetime is given by

$$\begin{aligned} \mathbf{g}_{RN}(m_0, Q_0) &= -\left(1 - \frac{2m_0}{r} + \frac{Q_0^2}{r^2}\right) dt^2 + \left(1 - \frac{2m_0}{r} + \frac{Q_0^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ &= -\Upsilon(m_0, Q_0) dt^2 + \Upsilon(m_0, Q_0)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned}$$

with initial mass m_0 and initial charge Q_0 , where $\Upsilon(m_0, Q_0) = 1 - \frac{2m_0}{r} + \frac{Q_0^2}{r^2}$. In particular the horizon and the photon sphere of the unperturbed spacetime are respectively given by

$$r_{\mathcal{H}}(m_0, Q_0) = m_0 + \sqrt{m_0^2 - Q_0^2}, \quad r_P(m_0, Q_0) = \frac{3m_0 + \sqrt{9m_0^2 - 8Q_0^2}}{2}$$

Defining the modified Hawking mass ϖ and the quasi-local charge e of the perturbed spacetime \mathcal{M} as in Section 3 of [2], then $|\varpi - m_0| \leq \epsilon_0$ and $|e - Q_0| \leq \epsilon_0$, for a small $\epsilon_0 > 0$, through the entire evolution. For the perturbed spacetime we can define $\Upsilon := 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2}$, using the quasi-local mass ϖ and charge e .

As in the construction of the perturbed spacetime in [5], we assume that the spacetime \mathcal{M} contains a timelike hypersurface \mathcal{T} external to the horizon that separates the spacetime in two regions, such that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, with \mathcal{M}_2 bounded. The spacetime \mathcal{M} is foliated by 2-spheres S and comes equipped with two scalar functions r and u with u optical in \mathcal{M}_1 normalized on the last incoming slice $\underline{\mathcal{C}}_*$, and r being the area radius of the spheres S .

We divide \mathcal{M} in the following regions:

1. The red shift region $\mathcal{M}_{red} = \mathcal{M}_2$:

$$r \in [r_A := r_{\mathcal{H}}(m_0, Q_0)(1 - \delta_0), r_{\mathcal{H}}(m_0, Q_0)(1 + \delta_0)]$$

for a small $\delta_0 > 0$.

2. Trapping region \mathcal{M}_{trap} :

$$\frac{5}{6}r_P(m_0, Q_0) \leq r \leq \frac{7}{6}r_P(m_0, Q_0)$$

3. The far region \mathcal{M}_{far} :

$$r \geq R_0$$

with R_0 a fixed number $R_0 \gg \frac{7}{6}r_P(m_0, Q_0)$.

For fixed R we denote by $\mathcal{M}_{\leq R}$ and $\mathcal{M}_{\geq R}$ the regions defined by $r \leq R$ and $r \geq R$. We denote by $\mathcal{M}_{trq\phi}$ any region outside the trapping region \mathcal{M}_{trap} .

We foliate our spacetime domain \mathcal{M} by \mathbf{Z} invariant hypersurfaces $\Sigma(\tau)$ which are:

1. Incoming null in \mathcal{M}_{red} , with e_3 as null incoming generator. We denote this portion $\Sigma_{red}(\tau)$.

2. Strictly spacelike in \mathcal{M}_{trap} . We denote this portion by Σ_{trap} .
3. Outgoing null in $\mathcal{M}_{far} = \mathcal{M}_{r \geq R_0}$ with e_4 as null outgoing generator. We denote this portion by Σ_{far} or $\Sigma_{\geq R_0}$. We can in fact assume that $\Sigma_{r \geq R_0}$ are the level hypersurfaces of the optical function u .

We introduce the following vectorfields, used in the derivation of the estimates:

- $T = \frac{1}{2}(e_4 + \Upsilon e_3)$,
- $R = \frac{1}{2}(e_4 - \Upsilon e_3)$,
- The redshift vectorfield $Y_{\mathcal{H}}$ supported in the region $|\Upsilon| \leq 2\delta_{\mathcal{H}}$.
- Let θ a smooth bump function equal 1 on $|\Upsilon| \leq \delta_{\mathcal{H}}$ vanishing for $|\Upsilon| \geq 2\delta_{\mathcal{H}}$ and define the vectorfields,

$$\begin{aligned}\check{R} &:= \theta \frac{1}{2}(e_4 - e_3) + (1 - \theta)\Upsilon^{-1}R = \frac{1}{2}[\check{\theta}e_4 - e_3] \\ \check{T} &:= \theta \frac{1}{2}(e_4 + e_3) + (1 - \theta)\Upsilon^{-1}T = \frac{1}{2}[\check{\theta}e_4 + e_3]\end{aligned}\tag{17}$$

where $\check{\theta} = \theta + \Upsilon^{-1}(1 - \theta)$. Note that $2(|\check{R}\Psi|^2 + |\check{T}\Psi|^2) = |e_3\Psi|^2 + \check{\theta}^2|e_4\Psi|^2$.

- $\check{e}_4 = e_4 + \frac{1}{r}$

We introduce the following quantities for a symmetric traceless S -horizonthal tensor Ψ in regions $\mathcal{M}(\tau_1, \tau_2) \subset \mathcal{M}$ in the past of $\Sigma(\tau_2)$ and in the future of $\Sigma(\tau_1)$.

1. Energy quantities on $\Sigma(\tau)$:

- Basic energy quantity

$$E[\Psi](\tau) = \int_{\Sigma(\tau)} \left(\frac{1}{2}(N_{\Sigma}, e_3)^2 |e_4\Psi|^2 + \frac{1}{2}(N_{\Sigma}, e_4)^2 |e_3\Psi|^2 + |\not{V}\Psi|^2 + r^{-2}|\Psi|^2 \right)\tag{18}$$

- Weighted energy quantities in the far away region

$$\begin{aligned}\dot{E}_{p;R}[\Psi](\tau) &:= \int_{\Sigma_{\geq R}(\tau)} r^p |\check{e}_4\Psi|^2, \\ \dot{E}'_{p;R}[\Psi](\tau) &:= \int_{\Sigma_{\geq R}(\tau)} r^p (|\check{e}_4\Psi|^2 + r^{-2}|\Psi|^2)\end{aligned}\tag{19}$$

- Weighted energy quantities

$$\begin{aligned}E_p[\Psi](\tau) &:= E[\Psi](\tau) + \dot{E}_{p;R}[\Psi](\tau) \\ E'_p[\Psi](\tau) &:= E[\Psi](\tau) + \dot{E}'_{p;R}[\Psi](\tau)\end{aligned}\tag{20}$$

2. Morawetz bulk quantities in $\mathcal{M}(\tau_1, \tau_2)$:

- Basic Morawetz bulk

$$\text{Mor}[\Psi](\tau_1, \tau_2) := \int_{\mathcal{M}(\tau_1, \tau_2)} \frac{\varpi^2}{r^3} |\check{R}(\Psi)|^2 + \frac{\varpi}{r^4} |\Psi|^2 + \frac{(r^2 - 3\varpi r + 2e^2)^2}{r^5} \left(|\check{\mathcal{V}}\Psi|^2 + \frac{\varpi^2}{r^2} |\check{T}\Psi|^2 \right) \quad (21)$$

- Improved Morawetz bulk

$$\begin{aligned} \text{Morr}[\Psi](\tau_1, \tau_2) &:= \text{Mor}[\Psi](\tau_1, \tau_2) + \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-1-\delta} |e_3(\Psi)|^2 \\ &= \int_{\mathcal{M}_{trap}} |R(\Psi)|^2 + r^{-2} |\Psi|^2 + \frac{(r^2 - 3\varpi r + 2e^2)^2}{r^4} \left(|\check{\mathcal{V}}\Psi|^2 + \frac{D}{r^2} |T\Psi|^2 \right) \\ &\quad \int_{\mathcal{M}_{trap}} r^{-3} (|e_4\Psi|^2 + r^{-1} |\Psi|^2) + r^{-1} |\check{\mathcal{V}}\Psi|^2 + r^{-1-\delta} |e_3(\Psi)|^2 \end{aligned} \quad (22)$$

- Weighted bulk norms in the far away region

$$\dot{\mathcal{M}}_{p;R}[\Psi](\tau_1, \tau_2) := \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^{p-1} (p|\check{e}_4(\Psi)|^2 + (2-p)|\check{\mathcal{V}}\Psi|^2 + r^{-2} |\Psi|^2) \quad (23)$$

- Weighted bulk norms

$$\mathcal{M}_p[\Psi](\tau_1, \tau_2) := \text{Morr}[\Psi](\tau_1, \tau_2) + \dot{\mathcal{M}}_{p;R}[\Psi](\tau_1, \tau_2) \quad (24)$$

- Non-degenerate bulk norm

$$\hat{\mathcal{M}}_p[\Psi](\tau_1, \tau_2) := \int_{\mathcal{M}(\tau_1, \tau_2)} r^{-3+p} |\Psi|^2 + r^{-1+p} |e_4(\Psi)|^2 + r^{-1+p} |\check{\mathcal{V}}\Psi|^2 \quad (25)$$

3. Flux through $\underline{\mathcal{C}}_*(\tau_1, \tau_2) = \underline{\mathcal{C}}_* \cap \mathcal{M}(\tau_1, \tau_2)$:

- Basic flux quantity

$$F[\Psi](\tau_1, \tau_2) = \int_{\underline{\mathcal{C}}_*(\tau_1, \tau_2)} (|e_3\Psi|^2 + |\check{\mathcal{V}}\Psi|^2 + r^{-2} |\Psi|^2) \quad (26)$$

- Weighted Flux quantity

$$\dot{F}_p[\Psi](\tau_1, \tau_1) := \int_{\underline{\mathcal{C}}_*(\tau_1, \tau_2)} r^p (|\check{\mathcal{V}}\Psi|^2 + r^{-2} |\Psi|^2) \quad (27)$$

$$F_p[\Psi](\tau_1, \tau_1) := F[\Psi](\tau_1, \tau_1) + \dot{F}_p[\Psi](\tau_1, \tau_1)$$

4. Norm for $\mathbf{M}[\Psi]$: for $p \geq \delta$,

$$\mathcal{I}_p[\Psi, \mathbf{M}](\tau_1, \tau_2) := \int_{\mathcal{M}_{trap}} r^{1+p} |\mathbf{M}|^2 + \int_{\mathcal{M}_{far}} (|R\Psi| + r^{-1} |\Psi|) |\mathbf{M}| \quad (28)$$

5. Quadratic error term:

$$\text{Err}_{\epsilon, \delta}(\tau_1, \tau_2)[\Psi] = O(\epsilon) \int_{\mathcal{M}_{\geq R_0}(\tau_1, \tau_2)} r^{-1+\delta} (|e_4\Psi|^2 + |\check{\mathcal{V}}\Psi|^2 + r^{-2} |\Psi|^2) \quad (29)$$

Remark 1.1. Notice that the Morawetz bulk $\mathcal{M}_p[\Psi]$ is degenerate in the trapping region. Notice that the norm for $\mathbf{M}[\Psi]$ given by $\mathcal{I}_p[\Psi, \mathbf{M}](\tau_1, \tau_2)$ does not contain the term $\int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{\text{trap}}(\tau)} |T\Psi| |\mathbf{M}|$. Indeed, this term has to be kept in the estimates with its sign, in order to be cancelled out in the analysis of the coupling terms.

For each of these quantities, we define their higher derivative version. We denote $\mathfrak{d} = \{e_3, re_4, \mathfrak{d}\}$, where $\mathfrak{d} = r\mathcal{V}$ denotes angular derivative.¹ We define

1. Higher derivative energies:

$$\begin{aligned} E^s[\Psi] &= \sum_{0 \leq k \leq s} E[\mathfrak{d}^k \Psi], & \dot{E}_{p;R}^s[\Psi] &= \sum_{0 \leq k \leq s} \dot{E}_{p;R}[\mathfrak{d}^k \Psi], & \dot{E}'_{p;R}{}^s[\Psi] &= \sum_{0 \leq k \leq s} \dot{E}'_{p;R}[\mathfrak{d}^k \Psi], \\ E_p^s[\Psi] &= E^s[\Psi] + \dot{E}_{p;R}^s[\Psi], & E_p'^s[\Psi] &= E^s[\Psi] + \dot{E}'_{p;R}{}^s[\Psi], \end{aligned}$$

2. Higher derivative Morawetz bulks:

$$\begin{aligned} \text{Mor}^s[\Psi] &= \sum_{0 \leq k \leq s} \text{Mor}[\mathfrak{d}^k \Psi], & \text{Mor}^s_p[\Psi] &= \sum_{0 \leq k \leq s} \text{Mor}_p[\mathfrak{d}^k \Psi], & \dot{\mathcal{M}}_{p;R}^s[\Psi] &= \sum_{0 \leq k \leq s} \dot{\mathcal{M}}_{p;R}[\mathfrak{d}^k \Psi], \\ \mathcal{M}_p^s[\Psi] &= \text{Mor}^s[\Psi] + \dot{\mathcal{M}}_{p;R}^s[\Psi], & \hat{\mathcal{M}}_p^s[\Psi] &= \sum_{0 \leq k \leq s} \hat{\mathcal{M}}_p[\mathfrak{d}^k \Psi] \end{aligned}$$

3. Higher derivative fluxes:

$$F^s[\Psi] = \sum_{0 \leq k \leq s} F[\mathfrak{d}^k \Psi], \quad \dot{F}_p^s[\Psi] = \sum_{0 \leq k \leq s} \dot{F}_p[\mathfrak{d}^k \Psi], \quad F_p^s[\Psi] = F^s[\Psi] + \dot{F}_p^s[\Psi]$$

4. Higher derivative norm for $\mathcal{M}[\Psi]$:

$$\mathcal{I}_p^s[\Psi, \mathbf{M}] = \mathcal{I}_p[\mathfrak{d}^s \Psi, \mathfrak{d}^s \mathbf{M}]$$

5. Higher derivative error term:

$$\text{Err}_{\epsilon, \delta}^s(\tau_1, \tau_2)[\Psi] = \sum_{0 \leq k \leq s} \text{Err}_{\epsilon, \delta}(\tau_1, \tau_2)[\mathfrak{d}^k \Psi]$$

These quantities will appear in the energy-Morawetz and r^p weighted estimates of the system (7).

In the estimates below we will denote $\mathcal{A} \lesssim \mathcal{B}$ if there exists a constant C independent of ϵ such that $\mathcal{A} \leq C\mathcal{B}$.

2 Energy estimates for the system of coupled wave equations

In this section we prove the energy boundedness, weighted r^p -estimates and integrated energy decay for the system (7).

¹Notice the different weight in r between e_3 and e_4 : this is consistent with the asymmetric definitions of the norms.

Theorem 2.1 (Energy-Morawetz and r^p estimates for the coupled system for \mathbf{q} and $\mathbf{q}^{\mathbf{F}}$). *Let $\psi_0, \psi_1, \mathbf{q}, \psi_3, \mathbf{q}^{\mathbf{F}}$ be curvature terms, as defined in (3), of an axially symmetric polarized spacetime which is a $O(\epsilon)$ -perturbation of Reissner-Nordström spacetime. Then, there exist constants $C > 0$, independent of ϵ , and a fixed, arbitrarily small, $\delta > 0$ with $0 < \epsilon \ll \delta$, such that, if $e \ll \varpi$, the following estimates hold true in $\mathcal{M}(\tau_1, \tau_2)$:*

- For $\delta \leq p \leq 2 - \delta$, we have

$$\begin{aligned} & (E_p^s[\mathbf{q}] + E_p^{s+1}[\mathbf{q}^{\mathbf{F}}]) (\tau_2) + (\mathcal{M}_p^s[\mathbf{q}] + \mathcal{M}_p^{s+1}) (\tau_1, \tau_2) + (F_p^s[\mathbf{q}] + F_p^{s+1}[\mathbf{q}]) (\tau_1, \tau_2) \\ & \leq C \left((E_p^s[\mathbf{q}] + E_p^{s+1}[\mathbf{q}^{\mathbf{F}}] + E_p^s[\psi_0] + E_p^s[\psi_1] + E_p^s[\psi_2]) (\tau_1) + Err \right) \end{aligned} \quad (30)$$

- For $\delta \leq p \leq 1 - \delta$, we have

$$\begin{aligned} & (E_p^{s'}[\mathbf{q}] + E_p^{s'+1}[\mathbf{q}^{\mathbf{F}}]) (\tau_2) + (\mathcal{M}_p^{s'}[\mathbf{q}] + \mathcal{M}_p^{s'+1}) (\tau_1, \tau_2) + (F_p^{s'}[\mathbf{q}] + F_p^{s'+1}[\mathbf{q}]) (\tau_1, \tau_2) \\ & \leq C \left((E_p^{s'}[\mathbf{q}] + E_p^{s'+1}[\mathbf{q}^{\mathbf{F}}] + E_p^{s'}[\psi_0] + E_p^{s'}[\psi_1] + E_p^{s'}[\psi_2]) (\tau_1) + Err \right) \end{aligned} \quad (31)$$

where Err are quadratic terms given by

$$\begin{aligned} Err &= Err_{\epsilon, \delta}(\tau_1, \tau_2)[\mathbf{q}] + Err_{\epsilon, \delta}^1(\tau_1, \tau_2)[\mathbf{q}^{\mathbf{F}}] + \mathcal{I}_p[\mathbf{q}, N_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) \\ & - \int_{\mathcal{M}_{trap}} \Lambda T(\mathbf{q}) N_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] + \Lambda T T \mathbf{q}^{\mathbf{F}} \cdot T(N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) + \Lambda T R \mathbf{q}^{\mathbf{F}} \cdot R(N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) + \Lambda T \not\psi \mathbf{q}^{\mathbf{F}} \cdot \not\psi(N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) \end{aligned}$$

Proof. According to Theorem 0.1, the curvature quantities $\psi_0, \psi_1, \mathbf{q}, \psi_3, \mathbf{q}^{\mathbf{F}}$ of an $O(\epsilon)$ -perturbation of Reissner-Nordström spacetime verify the system (7), composed of the two equations (15) and (16). We will proof estimate (30) in the case of $s = 0$, and for higher derivatives the proof can be extended, relying on higher order derivative separated estimates. We summarize here the main steps to the proof, assuming the intermediate steps which will be proved in Sections 2.1-2.3. We will consider the case of $\delta \leq p \leq 2 - \delta$. The stronger estimate for $\delta \leq p \leq 1 - \delta$ is treated in the same way.

1. Treating the two equations separately, we derive energy-Morawetz and r^p -estimates with standard techniques, as summarized in Theorems 2.3 and 2.4. The 0th order estimate (43) for \mathbf{q} , with $\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] = e C_1[\mathbf{q}^{\mathbf{F}}] + e L_1[\mathbf{q}^{\mathbf{F}}] + e^2 L_1[\mathbf{q}] + N_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$, reads

$$\begin{aligned} & E_p[\mathbf{q}](\tau_2) + \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2) + F_p[\mathbf{q}](\tau_1, \tau_2) \\ & \lesssim E_p[\mathbf{q}](\tau_1) + (\mathcal{I}_p[\mathbf{q}, e C_1[\mathbf{q}^{\mathbf{F}}]] + \mathcal{I}_p[\mathbf{q}, e L_1[\mathbf{q}^{\mathbf{F}}]] + \mathcal{I}_p[\mathbf{q}, e^2 L_1[\mathbf{q}]]) (\tau_1, \tau_2) \\ & - \int_{\mathcal{M}_{trap}} \Lambda T(\mathbf{q}) \cdot (e C_1[\mathbf{q}^{\mathbf{F}}] + e L_1[\mathbf{q}^{\mathbf{F}}] + e^2 L_1[\mathbf{q}]) + Err_0[\mathbf{q}] \end{aligned} \quad (32)$$

where $Err_0[\mathbf{q}] := Err_{\epsilon, \delta}(\tau_1, \tau_2)[\mathbf{q}] + \mathcal{I}_p[\mathbf{q}, N_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) - \int_{\mathcal{M}_{trap}} \Lambda T(\mathbf{q}) N_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$ are quadratic terms.

The 1-st order estimate (49) for $\mathbf{q}^{\mathbf{F}}$, with $\mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] = e C_2[\mathbf{q}] + e^2 L_2[\mathbf{q}^{\mathbf{F}}] + N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$ is

$$\begin{aligned}
E_p^1[\mathbf{q}^{\mathbf{F}}](\tau_2) + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) + F_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) &\lesssim E_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1) + (\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, eC_2[\mathbf{q}]] + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, e^2 L_2[\mathbf{q}^{\mathbf{F}}]])(\tau_1, \tau_2) \\
&\quad - \int_{\mathcal{M}_{trap}} \Lambda TT \mathbf{q}^{\mathbf{F}} \cdot T(e C_2[\mathbf{q}] + e^2 L_2[\mathbf{q}^{\mathbf{F}}]) \\
&\quad - \int_{\mathcal{M}_{trap}} \Lambda TR \mathbf{q}^{\mathbf{F}} \cdot R(e C_2[\mathbf{q}] + e^2 L_2[\mathbf{q}^{\mathbf{F}}]) \\
&\quad - \int_{\mathcal{M}_{trap}} \Lambda T \not\psi \mathbf{q}^{\mathbf{F}} \cdot \not\psi(e C_2[\mathbf{q}] + e^2 L_2[\mathbf{q}^{\mathbf{F}}]) + \text{Err}_1[\mathbf{q}^{\mathbf{F}}]
\end{aligned} \tag{33}$$

where $\text{Err}_1[\mathbf{q}^{\mathbf{F}}] := \text{Err}_{\epsilon, \delta}^1(\tau_1, \tau_2)[\mathbf{q}^{\mathbf{F}}] + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) - \int_{\mathcal{M}_{trap}} \Lambda TT \mathbf{q}^{\mathbf{F}} \cdot T(N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) + \Lambda TR \mathbf{q}^{\mathbf{F}} \cdot R(N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) + \Lambda T \not\psi \mathbf{q}^{\mathbf{F}} \cdot \not\psi(N_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}])$ are quadratic terms.

This is done in Section 2.1.

We sum a multiple of the first estimate to the second estimate. In particular, taking $2 \times (32) + (33)$, and keeping the constant explicit for the integrals in the trapping region on the right hand side we obtain

$$\begin{aligned}
&(E_p[\mathbf{q}] + E_p^1[\mathbf{q}^{\mathbf{F}}])(\tau_2) + (\mathcal{M}_p[\mathbf{q}] + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}])(\tau_1, \tau_2) + (F_p[\mathbf{q}] + F_p^1[\mathbf{q}^{\mathbf{F}}])(\tau_1, \tau_2) \\
&\lesssim (E_p[\mathbf{q}] + E_p^1[\mathbf{q}^{\mathbf{F}}])(\tau_1) + \text{Err}_0[\mathbf{q}] + \text{Err}_1[\mathbf{q}^{\mathbf{F}}] \\
&\quad + (\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]] + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, eC_2[\mathbf{q}]]) (\tau_1, \tau_2) - \int_{\mathcal{M}_{trap}} \Lambda TR \mathbf{q}^{\mathbf{F}} \cdot R(e C_2[\mathbf{q}]) \\
&\quad - e\Lambda \int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot C_1[\mathbf{q}^{\mathbf{F}}] + TT \mathbf{q}^{\mathbf{F}} \cdot T(C_2[\mathbf{q}]) + T \not\psi \mathbf{q}^{\mathbf{F}} \cdot \not\psi(C_2[\mathbf{q}]) \\
&\quad + (\mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}^{\mathbf{F}}]] + \mathcal{I}_p[\mathbf{q}, e^2 L_1[\mathbf{q}]] + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, e^2 L_2[\mathbf{q}^{\mathbf{F}}]])(\tau_1, \tau_2) \\
&\quad - e\Lambda \int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot (L_1[\mathbf{q}^{\mathbf{F}}] + eL_1[\mathbf{q}]) - eTT \mathbf{q}^{\mathbf{F}} \cdot T(L_2[\mathbf{q}^{\mathbf{F}}]) - eTR \mathbf{q}^{\mathbf{F}} \cdot R(L_2[\mathbf{q}^{\mathbf{F}}]) - eT \not\psi \mathbf{q}^{\mathbf{F}} \cdot \not\psi(L_2[\mathbf{q}^{\mathbf{F}}])
\end{aligned} \tag{34}$$

The third and fourth line of estimate (34) concern the coupling terms, while the last two lines contain the lower order terms, and they are all $O(e)$. We will be able to absorb completely the coupling terms by the left hand side of the estimate for small charge and using a cancellation, as shown in Step 2. We will absorb the lower order terms modulo a necessary bound on initial energy for the lower order quantities ψ_0, ψ_1, ψ_3 , as done in Step 3.

2. Because of the structure of the coupling terms, we are able to absorb most of those terms appearing on the right hand side of estimate (34) by the Morawetz bulks and the energies of \mathbf{q} and $\mathbf{q}^{\mathbf{F}}$ on the left hand side. In particular, outside the trapping region, this absorption is straightforward for all terms. In the trapping region though, there are problems due to the degeneracy of the Morawetz bulks, in T and $\not\psi$. Specifically, the terms $\int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot C_1[\mathbf{q}^{\mathbf{F}}] - TT \mathbf{q}^{\mathbf{F}} \cdot T(C_2[\mathbf{q}]) - T \not\psi \mathbf{q}^{\mathbf{F}} \cdot \not\psi(C_2[\mathbf{q}])$ cannot be absorbed by the left hand side. Nevertheless, due to the particular structure of the higher order coupling terms, which is related to the fact that the system composed with higher order terms is diagonalizable, these terms get cancelled out (see Remark 2.7). In Proposition 2.5 we obtain

$$\begin{aligned}
&(\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]] + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, eC_2[\mathbf{q}]]) (\tau_1, \tau_2) - e\Lambda \int_{\mathcal{M}_{trap}} TR \mathbf{q}^{\mathbf{F}} \cdot R(C_2[\mathbf{q}]) \\
&\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} (E[\mathbf{q}](\tau) + E^1[\mathbf{q}^{\mathbf{F}}](\tau)) + e (\mathcal{M}_p[\mathbf{q}] + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}])(\tau_1, \tau_2)
\end{aligned} \tag{35}$$

and in Proposition 2.6 we obtain

$$\begin{aligned}
& -e\Lambda \int_{\mathcal{M}_{\text{trap}}} 2T(\mathfrak{q}) \cdot C_1[\mathfrak{q}^{\mathbf{F}}] + TT\mathfrak{q}^{\mathbf{F}} \cdot T(C_2[\mathfrak{q}]) + T\mathfrak{q}^{\mathbf{F}} \cdot \mathfrak{p}(C_2[\mathfrak{q}]) \\
& \lesssim e \sup_{\tau \in [\tau_1, \tau_2]} (E[\mathfrak{q}](\tau) + E^1[\mathfrak{q}^{\mathbf{F}}](\tau)) + e(\mathcal{M}_p[\mathfrak{q}] + \mathcal{M}_p^1[\mathfrak{q}^{\mathbf{F}}] + \hat{\mathcal{M}}[\psi_3])(\tau_1, \tau_2)
\end{aligned} \tag{36}$$

This is done in Section 2.2.

3. In order to control the lower order terms of the equations, we use the differential relations existing between the quantities $\psi_0, \psi_1, \psi_3, \mathfrak{q}^{\mathbf{F}}$. We derive in Proposition 2.8 transport estimates for ψ_0, ψ_1, ψ_3 and in Proposition 2.9 we estimate the norms involving the lower order terms. We arrive to the following estimate in Corollary 2.10:

$$\begin{aligned}
& (\mathcal{I}_p[\mathfrak{q}, eL_1[\mathfrak{q}^{\mathbf{F}}]] + \mathcal{I}_p[\mathfrak{q}, eL_1[\mathfrak{q}]] + \mathcal{I}_p^1[\mathfrak{q}^{\mathbf{F}}, e^2L_2[\mathfrak{q}^{\mathbf{F}}]])(\tau_1, \tau_2) \\
& - e\Lambda \int_{\mathcal{M}_{\text{trap}}} 2T(\mathfrak{q}) \cdot (L_1[\mathfrak{q}^{\mathbf{F}}] + eL_1[\mathfrak{q}]) - eTT\mathfrak{q}^{\mathbf{F}} \cdot T(L_2[\mathfrak{q}^{\mathbf{F}}]) - eTR\mathfrak{q}^{\mathbf{F}} \cdot R(L_2[\mathfrak{q}^{\mathbf{F}}]) - eT\mathfrak{q}^{\mathbf{F}} \cdot \mathfrak{p}(L_2[\mathfrak{q}^{\mathbf{F}}]) \\
& \lesssim e \sup_{\tau \in [\tau_1, \tau_2]} (E[\mathfrak{q}](\tau) + E^1[\mathfrak{q}^{\mathbf{F}}](\tau)) + e\mathcal{M}_p^1[\mathfrak{q}^{\mathbf{F}}](\tau_1, \tau_2) + (E_p[\psi_0] + E_p[\psi_1] + E_p[\psi_3])(\tau_1)
\end{aligned} \tag{37}$$

This is done in section 2.3.

Back to estimate (34), using (35) and (36) for the coupling terms and (37) for the lower order terms, we obtain

$$\begin{aligned}
& (E_p[\mathfrak{q}] + E_p^1[\mathfrak{q}^{\mathbf{F}}])(\tau_2) + (\mathcal{M}_p[\mathfrak{q}] + \mathcal{M}_p^1[\mathfrak{q}^{\mathbf{F}}])(\tau_1, \tau_2) + (F_p[\mathfrak{q}] + F_p^1[\mathfrak{q}^{\mathbf{F}}])(\tau_1, \tau_2) \\
& \lesssim (E_p[\mathfrak{q}] + E_p^1[\mathfrak{q}^{\mathbf{F}}] + E_p[\psi_0] + E_p[\psi_1] + E_p[\psi_3])(\tau_1) + \text{Err} \\
& + e \sup_{\tau \in [\tau_1, \tau_2]} (E[\mathfrak{q}](\tau) + E^1[\mathfrak{q}^{\mathbf{F}}](\tau)) + e(\mathcal{M}_p[\mathfrak{q}] + \mathcal{M}_p^1[\mathfrak{q}^{\mathbf{F}}])(\tau_1, \tau_2)
\end{aligned} \tag{38}$$

where the $\hat{\mathcal{M}}[\psi_3]$ appearing in the coupling term was absorbed using the transport estimates. For $e \ll \varpi$, we can absorb the energies and the bulks of \mathfrak{q} and $\mathfrak{q}^{\mathbf{F}}$ on the left hand side, which proves (30). The proof of (31) is identical. \square

In what follows, we prove the estimates in the intermediate steps of the proof of Theorem 2.1.

2.1 Energy estimates for the two equations separately

The derivation of the Morawetz and r^p -weighted estimates for equations (15) and (16) is based on the following Proposition.

Proposition 2.2. *Consider symmetric traceless S -horizonal tensor Ψ_{AB} and $\mathbf{M}[\Psi]_{AB}$ verifying the spacetime equation*

$$\dot{\square}_{\mathbf{g}}\Psi = V\Psi + \mathbf{M}[\Psi] \tag{39}$$

Define the energy momentum tensor associated to the equation (39) as

$$\mathcal{Q}_{\mu\nu} : = \dot{\mathbf{D}}_{\mu}\Psi \cdot \dot{\mathbf{D}}_{\nu}\Psi - \frac{1}{2}\mathbf{g}_{\mu\nu}(\dot{\mathbf{D}}_{\lambda}\Psi \cdot \dot{\mathbf{D}}^{\lambda}\Psi + V\Psi \cdot \Psi) \tag{40}$$

$$= \dot{\mathbf{D}}_{\mu}\Psi \cdot \dot{\mathbf{D}}_{\nu}\Psi - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{L}(\Psi) \tag{41}$$

Let $X = a(r)e_3 + b(r)e_4$ be a vectorfield, w a scalar function and M a one form. Defining

$$\mathcal{P}_\mu[X, w, M] = \mathcal{Q}_{\mu\nu}X^\nu + \frac{1}{2}w\Psi\dot{\mathbf{D}}_\mu\Psi - \frac{1}{4}\Psi^2\partial_\mu w + \frac{1}{4}\Psi^2M_\mu,$$

then, in an axially symmetric polarized spacetime,

$$\begin{aligned} \mathbf{D}^\mu\mathcal{P}_\mu[X, w, M] &= \frac{1}{2}\mathcal{Q} \cdot {}^{(X)}\pi - \frac{1}{2}X(V)\Psi \cdot \Psi + \frac{1}{2}w\mathcal{L}[\Psi] - \frac{1}{4}\Psi^2\Box_g w \\ &+ \frac{1}{4}\dot{\mathbf{D}}^\mu(\Psi^2M_\mu) + \left(X(\Psi) + \frac{1}{2}w\Psi\right) \cdot \mathbf{M}[\Psi] \end{aligned} \quad (42)$$

Proof. Standard computations. \square

We summarize in this section the energy-Morawetz and the r^p -weighted estimates for the two equations separately. For both equations of the system (7), we can derive higher derivative energy-Morawetz and higher weighted r^p estimates. Nevertheless, because of the structure and dependence of derivatives of $\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$ and $\mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$, we will need to combine the 0th order estimate for the first equation (15) with the 1st order estimate for the second equation (16). In general we will need to combine the s -th order estimate for the first with the $(s+1)$ -th order estimate for the second.

To obtain the cancellation that will be needed in the absorption of the coupling terms, we keep a part of the term $(X(\Psi) + \frac{1}{2}w\Psi) \cdot \mathbf{M}[\Psi]$ as it is in the estimates, with its sign, and we bound the rest of it with the norm $\mathcal{I}_p[\Psi, \mathbf{M}]$.

Theorem 2.3 (0th order energy-Morawetz and r^p weighted estimates for \mathbf{q}). *Consider the equation*

$$\Box_g \mathbf{q} = V_1 \mathbf{q} + \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$$

in a $O(\epsilon)$ -perturbation of Reissner-Nordström spacetime, where $V_1 = -\kappa_{\mathbf{K}} + 10^{(F)}\rho^2$. Then there exists a fixed, arbitrarily small, $\delta > 0$ with $0 < \epsilon \ll \delta$, and $\Lambda > \delta^{-1}$ such that the following estimates hold true in $\mathcal{M}(\tau_1, \tau_2)$:

- For $\delta \leq p \leq 2 - \delta$, we have

$$\begin{aligned} E_p[\mathbf{q}](\tau_2) + \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2) + F_p[\mathbf{q}](\tau_1, \tau_2) &\lesssim E_p[\mathbf{q}](\tau_1) + \mathcal{I}_p[\mathbf{q}, \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]] + Err_{\epsilon, \delta}(\tau_1, \tau_2)[\mathbf{q}] \\ &- \int_{\mathcal{M}_{trap}} \Lambda T(\mathbf{q}) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] \end{aligned} \quad (43)$$

- For $\delta \leq p \leq 1 - \delta$, we have

$$\begin{aligned} E'_p[\mathbf{q}](\tau_2) + \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2) + F_p[\mathbf{q}](\tau_1, \tau_2) &\lesssim E'_p[\mathbf{q}](\tau_1) + \mathcal{I}_p[\mathbf{q}, \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]] + Err_{\epsilon, \delta}(\tau_1, \tau_2)[\mathbf{q}] \\ &- \int_{\mathcal{M}_{trap}} \Lambda T(\mathbf{q}) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] \end{aligned} \quad (44)$$

Proof. We sketch here the proof, to appear in [3]. We apply Proposition 2.2 to equation $\Box_g \mathbf{q} = V_1 \mathbf{q} + \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$ with triplet

$$(X, w, M) := (f_\delta R, w_\delta, 2hR) + \epsilon_{\mathcal{H}}(Y_{\mathcal{H}}, 0, 0) + (0, 0, 2h\check{R}) + (\Lambda T, 0, 0) + (f_p e_4, \frac{2f_p}{r}, \frac{2f'_p}{r} e_4)$$

for well-chosen functions $f_\delta, w_\delta, h, f_p = \theta(r)r^p$ and well chosen constants $\epsilon_{\mathcal{H}}, \Lambda$. Defining

$$\mathcal{E}[X, w, M](\mathbf{q}) := \mathbf{D}^\mu \mathcal{P}_\mu[X, w, M] - \left(X(\mathbf{q}) + \frac{1}{2}w\mathbf{q} \right) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$$

applying divergence theorem we obtain

$$\int_{r=r_A} \mathcal{P} \cdot N_A + \int_{\Sigma_2} \mathcal{P} \cdot N_\Sigma + \int_{\mathcal{M}(\tau_1, \tau_2)} \mathcal{E} + \int_{\underline{\mathcal{L}}_*} \mathcal{P} \cdot e_3 = \int_{\Sigma_1} \mathcal{P} \cdot N_\Sigma - \int_{\mathcal{M}(\tau_1, \tau_2)} \left(X(\mathbf{q}) + \frac{1}{2}w\mathbf{q} \right) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] \quad (45)$$

where $\mathcal{E} = \mathcal{E}[X, w, M](\mathbf{q})$ and N_A is the unit normal to the spacelike hypersurface $r = r_A$. We can show that, for $C \gg \delta^{-1}$ independent of ϵ ,

$$\begin{aligned} \int_{\mathcal{M}(\tau_1, \tau_2)} \mathcal{E} &\geq C^{-1} \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2) + O(\epsilon\Lambda) \text{Err}_{\epsilon, \delta}[\mathbf{q}](\tau_1, \tau_2), \\ \int_{r=r_A} \mathcal{P} \cdot N_A &\geq 0, \\ \int_{\Sigma(\tau)} \mathcal{P} \cdot N_\Sigma &\geq E_p[\Psi](\tau), \\ \int_{\underline{\mathcal{L}}_*(\tau_1, \tau_2)} \mathcal{P} \cdot e_3 &\geq \frac{\Lambda}{4} F[\Psi](\tau_1, \tau_2) \end{aligned} \quad (46)$$

We analyze the inhomogeneous term $-\int_{\mathcal{M}(\tau_1, \tau_2)} \left(X(\mathbf{q}) + \frac{1}{2}w\mathbf{q} \right) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$. Since the vectorfield X is given by $X = f_\delta R + \epsilon_{\mathcal{H}} Y_{\mathcal{H}} + \Lambda T + f_p e_4$, and f_p is supported in $r \geq R$, we separate the term involving $T\mathbf{q}$ and bound the other ones with their absolute value:

$$\begin{aligned} - \int_{\mathcal{M}(\tau_1, \tau_2)} \left(X(\mathbf{q}) + \frac{1}{2}w\mathbf{q} \right) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] &\leq - \int_{\mathcal{M}(\tau_1, \tau_2)} \Lambda T(\mathbf{q}) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] + C \int_{\mathcal{M}(\tau_1, \tau_2)} (|\check{R}\mathbf{q}| + r^{-1}|\mathbf{q}|) |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]| \\ &\quad + C \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^p |e_4 \mathbf{q}| |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]| \end{aligned} \quad (47)$$

We separate the first and second integral on the right of (47) in the integral in the trapping region and outside the trapping region. Outside the trapping region, the integral $-\int_{\mathcal{M}(\tau_1, \tau_2)} \Lambda T(\mathbf{q}) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$ can also be bounded by its absolute value:

$$\begin{aligned} - \int_{\mathcal{M}(\tau_1, \tau_2)} \left(X(\mathbf{q}) + \frac{1}{2}w\mathbf{q} \right) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] &\lesssim - \int_{\mathcal{M}_{\text{trap}}} \Lambda T(\mathbf{q}) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] + \int_{\mathcal{M}_{\text{trap}}} (|\check{R}\mathbf{q}| + r^{-1}|\mathbf{q}|) |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]| \\ &\quad + \int_{\mathcal{M}_{\text{trap}}^c} (|\check{R}\mathbf{q}| + |\check{T}\mathbf{q}| + r^{-1}|\mathbf{q}|) |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]| \\ &\quad + \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^p |e_4 \mathbf{q}| |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]| \end{aligned} \quad (48)$$

The integrals outside the trapping region the integral can be separated using Cauchy-Schwarz:

$$\begin{aligned} \int_{\mathcal{M}_{\text{trap}}^c(\tau_1, \tau_2)} (|\check{R}\mathbf{q}| + |\check{T}\mathbf{q}| + r^{-1}|\mathbf{q}|) |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]| &\leq \lambda \int_{\mathcal{M}_{\text{trap}}^c} r^{-1-\delta} (|\check{R}\mathbf{q}|^2 + |\check{T}\mathbf{q}|^2 + r^{-2}|\mathbf{q}|^2) + \lambda^{-1} \int_{\mathcal{M}_{\text{trap}}^c} r^{1+\delta} |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]|^2, \\ \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^p |e_4 \mathbf{q}| |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]| &\leq \lambda \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^{p-1} |e_4 \mathbf{q}|^2 + \lambda^{-1} \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^{p+1} |\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]|^2 \end{aligned}$$

and for λ small enough the first integrals on the right can be absorbed in the Morawetz bulk $\mathcal{M}_p[\mathbf{q}]$. We thus arrive to the estimate,

$$\begin{aligned} E_p[\mathbf{q}](\tau_2) + \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2) + F_p[\mathbf{q}](\tau_1, \tau_2) &\lesssim E_p[\mathbf{q}](\tau_2) + \text{Err}_\epsilon(\tau_1, \tau_2) - \int_{\mathcal{M}_{\text{trap}}} \Lambda T(\mathbf{q}) \cdot \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}] \\ &\quad + \int_{\mathcal{M}_{\text{trap}}} (|\check{R}\mathbf{q}| + r^{-1}|\mathbf{q}|)|\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]| + \int_{\mathcal{M}_{\text{trap}^c}} r^{1+p}|\mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]|^2 \end{aligned}$$

and recalling the definition of $\mathcal{I}_p[\mathbf{q}, \mathbf{M}_1[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]]$ in (28), we obtain the desired estimate. \square

With the same techniques, we derive the estimates for $\mathbf{q}^{\mathbf{F}}$, independently of the ones for \mathbf{q} .

Theorem 2.4 (1st order energy-Morawetz and r^p weighted estimates for $\mathbf{q}^{\mathbf{F}}$). *Consider the equation*

$$\square_{\mathbf{g}}\mathbf{q}^{\mathbf{F}} = V_2\mathbf{q}^{\mathbf{F}} + \mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$$

in a $O(\epsilon)$ -perturbation of Reissner-Nordström spacetime, where $V_2 = -\kappa_{\underline{K}} - 3\rho$. Then there exists a fixed, arbitrarily small, $\delta > 0$ with $0 < \epsilon \ll \delta$, such that the following estimates hold true in $\mathcal{M}(\tau_1, \tau_2)$:

- For $\delta \leq p \leq 2 - \delta$, we have

$$\begin{aligned} E_p^1[\mathbf{q}^{\mathbf{F}}](\tau_2) + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) + F_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) &\lesssim E_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1) + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, \mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) + \text{Err}_{\epsilon, \delta}^1(\tau_1, \tau_2)[\mathbf{q}^{\mathbf{F}}] \\ &\quad - \int_{\mathcal{M}_{\text{trap}}} \Lambda T T \mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) \\ &\quad - \int_{\mathcal{M}_{\text{trap}}} \Lambda T R \mathbf{q}^{\mathbf{F}} \cdot R(\mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) \\ &\quad - \int_{\mathcal{M}_{\text{trap}}} \Lambda T \not\partial \mathbf{q}^{\mathbf{F}} \cdot \not\partial(\mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) \end{aligned} \tag{49}$$

- For $\delta \leq p \leq 1 - \delta$, we have

$$\begin{aligned} E_p^1[\mathbf{q}^{\mathbf{F}}](\tau_2) + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) + F_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) &\lesssim E_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1) + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, \mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) + \text{Err}_{\epsilon, \delta}^1(\tau_1, \tau_2)[\mathbf{q}^{\mathbf{F}}] \\ &\quad - \int_{\mathcal{M}_{\text{trap}}} \Lambda T T \mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) \\ &\quad - \int_{\mathcal{M}_{\text{trap}}} \Lambda T R \mathbf{q}^{\mathbf{F}} \cdot R(\mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) \\ &\quad - \int_{\mathcal{M}_{\text{trap}}} \Lambda T \not\partial \mathbf{q}^{\mathbf{F}} \cdot \not\partial(\mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]) \end{aligned} \tag{50}$$

Proof. The proof proceeds as in Theorem 2.3 after commuting the equation $\square_{\mathbf{g}}\mathbf{q}^{\mathbf{F}} = V_2\mathbf{q}^{\mathbf{F}} + \mathbf{M}_2[\mathbf{q}, \mathbf{q}^{\mathbf{F}}]$ with T , R and $\not\partial$. To appear in [3]. \square

Theorems 2.3 and 2.4 have technical but standard proofs which will be postponed to appear in [3] for the sake of the clarity of the combined estimates of the system.

2.2 Estimates for the coupling terms

The aim of this section is to prove (35) and (36) in Step 2 of the proof of Theorem 0.2.

By (4), (6) and (8), the coupling terms are given by

$$\begin{aligned} C_1[\mathbf{q}^{\mathbf{F}}] &= \frac{2}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}} - \frac{2}{r} \underline{\kappa} e_4 \mathbf{q}^{\mathbf{F}} - \frac{2}{r} \kappa e_3 \mathbf{q}^{\mathbf{F}} + \frac{1}{r} (3\kappa \underline{\kappa} + 8\rho + 4 \binom{F}{\rho}) \mathbf{q}^{\mathbf{F}}, \\ C_2[\mathbf{q}] &= -\frac{2}{r^3} \mathbf{q} \end{aligned}$$

Recall that, in an $O(\epsilon)$ perturbation of Reissner-Nordström spacetime, $\kappa - \overline{\kappa}, \underline{\kappa} - \overline{\underline{\kappa}}, \rho - \overline{\rho}, \binom{F}{\rho} - \overline{\binom{F}{\rho}} = O(\epsilon)$, therefore we can write schematically

$$\kappa = \frac{1}{r} + O(\epsilon), \quad \underline{\kappa} = \frac{1}{r} + O(\epsilon), \quad \rho = \frac{\overline{\omega}}{r^3} + O(\epsilon), \quad \binom{F}{\rho} = \frac{e}{r^2} + O(\epsilon)$$

Outside the trapping region we will schematize² the coupling terms as

$$C_1[\mathbf{q}^{\mathbf{F}}] = \left[\frac{1}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}}, \frac{1}{r^2} e_4 \mathbf{q}^{\mathbf{F}}, \frac{1}{r^2} e_3 \mathbf{q}^{\mathbf{F}}, \frac{1}{r^3} \mathbf{q}^{\mathbf{F}} \right], \quad (51)$$

$$C_2[\mathbf{q}] = \left[\frac{1}{r^3} \mathbf{q} \right] \quad (52)$$

The coupling terms appear in the equations with a good decay in r , therefore outside the trapping region they can be easily absorbed by the Morawetz bulks of \mathbf{q} and $\mathbf{q}^{\mathbf{F}}$. In the trapping region though the Morawetz bulks are degenerate, therefore they can't absorb all second derivatives of $\mathbf{q}^{\mathbf{F}}$, in particular $\not\Delta \mathbf{q}^{\mathbf{F}}$. We need to use the equations to absorb the terms we need. It turns out that the structure of the coupling terms allows a very convenient cancellation of the bad terms.

We first prove the absorption of the non-degenerate terms, as in (35).

Proposition 2.5. *With the notations above, for all $\delta \leq p \leq 2 - \delta$, we have*

$$\begin{aligned} & (\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]] + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, eC_2[\mathbf{q}]]) (\tau_1, \tau_2) - e\Lambda \int_{\mathcal{M}_{trap}} TR \mathbf{q}^{\mathbf{F}} \cdot R(C_2[\mathbf{q}]) \\ & \lesssim e \sup_{\tau \in [\tau_1, \tau_2]} (E[\mathbf{q}](\tau) + E^1[\mathbf{q}^{\mathbf{F}}]) + e(\mathcal{M}_p[\mathbf{q}] + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}]) (\tau_1, \tau_2) \end{aligned}$$

Proof. Recall that, by definition, the Morawetz bulks $\mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2)$ and $\mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)$ are given by

$$\begin{aligned} \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2) &= \int_{\mathcal{M}_{trap}} |R(\mathbf{q})|^2 + r^{-2} |\mathbf{q}|^2 + \frac{(r^2 - 3\overline{\omega}r + 2e^2)^2}{r^4} \left(|\not\mathcal{V}\mathbf{q}|^2 + \frac{D}{r^2} |T\mathbf{q}|^2 \right) \\ &+ \int_{\mathcal{M}_{trap}} r^{-3} (|e_4 \mathbf{q}|^2 + r^{-1} |\mathbf{q}|^2) + r^{-1} |\not\mathcal{V}\mathbf{q}|^2 + r^{-1-\delta} |e_3(\mathbf{q})|^2 \\ &+ \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^{p-1} (p|\check{e}_4(\mathbf{q})|^2 + (2-p)|\not\mathcal{V}\mathbf{q}|^2 + r^{-2} |\mathbf{q}|^2) \end{aligned}$$

²Ignoring the quadratic terms

and

$$\begin{aligned}
\mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) &= \int_{\mathcal{M}_{trap}} |R(\mathbf{q}^{\mathbf{F}})|^2 + r^{-2}|\mathbf{q}^{\mathbf{F}}|^2 + \frac{(r^2 - 3\varpi r + 2e^2)^2}{r^4} \left(|\check{\Psi}\mathbf{q}^{\mathbf{F}}|^2 + \frac{D}{r^2}|T\mathbf{q}^{\mathbf{F}}|^2 \right) \\
&+ \int_{\mathcal{M}_{trap}} r^{-3}(|e_4\mathbf{q}^{\mathbf{F}}|^2 + r^{-1}|\mathbf{q}^{\mathbf{F}}|^2) + r^{-1}|\check{\Psi}\mathbf{q}^{\mathbf{F}}|^2 + r^{-1-\delta}|e_3(\mathbf{q}^{\mathbf{F}})|^2 \\
&+ \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^{p-1} (p|\check{e}_4(\mathbf{q}^{\mathbf{F}})|^2 + (2-p)|\check{\Psi}\mathbf{q}^{\mathbf{F}}|^2 + r^{-2}|\mathbf{q}^{\mathbf{F}}|^2) \\
&+ \int_{\mathcal{M}_{trap}} |R(\partial\mathbf{q}^{\mathbf{F}})|^2 + r^{-2}|\partial\mathbf{q}^{\mathbf{F}}|^2 + \frac{(r^2 - 3\varpi r + 2e^2)^2}{r^4} \left(|\check{\Psi}\partial\mathbf{q}^{\mathbf{F}}|^2 + \frac{D}{r^2}|T\partial\mathbf{q}^{\mathbf{F}}|^2 \right) \\
&+ \int_{\mathcal{M}_{trap}} r^{-3}(|e_4\partial\mathbf{q}^{\mathbf{F}}|^2 + r^{-1}|\partial\mathbf{q}^{\mathbf{F}}|^2) + r^{-1}|\check{\Psi}\partial\mathbf{q}^{\mathbf{F}}|^2 + r^{-1-\delta}|e_3(\partial\mathbf{q}^{\mathbf{F}})|^2 \\
&+ \int_{\mathcal{M}_{\geq R}(\tau_1, \tau_2)} r^{p-1} (p|\check{e}_4(\partial\mathbf{q}^{\mathbf{F}})|^2 + (2-p)|\check{\Psi}\partial\mathbf{q}^{\mathbf{F}}|^2 + r^{-2}|\partial\mathbf{q}^{\mathbf{F}}|^2)
\end{aligned}$$

Consider $\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2)$, where we can use the schematic version $C_1[\mathbf{q}^{\mathbf{F}}] = \left[\frac{1}{r}\check{\mathcal{A}}_2\mathbf{q}^{\mathbf{F}}, \frac{1}{r^2}e_4\mathbf{q}^{\mathbf{F}}, \frac{1}{r^2}e_3\mathbf{q}^{\mathbf{F}}, \frac{1}{r^3}\mathbf{q}^{\mathbf{F}} \right]$. We consider the spacetime integral in the three region in which the spacetime is divided: \mathcal{M}_{red} , \mathcal{M}_{trap} and \mathcal{M}_{far} .

In the redshift region \mathcal{M}_{red} , the Morawetz bulk $\mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}]$ contains all second derivatives, and powers of r don't matter. Therefore

$$\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]] \simeq e^2 \int_{\mathcal{M}_{red}(\tau_1, \tau_2)} |\check{\Psi}^2\mathbf{q}^{\mathbf{F}}|^2 + |\partial\mathbf{q}^{\mathbf{F}}|^2 + |\mathbf{q}^{\mathbf{F}}|^2 \lesssim e^2 \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)$$

In the trapping region \mathcal{M}_{trap} , we have

$$\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]] \simeq e \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} (|R\mathbf{q}| + r^{-1}|\mathbf{q}|)(|\check{\mathcal{A}}_2\mathbf{q}^{\mathbf{F}}| + |e_4\mathbf{q}^{\mathbf{F}}| + |e_3\mathbf{q}^{\mathbf{F}}| + |\mathbf{q}^{\mathbf{F}}|)$$

All the terms with up to one derivative of $\mathbf{q}^{\mathbf{F}}$ can be absorbed by

$$\begin{aligned}
e \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} (|R\mathbf{q}| + r^{-1}|\mathbf{q}|)(|e_4\mathbf{q}^{\mathbf{F}}| + |e_3\mathbf{q}^{\mathbf{F}}| + |\mathbf{q}^{\mathbf{F}}|) &\lesssim e \int_{\tau_1}^{\tau_2} E[\mathbf{q}](\tau)^{1/2} \left(\int_{\Sigma_{trap}(\tau)} |\partial\mathbf{q}^{\mathbf{F}}|^2 \right)^{1/2} \\
&\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e\mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)
\end{aligned}$$

Similarly, since $R\mathbf{q}$ does not degenerate in the trapping region, we can bound

$$\begin{aligned}
e \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} (|R\mathbf{q}| + r^{-1}|\mathbf{q}|)|\check{\mathcal{A}}_2\mathbf{q}^{\mathbf{F}}| &\lesssim e \int_{\tau_1}^{\tau_2} E^1[\mathbf{q}^{\mathbf{F}}](\tau)^{1/2} \left(\int_{\Sigma_{trap}(\tau)} |\check{R}\mathbf{q}|^2 + r^{-1}|\mathbf{q}|^2 \right)^{1/2} \\
&\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E^1[\mathbf{q}^{\mathbf{F}}](\tau) + e\mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2)
\end{aligned}$$

In the far-away region \mathcal{M}_{far} , we write

$$\begin{aligned}
\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) &= e^2 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{1+p} \left| \frac{1}{r}\check{\mathcal{A}}_2\mathbf{q}^{\mathbf{F}}, \frac{1}{r^2}e_4\mathbf{q}^{\mathbf{F}}, \frac{1}{r^2}e_3\mathbf{q}^{\mathbf{F}}, \frac{1}{r^3}\mathbf{q}^{\mathbf{F}} \right|^2 \\
&= e^2 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-5+p} |\check{\Psi}^2\mathbf{q}^{\mathbf{F}}|^2 + r^{-3+p} |e_4\mathbf{q}^{\mathbf{F}}|^2 + r^{-3+p} |e_3\mathbf{q}^{\mathbf{F}}|^2 + r^{-5+p} |\mathbf{q}^{\mathbf{F}}|^2
\end{aligned}$$

On the other hand in the far-away region, for $\delta \leq p \leq 2 - \delta$, the Morawetz bulk simplifies to

$$\begin{aligned} \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) &\simeq \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-3+p} |\not\partial \mathbf{q}^{\mathbf{F}}|^2 + r^{-1-\delta} |e_3 \not\partial \mathbf{q}^{\mathbf{F}}|^2 + r^{-1+p} |e_4 \not\partial \mathbf{q}^{\mathbf{F}}|^2 \\ &\quad + \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-1+p} |e_4 \mathbf{q}^{\mathbf{F}}|^2 + r^{-1-\delta} |e_3 \mathbf{q}^{\mathbf{F}}|^2 + r^{-1+p} |\not\nabla \mathbf{q}^{\mathbf{F}}|^2 + r^{-3+p} |\mathbf{q}^{\mathbf{F}}|^2 \end{aligned}$$

Since the powers of r of $\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]]$ decay all faster than the respective ones in $\mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}]$, we have that, for $r \geq R_0$,

$$\mathcal{I}_p[\mathbf{q}, eC_1[\mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) \lesssim e^2 \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)$$

Consider $\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, C_2[\mathbf{q}]]$ with $C_2[\mathbf{q}] = \frac{1}{r^3} \mathbf{q}$. We separate this norm in the three regions. In the redshift region \mathcal{M}_{red} , we have

$$\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, eC_2[\mathbf{q}]](\tau_1, \tau_2) \simeq e^2 \int_{\mathcal{M}_{red}(\tau_1, \tau_2)} |\mathbf{q}|^2 + e^2 \int_{\mathcal{M}_{red}(\tau_1, \tau_2)} |\not\partial \mathbf{q}|^2 \lesssim e^2 \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2)$$

In the trapping region \mathcal{M}_{trap} , we have

$$\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, eC_2[\mathbf{q}]](\tau_1, \tau_2) = e \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} (|R \not\partial \mathbf{q}^{\mathbf{F}}| + r^{-1} |\not\partial \mathbf{q}^{\mathbf{F}}|) |\not\partial \mathbf{q}|$$

The last term, with only one derivative of $\mathbf{q}^{\mathbf{F}}$ can be easily bounded:

$$\begin{aligned} e \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} |\not\partial \mathbf{q}^{\mathbf{F}}| |\not\partial \mathbf{q}| &\lesssim e \int_{\tau_1}^{\tau_2} E[\mathbf{q}](\tau)^{1/2} \left(\int_{\Sigma_{trap}(\tau)} |\not\partial \mathbf{q}^{\mathbf{F}}|^2 \right)^{1/2} \\ &\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau)^{1/2} \int_{\tau_1}^{\tau_2} \left(\int_{\Sigma_{trap}(\tau)} |\not\partial \mathbf{q}^{\mathbf{F}}|^2 \right)^{1/2} \\ &\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) \end{aligned}$$

Similarly, the term $|R(\not\partial \mathbf{q}^{\mathbf{F}})|^2$ is present in the Morawetz bulk of $\mathbf{q}^{\mathbf{F}}$ without degeneracy, therefore as before

$$e \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} |\check{R} \not\partial \mathbf{q}^{\mathbf{F}}| |\not\partial \mathbf{q}| \lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)$$

In the far-away region \mathcal{M}_{far} , we need to take into account the power of r in the structure of the coupling terms (51). We have

$$\begin{aligned} \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, eC_2[\mathbf{q}]] &= e^2 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{1+p} \left| \frac{1}{r^3} \mathbf{q} \right|^2 + e^2 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{1+p} \left| \frac{1}{r^3} \not\partial \mathbf{q} \right|^2 \\ &= e^2 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-5+p} |\mathbf{q}|^2 + r^{-5+p} |e_3 \mathbf{q}|^2 + r^{-3+p} |e_4 \mathbf{q}|^2 + r^{-3+p} |\not\nabla \mathbf{q}|^2 \end{aligned}$$

On the other hand in the far-away region, for $\delta \leq p \leq 2 - \delta$, the Morawetz bulk simplifies to

$$\begin{aligned} \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2) &= \int_{\mathcal{M}_{far}} r^{-3} (|e_4 \mathbf{q}|^2 + r^{-1} |\mathbf{q}|^2) + r^{-1} |\not\nabla \mathbf{q}|^2 + r^{-1-\delta} |e_3(\mathbf{q})|^2 \\ &\quad + \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{p-1} (p |\check{e}_4(\mathbf{q})|^2 + (2-p) |\not\nabla \mathbf{q}|^2 + r^{-2} |\mathbf{q}|^2) \\ &\simeq \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-3+p} |\mathbf{q}|^2 + r^{-1-\delta} |e_3 \mathbf{q}|^2 + r^{-1+p} |e_4 \mathbf{q}|^2 + r^{-1+p} |\not\nabla \mathbf{q}|^2 \end{aligned}$$

Since the powers of r of $\mathcal{I}_p^1[C_2[\mathbf{q}]]$ decay all faster than the respective ones in $\mathcal{M}_p[\mathbf{q}]$, we have that, for $r \geq R_0$,

$$\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, eC_2[\mathbf{q}]](\tau_1, \tau_2) \lesssim e^2 \mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2)$$

Finally we bound the last term commuting T and R :

$$-e\Lambda \int_{\mathcal{M}_{trap}} TR\mathbf{q}^{\mathbf{F}} \cdot R(C_2[\mathbf{q}]) \lesssim -e\Lambda \int_{\mathcal{M}_{trap}} (RT\mathbf{q}^{\mathbf{F}} + \mathfrak{d}\mathbf{q}^{\mathbf{F}}) \cdot R(\mathbf{q}) \lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e\mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)$$

as before. Combining all the above bounds we get the desired estimate. \square

Observe that the proof of Proposition 2.5 fails in the trapping region for the terms involving $T\mathbf{q} \not\Delta \mathbf{q}^{\mathbf{F}}$, $TT\mathbf{q}^{\mathbf{F}} T\mathbf{q}$ and $T\check{\nabla} \mathbf{q}^{\mathbf{F}} \check{\nabla} \mathbf{q}$, because these terms are degenerate in the Morawetz bulks. To resolve this problem, we will make use of the equations, and the particular structure of the coupling terms implies a cancellation of the degenerate terms.

Proposition 2.6. *With the notations above, for all $\delta \leq p \leq 2 - \delta$, we have*

$$\begin{aligned} & -e\Lambda \int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot C_1[\mathbf{q}^{\mathbf{F}}] + TT\mathbf{q}^{\mathbf{F}} \cdot T(C_2[\mathbf{q}]) + T\check{\nabla} \mathbf{q}^{\mathbf{F}} \cdot \check{\nabla}(C_2[\mathbf{q}]) \\ & \lesssim e \sup_{\tau \in [\tau_1, \tau_2]} (E[\mathbf{q}](\tau) + E^1[\mathbf{q}^{\mathbf{F}}](\tau)) + e(\mathcal{M}_p[\mathbf{q}] + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}] + \hat{\mathcal{M}}[\psi_3])(\tau_1, \tau_2) \end{aligned}$$

Proof. We write explicetly the terms $C_1[\mathbf{q}^{\mathbf{F}}]$ and $C_2[\mathbf{q}]$ and since $T(r) = O(\epsilon)$ and $\check{\nabla}(r) = 0$, we have

$$\begin{aligned} & -e\Lambda \int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot C_1[\mathbf{q}^{\mathbf{F}}] + TT\mathbf{q}^{\mathbf{F}} \cdot T(C_2[\mathbf{q}]) + T\check{\nabla} \mathbf{q}^{\mathbf{F}} \cdot \check{\nabla}(C_2[\mathbf{q}]) \\ = & -e\Lambda \int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot \left(\frac{2}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}} - \frac{2}{r} \underline{\kappa} e_4 \mathbf{q}^{\mathbf{F}} - \frac{2}{r} \kappa e_3 \mathbf{q}^{\mathbf{F}} + \frac{1}{r} (3\underline{\kappa} + 8\rho + 4^{(F)}\rho^2) \mathbf{q}^{\mathbf{F}} \right) - \frac{2}{r^3} TT\mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{q}) - \frac{2}{r^3} T\check{\nabla} \mathbf{q}^{\mathbf{F}} \cdot \check{\nabla}(\mathbf{q}) \end{aligned}$$

The terms involving only one derivative of $\mathbf{q}^{\mathbf{F}}$ can be bounded by their absolute value and by the Morawetz bulk as before:

$$-e\Lambda \int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot \left(-\frac{2}{r} \underline{\kappa} e_4 \mathbf{q}^{\mathbf{F}} - \frac{2}{r} \kappa e_3 \mathbf{q}^{\mathbf{F}} + \frac{1}{r} (3\underline{\kappa} + 8\rho + 4^{(F)}\rho^2) \mathbf{q}^{\mathbf{F}} \right) \lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e\mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)$$

The higher order terms that are degenerate in the bulks are instead

$$\int_{\mathcal{M}_{trap}} -2T(\mathbf{q}) \cdot \left(\frac{2}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}} \right) + \frac{2}{r^3} TT\mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{q}) + \frac{2}{r} T\check{\nabla} \mathbf{q}^{\mathbf{F}} \cdot \check{\nabla}(\mathbf{q})$$

where $\check{\nabla} = r\check{\nabla}$. The last term can be integrated by parts twice estimating the boundary terms by the energy, obtaining

$$\begin{aligned} \int_{\mathcal{M}_{trap}} \frac{2}{r} T\check{\nabla} \mathbf{q}^{\mathbf{F}} \cdot \check{\nabla}(\mathbf{q}) & \lesssim \int_{\mathcal{M}_{trap}} -\frac{2}{r} \check{\nabla} \mathbf{q}^{\mathbf{F}} \cdot T\check{\nabla}(\mathbf{q}) + \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + \sup_{\tau \in [\tau_1, \tau_2]} E^1[\mathbf{q}^{\mathbf{F}}] \\ & \lesssim \int_{\mathcal{M}_{trap}} \frac{2}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{q}) + \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + \sup_{\tau \in [\tau_1, \tau_2]} E^1[\mathbf{q}^{\mathbf{F}}] \end{aligned}$$

Finally we can rewrite the second term using the wave equation for $\mathbf{q}^{\mathbf{F}}$. In fact we write \square_2 in terms of the vectorfields T and R , ignoring the quadratic terms:

$$\square_2 \Psi = -\frac{1}{\Upsilon} TT\Psi + \frac{1}{\Upsilon} RR\Psi + \not\Delta_2 \Psi + \frac{2}{r} R\Psi$$

Renormalizing T as $\tilde{T} = \frac{r}{\Upsilon^{1/2}} T$, we will obtain for $\mathbf{q}^{\mathbf{F}}$

$$\begin{aligned} \tilde{T}\tilde{T}\mathbf{q}^{\mathbf{F}} &= r^2 \not\Delta_2 \mathbf{q}^{\mathbf{F}} + \frac{r^2}{\Upsilon} RR\mathbf{q}^{\mathbf{F}} + 2rR\mathbf{q}^{\mathbf{F}} - r^2 \square_2 \mathbf{q}^{\mathbf{F}} \\ &= r^2 \not\Delta_2 \mathbf{q}^{\mathbf{F}} + \frac{r^2}{\Upsilon} RR\mathbf{q}^{\mathbf{F}} + 2rR\mathbf{q}^{\mathbf{F}} - r^2 V_2 \mathbf{q}^{\mathbf{F}} + e\frac{2}{r}\mathbf{q} - e^2\frac{4}{r}\psi_3 \end{aligned}$$

Without loss of generality, we can derive all the above estimate with the normalization \tilde{T} in the trapping region. In particular, in Proposition 2.4, we can commute the equation with a deformation of T that coincides with \tilde{T} in the trapping region. With an abuse of notation, we denote this deformation again T . Using the equation, and estimating the other terms by the Morawetz bulks we have

$$\begin{aligned} \int_{\mathcal{M}_{trap}} \frac{2}{r^3} TT\mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{q}) &= \int_{\mathcal{M}_{trap}} \frac{2}{r^3} \left(r^2 \not\Delta_2 \mathbf{q}^{\mathbf{F}} + \frac{r^2}{\Upsilon} RR\mathbf{q}^{\mathbf{F}} + 2rR\mathbf{q}^{\mathbf{F}} - r^2 V_2 \mathbf{q}^{\mathbf{F}} + e\frac{2}{r}\mathbf{q} - e^2\frac{4}{r}\psi_3 \right) \cdot T(\mathbf{q}) \\ &\lesssim \int_{\mathcal{M}_{trap}} \frac{2}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{q}) \\ &\quad + \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e\mathcal{M}_p[\mathbf{q}](\tau_1, \tau_2) + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) + e^2 \hat{\mathcal{M}}[\psi_3](\tau_1, \tau_2) \end{aligned}$$

We observe the cancellation for the higher order terms $\int_{\mathcal{M}_{trap}} -\frac{4}{r} T(\mathbf{q}) \cdot \not\Delta_2 \mathbf{q}^{\mathbf{F}} + \frac{2}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{q}) + \frac{2}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{q})$, therefore

$$\begin{aligned} e\Lambda \int_{\mathcal{M}_{trap}} -2T(\mathbf{q}) \cdot \left(\frac{2}{r} \not\Delta_2 \mathbf{q}^{\mathbf{F}} \right) + \frac{2}{r^3} TT\mathbf{q}^{\mathbf{F}} \cdot T(\mathbf{q}) + \frac{2}{r} T\nabla\mathbf{q}^{\mathbf{F}} \cdot \nabla(\mathbf{q}) \\ \lesssim e \sup_{\tau \in [\tau_1, \tau_2]} \left(E[\mathbf{q}](\tau) + E^1[\mathbf{q}^{\mathbf{F}}](\tau) \right) + e \left(\mathcal{M}_p[\mathbf{q}] + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}] + \hat{\mathcal{M}}[\psi_3] \right) (\tau_1, \tau_2) \end{aligned}$$

which gives the desired estimate. \square

Remark 2.7. *The above cancellation is related to the structure of the equations, in particular to the fact that the system formed by the higher order terms is diagonalizable, as observed by Pei-Ken Hung. In fact consider the higher order terms of the system (7):*

$$\begin{cases} \square_2 \mathbf{q} = e\frac{2}{r^3} \not\Delta^2 \mathbf{q}^{\mathbf{F}}, \\ \square_2 \mathbf{q}^{\mathbf{F}} = -e\frac{2}{r^3} \mathbf{q} \end{cases} \quad (53)$$

Commuting the second equation with $\not\Delta$ and denoting $\mathbf{p} = \not\Delta \mathbf{q}^{\mathbf{F}}$ we obtain the system, up to lower order terms,

$$\begin{cases} \square_2 \mathbf{q} = e\frac{2}{r^3} \not\Delta \mathbf{p}, \\ \square_2 \mathbf{p} = -e\frac{2}{r^3} \not\Delta \mathbf{q} \end{cases}$$

Therefore constructing the stress-energy tensor as $\mathcal{Q}_{ab} = \mathcal{Q}[\mathbf{q}]_{ab} + \mathcal{Q}[\mathbf{p}]_{ab} + e\frac{2}{r^3}\partial\mathbf{q}\cdot\mathbf{p}\mathbf{g}_{ab}$, in taking the divergence of $\mathcal{Q}_{ab}X^b$, for any vector field X , upon integration we will get a cancellation of the higher order terms of the equation with the divergence of the added term $e\frac{2}{r^3}\partial\mathbf{q}\cdot\mathbf{p}\mathbf{g}_{ab}$. Observe that this is the same structure observed in [4] for the system governing the odd part of the gravitational perturbation of Schwarzschild spacetime using harmonic gauge. A very interesting feature of this structure is that it doesn't need smallness of the right hand side, because the cancellation holds at the level of the stress-energy tensor. Nevertheless the system (7) contains lower order terms which don't have the same diagonalizable structure. Because of the presence of the first derivatives on the right hand side of the first equation, we need to commute the second equation with all derivatives, and at the present day we weren't able to find a way to cancel those terms as in the simpler case of (53). We consider instead the stress-energy tensor of the separated equations and get the cancellation of the higher order terms after obtaining the estimates, as shown in Proposition 2.5. In this approach though we need smallness of the charge to absorb the terms on the right hand side. We expect that with a more careful analysis of the lower order terms, this hypothesis could be eliminated.

2.3 Transport estimates for the lower order terms

The aim of this section is to prove estimate (37) in Step 3 of the proof of Theorem 2.1.

By (8), the lower order terms of the equations are schematically given by

$$\begin{aligned} L_1[\mathbf{q}] &= \left[\frac{1}{r^3}\psi_1, \frac{1}{r^2}\psi_0 \right], \\ L_1[\mathbf{q}^{\mathbf{F}}] = L_2[\mathbf{q}^{\mathbf{F}}] &= \left[\frac{1}{r^3}\psi_3 \right] \end{aligned}$$

Ignoring quadratic terms, the operator \underline{P} can be written as $\underline{P}f = \underline{\kappa}^{-1}e_3(rf)$, therefore by (3), we have

$$\begin{aligned} e_3(r\psi_0) &= -\frac{2}{r}\psi_1, \\ e_3(r\psi_1) &= -\frac{2}{r}\mathbf{q}, \\ e_3(r\psi_3) &= -\frac{2}{r}\mathbf{q}^{\mathbf{F}} \end{aligned} \tag{54}$$

We derive transport estimates for ψ_0 , ψ_1 and ψ_3 using the above differential relations, in the same way as done in [1].

Proposition 2.8. *Let ψ_0 , ψ_1 , \mathbf{q} , ψ_3 , $\mathbf{q}^{\mathbf{F}}$ be defined as in (3). Then, for all $\delta \leq p \leq 2 - \delta$, we have*

$$\begin{aligned} (E_p[\psi_0] + E_p[\psi_1])(\tau_2) + (\hat{\mathcal{M}}_p[\psi_1] + \hat{\mathcal{M}}_p[\psi_0])(\tau_1, \tau_2) &\lesssim (E_p[\psi_0] + E_p[\psi_1])(\tau_1) + \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) \\ E_p[\psi_3](\tau_2) + \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2) &\lesssim E_p[\psi_3](\tau_1) + \sup_{\tau \in [\tau_1, \tau_2]} E^1[\mathbf{q}^{\mathbf{F}}](\tau) \end{aligned}$$

Proof. We first consider estimates for ψ_3 . From $e_3(r\psi_3) = -\frac{2}{r}\mathbf{q}^{\mathbf{F}}$, we derive

$$\begin{aligned} \operatorname{div}(r^n|r\psi_3|^2 e_3) &= e_3(r^n|r\psi_3|^2) + r^n|r\psi_3|^2 \operatorname{div}(e_3) \\ &= -nr^{n-1}|r\psi_3|^2 - 4r^{n-1}\mathbf{q}^{\mathbf{F}}|r\psi_3| + \frac{1}{2}r^n|r\psi_3|^2 \operatorname{tr}\pi^{(e_3)} \\ &= -(n+2)r^{n-1}|r\psi_3|^2 - 4r^{n-1}\mathbf{q}^{\mathbf{F}}|r\psi_3| \end{aligned} \tag{55}$$

Similarly, commuting (54) with re_4 and $r\check{\nabla}$ we get

$$\begin{aligned} \operatorname{div} (r^n |re_4(r\psi_3)|^2 e_3) &= -(n+2)r^{n-1}|re_4(r\psi_3)|^2 + 2r^n |re_4(r\psi_3)|re_4(-\frac{2}{r}\mathbf{q}^{\mathbf{F}}) + 2r^n |re_4(r\psi_3)|[e_3, re_4](r\psi_3) \\ &= -(n+4)r^{n-1}|re_4(r\psi_3)|^2 + r^n |re_4(r\psi_3)| \left(\frac{4}{r} - \frac{16\varpi}{r^2} + \frac{12e^2}{r^3} \right) \mathbf{q}^{\mathbf{F}} - 4r^n |re_4(r\psi_3)|e_4(\mathbf{q}^{\mathbf{F}}), \quad (56) \\ \operatorname{div} (r^n |r\check{\nabla}(r\psi_3)|^2 e_3) &= -(n+2)r^{n-1}|r\check{\nabla}(r\psi_3)|^2 - 4r^n |r\check{\nabla}(r\psi_3)|\check{\nabla}(\mathbf{q}^{\mathbf{F}}) \end{aligned}$$

We add (55) to (56) and integrate in $\mathcal{M}(\tau_1, \tau_2)$, with $n = p - 4$. When integrating, in the trapping we bound:

$$\begin{aligned} \int_{M_{\text{trap}}} |\mathbf{q}^{\mathbf{F}} + \mathfrak{d}\mathbf{q}^{\mathbf{F}}| |\mathfrak{d}\psi_3| &\lesssim \int_{\tau_1}^{\tau_2} E[\mathbf{q}](\tau)^{1/2} \left(\int_{\Sigma_{\text{trap}}(\tau)} |\mathfrak{d}\psi_3|^2 \right)^{1/2} \\ &\lesssim \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}^{\mathbf{F}}](\tau)^{1/2} \int_{\tau_1}^{\tau_2} \left(\int_{\Sigma_{\text{trap}}(\tau)} |\mathfrak{d}\psi_3|^2 \right)^{1/2} \\ &\lesssim \lambda \sup_{\tau \in [\tau_1, \tau_2]} E^1[\mathbf{q}^{\mathbf{F}}](\tau) + \lambda^{-1} \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2) \end{aligned}$$

and for λ big enough, the non-degenerate bulk $\hat{\mathcal{M}}[\psi_3]$ can be absorbed by the left hand side of (56). We arrive therefore to

$$E_p[\psi_3](\tau_2) + \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2) \lesssim E_p[\psi_3](\tau_1) + \sup_{\tau \in [\tau_1, \tau_2]} E^1[\mathbf{q}^{\mathbf{F}}](\tau)$$

Similarly, using the differential relation $e_3(r\psi_1) = -\frac{2}{r}\mathbf{q}$, we derive transport estimates for ψ_1 :

$$E_p[\psi_1](\tau_2) + \hat{\mathcal{M}}_p[\psi_1](\tau_1, \tau_2) \lesssim E_p[\psi_1](\tau_1) + \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) \quad (57)$$

Finally, using the differential relation $e_3(r\psi_0) = -\frac{2}{r}\psi_1$, and bounding with Cauchy-Schwarz we obtain

$$E_p[\psi_0](\tau_2) + \hat{\mathcal{M}}[\psi_0](\tau_1, \tau_2) \lesssim E_p[\psi_0](\tau_1) + \int_{\mathcal{M}(\tau_1, \tau_2)} r^{-3+p} (|\psi_1|^2 + |\mathfrak{d}\psi_1|^2) \quad (58)$$

Multiplying (58) by a constant A and summing to (57), choosing $A \ll 1$, we can absorb the integral on the right hand side of (58) by the non-degenerate bulk of ψ_1 . We obtain

$$(E_p[\psi_0] + E_p[\psi_1])(\tau_2) + (\hat{\mathcal{M}}_p[\psi_1] + \hat{\mathcal{M}}_p[\psi_0])(\tau_1, \tau_2) \lesssim (E_p[\psi_0] + E_p[\psi_1])(\tau_1) + \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau)$$

as desired. \square

We derive now the estimates for the terms of the main estimates involving the lower order terms.

Proposition 2.9. *With the notations above, for all $\delta \leq p \leq 2 - \delta$, we have*

$$\begin{aligned} &(\mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}^{\mathbf{F}}]] + \mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}]] + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, e^2 L_2[\mathbf{q}^{\mathbf{F}}]]) (\tau_1, \tau_2) \\ &- e\Lambda \int_{M_{\text{trap}}} 2T(\mathbf{q}) \cdot (L_1[\mathbf{q}^{\mathbf{F}}] + eL_1[\mathbf{q}]) - eTT\mathbf{q}^{\mathbf{F}} \cdot T(L_2[\mathbf{q}^{\mathbf{F}}]) - eTR\mathbf{q}^{\mathbf{F}} \cdot R(L_2[\mathbf{q}^{\mathbf{F}}]) - eT\check{\nabla}\mathbf{q}^{\mathbf{F}} \cdot \check{\nabla}(L_2[\mathbf{q}^{\mathbf{F}}]) \\ &\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} (E[\mathbf{q}](\tau) + E^1[\mathbf{q}^{\mathbf{F}}](\tau)) + e(\hat{\mathcal{M}}[\psi_0] + \hat{\mathcal{M}}[\psi_1] + \hat{\mathcal{M}}[\psi_3] + \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}]) (\tau_1, \tau_2) \end{aligned}$$

Proof. Consider $\mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}^{\mathbf{F}}]]$, and recall that $L_1[\mathbf{q}^{\mathbf{F}}] = \frac{1}{r^3}\psi_3$. We consider the spacetime integral in the three region in which the spacetime is divided: \mathcal{M}_{red} , \mathcal{M}_{trap} and \mathcal{M}_{far} .

In the redshift region \mathcal{M}_{red} ,

$$\mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}^{\mathbf{F}}]] \simeq e^2 \int_{\mathcal{M}_{red}(\tau_1, \tau_2)} |\psi_3|^2 \lesssim e^2 \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2)$$

In the trapping region \mathcal{M}_{trap} , we don't have issue with degeneracy, since the bulk term $\hat{M}[\psi_3]$ is not degenerate at the trapping region:

$$\begin{aligned} \mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}^{\mathbf{F}}]] &\simeq e \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} (|R\mathbf{q}| + r^{-1}|\mathbf{q}|)(|\psi_3|) \lesssim e \int_{\tau_1}^{\tau_2} E[\mathbf{q}](\tau)^{1/2} \left(\int_{\Sigma_{trap}(\tau)} |\psi_3|^2 \right)^{1/2} \\ &\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2) \end{aligned}$$

In the far-away region \mathcal{M}_{far} ,

$$\begin{aligned} \mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) &= e^2 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{1+p} \left| \frac{1}{r^3} \psi_3 \right|^2 = e^2 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-5+p} |\psi_3|^2 \\ &\lesssim e^2 \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2) \end{aligned}$$

which is absorbed by the non-degenerate bulk, which decays as $r^{-1+p}|\psi_3|^2$.

Consider $\mathcal{I}_p[\mathbf{q}, e^2 L_1[\mathbf{q}]]$ and recall that $L_1[\mathbf{q}] = \left[\frac{1}{r^3} \psi_1, \frac{1}{r^2} \psi_0 \right]$.

In the redshift region \mathcal{M}_{red} ,

$$\mathcal{I}_p[\mathbf{q}, e^2 L_1[\mathbf{q}]] \simeq e^4 \int_{\mathcal{M}_{red}(\tau_1, \tau_2)} |\psi_1|^2 + |\psi_0|^2 \lesssim e^4 \hat{\mathcal{M}}_p[\psi_1](\tau_1, \tau_2) + e^4 \hat{\mathcal{M}}_p[\psi_0](\tau_1, \tau_2)$$

In the trapping region \mathcal{M}_{trap} , we have

$$\begin{aligned} \mathcal{I}_p[\mathbf{q}, e^2 L_1[\mathbf{q}]] &\simeq e^2 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} (|R\mathbf{q}| + r^{-1}|\mathbf{q}|)(|\psi_1| + |\psi_0|) \\ &\lesssim e^2 \int_{\tau_1}^{\tau_2} E[\mathbf{q}](\tau)^{1/2} \left(\int_{\Sigma_{trap}(\tau)} |\psi_1|^2 + |\psi_0|^2 \right)^{1/2} \\ &\lesssim e^2 \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e^2 \hat{\mathcal{M}}_p[\psi_1](\tau_1, \tau_2) + e^2 \hat{\mathcal{M}}_p[\psi_0](\tau_1, \tau_2) \end{aligned}$$

In the far-away region \mathcal{M}_{far} ,

$$\begin{aligned} \mathcal{I}_p[\mathbf{q}, e^2 L_1[\mathbf{q}]](\tau_1, \tau_2) &= e^4 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{1+p} \left| \frac{1}{r^3} \psi_1, \frac{1}{r^2} \psi_0 \right|^2 \\ &= e^4 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-5+p} |\psi_1|^2 + r^{-3+p} |\psi_0|^2 \\ &\lesssim e^4 \hat{\mathcal{M}}_p[\psi_1](\tau_1, \tau_2) + e^4 \hat{\mathcal{M}}_p[\psi_0](\tau_1, \tau_2) \end{aligned}$$

which is absorbed by the non-degenerate bulks, which decay as $r^{-1+p}(|\psi_1|^2 + |\psi_0|^2)$.

Consider $\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, e^2 L_2[\mathbf{q}^{\mathbf{F}}]]$, and recall that $L_2[\mathbf{q}^{\mathbf{F}}] = \frac{1}{r^3}\psi_3$. We have in the redshift region \mathcal{M}_{red} ,

$$\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, e^2 L_2[\mathbf{q}^{\mathbf{F}}]] \simeq e^4 \int_{\mathcal{M}_{red}(\tau_1, \tau_2)} |\partial\psi_3|^2 \lesssim e^4 \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2) + e^4 \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)$$

In the trapping region \mathcal{M}_{trap} , we have

$$\begin{aligned}
\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, e^2 L_2[\mathbf{q}^{\mathbf{F}}]] &\simeq e^2 \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_{trap}(\tau)} (|R\mathfrak{d}\mathbf{q}^{\mathbf{F}}| + r^{-1}|\mathfrak{d}\mathbf{q}^{\mathbf{F}}|)(|\mathfrak{d}\psi_3|) \\
&\lesssim e^2 \int_{\tau_1}^{\tau_2} E^1[\mathbf{q}^{\mathbf{F}}](\tau)^{1/2} \left(\int_{\Sigma_{trap}(\tau)} |\mathfrak{d}\psi_3|^2 \right)^{1/2} \\
&\lesssim e^2 \sup_{\tau \in [\tau_1, \tau_2]} E^1[\mathbf{q}^{\mathbf{F}}](\tau) + e^2 \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2) + e^4 \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)
\end{aligned}$$

In the far-away region \mathcal{M}_{far} ,

$$\mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, e^2 L_2[\mathbf{q}^{\mathbf{F}}]](\tau_1, \tau_2) = e^4 \int_{\mathcal{M}_{far}(\tau_1, \tau_2)} r^{-5+p} |\mathfrak{d}\psi_3|^2 \lesssim e^4 \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2) + e^4 \mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2)$$

For the remaining terms we have similarly

$$\begin{aligned}
-e\Lambda \int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot (L_1[\mathbf{q}^{\mathbf{F}}] + eL_1[\mathbf{q}]) &\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} E[\mathbf{q}](\tau) + e^2 \hat{\mathcal{M}}_p[\psi_1](\tau_1, \tau_2) \\
&\quad + e^2 \hat{\mathcal{M}}_p[\psi_0](\tau_1, \tau_2) + e \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2), \\
e^2 \Lambda \int_{\mathcal{M}_{trap}} TT\mathbf{q}^{\mathbf{F}} \cdot T(L_2[\mathbf{q}^{\mathbf{F}}]) + TR\mathbf{q}^{\mathbf{F}} \cdot R(L_2[\mathbf{q}^{\mathbf{F}}]) + T\mathcal{V}\mathbf{q}^{\mathbf{F}} \cdot \mathcal{V}(L_2[\mathbf{q}^{\mathbf{F}}]) \\
&\lesssim e^2 \sup_{\tau \in [\tau_1, \tau_2]} E^1[\mathbf{q}^{\mathbf{F}}](\tau) + e^2 \hat{\mathcal{M}}_p[\psi_3](\tau_1, \tau_2)
\end{aligned}$$

and combining all the above, we obtain the desired estimate. \square

Corollary 2.10. *With the notations above, for all $\delta \leq p \leq 2 - \delta$, we have*

$$\begin{aligned}
&(\mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}^{\mathbf{F}}]] + \mathcal{I}_p[\mathbf{q}, eL_1[\mathbf{q}]] + \mathcal{I}_p^1[\mathbf{q}^{\mathbf{F}}, e^2 L_2[\mathbf{q}^{\mathbf{F}}]])(\tau_1, \tau_2) \\
&-e\Lambda \int_{\mathcal{M}_{trap}} 2T(\mathbf{q}) \cdot (L_1[\mathbf{q}^{\mathbf{F}}] + eL_1[\mathbf{q}]) - eTT\mathbf{q}^{\mathbf{F}} \cdot T(L_2[\mathbf{q}^{\mathbf{F}}]) - eTR\mathbf{q}^{\mathbf{F}} \cdot R(L_2[\mathbf{q}^{\mathbf{F}}]) - eT\mathcal{V}\mathbf{q}^{\mathbf{F}} \cdot \mathcal{V}(L_2[\mathbf{q}^{\mathbf{F}}]) \\
&\lesssim e \sup_{\tau \in [\tau_1, \tau_2]} (E[\mathbf{q}](\tau) + E^1[\mathbf{q}^{\mathbf{F}}](\tau)) + e\mathcal{M}_p^1[\mathbf{q}^{\mathbf{F}}](\tau_1, \tau_2) + (E_p[\psi_0] + E_p[\psi_1] + E_p[\psi_3])(\tau_1)
\end{aligned}$$

Proof. Straightforward consequence of Proposition 2.9 and the transport estimates in Proposition 2.8. \square

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