

# Extremal functions for an embedding from some anisotropic space, and partial differential equation involving the "one Laplacian"

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## Abstract

In this paper, we prove the existence of extremal functions for the best constant of embedding from anisotropic space, allowing some of the Sobolev exponents to be equal to 1. We prove also that the extremal functions satisfy a partial differential equation involving the 1 Laplacian.

## 1 Introduction

Anisotropic Sobolev spaces have been studied for a long time, with different purposes. Let us recall that for  $\vec{p} = (p_1, \dots, p_N)$ , and the  $p_i \geq 1$  the space  $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ , denotes the closure of  $\mathcal{D}(\mathbb{R}^N)$  for the norm  $\sum_i |\partial_i u|_{p_i}$ . The existence of a critical embedding from  $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$  into  $L^{p^*}$ , with  $p^* = \frac{N}{\sum_i \frac{1}{p_i} - 1}$  when  $\sum_i \frac{1}{p_i} > 1$  is due to Troisi, [33].

There is by now a large number of papers and an increasing interest about anisotropic problems. With no hope of being complete, let us mention some pioneering works on anisotropic Sobolev spaces [23], [29] and some more recent regularity results for minimizers of anisotropic functionals, that we will cite below.

Let us note that anisotropic operators bring new problems, essentially when one wants to prove regularity properties. As an example the property that  $\Omega$  be Lipschitz does not ensure the embedding  $W_o^{1,\vec{p}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ . This is linked to the fact that in the absence of further geometric properties of  $\Omega$ , one cannot provide a continuous extension operator from  $W^{1,\vec{p}}(\Omega)$  in  $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ . To illustrate this, see the counterexample in [22], see also [13] for one example when some of the  $p_i$  are equal to 1, in the context of the present article.

Let us say a few words about the existence and regularity results of solutions to  $-\sum_i \partial_i(|\partial_i u|^{p_i-2} \partial_i u) = f$ ,  $u = 0$  on  $\partial\Omega$  when  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ .

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Assuming a convenient assumption on  $f$ , the existence of solutions can generally easily be obtained by the use of classical methods in the calculus of variations. But, as a first step in the regularity of such solutions, the local boundedness of the solutions, can fail if the supremum of the  $p_i$  is too large, let us cite to that purpose [18] and [26] where the author exhibits a counterexample to the local boundedness when  $p_i = 2$  for  $i \leq N - 1$  and  $p_N > 2\frac{N-1}{N-3}$ . This restriction on  $\vec{p}$ , to ensure the local boundedness is confirmed by the results obtained later : let us cite in a non exhaustive way [7], [26], [4]. From all these papers it emanates in a first time that a sufficient condition for a local minimizer to be locally bounded is that the supremum of the  $p_i$  be strictly less than the critical exponent  $p^*$ . This local boundedness is extended by Fusco Sbordone in [17] to the case where  $\sup p_i = p^*$ . For further regularity properties of the solutions, as the local higher integrability of the local minimizers for some genarized functionals, see Marcellini in [27], and Esposito Leonetti Mingione [14, 15]

Coming back to  $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ , and concerning extremal functions, let us recall that in the isotropic case, the first results concerned the case where  $p_i = 2$  for all  $i$ , in which case the extremal functions are solutions of  $-\Delta u = u^{2^*-1}$ . The existence and the explicit form of them is completely solved by Aubin [3], and Talenti, [31]. For  $W^{1,p}$  and the isotropic  $p$  Laplacian, say  $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$  the explicit form is also known as the family of radial functions  $u_{a,b}(r) = (a + br^{\frac{p}{p-1}})^{\frac{p-N}{p}}$ , while for the  $p$ -Laplacian non isotropic, say for the equation  $-\sum_{i=1}^N \partial_i(|\partial_i u|^{p-2}\partial_i u) = u^{p^*-1}$ , the explicit solutions are obtained by Alvino Ferrone Trombetti Lions [1] and are given by  $u_{a,b}(r) = (a + b \sum_{i=1}^N |x_i|^{\frac{p}{p-1}})^{\frac{p-N}{p}}$ . For further results about sharp embedding constant, and a new, elegant approach by using mass transportation the author can see [6].

Let us now consider the case where the  $p_i$  can be different from each others, and let us first cite the paper of Fragala Gazzola and Kawohl [16], where the authors prove the existence of extremal functions for some subcritical embeddings in the case of bounded domains.

For the case of  $\mathbb{R}^N$  and the critical case, the existence of extremal functions is proved in [21], when all the  $p_i > 1$ , and  $p^+ := \sup p_i < p^*$ . The authors provide also some properties of the extremal functions, as the  $L^\infty$  behaviour, extending in that way the regularity results already obtained for solutions of anisotropic partial differential equation in a bounded domain, with a right hand side sub-critical as in [16], to the critical one. The method uses essentially the concentration compactness theory of P. L. Lions [24, 25] adapted to this context, and some other tools developed also in a more general context in [20].

In the case where  $p^+ = p^*$  and for more general domains than  $\mathbb{R}^N$  the reader can see Vetois, [34]. In this article this author provides also some vanishing properties of the solutions, as well as some further regularity properties of the solutions.

When some of the  $p_i$  are equal to 1, let us cite the paper of Mercaldo, Rossi, Segura de leon, Trombetti, [28], which proved the existence of solutions in some anisotropic space, with some derivative in the space of bounded measures, for the  $\vec{p}$ -Laplace equation in bounded domains, using the definition of the one Laplacian with respect to the coordinates for which  $p_i = 1$ . For the existence of extremal functions in the case of  $\mathbb{R}^N$ , and in the best of our knowledge, nothing has been done in the case where some of the  $p_i$  are equal to 1. Of course in that case these extremal functions have their corresponding derivative in the space  $M^1(\mathbb{R}^N)$  of bounded measures on  $\mathbb{R}^N$ . Even if the existence of such extremal can be obtained following the lines in the proof of [21], the partial differential equation satisfied by the extremal cannot be obtained by this existence's result. In order to get it, we are led to consider a sequence of extremal functions for the embedding of  $\mathcal{D}^{1,\vec{p}_\epsilon}(\mathbb{R}^N)$  in  $L^{p_\epsilon^*}(\mathbb{R}^N)$  where in  $\vec{p}_\epsilon$ , all the  $p_i^\epsilon > p_i$  and tend to them as  $\epsilon$  goes to zero. Note that one of the difficulties raised by this approximation is that, due to the unboundedness of  $\mathbb{R}^N$ ,  $\mathcal{D}^{1,\vec{p}_\epsilon}(\mathbb{R}^N)$  is not a subspace of  $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ , a problem which does not appear when one works with bounded domains, see [13]. In particular this does not allow to use directly the concentration compactness theory of P.L. Lions, [24]. We will prove both that the best constant for the embedding from  $\mathcal{D}^{1,\vec{p}_\epsilon}(\mathbb{R}^N)$  in  $L^{p_\epsilon^*}(\mathbb{R}^N)$  converges to the best constant for the embedding of  $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$  into  $L^{p^*}(\mathbb{R}^N)$ , and that some extremal  $u_\epsilon$  converge sufficiently tightly to some  $u$ . Passing to the limit in the partial differential equation satisfied by  $u_\epsilon$  one obtains that  $u$  is extremal and satisfies the required partial differential equation.

## 2 Notations, and previous results

### 2.1 Some measure Theory, definition and properties of the space $BV^{\vec{p}}$

**Definition 2.1.** Let  $\Omega$  be an open set in  $\mathbb{R}^N$ , and  $M(\Omega)$ , the space of scalar Radon measures, i.e. the dual of  $\mathcal{C}_c(\Omega)$ . Let  $M^1(\mathbb{R}^N)$  be the space of scalar bounded Radon measures or equivalently the subspace of  $\mu \in M(\Omega)$  which satisfy  $\int_\Omega |\mu| = \sup_{\varphi \in \mathcal{C}_c(\Omega)} \langle \mu, \varphi \rangle < \infty$ .

$M^+(\Omega)$  is the space of non negative bounded measures on  $\mathbb{R}^N$ .

**Definition 2.2.** When  $\mu = (\mu_1, \dots, \mu_n)$  we define  $|\mu| = (\sum \mu_i^2)^{\frac{1}{2}}$  as the measure : For  $\varphi \geq 0$  in  $\mathcal{C}_c(\Omega)$ ,  $\langle |\mu|, \varphi \rangle = \sup_{\psi \in \mathcal{C}_c(\Omega, \mathbb{R}^N), \sum_1^N \psi_i^2 \leq \varphi^2} \sum \langle \mu_i, \psi_i \rangle$ .

Let us recall that

**Definition 2.3.**  $\mu_n \rightharpoonup \mu$  vaguely or weakly in  $M(\Omega)$  if for any  $\varphi \in \mathcal{C}_c(\Omega)$ ,  $\langle \mu_n, \varphi \rangle \rightarrow \langle \mu, \varphi \rangle$ .

When  $\mu_n$  and  $\mu$  are in  $M^1(\Omega)$  we will say that  $\mu_n$  converges tightly to  $\mu$  if for any  $\varphi \in \mathcal{C}(\mathbb{R}^N)$  and bounded,  $\langle \mu_n, \varphi \rangle \rightarrow \langle \mu, \varphi \rangle$ .

**Remark 2.4.** When  $\mu_n \geq 0$ , the tight convergence of  $\mu_n$  to  $\mu$  is equivalent to both the two conditions 1)  $\mu_n \rightharpoonup \mu$  vaguely and 2)  $\int_{\Omega} \mu_n \rightarrow \int_{\Omega} \mu$ .

We will frequently use the following density result:

**Proposition 2.5.** If  $\vec{\mu} \in M^1(\Omega, \mathbb{R}^N)$  there exists  $u_n \in \mathcal{D}(\Omega, \mathbb{R}^N)$  such that  $(u_n)_i^{\pm}$ , respectively  $|u_n|$  converges tightly to  $\mu_i^{\pm}$  (respect.  $|\mu|$ ).

The reader is referred to [12], [11], for further properties on convergence of measures and density of regular functions for the vague and tight topology.

Let  $N_1 \leq N \in \mathbb{N}$ , and  $\vec{p} := (p_1, \dots, p_N) \in \mathbb{R}^N$  such that  $p_i = 1$  for all  $1 \leq i \leq N_1$ , and  $p_i > 1$  for all  $N_1 + 1 \leq i \leq N$ .

Let  $p^+ = \sup p_i$ , and

$$p^* := \frac{N}{N_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} - 1}.$$

In all the paper we will suppose that  $p^+ < p^*$ . Let  $\mathcal{D}^{1, \vec{p}}(\mathbb{R}^N)$  be the completion of  $\mathcal{D}(\mathbb{R}^N)$  with respect to the norm

$$|u|_{\vec{p}} = \left( \sum_{i=1}^{N_1} (\partial_i u)^2 \right)^{\frac{1}{2}} + \sum_{i=N_1+1}^N |\partial_i u|_{p_i} := |\nabla_1 u|_1 + \sum_{i=N_1+1}^N |\partial_i u|_{p_i} \quad (2.1)$$

where  $\nabla_1 u$  is the  $N_1$  vector  $(\partial_1 u, \dots, \partial_{N_1} u)$ , and  $|u|_{p_i}$  denotes for  $i \geq N_1 + 1$  the usual  $L^{p_i}(\mathbb{R}^N)$  norm.

**Remark 2.6.** Of course by the equivalence of norms in  $\mathbb{R}^{N_1}$  this completion coincides with the completion for the norm  $\sum_{i=1}^N |\partial_i u|_{p_i}$ .

We now recall the existence of the embedding from  $\mathcal{D}^{1, \vec{p}}(\mathbb{R}^N)$  in  $L^{p^*}(\mathbb{R}^N)$ , a particular case of the result of Troisi, [33].

**Theorem 2.7.**

$$\mathcal{D}^{1, \vec{p}}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N),$$

and there exists some constant  $T_0$  depending only on  $\vec{p}$ , and  $N$  such that

$$T_0|u|_{p^*} \leq \prod_{i=1}^N |\partial_i u|_{p_i}^{\frac{1}{N}}, \text{ and } |u|_{p^*} \leq \frac{1}{T_0 N} \left( \sqrt{N_1} |\nabla_1 u|_1 + \sum_{i=N_1+1}^N |\partial_i u|_{p_i} \right), \quad (2.2)$$

for all  $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ .

We now introduce a weak closure of  $\mathcal{D}(\mathbb{R}^N)$  for the norm (2.1). Set

$$BV^{\vec{p}}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N), \partial_i u \in M^1(\mathbb{R}^N) \text{ for } 1 \leq i \leq N_1, \text{ and } \partial_i u \in L^{p_i}(\mathbb{R}^N) \text{ for } N_1 + 1 \leq i \leq N\}.$$

We also define

$$BV_{loc}^{\vec{p}}(\mathbb{R}^N) = \{u \in \mathcal{D}'(\mathbb{R}^N), \varphi u \in BV^{\vec{p}}(\mathbb{R}^N), \text{ for any } \varphi \in \mathcal{D}(\mathbb{R}^N)\}$$

**Definition 2.8.** We will say that  $u_n \in BV^{\vec{p}}(\mathbb{R}^N)$  converges weakly to  $u$  if  $u_n \rightharpoonup u$  (weakly) in  $L^{p^*}$ ,  $\partial_i u_n$  converges vaguely to  $\partial_i u$  in  $M^1(\mathbb{R}^N)$  when  $i \leq N_1$ , and  $\partial_i u_n \rightharpoonup \partial_i u$  (weakly) in  $L^{p_i}$ , when  $i > N_1$ .

The convergence is said to be tight if furthermore  $\int_{\mathbb{R}^N} |\partial_i u_n|^{p_i} \rightarrow \int_{\mathbb{R}^N} |\partial_i u|^{p_i}$  for any  $i \geq N_1$ , and  $\int_{\mathbb{R}^N} |\nabla_1 u_n| \rightarrow \int_{\mathbb{R}^N} |\nabla_1 u|$ .

**Remark 2.9.** If  $u_n$  converges weakly to  $u$ , since  $(u_n)$  is bounded in  $L^{p^*}$ , it converges strongly in  $L_{loc}^q$  for a subsequence, when  $q < p^*$  and then for a subsequence it converges almost everywhere.

**Proposition 2.10.** It is equivalent to say that

1.  $u \in BV^{\vec{p}}(\mathbb{R}^N)$
2. There exists  $u_n \in \mathcal{D}(\mathbb{R}^N)$  which converges tightly to  $u$ .
3. There exists  $u_n \in \mathcal{D}(\mathbb{R}^N)$  which converges weakly to  $u$ .

**Remark 2.11.** Following the lines in the proof below, but using strong convergence in  $L^1$  of  $\partial_i u_n$  for  $i \leq N_1$ , in place of tight convergence, it is clear that  $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N), \partial_i u \in L^1(\mathbb{R}^N), i \leq N_1, \partial_i u \in L^{p_i}(\mathbb{R}^N), i \geq N_1 + 1\}$ .

*Proof.* Suppose that 1) holds.

We begin by a truncature. For  $1 \leq i \leq N$  let  $\alpha_i$  defined as

$$\alpha_i = \frac{p^*}{p_i} - 1.$$

Let  $\varphi \in \mathcal{D}([-2, 2])$ ,  $\varphi = 1$  on  $[-1, 1]$ , and for all  $n \in \mathbf{N}$ ,

$$u_n(x) = \prod_{i=1}^N \varphi\left(\frac{x_i}{n^{\alpha_i}}\right) u(x).$$

We denote  $C_n = \prod_{i=1}^N [-2n^{\alpha_i}, 2n^{\alpha_i}]$ , note that  $|C_n| = 4^N n^{\sum_{i=1}^N \alpha_i}$ . We need to prove that  $\partial_i u_n \rightarrow \partial_i u$  in  $L^{p_i}(\mathbb{R}^N)$  for all  $i \in [N_1 + 1, N]$ . Since

$$\partial_i u_n(x) = u(x) \partial_i \left( \prod_{j=1}^N \varphi\left(\frac{x_j}{n^{\alpha_j}}\right) \right) + \prod_{j=1}^N \varphi\left(\frac{x_j}{n^{\alpha_j}}\right) \partial_i u(x),$$

it is sufficient to prove that  $u \partial_i \left( \prod_{j=1}^N \varphi\left(\frac{x_j}{n^{\alpha_j}}\right) \right) \rightarrow 0$  in  $L^{p_i}(\mathbb{R}^N)$ . By Hölder's inequality

$$\begin{aligned} \int_{\mathbb{R}^N} |u \partial_i \left( \prod_{j=1}^N \varphi\left(\frac{x_j}{n^{\alpha_j}}\right) \right)|^{p_i} &\leq \frac{c}{n^{\alpha_i p_i}} \left( \int_{\mathbb{R}^{N-1}} \int_{n^{\alpha_i} \leq |x_i| \leq 2n^{\alpha_i}} |u|^{p^*} \right)^{\frac{p_i}{p^*}} |C_n|^{1 - \frac{p_i}{p^*}} \\ &\leq c' n^{-\alpha_i p_i + (1 - \frac{p_i}{p^*}) \sum_{j=1}^N \alpha_j} o(1) \end{aligned}$$

which tends to zero, since  $u \in L^{p^*}(\mathbb{R}^N)$  implies that  $\int_{\mathbb{R}^{N-1}} \int_{n^{\alpha_i} \leq |x_i| \leq 2n^{\alpha_i}} |u|^{p^*} \xrightarrow{n \rightarrow \infty} 0$ , and for any  $i$  by the definition of  $\alpha_i$ ,  $\alpha_i p_i \geq (1 - \frac{p_i}{p^*}) \sum_{j=1}^N \alpha_j$ . In the same manner we have  $\int_{\mathbb{R}^N} |\nabla_1 u_n - \nabla_1 u| \rightarrow 0$ . The second step classically uses a regularisation process. Recall that when  $\bar{\mu}$  is a compactly supported measure in  $\mathbb{R}^N$ , with values in  $\mathbb{R}^N$ , when  $\rho \in \mathcal{D}(\mathbb{R}^N)$ ,  $\int \rho = 1$ ,  $\rho \geq 0$ , and  $\rho_\epsilon = \frac{1}{\epsilon^N} \rho\left(\frac{x}{\epsilon}\right)$ ,  $\rho_\epsilon \star |\bar{\mu}|$  converges tightly to  $|\bar{\mu}|$ ,  $\rho_\epsilon \star \mu_i^\pm$  converges tightly to  $\mu_i^\pm$ . From this one derives the tight convergence when  $\epsilon$  goes to zero and  $n$  to  $\infty$  of  $|\nabla_1(\rho_\epsilon \star u_n)|$  towards  $|\nabla_1 u|$ .

2) implies 3) is obvious. To prove that 3) implies 1), note that if  $(u_n)$  is weakly convergent to  $u$ , one has the existence of some constant independent on  $n$  so that  $|\nabla_1 u_n|_1 + \sum_{i=N_1+1}^N |\partial_i u_n|_{p_i} \leq C$ . Then by the embedding in Theorem 2.7,  $(u_n)$  is bounded in  $L^{p^*}$ , and by extracting subsequences from  $\nabla_1 u_n$  in  $M^1(\mathbb{R}^N, \mathbb{R}^{N_1})$  weakly and from  $\partial_i u_n$  in  $L^{p_i}$  weakly for  $i \geq N_1 + 1$ , one gets that the limit  $u \in BV^{\vec{p}}(\mathbb{R}^N)$ .  $\square$

**Remark 2.12.** *Using the last proposition, one sees that (2.2) extends to the functions in  $BV^{\vec{p}}(\mathbb{R}^N)$ .*

We now enounce a result which extends the definition of the "Anzelotti pairs", [2], see also Temam [32], Strang Temam in [30], and [8, 9, 10].

**Theorem 2.13.** *Let  $\sigma$  a function with values in  $\mathbb{R}^N$ , such that its projection  $\sigma^1$  on the first  $N_1$  coordinates, belongs to  $L_{loc}^\infty(\mathbb{R}^N, \mathbb{R}^{N_1})$ , and suppose that for any  $i \geq N_1 + 1$ ,  $\sigma \cdot e_i \in L_{loc}^{p'_i}$ , and that  $\operatorname{div} \sigma \in L_{loc}^{\frac{p^*}{p^*-1}}$ . Then if  $u \in BV_{loc}^{\vec{p}}(\mathbb{R}^N)$ , one can define a distribution  $\sigma \cdot \nabla u$  in the following manner, for  $\varphi \in \mathcal{D}(\mathbb{R}^N)$ ,*

$$\langle \sigma \cdot \nabla u, \varphi \rangle = - \int_{\mathbb{R}^N} \operatorname{div} \sigma(u \varphi) - \int_{\mathbb{R}^N} (\sigma \cdot \nabla \varphi) u.$$

Then  $\sigma \cdot \nabla u$  is a measure, and  $\sigma^1 \cdot \nabla_1 u := \sigma \cdot \nabla u - \sum_{i=N_1+1}^N \sigma_i \partial_i u$  is a measure absolutely continuous with respect to  $|\nabla_1 u|$ , with for  $\varphi \geq 0$  in  $\mathcal{C}_c(\mathbb{R}^N)$  :

$$\langle |\sigma^1 \cdot \nabla_1 u|, \varphi \rangle \leq |\sigma^1|_{L^\infty(\text{Suppt}\varphi)} \langle |\nabla_1 u|, \varphi \rangle. \quad (2.3)$$

Furthermore when  $\sigma^1 \in L^\infty(\mathbb{R}^N, \mathbb{R}^{N_1})$  and  $\sigma_i \in L^{p'_i}(\mathbb{R}^N)$ , for any  $i \geq N_1 + 1$ ,  $\text{div} \sigma \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N)$  and  $u \in BV^{\vec{p}}(\mathbb{R}^N)$ ,  $\sigma \cdot \nabla u$  and  $\sigma^1 \cdot \nabla_1 u$  are bounded measures on  $\mathbb{R}^N$  and one has

$$\int_{\mathbb{R}^N} \sigma \cdot \nabla u = - \int_{\mathbb{R}^N} \text{div}(\sigma)u \quad (2.4)$$

and

$$|\sigma^1 \cdot \nabla_1 u| \leq |\sigma^1|_\infty |\nabla_1 u|$$

*Proof.* Take  $\psi \in \mathcal{D}(\mathbb{R}^N)$ ,  $\psi = 1$  on  $\text{Suppt} \varphi$ . Then if  $u \in BV_{loc}^{\vec{p}}(\mathbb{R}^N)$ ,  $\psi u \in BV^{\vec{p}}(\mathbb{R}^N)$ . By Proposition 2.10, there exists  $u_n \in \mathcal{D}(\mathbb{R}^N)$  such that  $u_n$  converges tightly to  $\psi u$  in  $BV^{\vec{p}}(\mathbb{R}^N)$ . By the classical Green's formula

$$\int_{\mathbb{R}^N} \sigma \cdot \nabla u_n \varphi = - \int_{\mathbb{R}^N} \text{div}(\sigma)(u_n \varphi) - \int_{\mathbb{R}^N} (\sigma \cdot \nabla \varphi) u_n.$$

Using the weak convergence of  $u_n$  towards  $\psi u$  one gets that  $\int (\sigma \cdot \nabla u_n) \varphi$  converges to  $\langle \sigma \cdot \nabla u, \varphi \rangle$ . By the assumptions on  $\sigma_i$  and  $\partial_i u_n$ , one has  $\int \sigma_i \partial_i u_n \varphi \rightarrow \int \sigma_i \partial_i u \varphi$ , for  $i \geq N_1 + 1$ , hence  $\int (\sigma^1 \cdot \nabla_1 u_n) \varphi \rightarrow \langle \sigma^1 \cdot \nabla_1 u, \varphi \rangle$ . Furthermore, using for  $\varphi \geq 0$ ,  $|\int \sigma^1 \cdot \nabla_1 u_n \varphi| \leq |\sigma^1|_{L^\infty(\text{Suppt}\varphi)} \int |\nabla_1 u_n| \varphi \rightarrow |\sigma^1|_{L^\infty(\text{Suppt}\varphi)} \int |\nabla_1 u| \varphi$ , one gets (2.3). The identity (2.4) is easily obtained by letting  $\varphi$  go to  $1_{\mathbb{R}^N}$ , since all the measures involved are bounded measures.  $\square$

## 2.2 The approximated space $\mathcal{D}^{1, \vec{p}_\epsilon}(\mathbb{R}^N)$

Let  $\epsilon > 0$  small, define

$$a_i^\epsilon = \frac{(p_i - 1)p_i \epsilon^2}{1 - \epsilon(p_i - 1)}, \quad \epsilon_i = p_i \epsilon + a_i^\epsilon \text{ and } p_i^\epsilon = p_i(1 + \epsilon_i). \quad (2.5)$$

Note that one has for all  $i \geq N_1 + 1$ ,  $\frac{p_i(1+\epsilon_i)}{\epsilon_i} = \frac{1+\epsilon}{\epsilon}$ . We define  $\mathcal{D}^{1, \vec{p}_\epsilon}(\mathbb{R}^N)$  as the closure of  $\mathcal{D}(\mathbb{R}^N)$  for the norm  $|\nabla_1 v|_{1+\epsilon} + \sum_{i=N_1+1}^N |\partial_i v|_{p_i^\epsilon}$ . Then the critical exponent  $p_\epsilon^*$  for this space is defined by  $\frac{N}{p_\epsilon^*} = \frac{N_1}{1+\epsilon} + \sum_i \frac{1}{p_i^\epsilon} - 1$ . Note that  $p_\epsilon^*$  satisfies

$$\frac{N}{p_\epsilon^*} = \frac{N}{p^*} - \frac{\epsilon N}{1 + \epsilon},$$

and as soon as  $\epsilon$  is small enough,  $p_\epsilon^+ < p_\epsilon^*$ . Let us finally define

$$\lambda_\epsilon = \frac{p_\epsilon^* \epsilon}{1 + \epsilon} + 1, \quad (2.6)$$

and note for further purposes that  $\lambda_\epsilon p^* = p_\epsilon^*$ .

Recall that as a consequence of the embedding of Troisi, [33] one has

$$\mathcal{D}^{1, \vec{p}_\epsilon}(\mathbb{R}^N) \hookrightarrow L^{p_\epsilon^*}(\mathbb{R}^N),$$

and there exists some  $T_0^\epsilon > 0$ , such that for all  $u \in \mathcal{D}^{1, \vec{p}_\epsilon}(\mathbb{R}^N)$ ,

$$T_0^\epsilon |u|_{p_\epsilon^*} \leq \prod_{i=1}^N |\partial_i u|_{p_i^\epsilon}^{\frac{1}{N}}, \text{ and then } |u|_{p_\epsilon^*} \leq \frac{1}{NT_0^\epsilon} (\sqrt{N_1} |\nabla_1 u|_{1+\epsilon} + \sum_{i=N_1+1}^N |\partial_i u|_{p_i^\epsilon})$$

for all  $u \in \mathcal{D}^{1, \vec{p}_\epsilon}(\mathbb{R}^N)$ .

Let us define

$$\mathcal{K}_\epsilon = \inf_{u \in \mathcal{D}^{1, \vec{p}_\epsilon}(\mathbb{R}^N), |u|_{p_\epsilon^*} = 1} \left[ \frac{1}{1 + \epsilon} |\nabla_1 u|_{1+\epsilon}^{1+\epsilon} + \sum_{i=N_1+1}^N \frac{1}{p_i^\epsilon} |\partial_i u|_{p_i^\epsilon}^{p_i^\epsilon} \right]$$

and

$$\mathcal{K} = \inf_{u \in \mathcal{D}^{1, \vec{p}}(\mathbb{R}^N), |u|_{p^*} = 1} \left[ |\nabla_1 u|_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i u|_{p_i}^{p_i} \right]$$

It is clear by Proposition 2.10 that

$$\mathcal{K} = \inf_{u \in BV^{\vec{p}}(\mathbb{R}^N), |u|_{p^*} = 1} \left[ \int |\nabla_1 u| + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i u|_{p_i}^{p_i} \right]$$

Adapting the proof in [21] one has the following result

**Theorem 2.14.** *There exists  $u_\epsilon \in \mathcal{D}^{1, \vec{p}_\epsilon}(\mathbb{R}^N)$  non negative which satisfies  $|u_\epsilon|_{p_\epsilon^*} = 1$  and*

$$\mathcal{K}_\epsilon = \frac{1}{1 + \epsilon} |\nabla_1 u_\epsilon|_{1+\epsilon}^{1+\epsilon} + \sum_{i=N_1+1}^N \frac{1}{p_i^\epsilon} |\partial_i u_\epsilon|_{p_i^\epsilon}^{p_i^\epsilon}.$$

Furthermore there exists  $l_\epsilon > 0$ , so that

$$- \sum_{i=1}^{N_1} \partial_i (|\nabla_1 u_\epsilon|^{\epsilon-1} \partial_i u_\epsilon) - \sum_{i=N_1+1}^N \partial_i (|\partial_i u_\epsilon|^{p_i^\epsilon-2} \partial_i u_\epsilon) = l_\epsilon u_\epsilon^{p_\epsilon^*-1}. \quad (2.7)$$



In the sequel we will use the notation  $\operatorname{div}_1$  as the divergence of some  $N_1$  vector with respect to the  $N_1$  first variables.

By multiplying equation ( 2.7) by  $u_\epsilon$  and integrating one has  $\mathcal{K}_\epsilon \leq l_\epsilon \leq p_\epsilon^+ \mathcal{K}_\epsilon$ , and as we will see in Proposition 3.4 that  $\limsup \mathcal{K}_\epsilon \leq \mathcal{K}$ , if  $u_\epsilon$  is an extremal function for  $\mathcal{K}_\epsilon$ ,  $|\nabla_1 u_\epsilon|_{1+\epsilon}$  and  $|\partial_i u_\epsilon|_{p_i^\epsilon}$  are bounded independently on  $\epsilon$ , hence one can extract from it a subsequence which converges weakly in  $BV^{\vec{p}}$ . In the sequel we will prove that by choosing conveniently the sequence  $u_\epsilon$ , it converges up to subsequence to an extremal function for  $\mathcal{K}$ .

### 3 The main results

The main result of this paper is the following :

**Theorem 3.1.** *1) There exists  $v_\epsilon \in \mathcal{D}^{1,\vec{p}_\epsilon}(\mathbb{R}^N)$ ,  $|v_\epsilon|_{p_\epsilon^*} = 1$ , an extremal function for  $\mathcal{K}_\epsilon$ , which converges in the following sense to  $v \in BV^{\vec{p}}(\mathbb{R}^N)$ :  $v_\epsilon$  converges to  $v$  in the distribution sense, and almost everywhere,  $\int_{\mathbb{R}^N} |\nabla_1 v_\epsilon|^{1+\epsilon} \rightarrow \int_{\mathbb{R}^N} |\nabla_1 v|$ ,  $\int_{\mathbb{R}^N} |\partial_i v_\epsilon|^{p_i^\epsilon} \rightarrow \int_{\mathbb{R}^N} |\partial_i v|^{p_i}$  for all  $i \geq N_1 + 1$ , and  $|v|_{p^*} = 1$ . Furthermore  $\lim \mathcal{K}_\epsilon = \mathcal{K}$ . As a consequence  $v$  is an extremal function for  $\mathcal{K}$ .*

*2)  $v$  satisfies the partial differential equation :*

$$-\operatorname{div}_1(\sigma^1) - \sum_{i=N_1+1}^N \partial_i(|\partial_i v|^{p_i-2} \partial_i v) = l v^{p^*-1}, \quad \sigma^1 \cdot \nabla_1 v = |\nabla_1 v| \quad (3.1)$$

where  $\mathcal{K} \leq l \leq p^+ \mathcal{K}$ .

The proof of Theorem 3.1 is given in the next subsection, and it relies of course on a convenient adaptation of the PL Lions compactness concentration theory. However, due to the fact that the exponents of the derivatives and the critical exponent vary with  $\epsilon$ , we are led to introduce a power  $v_\epsilon^{\lambda_\epsilon}$  of some convenient extremal function, - where  $\lambda_\epsilon$  has been defined in (2.6)-, and to analyze the behaviour of this new sequence, which belongs to  $BV^{\vec{p}}(\mathbb{R}^N)$ , and is bounded in that space, independently on  $\epsilon$ , as we will see later.

In a second time we prove that

**Theorem 3.2.** *Let  $v$  be given by Theorem 3.1. Then  $v \in L^\infty(\mathbb{R}^N)$  and there exists some constant  $C(|v|_{p^*})$  depending on the  $L^{p^*}$  norm of  $v$  and on universal constants, such that  $|v|_\infty \leq C(|v|_{p^*})$ .*

### 3.1 Proof of Theorem 3.1

The proof is the consequence of several lemmata and propositions.

**Lemma 3.3.** *Suppose that  $u \in BV^{\vec{p}}(\mathbb{R}^N)$ , and that  $|u|_{p^*} \leq 1$ , then*

$$\mathcal{K}|u|_{p^*}^{p^+} \leq |\nabla_1 u|_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i u|_{p_i}^{p_i}$$

and analogously if  $u \in \mathcal{D}^{1,\vec{p}_\epsilon}(\mathbb{R}^N)$ ,  $|u|_{p_\epsilon^*} \leq 1$

$$\mathcal{K}_\epsilon |u|_{p_\epsilon^*}^{p_\epsilon^+} \leq \frac{1}{1+\epsilon} |\nabla_1 u|_{1+\epsilon}^{1+\epsilon} + \sum_{i=N_1+1}^N \frac{1}{p_i^\epsilon} |\partial_i u|_{p_i^\epsilon}^{p_i^\epsilon}.$$

Hint of the proof :

Use  $\frac{u}{|u|_{p^*}}$  in the definition of  $\mathcal{K}$  and the fact that if  $|u|_{p^*} \leq 1$ ,  $|u|_{p^*}^{p_i} \geq |u|_{p^*}^{p^+}$ .

**Proposition 3.4.** *One has*

$$\limsup \mathcal{K}_\epsilon \leq \mathcal{K}.$$

As a consequence any sequence  $(v_\epsilon)$  of extremal functions for  $\mathcal{K}_\epsilon$  is bounded independently on  $\epsilon$ , more precisely there exists some positive constant  $c$  so that, for all  $\epsilon > 0$ ,  $|\nabla_1 v_\epsilon|_{1+\epsilon}, |\partial_i v_\epsilon|_{p_i^\epsilon} \leq c$ , for all  $i \geq N_1 + 1$ .

*Proof.* Let  $\delta > 0$ ,  $\delta < \frac{1}{2}$  and let  $u_\delta \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ , ( or  $BV^{\vec{p}}(\mathbb{R}^N)$ ), so that  $|u_\delta|_{p^*} = 1$  and

$$|\nabla_1 u_\delta|_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i u_\delta|_{p_i}^{p_i} \leq \mathcal{K} + \delta.$$

By definition of  $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ , there exists  $v_\delta \in \mathcal{D}(\mathbb{R}^N)$  such that  $||v_\delta|_{p^*} - 1| \leq \delta$ ,

$$|\nabla_1 v_\delta|_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i v_\delta|_{p_i}^{p_i} \leq \mathcal{K} + 2\delta.$$

For  $\epsilon$  small enough one has  $||v_\delta|_{p_\epsilon^*} - 1| \leq 2\delta$ . By considering  $w_\delta^\epsilon = \frac{v_\delta}{|v_\delta|_{p_\epsilon^*}}$ , one sees that  $w_\delta^\epsilon \in \mathcal{D}(\mathbb{R}^N)$ ,  $|w_\delta^\epsilon|_{p_\epsilon^*} = 1$ , and

$$\begin{aligned} |\nabla_1 w_\delta^\epsilon|_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i w_\delta^\epsilon|_{p_i}^{p_i} &\leq \frac{|\nabla_1 v_\delta|_1}{1-2\delta} + \sum_{i=N_1+1}^N \frac{1}{p_i} \frac{|\partial_i v_\delta|_{p_i}^{p_i}}{(1-2\delta)^{p_i}} \\ &\leq \frac{1}{(1-2\delta)^{p^+}} (\mathcal{K} + 2\delta). \end{aligned}$$

By the Lebesgue's dominated convergence theorem,  $|\nabla_1 w_\delta^\epsilon|_{1+\epsilon}^{1+\epsilon} \rightarrow \frac{|\nabla_1 v_\delta|_1}{|v_\delta|_{p^*}}$ , and  $|\partial_i w_\delta^\epsilon|_{p_i^\epsilon}^{p_i^\epsilon} \rightarrow \frac{|\partial_i v_\delta|_{p_i}}{|v_\delta|_{p^*}}^{p_i}$  for all  $i \geq N_1 + 1$  when  $\epsilon \rightarrow 0$ , hence we get

$$\limsup_{\epsilon \rightarrow 0} \mathcal{K}_\epsilon \leq \limsup_{\epsilon \rightarrow 0} \left[ \frac{1}{1+\epsilon} |\nabla_1 w_\delta^\epsilon|_{1+\epsilon}^{1+\epsilon} + \sum_{i=N_1+1}^N \frac{1}{p_i^\epsilon} |\partial_i w_\delta^\epsilon|_{p_i^\epsilon}^{p_i^\epsilon} \right] \leq \frac{\mathcal{K} + 2\delta}{(1-2\delta)^{p^+}},$$

which concludes the proof since  $\delta$  is arbitrary.  $\square$

**Proposition 3.5.** *Suppose that  $w_\epsilon \in BV^{\vec{p}}(\mathbb{R}^N)$  satisfies  $|w_\epsilon|_{p^*} = 1$ , and that  $w_\epsilon \rightarrow v$  almost everywhere. Then for  $\epsilon$  small enough  $|w_\epsilon - v|_{p^*} \leq 1$ .*

*Proof.* If  $v \equiv 0$ , there is nothing to prove. If  $v \neq 0$ , using Brezis Lieb Lemma, [5] one has  $|w_\epsilon - v|_{p^*} - (|w_\epsilon|_{p^*} - |v|_{p^*}) \rightarrow 0$  which implies that  $\limsup |w_\epsilon - v|_{p^*} < 1$ , hence the result holds. This lemma will be used for  $w_\epsilon = v_\epsilon^{\lambda_\epsilon}$ , where  $v_\epsilon$  is some convenient extremal function, given in Lemma 3.6 below, and  $\lambda_\epsilon$  has been defined in (2.6).  $\square$

**Lemma 3.6.** *Let  $u_\epsilon$  be a non negative extremal function for  $\mathcal{K}_\epsilon$ , so that  $|u_\epsilon|_{p_\epsilon^*} = 1$ . There exists  $v_\epsilon \geq 0$  which satisfies*

$$\begin{aligned} |u_\epsilon|_{p_\epsilon^*} &= |v_\epsilon|_{p_\epsilon^*} = 1, \quad |\nabla_1 u_\epsilon|_{1+\epsilon} = |\nabla_1 v_\epsilon|_{1+\epsilon}, \quad \text{and} \quad |\partial_i u_\epsilon|_{p_i^\epsilon} = |\partial_i v_\epsilon|_{p_i^\epsilon}, \quad \text{for all } i \geq N_1 + 1, \\ &\text{and} \quad \int_{B(0,1)} v_\epsilon^{p_\epsilon^*} = \frac{1}{2}. \end{aligned}$$

*Proof.* This proof is as in [21], but we reproduce it here for the reader's convenience. Let  $\alpha_i^\epsilon = \frac{p_\epsilon^*}{p_i^\epsilon} - 1, i = 1, \dots, N$ . For every  $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ , and for any  $u \in \mathcal{D}^{1, \vec{p}^\epsilon}(\mathbb{R}^N)$ , and  $t > 0$ , we set

$$u^{t,y}(x) = tu(t^{\alpha_1^\epsilon}(x_1 - y_1), \dots, t^{\alpha_N^\epsilon}(x_N - y_N)).$$

Then, we have

$$\begin{aligned} |u|_{p_\epsilon^*} &= |u^{t,y}|_{p_\epsilon^*}, \\ |\partial_i u|_{p_i^\epsilon} &= |\partial_i u^{t,y}|_{p_i^\epsilon}, \quad \text{for all } 1 \leq i \leq N, \\ |\nabla_1 u|_{1+\epsilon} &= |\nabla_1 u^{t,y}|_{1+\epsilon}. \end{aligned}$$

Let  $u_\epsilon$  be an extremal function for  $\mathcal{K}_\epsilon$  so that  $|u_\epsilon|_{p_\epsilon^*} = 1$ . As in [21], [25], we recall the definition of the Levy concentration function, for  $t > 0$  :

$$Q_\epsilon(t) = \sup_{y \in \mathbb{R}^N} \int_{E(y, t^{\alpha_1^\epsilon}, \dots, t^{\alpha_N^\epsilon})} |u_\epsilon|_{p_\epsilon^*}^{p_\epsilon^*},$$

where  $E(y, t^{\alpha_1}, \dots, t^{\alpha_N})$  is the ellipse defined by

$$\{z = (z_1, \dots, z_N) \in \mathbb{R}^N, \sum_{i=1}^N \frac{(z_i - y_i)^2}{t^{2\alpha_i}} \leq 1\},$$

with  $y = (y_1, \dots, y_N)$ , and  $\alpha_i^\epsilon = \frac{p_i^*}{p_i} - 1$  for all  $i$ .

Since for every  $\epsilon > 0$ ,  $\lim_{t \rightarrow 0} Q_\epsilon(t) = 0$ , and  $\lim_{t \rightarrow \infty} Q_\epsilon(t) = 1$ , there exists  $t_\epsilon > 0$  such that  $Q_\epsilon(t_\epsilon) = \frac{1}{2}$ , and there exists  $y_\epsilon \in \mathbb{R}^N$  such that

$$\int_{E(y_\epsilon, t_\epsilon^{\alpha_1}, \dots, t_\epsilon^{\alpha_N})} |u_\epsilon|^{p_i^*}(x) dx = \frac{1}{2}.$$

Thus, by a change of variable one has for  $v_\epsilon = u_\epsilon^{t_\epsilon, y_\epsilon}$  :

$$\int_{B(0,1)} |v_\epsilon|^{p_i^*} = \frac{1}{2} = \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_\epsilon|^{p_i^*}.$$

Note for further purpose that  $v_\epsilon$  is also extremal for  $\mathcal{K}_\epsilon$ .

□

**Proposition 3.7.** *Let  $v_\epsilon \geq 0$  be in  $\mathcal{D}^{1, \vec{p}_\epsilon}(\mathbb{R}^N)$ , bounded in that space, independently on  $\epsilon$ . Then for  $\lambda_\epsilon$  defined in (2.6), the sequence  $w_\epsilon = v_\epsilon^{\lambda_\epsilon}$  is bounded in  $\mathcal{D}^{1, \vec{p}}(\mathbb{R}^N)$ .*

*Proof.* One has

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla_1(v_\epsilon^{\lambda_\epsilon})| &= \lambda_\epsilon \int_{\mathbb{R}^N} v_\epsilon^{\lambda_\epsilon - 1} |\nabla_1 v_\epsilon| \\ &\leq \lambda_\epsilon \left( \int_{\mathbb{R}^N} |\nabla_1 v_\epsilon|^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}} \left( \int_{\mathbb{R}^N} v_\epsilon^{\frac{(\lambda_\epsilon - 1)(1+\epsilon)}{\epsilon}} \right)^{\frac{\epsilon}{1+\epsilon}} \\ &= \lambda_\epsilon \left( \int_{\mathbb{R}^N} |\nabla_1 v_\epsilon|^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}} \left( \int_{\mathbb{R}^N} v_\epsilon^{p_i^*} \right)^{\frac{\epsilon}{1+\epsilon}} \end{aligned}$$

and for all  $i > N_1$ , using the definition in (2.5)

$$\begin{aligned} \int_{\mathbb{R}^N} |\partial_i(v_\epsilon^{\lambda_\epsilon})|^{p_i} &= \lambda_\epsilon^{p_i} \int_{\mathbb{R}^N} v_\epsilon^{(\lambda_\epsilon - 1)p_i} |\partial_i v_\epsilon|^{p_i} \\ &\leq \lambda_\epsilon^{p_i} \left( \int_{\mathbb{R}^N} |\partial_i v_\epsilon|^{p_i^\epsilon} \right)^{\frac{1}{1+\epsilon_i}} \left( \int_{\mathbb{R}^N} v_\epsilon^{\frac{(\lambda_\epsilon - 1)p_i^\epsilon}{\epsilon_i}} \right)^{\frac{\epsilon_i}{1+\epsilon_i}} \\ &= \lambda_\epsilon^{p_i} \left( \int_{\mathbb{R}^N} |\partial_i v_\epsilon|^{p_i^\epsilon} \right)^{\frac{1}{1+\epsilon_i}} \left( \int_{\mathbb{R}^N} v_\epsilon^{p_i^*} \right)^{\frac{\epsilon_i}{1+\epsilon_i}} \end{aligned}$$

Then  $\int_{\mathbb{R}^N} |\nabla_1(v_\epsilon^{\lambda_\epsilon})|$  and  $\int_{\mathbb{R}^N} |\partial_i(v_\epsilon^{\lambda_\epsilon})|^{p_i}$  for  $i \geq N_1 + 1$  are bounded independently on  $\epsilon$ , by the assumptions. □

Let  $v_\epsilon$  be given by Lemma 3.6. One has by the definition of  $\lambda_\epsilon$ ,

$$\int_{\mathbb{R}^N} |v_\epsilon|^{p_\epsilon^*} = 1 = \int_{\mathbb{R}^N} |v_\epsilon^{\lambda_\epsilon}|^{p^*}.$$

Let us define

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \int_{|x| > R} |v_\epsilon^{\lambda_\epsilon}|^{p^*} = \nu_\infty,$$

and

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \int_{|x| > R} \left( |\nabla_1(v_\epsilon^{\lambda_\epsilon})| + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i(v_\epsilon^{\lambda_\epsilon})|^{p_i} \right) = \mu_\infty$$

while

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \int_{|x| > R} \left( \frac{1}{1+\epsilon} |\nabla_1 v_\epsilon|^{1+\epsilon} + \sum_{i=N_1+1}^N \frac{1}{p_i^\epsilon} |\partial_i v_\epsilon|^{p_i^\epsilon} \right) = \tilde{\mu}_\infty,$$

**Remark 3.8.** Note that since  $\int_{B(0,1)} |v_\epsilon^{\lambda_\epsilon}|^{p^*} = \frac{1}{2}$ , and  $\int_{\mathbb{R}^N} |v_\epsilon^{\lambda_\epsilon}|^{p^*} = 1$ ,  $\nu_\infty \leq \frac{1}{2}$ .

**Theorem 3.9.** Let  $v_\epsilon \in \mathcal{D}^{1, \bar{p}_\epsilon}(\mathbb{R}^N)$ , be given by Lemma 3.6, and  $\lambda_\epsilon$  be defined in (2.6). There exist positive bounded measures on  $\mathbb{R}^N$  :  $\tau, \tilde{\tau}, \mu^i, \tilde{\mu}^i$ , for  $N_1+1 \leq i \leq N$ , and  $\nu$ , a sequence of points  $x_j \in \mathbb{R}^N$ , and some positif reals  $\nu_j, \mu_j^i, \tau_j, \tilde{\tau}_j, j \in \mathbf{N}$ , so that for a subsequence

1.  $v_\epsilon$ , and  $v_\epsilon^{\lambda_\epsilon}$  converge both to  $v$ , almost everywhere and strongly in every  $L^q_{loc}$ ,  $q < p^*$ , and  $v \in BV^{\bar{p}}(\mathbb{R}^N)$ .
2.  $|\nabla_1(v_\epsilon^{\lambda_\epsilon})| \rightharpoonup |\nabla_1 v| + \tau$ ,  $|\nabla_1 v_\epsilon|^{1+\epsilon} \rightharpoonup |\nabla_1 v| + \tilde{\tau}$ , with  $\tilde{\tau} \geq \tau$ , in  $M^1(\mathbb{R}^N)$  weakly.
3.  $|\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} \rightharpoonup |\partial_i v|^{p_i} + \mu^i$  for all  $i \geq N_1+1$ ,  $|\partial_i v_\epsilon|^{p_i^\epsilon} \rightharpoonup |\partial_i v|^{p_i} + \tilde{\mu}^i$  with  $\tilde{\mu}^i \geq \mu^i$ , in  $M^1(\mathbb{R}^N)$  weakly.
4.  $|v_\epsilon^{\lambda_\epsilon}|^{p^*} = |v_\epsilon|^{p_\epsilon^*} \rightharpoonup |v|^{p^*} + \nu := |v|^{p^*} + \sum_j \nu_j \delta_{x_j}$  in  $M^1(\mathbb{R}^N)$  weakly.
5. One has  $\tau \geq \sum_j \tau_j \delta_{x_j}$ ,  $\mu^i \geq \sum_j \mu_j^i \delta_{x_j}$ , for all  $i \geq N_1+1$ , and for any  $j \in \mathbf{N}$ ,  $\nu_j^{\frac{p^*}{p^*+}} \leq \frac{1}{\bar{\kappa}} (\tau_j + \sum_i \frac{1}{p_i} \mu_j^i)$ , and  $\nu_\infty^{\frac{p^*}{p^*+}} \leq \frac{1}{\bar{\kappa}} \mu_\infty$ .
- 6.

$$\begin{aligned} |\nabla_1(v_\epsilon^{\lambda_\epsilon})|_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} &\rightarrow \int_{\mathbb{R}^N} |\nabla_1 v| + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i v|^{p_i} \\ &+ \int_{\mathbb{R}^N} \left[ \tau + \sum_{i=N_1+1}^N \frac{1}{p_i} \mu^i \right] + \mu_\infty. \end{aligned}$$

7.

$$\begin{aligned} \frac{1}{1+\epsilon} |\nabla_1 v_\epsilon|_{1+\epsilon}^{1+\epsilon} + \sum_{i=N_1+1}^N \frac{1}{p_i^\epsilon} |\partial_i v_\epsilon|_{p_i^\epsilon}^{p_i^\epsilon} &\rightarrow \int_{\mathbb{R}^N} |\nabla_1 v| + \sum_{i=N_1+1}^N \frac{1}{p_i} \int |\partial_i v|^{p_i} \\ &+ \int_{\mathbb{R}^N} \left[ \tilde{\tau} + \sum_{i=N_1+1}^N \frac{1}{p_i} \tilde{\mu}^i \right] + \tilde{\mu}_\infty. \end{aligned}$$

8.

$$\int_{\mathbb{R}^N} |v_\epsilon|^{p_\epsilon^*} = 1 = \int_{\mathbb{R}^N} |v_\epsilon|^{\lambda_\epsilon p^*} \rightarrow \int_{\mathbb{R}^N} |v|^{p^*} + \int_{\mathbb{R}^N} \nu + \nu_\infty.$$

*Proof.* 1 The convergence of  $v_\epsilon^{\lambda_\epsilon}$  is clear by using the compactness of the embedding from  $BV^{\vec{p}}$  in  $L^q$  with  $q < p^* < p_\epsilon^*$ , on bounded sets of  $\mathbb{R}^N$ , the analogous for  $v_\epsilon$  is also true since  $q < \liminf p_\epsilon^*$ .

Let us prove the existence of  $\tilde{\tau}, \tau, \mu^i, \tilde{\mu}^i, N_1 + 1 \leq i \leq N$ , and  $\nu$ . Indeed one has by extracting a subsequence the existence of  $\tilde{\tau}$ , since we know that  $|\nabla_1 v| \leq \liminf |\nabla_1 v_\epsilon|^{1+\epsilon}$ . The existence of  $\tau$  is obtained from the same arguments. Furthermore, by Hölder's inequality

$$\int |\nabla_1(v_\epsilon^{\lambda_\epsilon})| \varphi \leq \lambda_\epsilon \left( \int |\nabla_1 v_\epsilon|^{1+\epsilon} \varphi \right)^{\frac{1}{1+\epsilon}} \left( \int v_\epsilon^{p_\epsilon^*} \varphi \right)^{\frac{\epsilon}{1+\epsilon}}.$$

Letting  $\epsilon$  go to zero, since  $\lambda_\epsilon$  goes to 1, one gets that  $\tilde{\tau} \geq \tau$ . We argue in the same manner to prove the analogous results for  $|\partial_i(v_\epsilon^{\lambda_\epsilon})|^{p_i}$  and  $|\partial_i v_\epsilon|^{p_i^\epsilon}$ . The existence of  $\nu$  is clear.

We prove in the lines which follow that  $\nu$  is purely atomic. This is classical, but we reproduce the proof for the convenience of the reader. Let

$$\mu = 2|\nabla_1 v| + \tau + \sum_{i=N_1+1}^N \frac{2^{p_i-1}}{p_i} (\mu^i + 2|\partial_i v|^{p_i})$$

**Claim 1** For all  $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$ ,

$$\left( \int |\varphi|^{p^*} d\nu \right)^{\frac{1}{p^*}} \leq (p^+)^{\frac{1}{N} + \frac{1}{p^*}} \left( \int \mu \right)^{\frac{1}{N} + \frac{1}{p^*} - \frac{1}{p^+}} \frac{1}{T_o} \left( \int |\varphi|^{p^+} d\mu \right)^{\frac{1}{p^+}} \quad (3.2)$$

To prove **Claim 1**, let us define  $h_\epsilon = (v_\epsilon^{\lambda_\epsilon} - v)$ . Using (2.2),

$$\left( \int |h_\epsilon \varphi|^{p^*} \right)^{\frac{1}{p^*}} \leq \frac{1}{T_o} \Pi_1^N \left( \int |\partial_i(h_\epsilon \varphi)|^{p_i} \right)^{\frac{1}{N p_i}}. \quad (3.3)$$

We have defined  $\nu$  and  $\mu^i$  by the following vague convergences :  $v_\epsilon^{\lambda_\epsilon p^*} \rightharpoonup v^{p^*} + \nu$ ,  $|\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} \rightharpoonup |\partial_i v|^{p_i} + \mu^i$ , and  $|\nabla_1 v_\epsilon^{\lambda_\epsilon}| \rightharpoonup |\nabla_1 v| + \tau$ . By Brezis Lieb's Lemma, one derives that

$$|h_\epsilon|^{p^*} \rightharpoonup \nu,$$

while

$$|\partial_i h_\epsilon|^{p_i} \leq 2^{p_i-1} (|\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} + |\partial_i v|^{p_i}) \rightharpoonup 2^{p_i-1} (2|\partial_i v|^{p_i} + \mu^i),$$

and

$$|\nabla_1 h_\epsilon| \leq |\nabla_1(v_\epsilon)^{\lambda_\epsilon}| + |\nabla_1 v| \rightharpoonup 2|\nabla_1 v| + \tau \text{ vaguely.}$$

Using the fact that  $h_\epsilon$  tends to 0 in  $L^{p_i}(\text{Suppt}\varphi)$ , for all  $i$ , since  $p_i < p^*$ , one has  $\int |h_\epsilon|^{p_i} |\partial_i \varphi|^{p_i} \rightarrow 0$ . Passing to the limit in (3.3), one gets

$$\left( \int |\varphi|^{p^*} d\nu \right)^{\frac{1}{p^*}} \leq \frac{1}{T_o} \left( \int |\varphi| d(2|\nabla v| + \tau) \right)^{\frac{N_1}{N}} \prod_{i=N_1+1}^N \left( \int |\varphi|^{p_i} d(2^{p_i-1}(2|\partial_i v|^{p_i} + \mu^i)) \right)^{\frac{1}{N p_i}}.$$

We then use for  $i \geq N_1 + 1$

$$\int |\varphi|^{p_i} d(2^{p_i-1}(2|\partial_i v|^{p_i} + \mu^i)) \leq p^+ \left( \int \mu \right)^{1-\frac{p_i}{p^+}} \left( \int |\varphi|^{p^+} d\mu \right)^{\frac{p_i}{p^+}},$$

and

$$\int |\varphi| d(2|\nabla v| + \tau) \leq p^+ \left( \int \mu \right)^{1-\frac{1}{p^+}} \left( \int |\varphi|^{p^+} d\mu \right)^{\frac{1}{p^+}}.$$

Taking the power  $\frac{1}{N p_i}$  and  $\frac{N_1}{N}$  and multiplying the inequalities, one derives **Claim 1**.

By (3.2) one sees that  $\nu$  is absolutely continuous with respect to  $\mu$ , with for some constant  $c$  and for any borelian set  $E$ ,

$$\nu(E) \leq c\mu(E)^{\frac{p^*}{p^+}}$$

Let then  $h \geq 0$  be  $\mu$  integrable so that  $\nu = h d\mu$ . Then if  $x$  is a density point for  $\mu$ , ie, so that  $\lim_{r \rightarrow 0} \mu(B(x, r)) = 0$ , one gets that  $\frac{\nu(B(x, r))}{\mu(B(x, r))} \rightarrow 0$ , hence if  $D$  is the at most numerable set where  $\mu(\{x_j\}) > 0$ , one has  $h = 0$  in  $\mathbb{R}^N \setminus D$ . This implies that  $\nu$  has only atoms that we will denote  $\{x_j\}_{j \in \mathbb{N}}$ .

We now prove 5. We still follow the lines in [21].

Let  $\delta > 0$  small,  $q_i = \frac{p_i p^*}{p^* - p_i}$ ,  $\alpha_i = \frac{1}{q_i}$ , (note that  $\sum_{i=1}^N \alpha_i = 1$ ), define for  $j \in \mathbf{N}$  fixe,  $\phi \in \mathcal{D}(B(0, 1))$ ,  $\phi(0) = 1$ ,  $0 \leq \phi \leq 1$  the function  $\phi_\delta$  as  $\phi_\delta(x) = \phi(\frac{x-x_j^1}{\delta^{\alpha_1}}, \dots, \frac{x-x_j^N}{\delta^{\alpha_N}})$ .  $\phi_\delta$  satisfies  $\int_{\mathbb{R}^N} |\partial_i \phi_\delta|^{q_i} = \int_{\mathbb{R}^N} |\partial_i \phi|^{q_i}$ . In particular for all  $i \leq N$ ,

$$\int_{\mathbb{R}^N} |\partial_i \phi_\delta|^{p_i} v^{p_i} \leq \left( \int_{\mathbb{R}^N} |\partial_i \phi_\delta|^{q_i} \right)^{\frac{p_i}{q_i}} \left( \int_{B(x_j, \max_i \delta^{\alpha_i})} v^{p^*} \right)^{\frac{p_i}{p^*}} \rightarrow 0, \quad (3.4)$$

when  $\delta$  goes to zero.

**Claim 2**

$$\mathcal{K} \nu_j^{\frac{p^+}{p^*}} \leq \limsup_{\delta \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \left( \phi_\delta |\nabla_1 v_\epsilon^{\lambda_\epsilon}| + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i (v_\epsilon^{\lambda_\epsilon})|^{p_i} \phi_\delta^{p_i} \right)$$

To prove **Claim 2**, we apply Lemma 3.3 with  $|v_\epsilon^{\lambda_\epsilon} \phi_\delta|_{p^*} \leq 1$

$$\mathcal{K} \left( \int_{\mathbb{R}^N} |v_\epsilon^{\lambda_\epsilon} \phi_\delta|^{p^*} \right)^{\frac{p^+}{p^*}} \leq \int_{\mathbb{R}^N} |\nabla_1 (v_\epsilon^{\lambda_\epsilon} \phi_\delta)| + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i (v_\epsilon^{\lambda_\epsilon} \phi_\delta)|^{p_i}.$$

We use

$$\begin{aligned} \left| |\nabla_1 (v_\epsilon^{\lambda_\epsilon} \phi_\delta)| - |\nabla_1 (v_\epsilon^{\lambda_\epsilon})| |\phi_\delta| \right| &\leq v_\epsilon^{\lambda_\epsilon} |\nabla_1 \phi_\delta| \\ &\leq |v_\epsilon^{\lambda_\epsilon} - v| |\nabla_1 \phi_\delta| + v |\nabla_1 \phi_\delta|, \end{aligned}$$

hence by (3.4) when  $p_i = 1$  and  $v_\epsilon^{\lambda_\epsilon} - v \rightarrow 0$  in  $L_{loc}^q$  for all  $q < p^*$ , this goes to zero in  $L^1$  when  $\epsilon$  and  $\delta$  go to zero. For  $i \geq N_1 + 1$ , by the mean value's theorem

$$\begin{aligned} \left| |\partial_i (v_\epsilon^{\lambda_\epsilon} \phi_\delta)|^{p_i} - |\partial_i (v_\epsilon^{\lambda_\epsilon}) \phi_\delta|^{p_i} \right| &\leq p_i |(\partial_i \phi_\delta) v_\epsilon^{\lambda_\epsilon}| \left| |\partial_i \phi_\delta| v_\epsilon^{\lambda_\epsilon} + |\partial_i (v_\epsilon^{\lambda_\epsilon}) \phi_\delta| \right|^{p_i-1} \\ &\leq p_i \left( |\partial_i \phi_\delta| |v_\epsilon^{\lambda_\epsilon} - v| + |\partial_i \phi_\delta| v \right) \left| |(\partial_i \phi_\delta) v_\epsilon^{\lambda_\epsilon}| + |\partial_i (v_\epsilon^{\lambda_\epsilon}) \phi_\delta| \right|^{p_i-1}. \end{aligned}$$

Using Holder's inequality, (3.4) for  $i \geq N_1 + 1$ , the fact that  $\left| |(\partial_i \phi_\delta) v_\epsilon^{\lambda_\epsilon}| + |\partial_i (v_\epsilon^{\lambda_\epsilon}) \phi_\delta| \right|^{p_i-1}$  is bounded in  $L^{\frac{p_i}{p_i-1}}$ , and  $v_\epsilon^{\lambda_\epsilon} - v \rightarrow 0$  in  $L_{loc}^q$  for all  $q < p^*$ , this goes to zero in  $L^1$ , when  $\epsilon$  and  $\delta$  go to zero. **Claim 2** is proved.

We can now conclude, using the fact that  $|\nabla_1 v|$  is orthogonal to Dirac masses, as a consequence of the results on the dimension of the support of  $|\nabla_1 v|^s$ , [19], and using the fact that  $|\partial_i v|^{p_i}$  belongs to  $L^1$ , for  $i \geq N_1 + 1$ , that

$$\mathcal{K} \nu_j^{\frac{p^+}{p^*}} \leq \limsup_{\delta \rightarrow 0} \left( \int_{\mathbb{R}^N} \tau \phi_\delta + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} \mu^i \phi_\delta^{p_i} \right)$$



Defining  $\tau_j = \limsup_{\delta \rightarrow 0} \int_{\mathbb{R}^N} \tau \phi_\delta$  and  $\mu_j^i = \limsup_{\delta \rightarrow 0} \int_{\mathbb{R}^N} \mu^i \phi_\delta^{p_i}$ , one gets the first part of 5.

To prove the last part of 5, let  $R > 0$  large and  $\psi_R$  some  $C^\infty$  function which is 0 on  $|x| < R$ , and equals 1 for  $|x| > R + 1$ ,  $0 \leq \psi_R \leq 1$ . It can easily be seen that for any  $i \geq N_1 + 1$  and for any  $\gamma_i \geq 1$

$$\int_{|x| > R+1} |\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} \leq \int_{\mathbb{R}^N} |\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} \psi_R^{\gamma_i} \leq \int_{|x| > R} |\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} \quad (3.5)$$

$$\int_{|x| > R+1} |\nabla_1 v_\epsilon^{\lambda_\epsilon}| \leq \int_{\mathbb{R}^N} |\nabla_1 v_\epsilon^{\lambda_\epsilon}| \psi_R^{\gamma_1} \leq \int_{|x| > R} |\nabla_1 v_\epsilon^{\lambda_\epsilon}| \quad (3.6)$$

and

$$\int_{|x| > R+1} |v_\epsilon^{\lambda_\epsilon}|^{p^*} \leq \int_{\mathbb{R}^N} |v_\epsilon^{\lambda_\epsilon}|^{p^*} \psi_R^{p^*} \leq \int_{|x| > R} |v_\epsilon^{\lambda_\epsilon}|^{p^*}. \quad (3.7)$$

And then by the definition of  $\mu_\infty$

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla_1 v_\epsilon^{\lambda_\epsilon}| \psi_R + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} \psi_R^{p_i} = \mu_\infty.$$

Let us remark that since  $v \in BV^{\vec{p}}$ , one has  $\lim_{R \rightarrow +\infty} \int |\nabla_1 v| \psi_R + \sum_{i=N_1+1}^N \frac{1}{p_i} \int |\partial_i v|^{p_i} \psi_R^{p_i} + \int_{\mathbb{R}^N} |v|^{p^*} \psi_R^{p^*} = 0$ .

We use once more  $h_\epsilon = v_\epsilon^{\lambda_\epsilon} - v$ , which goes to zero in  $L_{loc}^q$ . Note that since  $|h_\epsilon|_{p^*} \leq 1$ , one also has  $|h_\epsilon \psi_R|_{p^*} \leq 1$  and then applying Lemma 3.3

$$\mathcal{K} \left( \int |h_\epsilon \psi_R|^{p^*} \right)^{\frac{p^+}{p^*}} \leq \int |\nabla_1 (h_\epsilon \psi_R)| + \sum_{i=N_1+1}^N \frac{1}{p_i} \int |\partial_i (h_\epsilon \psi_R)|^{p_i}. \quad (3.8)$$

Since  $\nabla \psi_R$  is compactly supported in  $R < |x| < R + 1$ , and since  $p_i < p^*$  one has

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} h_\epsilon |\nabla_1 (\psi_R)| + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i \psi_R|^{p_i} h_\epsilon^{p_i} = 0.$$

Then

$$\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla_1 (h_\epsilon \psi_R)| + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i (h_\epsilon \psi_R)|^{p_i} = \mu_\infty.$$

Note also that  $\lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \mathcal{K} \left( \int_{\mathbb{R}^N} |h_\epsilon \psi_R|^{p^*} \right)^{\frac{p^+}{p^*}} = \mathcal{K} \nu_\infty^{\frac{p^+}{p^*}}$ , hence, taking the limit in

(3.8), one gets  $\mathcal{K} \nu_\infty^{\frac{p^+}{p^*}} \leq \mu_\infty$ .

To show 6. by the definition of  $\tau$  and  $\mu^i$ ,

$$\begin{aligned} \lim_{R \rightarrow +\infty} \limsup_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla_1 v_\epsilon^{\lambda_\epsilon}| (1 - \psi_R) + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i v_\epsilon^{\lambda_\epsilon}|^{p_i} (1 - \psi_R) \\ = \int_{\mathbb{R}^N} |\nabla_1 v| + \int_{\mathbb{R}^N} \tau + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} (|\partial_i v|^{p_i} + \mu^i). \end{aligned}$$

And then one gets 6. by writing  $1 = \psi_R + (1 - \psi_R)$  and using (3.5) and (3.6).

7 can be proved in the same manner. 8 is obtained by gathering 4. and (3.7).  $\square$

*Proof.* of Theorem 3.1 We take a subsequence  $v_{\epsilon'}$  so that

$$\frac{1}{1 + \epsilon'} \int_{\mathbb{R}^N} |\nabla_1 v_{\epsilon'}|^{1+\epsilon'} + \sum_{i=N_1+1}^N \frac{1}{p_i^{\epsilon'}} \int_{\mathbb{R}^N} |\partial_i v_{\epsilon'}|^{p_i^{\epsilon'}} = \mathcal{K}_{\epsilon'}$$

with  $\lim \mathcal{K}_{\epsilon'} = \liminf \mathcal{K}_{\epsilon}$ , in the sequel we will still denote it  $v_\epsilon$  for simplicity .

We are going to prove both that  $\limsup \mathcal{K}_\epsilon = \mathcal{K} = \liminf \mathcal{K}_\epsilon$ ,  $\nu_\infty = \mu_\infty = 0$ ,  $\mu_j^i = \nu_j = 0$ , for all  $j \in \mathbf{N}$ , that for all  $i$   $|\partial_i v_\epsilon|^{p_i^\epsilon} \rightarrow |\partial_i v|^{p_i}$ , tightly on  $\mathbb{R}^N$ , and that  $\lim |\nabla_1(v_\epsilon^{\lambda_\epsilon})| = \lim |\nabla_1 v_\epsilon|^{1+\epsilon} = |\nabla_1 v|$ , tightly on  $\mathbb{R}^N$ . Indeed, using the previous convergences in Theorem 3.9

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla_1 v| + \int_{\mathbb{R}^N} \tau + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i v|^{p_i} + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} \mu^i + \mu_\infty \\ \leq \int_{\mathbb{R}^N} |\nabla_1 v| + \int_{\mathbb{R}^N} \tilde{\tau} + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i v|^{p_i} + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} \tilde{\mu}^i + \tilde{\mu}_\infty \\ \leq \lim \frac{1}{1 + \epsilon} \int_{\mathbb{R}^N} |\nabla_1 v_\epsilon|^{1+\epsilon} + \sum_{i=N_1+1}^N \frac{1}{p_i^\epsilon} \int_{\mathbb{R}^N} |\partial_i(v_\epsilon)|^{p_i^\epsilon} \\ = \liminf \mathcal{K}_\epsilon = \liminf \mathcal{K}_\epsilon (|v|^{p^*} + \sum \nu_j + \nu_\infty)^{\frac{p^+}{p^*}} \\ \leq \liminf \mathcal{K}_\epsilon \left( (|v|_{p^*}^{p^*})^{\frac{p^+}{p^*}} + (\sum \nu_j)^{\frac{p^+}{p^*}} + \nu_\infty^{\frac{p^+}{p^*}} \right) \\ \leq \liminf \mathcal{K}_\epsilon \left( \int_{\mathbb{R}^N} |v|_{p^*}^{p^*} \right)^{\frac{p^+}{p^*}} + \frac{\liminf \mathcal{K}_\epsilon}{\mathcal{K}} \left[ \sum_j (\tau_j + \sum_{i=N_1+1}^N \frac{1}{p_i} \mu_j^i) + \mu_\infty \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\liminf \mathcal{K}_\epsilon}{\mathcal{K}} \left( \int_{\mathbb{R}^N} |\nabla_1 v| + \sum_{i=N_1+1}^N \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_i v|^{p_i} \right) \\
&+ \frac{\liminf \mathcal{K}_\epsilon}{\mathcal{K}} \left( \sum_j \tau_j + \sum_j \sum_{i=N_1+1}^N \frac{1}{p_i} \mu_j^i + \mu_\infty \right)
\end{aligned}$$

Using the fact that  $\limsup \mathcal{K}_\epsilon \leq \mathcal{K}$ ,  $\int_{\mathbb{R}^N} \tau \geq \sum_j \tau_j$ ,  $\int_{\mathbb{R}^N} \mu^i \geq \sum_j \mu_j^i$ , one gets that we have equalities in place of inequalities everywhere we used them. In particular  $(\int_{\mathbb{R}^N} |v|^{p^*} + \sum_j \nu_j + \nu_\infty)^{\frac{p^+}{p^*}} = (\int_{\mathbb{R}^N} |v|^{p^*})^{\frac{p^+}{p^*}} + \sum_j \nu_j^{\frac{p^+}{p^*}} + \nu_\infty^{\frac{p^+}{p^*}}$ , and then only one of the positive reals  $\int_{\mathbb{R}^N} |v|^{p^*}, \nu_j, \nu_\infty$ , can be different from zero. But this imposes that the only one which is  $\neq 0$  must be equal to one. By Remark 3.8, one then gets  $\nu_\infty = 0$ . On the other hand, let  $j \in \mathbf{N}$ , either  $x_j \notin B(0, 1)$  and then for  $\delta$  small enough  $\int_{B(x_j, \delta)} |v_\epsilon|^{p_\epsilon^*} + \int_{B(0, 1)} |v_\epsilon|^{p_\epsilon^*} \leq 1$ , hence  $\nu_j = 0$ , or  $x_j \in B(0, 1)$  and then  $\nu_j \leq \lim \int_{B(0, 1)} |v_\epsilon|^{p_\epsilon^*} = \frac{1}{2}$ , and once more  $\nu_j = 0$ . One then derives that  $1 = |v_\epsilon|_{p_\epsilon^*}^{p_\epsilon^*} \rightarrow |v|_{p^*}^{p^*}$ . By the definition of  $\mathcal{K}$  one has

$$\begin{aligned}
\mathcal{K} \leq |\nabla_1 v|_1 + \sum_{N_1+1}^N \frac{1}{p_i} |\partial_i v|^{p_i} &\leq |\nabla_1 v|_1 + \sum_{N_1+1}^N \frac{1}{p_i} |\partial_i v|^{p_i} + \tilde{\tau} + \sum_{N_1+1}^N \frac{1}{p_i} \tilde{\mu}^i + \tilde{\mu}_\infty \\
&\leq \liminf \mathcal{K}_\epsilon \leq \limsup \mathcal{K}_\epsilon \leq \mathcal{K}
\end{aligned}$$

and then  $\tilde{\tau} = \tau = \tilde{\mu}_\infty = \mu_\infty = \tilde{\mu}^i = \mu^i = 0$ ,  $\lim |\nabla_1 v_\epsilon|_{1+\epsilon}^{1+\epsilon} = \lim |\nabla_1 (v_\epsilon^{\lambda_\epsilon})|_1 = |\nabla_1 v|_1$ , and for all  $i \geq N_1 + 1$ , both  $|\partial_i (v_\epsilon^{\lambda_\epsilon})|_{p_i}^{p_i}$  and  $|\partial_i v_\epsilon|_{p_i}^{p_i}$  converge to  $|\partial_i v|_{p_i}^{p_i}$ . We have obtained that  $v$  is an extremal function, and  $\lim \mathcal{K}_\epsilon = \mathcal{K}$ .

We now prove that  $v$  satisfies (3.1). First recall that  $l_\epsilon \geq \mathcal{K}_\epsilon \geq \frac{1}{p^+} l_\epsilon$ , as we can see by multiplying (2.7) by  $v_\epsilon$  the equation, integrating, and using  $|v_\epsilon|_{p_\epsilon^*}^{p_\epsilon^*} = 1$ . In particular  $l_\epsilon$  is bounded. Let us extract from it a subsequence which converges to some  $l \geq 0$ .

Let us define  $\sigma^{1, \epsilon} = |\nabla_1 v_\epsilon|^{\epsilon-1} \nabla_1 v_\epsilon$ ,  $\sigma_i^\epsilon = |\partial_i v_\epsilon|^{p_i^\epsilon - 2} \partial_i v_\epsilon$  for  $i \geq N_1 + 1$ , and - with an obvious abuse of notation-  $\sigma_\epsilon = (\sigma^{1, \epsilon}, \sigma_{N_1+1}^\epsilon, \dots, \sigma_N^\epsilon)$ . Note that  $\sigma^{1, \epsilon}$  is bounded in  $L_{loc}^q$ , for any  $q < \infty$ . Indeed, let  $K$  be a compact set, one has by Holder's inequality  $\int_K |\sigma^{1, \epsilon}|^q = \int_K |\nabla_1 v_\epsilon|^{\epsilon q} \leq (\int_K |\nabla_1 v_\epsilon|^{1+\epsilon})^{\frac{q\epsilon}{1+\epsilon}} |K|^{1-\frac{q\epsilon}{1+\epsilon}}$  and then  $(\int_K |\sigma^{1, \epsilon}|^q)^{\frac{1}{q}} \leq ((1+\epsilon)\mathcal{K}_\epsilon)^{\frac{\epsilon}{1+\epsilon}} |K|^{\frac{1}{q} - \frac{\epsilon}{1+\epsilon}}$ . Using the boundedness of  $\mathcal{K}_\epsilon$  one gets that  $\sigma^{1, \epsilon}$  is bounded in  $L_{loc}^q$ , hence converges up to subsequence weakly in  $L_{loc}^q$  to some

$\sigma^1$  which satisfies for any compact set  $K$   $|\sigma^1|_{L^q(K)} \leq |K|^{\frac{1}{q}}$ , hence  $\sigma^1 \in L^\infty(\mathbb{R}^N, \mathbb{R}^{N_1})$  and  $|\sigma^1|_\infty \leq 1$ . Furthermore, the strong convergence of  $|\partial_i v_\epsilon|^{p_i^\epsilon}$  towards  $|\partial_i v|^{p_i}$  in  $L^1$  when  $i \geq N_1 + 1$  ensures that  $\sigma_i = |\partial_i v|^{p_i-2} \partial_i v$ . From these convergences, one gets that defining  $\sigma = (\sigma^1, \sigma_{N_1+1}, \dots, \sigma_N)$ , by the definition in Theorem 2.13,  $\sigma^\epsilon \cdot \nabla v_\epsilon$  converges to  $\sigma \cdot \nabla v$  in the distribution sense. Using  $\sum_{N_1+1}^N \sigma_i^\epsilon \partial_i v_\epsilon \rightarrow \sum_{N_1+1}^N \sigma_i \partial_i v$  in  $L^1_{loc}$ , one derives that  $\sigma^{1,\epsilon} \cdot \nabla_1 v_\epsilon$  converges to  $\sigma^1 \cdot \nabla_1 v$  in  $\mathcal{D}'(\mathbb{R}^N)$ . Since  $\sigma^{1,\epsilon} \cdot \nabla_1 v_\epsilon$  is also bounded in  $L^1$ , this convergence is in fact vague. By lower semi-continuity for the vague topology, for any  $\varphi \geq 0$  in  $\mathcal{C}_c(\mathbb{R}^N)$

$$\int |\nabla_1 v| \varphi \leq \liminf_{\epsilon \rightarrow 0} \int |\nabla_1 v_\epsilon|^{1+\epsilon} \varphi = \liminf_{\epsilon \rightarrow 0} \int \sigma^{1,\epsilon} \cdot \nabla_1 v_\epsilon \varphi = \langle \sigma^1 \cdot \nabla_1 v, \varphi \rangle$$

This implies that  $|\nabla_1 v| \leq \sigma^1 \cdot \nabla_1 v$  in the sense of measures, and since one always has the reverse inequality, we have obtained that  $\sigma^1 \cdot \nabla_1 v = |\nabla_1 v|$ .

We get by passing to the limit in (2.7) that  $v$  satisfies the partial differential equation :

$$-\operatorname{div}_1(\sigma^1) - \sum_{i=N_1+1}^N \partial_i(|\partial_i v|^{p_i-2} \partial_i v) = l v^{p^*-1}$$

with

$$\int_{\mathbb{R}^N} v^{p^*} = 1, \text{ and } \sigma^1 \cdot \nabla_1 v = |\nabla_1 v|.$$

Furthermore, multiplying the equation by  $v$  and integrating, one gets  $l \geq \mathcal{K} > 0$ .  $\square$

### 3.2 Proof of Theorem 3.2

We will prove the  $L^\infty$  regularity when  $u$  is some extremal function which satisfies (3.1), with  $l = 1$ . Indeed one has

**Lemma 3.10.** *Let  $v_\epsilon$  and  $v$  be as in Theorem 3.1. Then*

$$u(x) = v(l^{-1}x_1, \dots, l^{-1}x_{N_1}, l^{-\frac{1}{p_{N_1+1}}}x_{N_1+1}, \dots, l^{\frac{-1}{p_N}}x_N)$$

and

$$u_\epsilon(x) = v_\epsilon(l_\epsilon^{\frac{-1}{1+\epsilon}}x_1, \dots, l_\epsilon^{\frac{-1}{1+\epsilon}}x_{N_1}, l_\epsilon^{-\frac{1}{p_{N_1+1}^\epsilon}}x_{N_1+1}, \dots, l_\epsilon^{\frac{-1}{p_N^\epsilon}}x_N)$$

satisfy respectively

$$-\operatorname{div}_1(\sigma^1(u)) - \sum_{i=N_1+1}^N \partial_i(|\partial_i u|^{p_i-2} \partial_i u) = u^{p^*-1}$$

with  $\sigma^1 \cdot \nabla_1 u = |\nabla_1 u|$ , and

$$-\operatorname{div}_1(|\nabla_1 u_\epsilon|^{\epsilon-1} \nabla_1 u_\epsilon) - \sum_{i=N_1+1}^N \partial_i(|\partial_i u_\epsilon|^{p_i^\epsilon-2} \partial_i u_\epsilon) = u_\epsilon^{p_\epsilon^*-1} \quad (3.9)$$

Furthermore  $u_\epsilon$  converges tightly to  $u$  in  $BV^{\vec{p}}(\mathbb{R}^N)$ .

We do not give the proof of this lemma, which is left to the reader.

In the sequel we will consider  $u$  and  $u_\epsilon$  as in Lemma 3.10.

**Lemma 3.11.** *Suppose that  $u \in BV^{\vec{p}}$  is as in Lemma 3.10. Suppose that  $g$  is Lipschitz continuous on  $\mathbb{R}$ , such that  $g(0) = 0$  and  $g' \geq 0$ , then  $g(u) \in BV^{\vec{p}}$ , with  $\sigma^1 \cdot \nabla_1(g(u)) = |\nabla_1(g(u))|$ . Furthermore one has the identity*

$$\int_{\mathbb{R}^N} |\nabla_1(g(u))| + \sum_{i=N_1+1}^N \int_{\mathbb{R}^N} g'(u) |\partial_i u|^{p_i} = \int_{\mathbb{R}^N} g(u) u^{p^*-1} \quad (3.10)$$

*Proof.* In the following lines, we will use "UTS" to say that the convergence holds up to subsequence .

Note that  $g(u_\epsilon) \in \mathcal{D}^{1, \vec{p}_i}(\mathbb{R}^N)$  by the mean value's theorem, since  $g' \in L^\infty$ , and  $(g(u_\epsilon))_\epsilon$  is bounded in that space by the assumptions on  $u_\epsilon$ , and then also in  $BV_{loc}^{\vec{p}}$ . Then since  $u_\epsilon$  converges to  $u$  almost everywhere "UTS" and  $g$  is continuous,  $g(u) \in BV^{\vec{p}}(\mathbb{R}^N)$ , and  $g(u_\epsilon)$  converges weakly to  $g(u)$  in  $BV_{loc}^{\vec{p}}$  "UTS" . In particular it converges to  $g(u)$  in  $L_{loc}^q$ , "UTS" for all  $q < p^*$ . Let us observe that the sequence of measures  $\sigma_\epsilon \cdot \nabla(g(u_\epsilon))$  converges "UTS" to  $\sigma \cdot \nabla(g(u))$  : Since  $\sigma_\epsilon \cdot \nabla g(u_\epsilon)$  is bounded in  $L^1$ , it is sufficient to prove that it converges in the distribution sense. To check this, let  $\varphi \in \mathcal{D}(\mathbb{R}^N)$ , take  $q < p^*$  so that for  $\epsilon$  small enough  $p_i^\epsilon < q$ , then  $\sigma_\epsilon \rightarrow \sigma$  "UTS" in  $L_{loc}^q$ . Using  $g(u_\epsilon) \rightarrow g(u)$  in  $L_{loc}^q$  strongly and "UTS" for all  $q < p^*$ , one has  $\int g(u_\epsilon) \sigma_\epsilon \cdot \nabla \varphi \rightarrow \int g(u) \sigma \cdot \nabla \varphi$ . Secondly note that  $u_\epsilon^{p_\epsilon^*-1} g(u_\epsilon) \leq |g'|_\infty |u_\epsilon|^{p_\epsilon^*}$ . By the strong convergence of  $(u_\epsilon)^{p_\epsilon^*}$  in  $L^1$  one can suppose that "UTS" is dominated by a function  $h$  in  $L^1$ , hence so does  $u_\epsilon^{p_\epsilon^*-1} g(u_\epsilon)$ . By the almost everywhere convergence "UTS" of  $u_\epsilon^{p_\epsilon^*-1} g(u_\epsilon)$  to  $u^{p^*-1} g(u)$  and the Lebesgue's dominated convergence theorem, one gets that for any  $\varphi \in \mathcal{D}(\mathbb{R}^N)$ ,  $\int u_\epsilon^{p_\epsilon^*-1} g(u_\epsilon) \varphi \rightarrow \int u^{p^*-1} g(u) \varphi$ . We have obtained that  $\int \sigma_\epsilon \cdot \nabla(g(u_\epsilon)) \varphi \rightarrow \int \sigma \cdot \nabla(g(u)) \varphi$ , for any  $\varphi$  in  $\mathcal{D}(\mathbb{R}^N)$ , hence also for  $\varphi$  in  $\mathcal{C}_c(\mathbb{R}^N)$ . Furthermore, by lower semicontinuity one has for all  $\varphi \geq 0$  in  $\mathcal{C}_c(\mathbb{R}^N)$ ,

$$\begin{aligned} \int |\nabla_1(g(u))| \varphi &\leq \liminf_{\epsilon \rightarrow 0} \int |\nabla_1(g(u_\epsilon))|^{1+\epsilon} \varphi \\ &= \liminf_{\epsilon \rightarrow 0} \int (g'(u_\epsilon))^{1+\epsilon} |\nabla_1 u_\epsilon|^{1+\epsilon} \varphi \\ &\leq \liminf_{\epsilon \rightarrow 0} |g'|_\infty^\epsilon \int (g'(u_\epsilon)) |\nabla_1 u_\epsilon|^{1+\epsilon} \varphi \end{aligned}$$

$$\begin{aligned}
&= \liminf_{\epsilon \rightarrow 0} \int \sigma^{1,\epsilon} \cdot \nabla_1(g(u_\epsilon))\varphi \\
&= \int \sigma^1 \cdot \nabla_1(g(u))\varphi
\end{aligned}$$

This implies since one also has  $\sigma^1 \cdot \nabla_1(g(u)) \leq |\nabla_1 g(u)|$ , that  $\sigma^1 \cdot \nabla_1(g(u)) = |\nabla_1(g(u))|$ .

To get identity (3.10) it is then sufficient to multiply the equation (3.9) by  $g(u_\epsilon)\varphi$ , and pass to the limit using the previous convergence. Next one can let  $\varphi$  go to  $1_{\mathbb{R}^N}$  since all the measures involved are bounded measures.  $\square$

**Corollary 3.12.** *Let  $u$  be as in Lemma 3.10. For any  $L$  and  $a > 0$ ,  $(u \min(u^a, L)) \in BV^{\vec{p}}(\mathbb{R}^N)$ ,  $\sigma^1 \cdot \nabla_1(u \min(u^a, L)) = |\nabla_1(u \min(u^a, L))|$ , and*

$$\int |\nabla_1(u \min(u^a, L))| + \sum_{i=N_1+1}^N \left( \frac{1}{1 + \frac{a}{p_i}} \right)^{p_i-1} \int |\partial_i(u \min(u^{\frac{a}{p_i}}, L))|^{p_i} \leq \int u^{p^*} \min(u^a, L).$$

*Proof.* We use Lemma 3.11 with  $g(u) = u \min(u^a, L)$  and equation (3.10). Then it is sufficient to observe that

$$\int g'(u) |\partial_i u|^{p_i} \geq \left( \frac{1}{1 + \frac{a}{p_i}} \right)^{p_i-1} \int |\partial_i(u \min(u^{\frac{a}{p_i}}, L))|^{p_i}.$$

$\square$

We now prove the following

**Proposition 3.13.** *Let  $u$  be as in Lemma 3.10, then  $u \in L^\infty$ .*

*Proof.* This proof follows the lines in [16] and [21]. Once more, we reproduce it here for the sake of completeness. We begin to prove that  $u \in L^q$  for all  $q < \infty$ . In the sequel,  $c$  denotes some positive constant which does not depend on  $k$  nor on  $a$ , which can vary from one line to another. Let  $k$  to choose later, and write for all  $p_j$ , (recall that  $p_j = 1$  for  $j \leq N_1$ ):

$$\begin{aligned}
\int u^{p^*} \min(u^{ap_j}, L^{p_j}) &= \int_{u \leq k} u^{p^*} (\min(u^a, L))^{p_j} + \int_{u \geq k} u^{p^*} (\min(u^a, L))^{p_j} \\
&\leq k^{ap_j} \int |u|^{p^*} + \left( \int_{u \geq k} u^{p^*} \right)^{1 - \frac{p_j}{p^*}} \left( \int (u \min(u^a, L))^{p^*} \right)^{\frac{p_j}{p^*}}.
\end{aligned}$$

Using the embedding from  $BV^{\vec{p}}$  in  $L^{p^*}$  one has

$$\left( \int (u \min(u^a, L))^{p^*} \right)^{\frac{1}{p^*}} \leq c \left( \int |\nabla_1(u \min(u^a, L))| + \sum_{j=N_1+1}^N \left( \int |\partial_j(u \min(u^a, L))|^{p_j} \right)^{\frac{1}{p_j}} \right). \quad (3.11)$$

Using Corollary 3.12, for  $u \min(u^{ap_j}, L)$  one gets for all  $j$

$$(1+a)^{-p_j+1} \int |\partial_j(u \min(u^a, L))|^{p_j} \leq \int u^{p^*} \min(u^{ap_j}, L^{p_j})$$

and then defining  $I_j = \left( \int |\partial_j(u \min(u^a, L))|^{p_j} \right)^{\frac{1}{p_j}}$  and  $\epsilon_k = \int_{u \geq k} u^{p^*}$ ,

$$I_j \leq c(1+a) \left( k^a \left( \int u^{p^*} \right)^{\frac{1}{p_j}} + \epsilon_k^{\frac{1}{p_j} - \frac{1}{p^*}} \left[ \int |\nabla_1(u \min(u^a, L))| + \sum_{i=N_1+1}^N I_i \right] \right)$$

and

$$\int |\nabla_1(u \min(u^a, L))| \leq c(1+a) \left( k^a \int u^{p^*} + \epsilon_k^{1 - \frac{1}{p^*}} \left[ \int |\nabla_1(u \min(u^a, L))| + \sum_{i=N_1+1}^N I_i \right] \right).$$

Summing over  $j$  one gets

$$\begin{aligned} \int |\nabla_1(u \min(u^a, L))| &+ \sum_{j=N_1+1}^N I_j \\ &\leq c(1+a) \left( k^a \sum_{j=1}^N |u|_{p^*}^{\frac{p^*}{p_j}} \right. \\ &\quad \left. + \sum_{j=1}^N \epsilon_k^{\frac{1}{p_j} - \frac{1}{p^*}} \left( \int |\nabla_1(u \min(u^a, L))| + \sum_{i=N_1+1}^N I_i \right) \right). \end{aligned}$$

Choosing  $k_a$  so that  $c(a+1) \sum \epsilon_k^{\frac{1}{p_j} - \frac{1}{p^*}} < \frac{1}{2}$ , (recall that  $p_j < p^*$  for all  $j$  and  $\epsilon_k \rightarrow 0$  when  $k \rightarrow +\infty$ ), we have obtained

$$\frac{1}{2} \left[ \int |\nabla_1(u \min(u^a, L))| + \sum_{j=N_1+1}^N I_j \right] \leq c(1+a) k_a^a \sum_{j=1}^N |u|_{p^*}^{\frac{p^*}{p_j}},$$

hence, coming back to (3.11)

$$|u \min(u^a, L)|_{p^*} \leq c(1+a)k_a^a \sum_{j=1}^N |u|_{p^*}^{\frac{p_j^*}{p^*}}.$$

Letting  $L$  go to  $\infty$  one gets  $|u^{a+1}|_{p^*} \leq C'(|u|_{p^*})(1+a)k_a^a$ , taking the power  $\frac{1}{a+1}$ , one has obtained that for  $q = p^*(a+1)$ ,

$$|u|_q \leq C'(|u|_{p^*})^{\frac{1}{1+a}}(1+a)^{\frac{1}{a+1}}k_a^{\frac{a}{a+1}},$$

and then  $u$  belongs to  $L^q$  for all  $q < \infty$ .

To prove that  $u \in L^\infty$ , we still follow the lines in [21].

Choose  $q > p^*$  so that  $\epsilon := \frac{-1}{p^*} + (1 - \frac{p^*}{q})(1 - \frac{1}{p^*})\frac{1}{p^* - 1} > 0$ . Let  $\varphi_k = (u - k)_+$ , and  $A_k = \{x, u(x) > k\}$ . Let us begin to note that  $A_k$  is of finite measure for all  $k > 0$ , since

$$|\{x, u(x) > k\}|k^{p^*} \leq \int_{u>k} |u|^{p^*} \leq |u|_{p^*}^{p^*}.$$

We then deduce that for  $k > 0$ ,  $(u - k)_+ \in L^1$ , since

$$\int (u - k)^+ \leq \int_{u \geq k} u \leq \int_{u \geq k} \frac{u^{p^*}}{k^{p^* - 1}}. \quad (3.12)$$

We now apply Lemma 3.11 with  $g(u) = (u - k)^+$ . Using (3.10) one gets

$$\begin{aligned} |\nabla_1 \varphi_k|_1 + \sum_{i=N_1+1}^N |\partial_i \varphi_k|_{p_i}^{p_i} &= \int u^{p^* - 1} (u - k)^+ \\ &\leq |u|_q^{p^* - 1} |A_k|^{(1 - \frac{p^*}{q})(1 - \frac{1}{p^*})} |\varphi_k|_{p^*} \\ &\leq c |A_k|^{(1 - \frac{p^*}{q})(1 - \frac{1}{p^*})} |\varphi_k|_{p^*}. \end{aligned}$$

We then have since  $|\varphi_k|_{p^*} \leq |u|_{p^*} = 1$ , by Lemma 3.3

$$\begin{aligned} |\varphi_k|_{p^*}^{p^*} &\leq c \left( |\nabla_1 \varphi_k|_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i \varphi_k|_{p_i}^{p_i} \right) \\ &\leq c |A_k|^{(1 - \frac{p^*}{q})(1 - \frac{1}{p^*})} |\varphi_k|_{p^*}. \end{aligned}$$

hence



$$|\varphi_k|_{p^*} \leq c|A_k|^{\epsilon + \frac{1}{p^*}},$$

and using Hölder's inequality, one derives  $\int_{\mathbb{R}^N} (u - k)_+ \leq |A_k|^{1 - \frac{1}{p^*}} |\varphi_k|_{p^*} \leq c|A_k|^{1 + \epsilon}$ .

Let  $y(k) = \int_k^\infty |A_\tau| d\tau$ , then  $y(k) = \int_{\mathbb{R}^N} (u - k)_+ \leq c(-y'(k))^{1 + \epsilon}$ , and integrating one obtains

$$-y^{\frac{\epsilon}{1 + \epsilon}}(u(s)) + y^{\frac{\epsilon}{1 + \epsilon}}(k) \geq \frac{\epsilon}{1 + \epsilon} c^{\frac{-1}{1 + \epsilon}} (u(s) - k)$$

hence for any  $s$ , recalling (3.12), for some constants  $b$  and  $\gamma > 0$ :

$$u(s) - k \leq \frac{1 + \epsilon}{\epsilon} c^{\frac{1}{1 + \epsilon}} \frac{|u|_{p^*}^{\frac{p^* \epsilon}{1 + \epsilon}}}{k^{\frac{(p^* - 1)\epsilon}{1 + \epsilon}}} \leq \frac{b}{k^\gamma}.$$

Optimizing with respect to  $k$ , ie taking the infimum one gets that

$$u(s) \leq c(|u|_{p^*})$$

□

**Remark 3.14.** Let  $q_1, \dots, q_m$  be such that  $\{p_{N_1+1}, \dots, p_N\} = \{q_1, \dots, q_m\}$ , and  $q_i \neq q_j$  when  $i \neq j$ .

Note that one could consider in place of  $\mathcal{K} = \inf_{u \in \mathcal{D}^{1, \vec{p}}, |u|_{p^*} = 1} |\nabla_1 u|_1 + \sum_{i=N_1+1}^N \frac{1}{p_i} |\partial_i u|_{p_i}^{p_i}$  the infimum

$$\tilde{\mathcal{K}} = \inf_{u \in \mathcal{D}^{1, \vec{p}}(\mathbb{R}^N), |u|_{p^*} = 1} |\nabla_1 u|_1 + \sum_{j=1}^m \left( \frac{1}{q_j} \int \left( \sum_{i, p_i = q_j} |\partial_i u|^2 \right)^{\frac{q_j}{2}} \right)$$

and prove the existence of an extremal function with obvious changes.

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