REVERSE AGMON ESTIMATES IN FORBIDDEN REGIONS

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ABSTRACT. Let (M, q) be a compact, Riemannian manifold and $V \in C^{\infty}(M; \mathbb{R})$. Given a regular energy level $E > \min V$, we consider L^2 -normalized eigenfunctions, u_h , of the Schrödinger operator $P(h) = -h^2 \Delta_g + V - E(h)$ with $P(h)u_h = 0$ and $E(h) = E + o(1)$ as $h \to 0^+$. The well-known Agmon-Lithner estimates [\[Hel88\]](#page-21-0) are exponential decay estimates (ie. upper bounds) for eigenfunctions in the forbidden region ${V > E}$. The decay rate is given in terms of the Agmon distance function d_E associated with the degenerate Agmon metric $(V - E)_+ g$ with support in the forbidden region.

The point of this note is to prove a reverse Agmon estimate (ie. exponential lower bound for the eigenfunctions) in terms of Agmon distance in the forbidden region under a control assumption on eigenfunction mass in the allowed region ${V < E}$ arbitrarily close to the caustic ${V = E}$. We then give some applications to hypersurface restriction bounds for eigenfunctions in the forbidden region along with corresponding nodal intersection estimates.

1. INTRODUCTION

Let (M, g) be a compact, C^{∞} Riemannian manifold and $V \in C^{\infty}(M; \mathbb{R})$ be a real-valued potential. We assume that E is a regular value of V so that $dV|_{V=E} \neq 0$. The corresponding classically allowed region is

$$
\Omega_E := \{ x \in M; V(x) \le E \} \tag{1.1}
$$

with boundary C^{∞} hypersurface (ie. boundary caustic)

$$
\Lambda_E := \{ x \in M; V(x) = E \}. \tag{1.2}
$$

The forbidden region is the complement $\Omega_E^c = \{x \in M; V(x) > E\}.$

1.0.1. Agmon-Lithner estimates. Let $P(h): C^{\infty}(M) \to C^{\infty}(M)$ be the Schrödinger operator $P(h) := -h^2 \Delta_g + V(x) - E(h)$

and $u_h \in C^{\infty}(M)$ be L^2 -normalized eigenfunctions with eigenvalue $E(h) = E + o(1)$ as $h \to 0^+$ so that $P(h)u_h = 0$. The Agmon metric associated with $P(h)$ is defined by

$$
g_E(x) := (V(x) - E)_+ g(x).
$$

The degenerate metric g_E is supported in the forbidden region Ω_E^c and we denote the corresponding Riemannian distance function by $d_E : \Omega_E^c \times \Omega_E^c \to \mathbb{R}^+$. By a slight abuse of notation, we define the associated distance function to Λ_E by

$$
d_E(x) := d_E(x, \Lambda_E) = \inf_{y \in \Lambda_E} d_E(x, y), \quad x \in \Omega_E^c.
$$
\n(1.3)

It is well-known [\[Hel88\]](#page-21-0)(3.2.2) that $d_E \in Lip(\Omega_E^c)$ and also, $|\nabla_x d_E|_g^2 \leq (V(x) - E)_+, a.e.$

Given an open subset, U, of the forbidden region Ω_E^c with $\overline{U} \subset \Omega_E^c$, the Agmon-Lithner estimate [\[Hel88\]](#page-21-0) (Prop. 3.3.1) says that for any $\delta > 0$,

$$
||e^{(1-\delta)d_E/h}u_h||_{H_h^1(U)} = O_\delta(1). \tag{1.4}
$$

where $||f||_{H_h^1}^2 = \int_U (|f|^2 + |h\partial f|^2)$. A standard argument with Sobolev estimates [\[Hel88\]](#page-21-0) (Prop. 3.3.4) then yields corresponding pointwise upper bounds: In terms of local coordinates, for any $\delta > 0$ and multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$,

$$
|D_x^{\alpha}u_h(x)| \le C_{\alpha,\delta}e^{-d_E(x)/h}e^{\delta/h}; \quad x \in \Omega_E^c.
$$
 (1.5)

Such estimates have widespread applications to tunnelling problems [\[CS81,](#page-21-1) [Sim84,](#page-21-2) [HS84\]](#page-21-3) and the theory of Morse-Witten complexes [\[Wit82\]](#page-21-4).

Our objective here is to establish a partial lower bound that is consistent with [\(1.4\)](#page-1-0) in a Fermi neighbourhood of the caustic Λ_E (see subsection [2\)](#page-3-0) under a suitable control assumption on eigenfunction mass in the allowed region Ω_E . This is precisely the point of Theorem [3.](#page-11-0) We then give applications to lower bounds for L^p -restrictions of eigenfunctions to hypersurfaces in the forbidden region (so-called goodness estimates in the terminology of Toth and Zelditch [\[TZ09\]](#page-21-5)). Finally, we apply these rather explicit bounds to improve on the nodal intersection bounds of Canzani and Toth [\[CT16\]](#page-21-6) for a large class of hypersurfaces in forbidden regions. We now describe our results in more detail.

Since by the monotonicity assumption, $\nabla V|_{V=E} \neq 0$, it follows that the caustic $\Lambda_E =$ $\{V = E\}$ is a C^{∞} hypersurface of M. Fix a constant $r_0 \in (0, \frac{\text{inj}(M,g)}{2})$ $\left(\frac{M,g_j}{2}\right)$ and let $U_E(r_0)$ be a Fermi neighbourhood (see [\[Gr\]](#page-21-7) Chapter 2, section 2.1) of the caustic Λ_E of width $2r_0$ with respect to the ambient metric g. We denote the Fermi defining function $y_n : M \to \mathbb{R}$ with the property that $y_n > 0$ in the forbidden part and $\Lambda_E = \{y_n = 0\}$. In terms of Fermi coordinates, the collar neighbourhood is $U_E(r_0) = \{y; |y_n| < r_0\}$. Consider an annular region in $U_E(r_0) \cap \{V > E\}$ given by $A(\delta_1, \delta_2) := \{y \in U_E(r_0); E + \delta_1 < V(y) < E + \delta_2\}$ with $0 < \delta_1 < \delta_2$. Our first result in Theorem [3](#page-11-0) is a partial reverse Agmon estimate consistent with [\(1.4\)](#page-1-0). First, we introduce a *control* assumption on the eigenfunctions u_h in the allowed region.

DEFINITION 1. We say that the eigenfunctions u_h satisfy the control assumption if for every $\varepsilon > 0$ there exists constants $C_N(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ so that for $h \in (0, h_0(\varepsilon)],$

$$
\int_{\{E-\frac{\varepsilon}{2}\le V(x)\le E\}} |u_h|^2 \, dv_g \ge C_N(\varepsilon) h^N \tag{1.6}
$$

for some $N > 0$. When [\(1.6\)](#page-1-1) is satisfied for a fixed $\varepsilon = \varepsilon_0 > 0$, we say that the eigenfunction sequence satisfies the ε_0 control assumption.

Roughly speaking, the control assumption in Definition [1](#page-1-2) says that in arbitrarily small (but independent of h) annular neighbourhoods of the caustic in the *allowed* region, eigenfunctions have at least polynomial mass in h.

Our second assumption involves a monotonicity condition on the potential V itself in the tubular neighbourhood $U_E(r_0)$. Specifically, we make the following

DEFINITION 2. Given $r_0 \in (0, \frac{inj(M,g)}{g^2})$ $\frac{M,g)}{2}$, we say that V satisfies the monotonicity assumption in $U_E(r_0)$ provided there exists $C(r_0) > 0$ such that

$$
\partial_{y_n} V(y) \ge C(r_0), \quad y \in U_E(r_0).
$$

We note that the control assumption is automatically satisfied in the 1D case as a consequence of the WKB asymptotics for the eigenfunctions. In section [5,](#page-17-0) we give examples of eigenfunction sequences satisfying this condition in arbitrary dimension.

As for the monotonicity condition in Definition [2,](#page-2-0) it is not hard to show (see section [2.1](#page-3-1) [\(2.5\)](#page-4-0)) that for $r_0 > 0$ sufficiently small, this condition is necessarily satisfied in $U_E(r_0)$ provided $\nabla V|_{\{V=E\}} \neq 0$ for the energy value E. We note that this condition can readily be written in a more geometrically intrinsic way in terms of the local normal foliation of the tubular neighbourhood of Λ_E (see subsection [2.1\)](#page-3-1).

Then, under the control and monotonicity assumptions in Definitions [1](#page-1-2) and [2,](#page-2-0) by using Carleman estimates to pass across the caustic hypersurface, in Theorem [3](#page-11-0) we prove that for any $\varepsilon > 0$ and $h \in (0, h_0(\varepsilon))$,

$$
||e^{\tau_0 d_E/h} u_h||_{H^1_h(A(\delta_1, \delta_2))} \ge C(\varepsilon, \delta_1, \delta_2) e^{-\beta(\varepsilon)/h}, \tag{1.7}
$$

where $\beta(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$ and

$$
\tau_0 := \left(\frac{\max_{y \in U_E(r_0)} |\partial_{y_n} V|}{\min_{y \in U_E(r_0) \cap \Omega_E^c} |\partial_{y_n} V|}\right)^{1/2}.
$$

Remark 1.

(i) The ε -dependence on the RHS of [\(1.7\)](#page-2-1) is a consequence of the control assumption on the eigenfunctions u_h in Definition [1.](#page-1-2)

(ii) It is easy to see that the control assumption in Definiton [1](#page-1-2) is necessary in Theorem 3 since simple counterexamples can be constructed otherwise arising from the natural occurence of additional effective potentials upon separation of variables (see section [5\)](#page-17-0).

(iii) The control assumption in Definition [1](#page-1-2) can be weakened slightly in that the polynomial mass assumption in h can be replaced by the following subexponential one: for all $\varepsilon > 0$,

$$
\liminf_{h \to 0^+} h \log \Big(\int_{\{E - \frac{\varepsilon}{2} \le V(x) \le E\}} |u_h|^2 \, dv_g \Big) = 0.
$$

It follows easily (see [\(2.21\)](#page-11-1)) that Theorem [3](#page-11-0) and consequently, Theorems [5](#page-14-0) and [6](#page-16-0) all hold under this slightly weaker assumption.

(iv) In the case where the eigenfunction sequence only satisfies the ε_0 -control assumption, the lower bound in [\(1.7\)](#page-2-1) is also satisfied, where the constant $\beta(\varepsilon_0) > 0$ appearing on the RHS of the inequality can be explicitly estimated in terms of the potential, V (see Remark [4\)](#page-12-0). The same is true for the subsequent results in Theorems [5](#page-14-0) and [6.](#page-16-0)

Clearly, the geometric constant $\tau_0 \geq 1$. At present, we are unable to prove that [\(1.7\)](#page-2-1) holds in the general setting above with optimal constant $\tau_0 = 1$, but we hope to return to this point elsewhere.

We note that in one-dimension, both upper and lower bounds for eigenfunctions are wellknown; indeed, classical WKB methods [\[E\]](#page-21-8) (Chapter 4, sections 4.3-4.7) provide explicit

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asymptotic formulas for the eigenfunctions. In higher dimensions, the situation is much more complicated and, to our knowledge, the problem of establishing lower bounds has only been addressed in special cases associated with low-lying eigenvalues (i.e. semi-excited states) [\[Hel88\]](#page-21-0) (section 4.4) and [\[Sim84\]](#page-21-2). For more general eigenfunctions (eg. excited states), while the upper bounds given by the Agmon estimates are well-known, to our knowledge explicit lower bounds have not been addressed in the literature. That is the main point of Theorem [3](#page-11-0) above.

In section [3](#page-12-1) we use the Carleman bounds in [\(1.7\)](#page-2-1) with shrinking annuli together with a Green's formula argument to get lower bounds for L^p eigenfunction restrictions to hypersurfaces smoothly isotopic in $U_E(r_0) \cap \{V > E\}$ to level sets $H = \{y_n = const.\}$ (see Definition [4\)](#page-12-2). In case of the level sets $H = \{y_n = const\}$, Theorem [5](#page-14-0) says that, under the same assumptions as in [\(1.7\)](#page-2-1), for any $\varepsilon > 0$ and $h \in (0, h_0(\varepsilon))$ and with

$$
d_E^H := \max_{y \in H} d_E(y), \quad d_E(H) := \min_{y \in H} d_E(y),
$$

$$
||u_h||_{L^p(H)} \ge C(p, \varepsilon) e^{-(2\tau_0 d_E^H - d_E(H))/h} e^{-\beta(\varepsilon)/h}, \quad p \ge 1,
$$
 (1.8)

where $\beta(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$.

The bounds in [\(1.8\)](#page-3-2) are *goodness* estimates in the terminology of Toth and Zelditch [\[TZ09\]](#page-21-5); the key novelty here is the rather explicit geometric rate $2\tau_0 d_E^H - d_E(H)$ appearing in [\(1.8\)](#page-3-2).

Finally, in section [4,](#page-14-1) we give an application of [\(1.8\)](#page-3-2) to nodal intersection bounds in for-bidden regions. In [\[CT16\]](#page-21-6), Canzani and Toth prove that for any C^{ω} separating hypersurface H in the forbidden region, with $Z_{u_h} = \{x \in M; u_h(x) = 0\},\$

$$
\# \{ Z_{u_h} \cap H \} \le C_H' h^{-1}.
$$

While this rate in h is easily seen to be sharp in general, the constant $C_H > 0$ is not explicitly controlled in [\[CT16\]](#page-21-6). The lower bound in [\(1.8\)](#page-3-2) allows us to give a more concrete upper bound for C_H' in the cases where H is a separating hypersurface that is smoothly isotopic to a level set of the defining function y_n in the forbidden region. This is essentially the content of Theorem [6.](#page-16-0)

Finally, we note that while all results are stated here for compact manifolds, the results in Theorems [3](#page-11-0)[-6](#page-16-0) extend to the case of Schrödinger operators on \mathbb{R}^n and the proofs are the same.

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2. Carleman estimates in a Fermi neighbourhood of the caustic

2.1. Collar neighbourhood of caustic and Fermi coordinates. For background on the construction of Fermi coordinates in a tubular neighbourhood of any embedded compact submanifold $P \subset M$, we refer the reader to $|G_r|$ Chapter 2 (see also [\[HHHZ\]](#page-21-9) section 2). Here, given a compact Riemannian manifold (M, g) , we are interested in the special case where P is a C^{∞} compact hypersurface; specifically, $P = \Lambda_E = \{V = E\}$. Let $\nu \in N\Lambda_E$ be the unit, oriented vector field normal to Λ_E . Then, by the tubular neighbourhood theorem, there exists an open neighbourhood, $U_E(r_0)$, of Λ_E such that

$$
\Lambda_E \times (-r_0, r_0) \ni (y', t) \mapsto \exp_{y'}(t\nu) \in U_E(r_0)
$$

is a C^{∞} diffeomorphism provided $r_0 < inj(M,g)/2$, where exp denotes the geodesic exponential map in the metric g. Then, the Fermi defining function $y_n: U_E(r_0) \to (-r_0, r_0)$ is given by

$$
y_n(\exp_{y'}(t\nu)) = t; \quad t \in (-r_0, r_0).
$$

Writing $\gamma_{\nu}(t, y') := \exp_{y'}(t\nu)$, it then follows that the monotonicity condition in Definition [2](#page-2-0) can be rewritten in the somewhat more geometric form

$$
\langle \nabla V, \dot{\gamma_{\nu}}(t, y') \rangle_g \ge C_0 > 0, \quad (t, y') \in (-r_0, r_0) \times \Lambda_E. \tag{2.1}
$$

We will assume from now on that the monotonicity assumption in Definition [2](#page-2-0) (or equiv-alently, [\(2.1\)](#page-4-1)) is satisfied (see Remark [2\)](#page-4-2). In the following, we identify points $y' \in \Lambda_E$ with their local coordinate representations in Λ_E . Then, in terms of Fermi coordinates (y', y_n) in the ambient metric q , we have

$$
g = dy_n^2 + \sum_{i,j=1}^{n-1} h_{ij}(y', y_n) dy'_i dy'_j, \quad y \in U_E(r_0)
$$

where $\sum_{i,j=1}^{n-1} h_{ij}(y',0) dy'_i dy'_j$ is the induced metric on Λ_E . In the following, we abuse notation somewhat and simply write $h(y', y_n)|dy'|^2 := \sum_{i,j=1}^{n-1} h_{ij}(y', y_n)dy'_i dy'_j$.

In the following, $y_n \in C^{\infty}(M;\mathbb{R})$, is the Fermi defining function for Λ_E defined above with

$$
\Lambda_E = \{ y_n = 0 \}, \quad dy_n |_{\Lambda_E} \neq 0.
$$

Since under the monotonicity assumption, $\partial_{y_n} V(y) > 0$, it is immediate that then $\nabla V(y) \neq 0$ for $y \in U_E(r_0)$. Thus, it follows that $V(y) - E$ is also a legitimate defining function in the tubular neighbourhood $U_F(r_0)$.

We choose our sign convention so that

$$
\{V > E\} \cap U_E(r_0) = \{y; 0 < y_n < r_0\} \text{ and } \{V < E\} \cap U_E(r_0) = \{y; -r_0 < y_n < 0\}. \tag{2.2}
$$

It will also be useful in the following to introduce the following annular domains in the forbidden region defined by

$$
A(\delta, \delta') := \{ x \in M; \delta < y_n(x) < \delta' \}, \quad 0 < \delta < \delta'. \tag{2.3}
$$

In terms of the Fermi coordinates $y = (y', y_n)$, the corresponding Agmon metric $g_E|_{U_E(r_0)}$ has the form

$$
g_E = (V(y) - E) (dy_n^2 + h(y', y_n)|dy'|^2), \quad y \in U_E(r_0), y_n > 0.
$$
 (2.4)

Remark 2. By first-order Taylor expansion,

$$
V(y) - E = y_n F(y', y_n),
$$
\n(2.5)

where

$$
F(y', y_n) = \int_0^1 (\partial_{y_n} V)(y', ty_n) dt.
$$

Differentiation of (2.5) in the y-variables, gives

$$
\partial_{y_n} V(y) = F(y', y_n) + y_n \partial_{y_n} F(y), \quad \partial_{y'} V(y) = y_n \partial_{y'} F(y', y_n), \quad F(y', 0) = \partial_{y_n} V(y', 0).
$$

Thus, $\partial_{y'}V(y',0) = 0$ and since $\nabla V|_{V=E} \neq 0$ it follows that $\partial_{y_n}V(y',0) \neq 0$. Then, by continuity of ∇V there exists $r_0 > 0$ so that $|\partial_{y_n} V(y)| \geq C > 0$ for $y \in U_E(r_0)$ and under the sign conventions in (2.2) ,

$$
\partial_{y_n} V(y) \ge C > 0, \quad y \in U_E(r_0).
$$

Consequently, when $E \in \mathbb{R}$ is a regular value of V, one can always find a constant $r_0 > 0$ so that the monotonicity assumption in Definition [2](#page-2-0) is satisfied.

Under the monotonicity condition, we then have $F(y) \geq C > 0$ for $y \in U_E(r_0)$ and from $(2.5),$ $(2.5),$

$$
\min_{y \in U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V(y) \le F(y) \le \max_{y \in U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V(y); \quad y \in U_E(r_0) \cap \Omega_E^c. \tag{2.6}
$$

Finally, we note that in terms of the decomposition [\(2.5\)](#page-4-0), the Agmon metric can be written in the form

$$
g_E = y_n F(y) (dy_n^2 + h(y', y_n)|dy'|^2), \quad y \in U_E(r_0), \ y_n > 0,
$$
\n
$$
(2.7)
$$

2.1.1. Locally minimal geodesics and Agmon distance. In the collar neighbourhood $U_E(r_0)$, given a point $(y', y_n) \in U_E(r_0) \cap \Omega_E^c$, there is a unique minimal (i.e. distance-minimizing) geodesic $\gamma : [0,1] \times \Lambda_E \to U_E(r_0)$ for the ambient metric g joining (y', y_n) to the caustic hypersurface Λ_E . Setting $\gamma_t(y') = \gamma(t, y')$ where $\gamma_0(y') = (y', 0) \in \Lambda_E$ and $\gamma_1(y') = (y', y_n)$, the minimal geodesic is just

$$
\gamma_t(y') = (y', ty_n); \quad 0 \le t \le 1.
$$

In terms of the discussion in subsection [2.1](#page-3-1) above, these are just the geodesic segments normal to Λ_E . It is easy to see that these normal geodesic segments to Λ_E are unfortunately not, in general, minimal geodesics for the conformally rescaled Agmon metric g_E ; indeed the latter can be quite complicated. Nevertheless, we will need the following elementary estimate for Agmon distance in terms of the natural Fermi defining function $y_n : M \to \mathbb{R}$ above.

LEMMA 2.1. Let E be a regular value of $V \in C^{\infty}(M)$ and assume that the monotonicity condition is satisfied in $U_E(r_0)$. Then,

$$
d_E(y) \ge \frac{2}{3} \left(\min_{y \in U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V(y) \right)^{1/2} y_n^{3/2}; \quad y \in U_E(r_0) \cap \Omega_E^c.
$$

Proof. Let $\gamma : [0,1] \in \Omega_E^c$ be a piecewise-C¹ minimal geodesic for the Agmon metric g_E joining $y = (y', y_n) \in U_E(r_0) \cap \Omega_E^c$ to Λ_E ; explicitly, $\gamma(0) = (y', y_n)$ and $\gamma(1) = (f(y', y_n), 0)$ where $f(y) \in \mathbb{R}^{n-1}$. Then, writing $\gamma = (\gamma', \gamma_n)$, with $\gamma' = (\gamma_1, ..., \gamma_{n-1})$,

$$
d_E(y) = \int_0^1 |d_t \gamma(t)|_{g_E} dt,
$$

and since

$$
|d_t\gamma(t)|_{g_E} = \left(F(\gamma(t)) \gamma_n(t) |d_t\gamma_n(t)|^2 + F(\gamma(t)) \gamma_n(t) \langle h(y(t)) d_t\gamma'(t), d_t\gamma'(t) \rangle \right)^{1/2},
$$

with $F, \gamma_n > 0$, and $0 \leq h \in GL(n-1, \mathbb{R})$, it follows that

$$
d_E(y) \ge \min_{y \in U_E(r_0) \cap \Omega_E^c} F(y)^{1/2} \cdot \int_0^1 \gamma_n(t)^{1/2} |d_t \gamma_n(t)| dt, \quad y \in U_E(r_0) \cap \Omega_E^c.
$$

Finally, by making the change of variables $t \mapsto s = \gamma_n(t)$ in the last integral, one gets

$$
d_E(y) \ge \min_{y \in U_E(r_0) \cap \Omega_E^c} F(y)^{1/2} \cdot \int_0^{y_n} s^{1/2} \, ds, \quad y \in U_E(r_0) \cap \Omega_E^c.
$$

and the lemma follows from this last estimate combined with (2.6) since $\min_{y \in U_E(r_0) \cap \Omega_E^c} F(y)^{1/2} \ge$ $\min_{y\in U_{E}(r_{0})\cap\Omega_{E}^{c}}\left(\ \partial_{y_{n}}V(y)\ \right)^{1/2}$.

2.2. Local control and Carleman bounds near the caustic Λ_E .

2.2.1. Model computation. Consider the model Airy operator $P_0(h) := (hD_y)^2 + y$ where $y \in \mathbb{R}$ where $V(y) = y$ and $E = 0$ with the corresponding Airy-type weight function in the forbidden region given by

$$
\varphi_0(y) = \frac{2}{3}y^{3/2}, \quad y > 0.
$$

Then, the symbol of the conjugated operator $e^{\varphi_0/h}P_0(h)e^{-\varphi_0/h}$ is

$$
p_{\varphi_0}(y,\xi) = \xi^2 - |\varphi'_0(y)|^2 + y + 2iy^{1/2}\xi, \ \ y > 0
$$

and

$$
Char(p_{\varphi_0}) = \{ (y, \xi) \in \mathbb{R}^2; \xi = 0, y > 0 \}.
$$

The latter follows since $(y,\xi) \in \text{Char}(\varphi_{\varphi_0})$ iff $0 = |\xi|^2 - |\varphi'_0(y)|^2 + y + 2iy^{1/2}\xi$ which in turn holds iff $\xi = 0$ provided $y > 0$, since $|\varphi'_0(y)|^2 - y = 0$.

We note that the weight function φ_0 is *borderline* for the Hörmander subelliptic condition in the sense that for $(y, 0) \in \text{Char}(p_{\varphi_0})$, we have

$$
\{\operatorname{Re} p_{\varphi_0}, \operatorname{Im} p_{\varphi_0}\} = 4\varphi_0''(y)|\varphi_0'(y)|^2 - 2\varphi_0'(y) \equiv 0, \quad y > 0. \tag{2.8}
$$

Of course, in this case, $\varphi_0(y) = \frac{2}{3}y^{3/2} = \int_0^y \tau^{1/2} d\tau$ is precisely the Agmon distance function $d_E(y)$, where by convention we have set $E = 0$. In view of [\(2.8\)](#page-6-0), it is reasonable to expect that one can construct an appropriate perturbation, φ_{ε} , of the Airy model weight function φ_0 , that is a Carleman weight with $\{ \text{Re } p_{\varphi_{\varepsilon}}, \text{Im } p_{\varphi_{\varepsilon}} \} > 0$ in a neighbourhood of the caustic Λ_E . Indeed, as we show in subsection [2.2.2](#page-7-0) below, one can readily construct such a φ_{ε} as a legitimate Carleman weight in a Fermi neighbourhood of Λ_E . Combined with the control assumption on the eigenfunctions u_h , we will prove Theorem [3](#page-11-0) by applying Carleman estimates in a Fermi neighbourhood of the caustic with weight φ_{ε} .

2.2.2. Construction of the weight function. Let $P(h) = -h^2 \Delta_{g} + V - E : C^{\infty}(M) \to C^{\infty}(M)$ and consider the conjugated operator $P_{\varphi}(h) = e^{\varphi/h} P(h) e^{-\varphi/h} : C^{\infty}(M) \to C^{\infty}(M)$ with principal symbol $p_{\varphi}(x,\xi) = |\xi|_g^2 - |\nabla_x \varphi|_g^2 + V(x) - E + 2i \langle \xi, \nabla_x \varphi \rangle_g$. The model case above suggests that to create subellipticity for $P_{\varphi}(h)$ in a Fermi neighbourhood of the caustic, it should suffice to slightly modify the model weight function φ_0 in the normal Fermi coordinate (y', y_n) . With this in mind, for $\varepsilon > 0$ arbitrarily small (for concreteness, assume $10\varepsilon < r_0$) and constant $\tau > 0$ to be determined later on, we now set in Fermi coordinates (y', y_n) : $U_E \to \mathbb{R}^{n-1} \times (-r_0, r_0),$

$$
\varphi_{\varepsilon}(y_n) := \left(\frac{2}{3} + \varepsilon\right) \tau\left(y_n + 10\varepsilon\right)^{3/2}, \quad y_n \in (-4\varepsilon, r_0). \tag{2.9}
$$

Remark 3. We recall here that $r_0 < inj(M, g)$ is fixed (but not necessarily small), whereas $\varepsilon > 0$ will be chosen arbitrary small (but independent of h) consistent with the control assumption on the eigenfunctions.

We abuse notation somewhat in the following and simply write $\varphi = \varphi_{\varepsilon}$, the dependence on ε being understood. Then, $\varphi \in C^{\infty}([-4\varepsilon, r_0])$ and plainly $\varphi : [-4\varepsilon, r_0] \to \mathbb{R}^+$ is strictlyconvex and monotone increasing with

$$
\min(\varphi'(y_n), \varphi''(y_n)) \ge C(\varepsilon) > 0, \quad y_n \in (-4\varepsilon, r_0).
$$

Let $\pi : T^*M \to M$ be the natural projection map $\pi(y,\xi) = y$. Then, the relevant characteristic variety is

$$
Char(p_{\varphi}) \cap \pi^{-1}(\{(y', y_n); y_n \in (-4\varepsilon, r_0)\}) = \{(y, \xi); |\xi|_y^2 - |\partial_{y_n} \varphi|^2 + F(y)y_n = 0, \xi_n = 0, y_n \in (-4\varepsilon, r_0)\}.
$$

Since $F(y) > 0$, it follows that this set is non-trivial; indeed for any $-4\varepsilon < y_n < 0$ (ie. a point in the allowed region),

$$
Char(p_{\varphi}) \cap \pi^{-1}(\{(y', y_n), y_n \in (-4\epsilon, 0)\}) \cong S^* M_{A(-4\epsilon, 0)} \cap \{\xi_n = 0\},\
$$

where $S^*M_{A(-4\varepsilon,0)} := \{(y,\xi) \in T^*M; |\xi|_y = 1, y \in A(-4\varepsilon,0)\}.$

Since Char(p_{φ}) is non-trivial, global ellipticity over the interval (-4ε , r_0) evidently fails. However, we claim that *subellipticity* is now satisfied in such an interval provided $\tau > 0$ is chosen large enough but depending only on the potential V . Indeed, since the normal Fermi coordinate is y_n and φ is a function of only y_n with $g_{n,n} = 1$, a direct computation gives,

$$
\begin{aligned} \{\operatorname{Re} p_{\varphi}, \operatorname{Im} p_{\varphi}\} &= \{\xi_n^2 + |\xi'|_y^2 - (\partial_{y_n}\varphi)^2 + V - E, \ 2\partial_{y_n}\varphi \cdot \xi_n\} \\ &= 4\partial_{y_n}^2 \varphi \left(|\partial_{y_n}\varphi|^2 + \xi_n^2 \right) - 2\partial_{y_n}\varphi \cdot \partial_{y_n} V \\ &\ge 2\partial_{y_n}\varphi \left(2\partial_{y_n}^2\varphi \cdot \partial_{y_n}\varphi - \partial_{y_n} V \right) \\ &\ge 2\tau C(\varepsilon) \left(2\partial_{y_n}^2\varphi \cdot \partial_{y_n}\varphi - \partial_{y_n} V \right), \quad y_n \in (-4\varepsilon, r_0). \end{aligned} \tag{2.10}
$$

From [\(2.9\)](#page-7-1), for any $\varepsilon > 0$ and for all $y_n \in (-4\varepsilon, r_0)$,

$$
2\partial_{y_n}^2 \varphi \cdot \partial_{y_n} \varphi \equiv \frac{9}{4} \tau^2 \Big(\frac{2}{3} + \varepsilon\Big)^2 > \tau^2.
$$

Choosing

$$
\tau = \|\partial_{y_n} V\|_{L^{\infty}(U_E(r_0))}^{1/2},\tag{2.11}
$$

it follows from [\(2.10\)](#page-8-0) that for all (y, ξ) with $y_n \in (-4\varepsilon, r_0)$,

$$
\{\operatorname{Re} p_{\varphi}, \operatorname{Im} p_{\varphi}\}(y,\xi) \ge C(\tau,\varepsilon) > 0.
$$

Consequently, $\varphi = \varphi_{\varepsilon}$ is a Carleman weight for $P(h)$ globally in the Fermi neighbourhood of the caustic where $-4\varepsilon < y_n < r_0$.

Now, let $\chi \in C_0^{\infty}(\mathbb{R}; [0, 1])$ be a cutoff satisfying

$$
\chi(y_n) = 1; \quad -\frac{3}{2}\varepsilon < y_n < \tilde{\delta}_1
$$

with

$$
\chi(y_n)=0;\quad y_n\in\mathbb{R}\setminus(-2\varepsilon,\tilde{\delta_2}).
$$

where $0 < \tilde{\delta_1} < \tilde{\delta_2} < r_0$.

Figure 1. cutoff function χ

We note that one can write

$$
\chi = \chi_{-}\,\chi_{+}
$$

where, $\chi_+ \in C^{\infty}(\mathbb{R}; [0, 1])$ satisfies

$$
\chi_+(y_n) = 1, \quad y_n < \tilde{\delta}_1,
$$
\n
$$
\chi_+(y_n) = 0, \quad y_n > \tilde{\delta}_2,
$$

and similarily, $\chi_-\in C^\infty(\mathbb{R};[0,1])$ satisfies

$$
\chi_{-}(y_n) = 1; \quad y_n > -\frac{3}{2}\varepsilon,
$$

$$
\chi_{-}(y_n) = 0; \quad y_n < -2\varepsilon.
$$

Then, it follows from Leibniz rule that

 $\text{supp }\partial\chi \subset \text{supp }\partial\chi_+ \cup \text{supp }\partial\chi_-,$

where supp $\partial \chi_-\subset [-2\varepsilon,-\frac{3}{2}]$ $\frac{3}{2}\varepsilon$ and supp $\partial \chi_+ \subset [\tilde{\delta_1}, \tilde{\delta_2}]$.

Set
$$
P_{\varphi}(h) := e^{\varphi/h} P(h) e^{-\varphi/h} : C_0^{\infty}(U) \to C_0^{\infty}(U)
$$
 and with $\chi = \chi(y_n)$ above,

$$
v_h := e^{\varphi/h} \chi u_h
$$

where $P(h) := -h^2 \Delta_g + V(x) - E(h)$ and

$$
P(h)u_h=0.
$$

Moreover, we assume throughout that the eigenfunctions u_h are L^2 -normalized with $||u_h||_{L^2(M,g)} = 1.$

In view of the subellipticity estimate in (2.10) and the support properties of the cutoff $\chi \in C_0^{\infty}$ it follows by the standard Carleman estimate [\[Zwo12,](#page-21-10) Theorem 7.7] that

$$
||P_{\varphi}(h)v_h||_{L^2}^2 \ge C_1(\varepsilon)h ||v_h||_{H_h^1}^2. \tag{2.12}
$$

Since $P(h)u_h = 0$ and $P_\varphi(h)$ is local with supp $\partial \chi_+ \cap \text{supp }\partial \chi_- = \emptyset$, it follows from [\(2.12\)](#page-9-0) that

$$
||e^{\varphi/h}[P(h), \chi]u_h||_{L^2}^2 \ge C_1(\varepsilon)h ||e^{\varphi/h}\chi u_h||_{H_h^1}^2
$$
\n(2.13)

Let supp $\partial \chi_{\pm}$ be arbitrarily small neighbourhoods of supp $\partial \chi_{\pm}$ respectively. Specifically, we can assume that $\text{supp }\partial \chi_{\pm} \supset \text{supp }\partial \chi_{\pm}$, and choose (δ_1, δ_2) with $0 < \delta_1 < \tilde{\delta_1} < \tilde{\delta_2} < \delta_2$ so that supp $\widetilde{\partial \chi_+} \subset [\delta_1, \delta_2]$ and in addition

$$
\operatorname{meas}\left(\operatorname{supp}\widetilde{\partial\chi_{\pm}}\setminus\operatorname{supp}\partial\chi_{\pm}\right)\leq\frac{\varepsilon}{10}.
$$

Since the support of the coefficients of the h-differential operator $[P(h), \chi]$ is contained in supp $\partial \chi_+ \cup \text{supp }\partial \chi_-,$ it follows from [\(2.13\)](#page-9-1) that

$$
||e^{\varphi/h}[P(h),\chi]u_h||_{L^2(\text{supp }\widetilde{\partial_{\chi_+}})}^2 + ||e^{\varphi/h}[P(h),\chi]u_h||_{L^2(\text{supp }\widetilde{\partial_{\chi_-}})}^2
$$

$$
\geq C_1(\varepsilon)h||e^{\varphi/h}\chi u_h||_{H_h^1}^2.
$$
 (2.14)

Since $\chi(y_n) = 1$ when $y_n \in (-\varepsilon/2, 0)$, it follows from [\(2.14\)](#page-9-2) that

$$
||e^{\varphi/h}[P(h),\chi]u_h||_{L^2(\text{supp }\widetilde{\partial_{\chi_+}})}^2 + ||e^{\varphi/h}[P(h),\chi]u_h||_{L^2(\text{supp }\widetilde{\partial_{\chi_-}})}^2
$$

$$
\geq C_1(\varepsilon)h||e^{\varphi/h}u_h||_{H_h^1(\{y;-\varepsilon/2\leq y_n\leq 0\})}^2.
$$
 (2.15)

Since $e^{\varphi/h}[P(h),\chi]e^{-\varphi/h} = hQ(h)$ where $Q(h) = a(x)h\partial_x + b(x), a, b \in C^{\infty}(M)$, is an h-differential operator of order one,

$$
||e^{\varphi/h}[P(h),\chi]u_h||_{L^2(\text{supp }\widetilde{\partial \chi_{\pm}})}^2 \le Ch^2 ||e^{\varphi/h}u_h||_{H_h^1(\text{supp }\widetilde{\partial \chi_{\pm}})}^2. \tag{2.16}
$$

where supp $\partial \chi_{\pm} \supseteq \text{supp }\partial \chi_{\pm}$. After relabelling $\partial \chi_{\pm}$ simply as $\partial \chi_{\pm}$, one gets from [\(2.15\)](#page-10-0) and [\(2.16\)](#page-10-1) that with an appropriate constant $C_2(\varepsilon) > 0$,

$$
h^{2} \|e^{\varphi/h} u_{h}\|_{H_{h}^{1}(\text{supp }\widetilde{\partial \chi_{+}})}^{2} + h^{2} \|e^{\varphi/h} u_{h}\|_{H_{h}^{1}(\text{supp }\widetilde{\partial \chi_{-}})}^{2}
$$

$$
\geq C_{2}(\varepsilon)h \|e^{\varphi/h} u_{h}\|_{H_{h}^{1}(\{y; -\varepsilon/2 \leq y_{n} \leq 0\})}^{2}, \tag{2.17}
$$

or equivalently,

$$
h^2 \|e^{\varphi/h} u_h\|_{H_h^1(\text{supp}\,\widetilde{\partial_{\chi_+}})}^2 \ge C_2(\varepsilon) h \|e^{\varphi/h} u_h\|_{H_h^1(\{y; -\varepsilon/2 \le y_n \le 0\})}^2 - h^2 \|e^{\varphi/h} u_h\|_{H_h^1(\text{supp}\,\widetilde{\partial_{\chi_-}})}^2, \quad (2.18)
$$

Since $\widetilde{\partial \chi_{-}}$ is supported in the classically allowed region where $y_n < 0$, we will now use the control assumption in Definition [1](#page-1-2) to get an effective lower bound for the RHS in [\(2.18\)](#page-10-2).

Computing in Fermi coordinates and using the fact that one can take supp $\partial \chi_-\subset$ $\{y; -3\varepsilon < y_n < -\varepsilon\}$, the RHS of [\(2.18\)](#page-10-2) is

$$
\geq C_2(\varepsilon) h \int_{\{y; y_n \in (-\frac{\varepsilon}{2}, 0)\}} e^{2\varphi(y_n)/h} (|u_h(y)|^2 + |h \partial_y u_h(y)|^2) dy' dy_n
$$

$$
- h^2 \int_{\{y; y_n \in (-3\varepsilon, -\varepsilon)\}} e^{2\varphi(y_n)/h} (|u_h(y)|^2 + |h \partial_y u_h(y)|^2) dy' dy_n.
$$
 (2.19)

Next we use strict monotonicity of the weight function $\varphi \in C^{\infty}([-3\varepsilon, r_0])$ in [\(2.9\)](#page-7-1). We set $m(\varepsilon) := \min_{y_n \in (-\frac{\varepsilon}{2}, 0)} \varphi(y_n) > 0$ and $M(\varepsilon) := \max_{y_n \in (-3\varepsilon, -\varepsilon)} \varphi(y_n) > 0$ (we note that both $m(\varepsilon)$ and $M(\varepsilon)$ are of order $\varepsilon^{3/2}$). Then, since φ is strictly increasing,

$$
m(\varepsilon) - M(\varepsilon) = C_3(\varepsilon) > 0.
$$

So, it follows that [\(2.19\)](#page-10-3) is bounded below by

$$
C_2(\varepsilon)e^{2m(\varepsilon)/h}\left(h\|u_h\|_{H_h^1(\{y;y_n\in(-\frac{\varepsilon}{2},0)\})}^2-C_2(\varepsilon)^{-1}h^2e^{2[M(\varepsilon)-m(\varepsilon)]/h}\|u_h\|_{H_h^1(\{y;y_n\in(-3\varepsilon,-\varepsilon)\})}^2\right). \tag{2.20}
$$

Finally, by standard elliptic estimates, $||u_h||_{H_h^1} = O(1)$ and by the control assumption in Definition [1,](#page-1-2) it follows that for any $\varepsilon > 0$,

$$
||u_h||_{H_h^1(\{y; y_n \in (-\frac{\varepsilon}{2}, 0)\})}^2 \geq C_{3,N}(\varepsilon)h^N.
$$

Consequently, from $(2.18)-(2.20)$ $(2.18)-(2.20)$ it follows that with $h \in (0, h_0(\varepsilon))$, there exist constants $C_{j,N}(\varepsilon) > 0, j = 3, 4, 5$, such that

$$
h^{2} \|e^{\varphi/h} u_{h}\|_{H_{h}^{1}(supp \,\widetilde{\partial_{\chi+}})}^{2} \geq C_{2}(\varepsilon) e^{2m(\varepsilon)/h} \left(h^{N+1} C_{3,N}(\varepsilon) + O_{\varepsilon}(e^{-2C_{3}(\varepsilon)/h}) \right) \geq C_{4,N}(\varepsilon) h^{N+1} e^{2m(\varepsilon)/h} \geq C_{5,N}(\varepsilon) e^{m(\varepsilon)/h}.
$$
\n(2.21)

Next, we relate the weight function φ_{ε} to Agmon distance d_E . From Lemma [2.1](#page-5-1) we recall that

$$
d_E(y) \ge \frac{2}{3} \left(\min_{y \in U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V \right)^{1/2} y_n^{3/2}, \quad y \in U_E(r_0) \cap \Omega_E^c \tag{2.22}
$$

$$
= \left(\frac{\min_{y \in U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V}{\max_{U_E(r_0)} \partial_{y_n} V}\right)^{1/2} \varphi_{\varepsilon}(y_n) + O(\varepsilon). \tag{2.23}
$$

The last estimate in [\(2.22\)](#page-11-2) follows since in the definition of the weight φ_{ε} (see [\(2.9\)](#page-7-1)), we choose $\tau = \max_{y \in U_E(r_0)} |\partial_{y_n} V|^{1/2}$ (see [\(2.11\)](#page-8-1)). Since from the monotonicity assumption in Definition [2,](#page-2-0) $\min_{y \in U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V \ge C(r_0) > 0$, it then follows that

$$
\varphi_{\varepsilon}(y_n) \le \left(\frac{\max_{y \in U_E(r_0)} \partial_{y_n} V}{\min_{y \in U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V}\right)^{1/2} d_E(y) + O(\varepsilon). \tag{2.24}
$$

Thus, in view of [\(2.21\)](#page-11-1) and [\(2.24\)](#page-11-3), we have proved the following reverse Agmon estimate for eigenfunctions satisfying the control assumption.

THEOREM 3. Let $r_0 > 0$ define the collar neighbourhood $U_E(r_0)$ of the hypersurface $\{V = E\}$ as above and consider an annular subdomain

$$
A(\delta_1, \delta_2) \subset \Big(\{V > E\} \cap U_E(r_0)\Big), \quad 0 < \delta_1 < \delta_2 < r_0.
$$

Then, under the control and monotonicity assumptions in Definitions [1](#page-1-2) and [2,](#page-2-0) it follows that for any $\varepsilon > 0$ and $h \in (0, h_0(\varepsilon))$, there exists a constant $C(\varepsilon, \delta_1, \delta_2) > 0$ such that

$$
||e^{\tau_0 d_E/h}u_h||_{H^1_h(A(\delta_1,\delta_2))} \geq C(\varepsilon,\delta_1,\delta_2) e^{-\beta(\varepsilon)/h},
$$

with

$$
\tau_0 = \left(\frac{\max_{U_E(r_0)} \partial_{y_n} V}{\min_{U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V}\right)^{1/2}
$$

and where $0 < \beta(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0^+$.

Remark 4. We note in the more general case where the eigenfunction sequence satisfies the ε_0 -control assumption, the estimate in Theorem [3](#page-11-0) is still valid (similarily for Theorems [5](#page-14-0) and [6\)](#page-16-0). In such a case, the constant $\beta(\varepsilon_0)$ can be readily estimated explicitly in terms of the potential from [\(2.22\)](#page-11-2) and [\(2.24\)](#page-11-3) above.

3. L^p restriction lower bounds in forbidden regions

Consider a C^{∞} hypersurface $H \subset \Omega_E^c$ in the forbidden region that bounds a domain $\Omega_H \subset \Omega_E^c$ and is *admissible* in the sense of Definition [4](#page-12-2) (ii) (see also Figure 2). The point of this section is to extend Theorem [3](#page-11-0) to lower bounds for L^2 -restrictions of eigenfunctions to hypersurfaces H in the forbidden region.

Let v be the unit exterior normal to H with $\langle \nabla V, \nu \rangle$ < 0. Then, under the separation assumption above, by Green's formula,

$$
\int_{\Omega_H} |h \nabla u_h|_g^2 dv_g + \int_{\Omega_H} (V - E(h)) |u_h|^2 dv_g = h^2 \int_H \partial_\nu u_h \cdot u_h d\sigma \tag{3.1}
$$

Using the fact that $V(x) - E \ge C > 0$ for all $x \in \Omega_H$, it follows from [\(3.1\)](#page-12-3) that with a constant $C_{\delta} = C(V, E, E', \delta) > 0$

$$
h^2 \int_H \partial_\nu u_h \cdot u_h \, d\sigma \ge C_\delta \|u_h\|_{H^1_h(\Omega_H)}^2. \tag{3.2}
$$

From the pointwise Agmon estimates in [\(1.5\)](#page-1-3), for any $\delta > 0$,

$$
||h\partial_{\nu}u_h||_{L^{\infty}(H)} = O_{\delta}(e^{[-d_E(H) + \delta]/h}), \quad d_E(H) := \min_{q \in H} d_E(q)
$$

together with the Hölder inequality,

$$
||u_h||_{L^p(H)} \ge C_\delta(p) e^{[d_E(H) - \delta]/h} ||u_h||_{H_h^1(\Omega_H)}^2, \quad p \ge 1.
$$
 (3.3)

DEFINITION 4. We say that the hypersurface $H \subset \{V > E\}$ is admissible provided:

(i) H is a separating hypersurface bounding a C^{∞} domain $\Omega_H \subset \{V > E\}.$

(ii) There exists $E' > E$ such that the hypersurface $\Lambda_{E'} = \{y_n = E' - E\}$ has the property that

$$
\Lambda_{E'} \subset \Omega_H \cap U_E(r_0).
$$

Figure 2. Red region is $\{V > E\} \cap U_E(r_0)$

Set

$$
E(H) := \inf \{ E' > E; \, \Lambda_{E'} \subset (\Omega_H \cap U_E(r_0)) \}. \tag{3.4}
$$

Since $\Lambda_{E'} \cap \Omega_H = \emptyset$ for any $E' > E$ sufficiently close to E, it follows that $E(H) > E$. Moreover, under the admissibility assumption, it follows that for any $\delta > 0$ sufficiently small (we use the same $\delta > 0$ here as in [\(3.3\)](#page-12-4))

$$
A(E(H), E(H) + \delta) \subset (\Omega_H \cap U_E(r_0))
$$

and so,

$$
||u_h||_{H_h^1(\Omega_H)}^2 \ge ||u_h||_{H_h^1(A(E(H), E(H) + \delta)}^2.
$$
\n(3.5)

From the Carleman estimate in Theorem [3,](#page-11-0)

$$
||e^{\tau_0 d_E/h}u_h||_{H_h^1(A(E(H), E(H) + \delta))}^2 \ge C(\delta, \varepsilon)e^{-\beta(\varepsilon)/h},\tag{3.6}
$$

where $\beta(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. Here, we recall that $\varepsilon > 0$ is the parameter appearing in the control condition in Definition [1.](#page-1-2)

Since $\delta > 0$ is arbitrary, we can set $\delta = \varepsilon$ in [\(3.3\)](#page-12-4)-[\(3.6\)](#page-13-0) and then, it follows that for any $\varepsilon > 0$, and with

$$
\tau_0 = \left(\frac{\max_{U_E(r_0)} \partial_{y_n} V}{\min_{U_E(r_0) \cap \Omega_E^c} \partial_{y_n} V}\right)^{1/2}, \quad d_E^H := \max_{q \in \Lambda_{E(H)}} d_E(q), \quad d_E(H) = \min_{q \in H} d_E(q), \tag{3.7}
$$

one has the following lower bound for L^p -restrictions of the u_h to H :

$$
||u_h||_{L^p(H)} \ge C(\varepsilon, p) e^{-2\tau_0 \cdot d_E^H/h} \cdot e^{d_E(H)/h} \cdot e^{-\tilde{\beta}(\varepsilon)/h}, \quad p \ge 1
$$

where $\tilde{\beta}(\varepsilon) := \beta(\varepsilon) + \varepsilon \to 0^+$ as $\varepsilon \to 0^+$. In summary, we have proved

THEOREM 5. Let H be an admissible hypersurface in sense of Definition [4.](#page-12-2) Then, under the control and monotonicity assumptions in Definitions [1](#page-1-2) and [2](#page-2-0) and with $E(H)$ in [\(3.4\)](#page-13-1) and $d_E^H, d_E(H), \tau_0$ in [\(3.7\)](#page-13-2), it follows that for any $\varepsilon > 0$ and with $h \in (0, h_0(\varepsilon)],$

$$
||u_h||_{L^p(H)} \ge C(\varepsilon, p) e^{-\left[2\tau_0 d_E^H - d_E(H) + \tilde{\beta}(\varepsilon)\right]/h}, \quad p \ge 1,
$$

where $\tilde{\beta}(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$.

Remark 5. We note that since $\tau_0 \geq 1$ and $d_E^H \geq d_E(H)$, it is clear that the constant $2\tau_0(H)d_E^H - d_E(H) > 0.$

4. Nodal intersection bounds in forbidden regions

Consider the special case where dim $M = 2$ and (M, g, V) are, in addition, real-analytic. Let $H \subset \Omega_E^c$ be a simple, closed, real-analytic curve in the forbidden region. In [\[CT16\]](#page-21-6) ,the authors obtained nodal intersection bounds for the nodal sets of the eigenfunctions u_h with the fixed curve, H. More precisely, given the nodal set

$$
Z_{u_h} = \{ x \in M; u_h(x) = 0 \},\
$$

the problem is to estimate the number of nodal intersections with H; that is $\#\{H \cap Z_{u_h}\}\$ which is just the cardinality of the intersection. Indeed, under an exponential lower bound on the L^2 -restrictions of the eigenfunctions (ie. a *goodness* bound), this intersection consists of a finite set of points.

Let $q:[0,2\pi] \to H$ be a C^{ω} , 2π -periodic, parametrization of H. To bound the number of zeros of $u_h \circ q : [0, 2\pi] \to \mathbb{R}$ we consider its holomorphic extension $(u_h \circ q)^\mathbb{C} : H_\rho^\mathbb{C} \to \mathbb{C}$ to

$$
H_{\rho}^{\mathbb{C}} := \{ t \in \mathbb{C} : \text{ Re } t \in [0, 2\pi], \ |\text{Im } t| < \rho \}.
$$

The zeros of $(u_h \circ q)^\mathbb{C}$ are studied using the Poincaré-Lelong formula:

$$
\partial \overline{\partial} \log |(u_h \circ q)^{\mathbb{C}}(z)|^2 = \sum_{z_k \in Z_{(u_h \circ q)^{\mathbb{C}}}} \delta_{z_k}(z).
$$

Consider the complex tube of radius $\rho > 0$ about H given by $A(\rho) := q^{\mathbb{C}}(H_{\rho}^{\mathbb{C}})$ and let G_{ρ} be the Dirichlet Green's function of $A(\rho)$ satisfying:

$$
\Delta_y G_{\rho}(x, y) = \delta_x(y), \quad (x, y) \in A(\rho) \times A(\rho),
$$

$$
G_{\rho}(x, q) = 0, \quad (x, q) \in A(\rho) \times \partial A(\rho).
$$

We define the constant

$$
C_H(\rho) := \Big| \max_{(z,w)\in A(\rho/2)\times A(\rho/2)} G_{\rho}(z,w) \Big|^{-1}.
$$
 (4.1)

We note here that by the maximum principle $\max_{(z,w)\in A(\rho/2)\times A(\rho/2)} G_{\rho}(z,w) < 0$, so that $C_H(\rho) > 0$ is finite. Then, by [\[TZ09,](#page-21-5) Proposition 10],

$$
\#\{Z_{u_h} \cap H\} \le \#\{Z_{(u_h \circ q)^{\mathbb{C}}} \cap H_{\rho}^{\mathbb{C}}\} \le C_H(\rho) \max_{t \in H_{\rho}^{\mathbb{C}}} \log |F_h^{\mathbb{C}}(t)|,
$$
\n(4.2)

where $F_h^{\mathbb{C}}$ $h^{\mathbb{C}}(t)$ with $t \in H^{\mathbb{C}}_{\rho}$ is the holomorphic continuation of the normalized eigenfunction traces

$$
F_h(t) := \frac{u_h(q(t))}{\|u_h\|_{L^2(H)}}.
$$
\n(4.3)

It follows that we shall need to control the complexification $F_h^{\mathbb{C}}$ $h_hⁿ(t)$ to obtain upper bounds on the complex counting function $\# \{Z_{\varphi_h} \cap H\}$. Without loss of generality we assume that $H \subset \text{int } \Omega_{\gamma}$ where $\Omega_{\gamma} \subset \Omega_{E}^{c}$ is a domain whose closure is contained in Ω_{E}^{c} and whose boundary is a closed C^{ω} curve that we call γ . In particular, one has $\gamma \cap H = \emptyset$ so that with $d_E(H, \gamma) := \inf_{(q,r) \in H \times \gamma} d_E(q, \gamma),$

$$
d_E(H, \gamma) > 0, \quad d_E(H) - d_E(\gamma) > 0.
$$

In [\[CT16\]](#page-21-6) (see section 2), the authors consider the h-elliptic operator $-h^2\Delta_{g_{\Omega_E}}+1$: $C^{\infty}(M) \to C^{\infty}(M)$ where g_{Ω_E} is a metric extension to M of the Agmon metric $g_E|_{\overline{\Omega}_{\gamma}}$.

After choosing a suitable cutoff $\chi_{\Omega_\gamma} \in C_0^{\infty}(\Omega_E^c)$ with $0 \leq \chi_{\Omega_\gamma} \leq 1$ and $\chi_{\Omega_\gamma}|_{\Omega_\gamma} = 1$, one can define the h-pseudodifferential operator $Q(h) = \chi_{\Omega_{\gamma}}(-h^2\Delta_{g_{\Omega_E}} + 1)^{-1}\chi_{\Omega_{\gamma}} : C_0^{\infty}(\Omega_E^c) \to$ $C_0^{\infty}(\Omega_E^c)$. The restriction of the Schwartz kernel of $Q(h)$ to $\gamma \times H$ given by $Q_{\gamma}^H(t, s, h) :=$ $Q(q(t), r(s), h)$ is then the kernel of an analytic h-Fourier integral operator $Q_{\gamma}^{H}(h)$: $C^{\infty}(\gamma) \rightarrow$ $C^{\infty}(H)$ where $Q_{\gamma}^{H}(t, s, h)$ is of the form

$$
(2\pi h)^{-2} \int_{\mathbb{R}^2} e^{i\langle q(t) - r(s), \xi \rangle / h} e^{-C|q(t) - r(s)|^2 \langle \xi \rangle / h} a(q(t), r(s), \xi; h) d\xi, \quad (q(t), r(s)) \in H \times \gamma, \tag{4.4}
$$

In [\(4.4\)](#page-15-0), a is an analytic semiclassical symbol of order zero and $C > 0$ is an appropriate constant. Moreover, the Schwartz kernel $Q(t, s, h)$ extends holomorphically in the outgoing t variable to $Q^{\mathbb{C}}(t,s)$ with $(t,s) \in H^{\mathbb{C}}_{\rho^*} \times \gamma$ where $\rho^* > 0$ is a suitable tube radius independent of h (see Remark [6](#page-16-1) (ii) and (iii) below). It then follows from a potential layer formula $[CT16]$ (14), that the eigenfunction restriction $u_h \circ q(t)$ also holomorphically continues to $(u_h \circ q)^{\mathbb{C}}(t)$ in the strip $H_{\rho^*}^{\mathbb{C}}$. Setting

$$
\mathbf{Q}(h,\rho^*):=\max_{(r(s),q^{\mathbb{C}}(t))\in\gamma\times H^{\mathbb{C}}_{\rho^*}}(|Q^{\mathbb{C}}(t,s)|,|\partial_{\nu(s)}Q^{\mathbb{C}}(t,s)||)
$$

and using the formula in (4.4) combined with some potential layer analysis, in [\[CT16\]](#page-21-6) (16) , it is proved that

$$
|F_h^{\mathbb{C}}(t)| \le C \mathbf{Q}(h,\rho^*) \left(\frac{\|u_h\|_{L^2(\gamma)}}{\|u_h\|_{L^2(H)}} + \frac{\|\partial_\nu u_h\|_{L^2(\gamma)}}{\|u_h\|_{L^2(H)}} \right), \quad t \in H_{\rho^*}^{\mathbb{C}}.
$$
 (4.5)

In [\[CT16\]](#page-21-6) Theorem 4, the authors show that by choosing $\rho^* > 0$ sufficiently small (see Remark [6](#page-16-1) (iii)) one can show that

$$
\mathbf{Q}(h,\rho^*) = O(1). \tag{4.6}
$$

Thus, from [\(4.5\)](#page-15-1) this gives (see Remark [6\)](#page-16-1)),

$$
|F_h^{\mathbb{C}}(t)| \le C \left(\frac{\|u_h\|_{L^2(\gamma)}}{\|u_h\|_{L^2(H)}} + \frac{\|\partial_\nu u_h\|_{L^2(\gamma)}}{\|u_h\|_{L^2(H)}} \right), \quad t \in H_{\rho^*}^{\mathbb{C}} \tag{4.7}
$$

It follows from the Agmon estimates in [\(1.4\)](#page-1-0) that for any $\varepsilon > 0$,

$$
\max\{\|u_h\|_{L^2(\gamma)},\|\partial_\nu u_h\|_{L^2(\gamma)}\}\leq C(\varepsilon)e^{[-d_E(\gamma)+\varepsilon]/h},\quad d_E(\gamma)=\min_{r\in\gamma}d_E(r)
$$

for all $h \in (0, h_0(\varepsilon))$. On the other hand, if H is admissible in sense of Definition [4,](#page-12-2) from Theorem [5,](#page-14-0) we have the lower bound

$$
||u_h||_{L^2(H)} \ge C(\varepsilon)e^{[-2\tau_0 d_E^H + d_E(H) - \tilde{\beta}(\varepsilon)]/h}, \quad d_E^H = \max_{q \in \Lambda_{E(H)}} d_E(q), \ \tilde{\beta}(\varepsilon) = o(1).
$$

Consequently, from [\(4.7\)](#page-15-2) we get that with $\tilde{\beta}(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$,

$$
|F_h^{\mathbb{C}}(t)| \le C(\varepsilon)e^{[\tilde{\beta}(\varepsilon) + \varepsilon]/h} \cdot e^{[2\tau_0 d_E^H - d_E(H) - d_E(\gamma)]/h}, \quad t \in H_{\rho^*}^{\mathbb{C}}.
$$
 (4.8)

Finally, taking log of both sides of [\(4.8\)](#page-16-2) gives for $h \in (0, h_0(\varepsilon))$,

$$
h \log |F_h^{\mathbb{C}}(t)| \le 2\tau_0 d_E^H - d_E(H) - d_E(\gamma) + \mu(\varepsilon), \tag{4.9}
$$

with $\mu(\varepsilon) := \tilde{\beta}(\varepsilon) + \varepsilon \to 0^+$ as $\varepsilon \to 0^+$. A combination of [\(4.2\)](#page-14-2) and [\(4.9\)](#page-16-3) then proves the following

THEOREM 6. Assume that dim $M = 2$ and (M, q, H, γ, V) are all real-analytic and suppose that H is an admissible hypersurface in sense of Definition μ . Then, under the control and monotonicity assumptions in Definitions [1](#page-1-2) and [2,](#page-2-0) for any fixed $\varepsilon > 0$, there are constants $\mu(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ such that for $h \in (0, h_0(\varepsilon))$,

$$
\#\{Z_{u_h} \cap H\} \le C_H(\rho^*) \cdot \left(2\tau_0 d_E^H - d_E(H) - d_E(\gamma) + \mu(\varepsilon)\right) h^{-1}.
$$

Here, $\mu(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$ and $\tau_0, d_E^H, d_E(H)$ are the constants defined in [\(3.7\)](#page-13-2). In addition, $\rho^* > 0$ is the tube radius and $C_H(\rho^*)$ is the corresponding Dirichlet Green's function constant defined in (4.1) .

Remark 6. (i) We note that in Theorem [6](#page-16-0) above, the constant on the RHS given by $C_H(\rho^*)\cdot \left(2\tau_0 d_E^H\!-\!d_E(H)\!-\!d_E(\gamma)\!+\!\mu(\varepsilon)\right)=C_H(\rho^*)\cdot \left(2(\tau_0 d_E^H\!-\!d_E(H))\!+\!d_E(H)\!-\!d_E(\gamma)\!+\!\mu(\varepsilon)\right)>$ 0. This follows since $\tau_0 d_E^H - d_E(H) \geq d_E^H - d_E(H) \geq 0$, $d_E(H) - d_E(\gamma) > 0$ and the Green's function constant $C_H(\rho^*) > 0$.

(ii) As we have pointed out above, tube radius $\rho^* > 0$ is determined directly from analytic continuation properties of the operator kernel $Q^H_\gamma(t,s,h)$ and, as such, depends on the analyticity properties of the data (M, g, V, H, γ) in a complicated way (see also (iii) below). However, ρ^* and consequently, the Green's function constant $C_H(\rho^*)$, is nevertheless universal; that is, it is determined independent of the eigenfunction sequence u_h . On the other hand, the part of the nodal bound in [\(4.2\)](#page-14-2) that depends on the u_h 's (i.e. $\max_{t \in H_{\rho*}^{\mathbb{C}}} \log |F_h^{\mathbb{C}}|$ $\binom{n}{h}(t)$) is bounded in (4.9) rather explicitly in terms of the Agmon distance function.

(iii) It follows from [\[CT16\]](#page-21-6) Theorem 4 that the bound in (4.6) can be improved to

$$
\mathbf{Q}(h, \rho^*) = O(e^{-C_1(\rho^*)/h})
$$
\n(4.10)

for some $C_1(\rho^*) > 0$. It follows from the holomorphic continuation properties of the phase in (4.4) that when $d_E(\gamma, H) = \inf_{(r,q)\in\gamma\times H} d_E(r,q)$ is sufficiently small, one can choose

$$
\rho^* = \frac{1}{C_2} d_E^2(\gamma, H)
$$

where $C_2 > 0$ is sufficiently large. For such a choice of ρ^* , the constant in [\(4.10\)](#page-17-1) can be taken to be

$$
C_1(\rho^*) = \frac{1}{C_3} d_E^2(\gamma, H),
$$

where $C_3 > 0$ is another constant.

However, since

$$
d_E(H) - d_E(\gamma) - \frac{1}{C_3} d_E^2(\gamma, H) \sim d_E(H) - d_E(\gamma) \quad \text{as } d_E(\gamma, H) \to 0^+,
$$

 (4.10) only gives a marginal improvement in Theorem [6.](#page-16-0) Moreover, both constants C_2 and C_3 above depend in a complicated way on the data (M, g, V, H) and are difficult to determine explicitly. Consequently, we decided to omit these considerations from the statement of Theorem [6.](#page-16-0)

5. Eigenfunction control condition: examples

5.1. Counterexample: lack of eigenfunction control. Here we show that without the control assumption in Definition [1,](#page-1-2) we can establish a Schrödinger model such that the corresponding eigenfunction decays much faster than $e^{-(1-\varepsilon)d_E/h}$ in $A(\delta,\delta')$ for δ' small enough. Such counterexample is essentially inspired by the paper ([\[CT16\]](#page-21-6)).

Consider a convex surface of revolution generated by rotating a curve $\gamma = \{(z, f(z)), z \in$ [-1, 1]} about the z-axis where $f \in C^{\infty}((-1,1), \mathbb{R}^+) \cap C^0([-1,1]; \mathbb{R}^+), f(1) = f(-1) = 0$ and both $f(z) > 0$ and $f''(z) < 0$ for all $z \in (-1, 1)$. Let M be the corresponding convex surface of revolution given by the parametrization

$$
\beta : [-1, 1] \times [0, 2\pi) \to \mathbb{R}^3,
$$

$$
\beta(z, \theta) = (f(z) \cos \theta, f(z) \sin \theta, z).
$$

In addition, we require here that

$$
f^2 \in C^{\infty}([-1, 1]; \mathbb{R})
$$
 and $\lim_{z \to \pm 1^{\pm}} (f^2)'(z) = 2 \lim_{z \to \pm 1^{\pm}} f(z)f'(z) \neq 0.$

The latter condition ensures that M is smooth near the poles. Indeed, since $f^2 \in$ $C^{\infty}([-1,1]),$ by the Whitney extension theorem [\[Wh\]](#page-21-11), it has a C^{∞} -extension in $(-1-\delta,1+\delta)$ for $\delta > 0$. We abuse notation slightly and denote the extended function also by f^2 . Then, since $x^2 + y^2 = f^2(z)$, by an application of the implicit function theorem, it follows that if $(f^2)'(\pm 1) = 2ff'(\pm 1) \neq 0$, there exist $F_{\pm} \in C^{\infty}_{loc}$ such that $z = F_{\pm}(x^2 + y^2)$ for $(x, y, z) \in M$

near the respective poles $(0, 0, \pm 1)$. In particular, M is then locally a smooth graph over the (x, y) -plane near the poles. Also, since $f^2(r) > 0$ for $r \in (-1, 1)$ and $f^2(\pm 1) = 0$, it follows that $\mp (f^2)'(\pm 1) > 0$.

As an example, in the case of the round sphere, $f(z) = \sqrt{1-z^2}$, $z \in [-1,1]$, so that $f^2(z) =$ $1-z^2$ and $\lim_{z\to\pm 1^{\mp}} f(z)f'(z) = \mp 1$. Written in terms of standard spherical coordinates, $z = \cos \varphi$ and $f(z) = \sin \varphi$, where $\varphi \in [0, \pi]$ is the polar angle with the z-axis.

Then, M inherits a Riemannian metric g given by

$$
g = w^2(z)dz^2 + f^2(z)d\theta^2,
$$

where $w(z) = \sqrt{1 + (f'(z))^2}$.

Let $E \in \mathbb{R}$ be a regular value of V and consider the Schrödinger equation on M given by

$$
(-h^2 \Delta_g + V)\varphi_h = E(h)\varphi_h,
$$

where $V \in C^{\infty}(M)$ and is axisymmetric, so that $V(z, \theta) = V(z)$. We also assume that $E(h) = E + o(1)$ and that $\{V = E\}$ is disjoint from the poles $(0, 0, \pm 1)$.

In the following, we will relabel the z-coordinate and write $r = z$ to indicate that this variable essentially measures distance into the forbidden region $\{V(r) > E\}$.

We seek eigenfunctions of the form $\varphi_h(r,\theta) = v_h(r)\psi_h(\theta)$. These are joint eigenfunctions of the quantum integrable system given by the commuting operators $H = -h^2 \Delta_g + V(r)$ and $P = hD_\theta$ [\[TZ03\]](#page-21-12). The Laplace operator in the coordinates (r, θ) has the following form

$$
\Delta_g = \frac{1}{w(r)f(r)} \frac{\partial}{\partial r} \left(\frac{f(r)}{w(r)} \frac{\partial}{\partial r} \right) + \frac{1}{f^2(r)} \frac{\partial^2}{\partial \theta^2}.
$$

It follows that $\psi_h(\theta)$ must satisfy the ODE

$$
-h_k^2 \frac{d^2}{d\theta^2} \psi_h(\theta) = h_k^2 m_{h_k}^2 \psi_h(\theta)
$$
\n(5.1)

Let $\{h_k\}$ be a decreasing sequence with $h_k \to 0^+$ as $k \to +\infty$ and $m_{h_k} = 1/h_k \in \mathbb{Z}$. Then, we choose a particular sequence of solutions of [\(5.1\)](#page-18-0) given by

$$
\psi_{h_k}(\theta) = e^{im_{h_k}\theta}.
$$

Then, $v_{h_k}(r)$ must satisfy

$$
- h_k^2 \frac{1}{f^2(r)} \left(\frac{f(r)}{w(r)} \frac{d}{dr} \right)^2 v_{h_k}(r) + \left[(V(r) - E) + \frac{1}{f^2(r)} \right] v_{h_k}(r) = 0, \quad r \in (-1, 1).
$$
 (5.2)

As for boundary conditions in (5.2) (recalling that $r = z$), the eigefunctions are of the form \cdots

$$
e^{im_{h_k}\theta}v_{h_k}(z) = (\cos\theta + i\sin\theta)^{m_{h_k}}v_{h_k}(z) = \frac{(x+iy)^{m_{h_k}}}{f(z)^{m_{h_k}}}v_{h_k}(z).
$$
(5.3)

Thus, in view of [\(5.3\)](#page-18-2), to ensure that the eigenfunctions $\varphi_{h_k}(z,\theta) = v_{h_k}(z)\psi_{h_k}(\theta)$ are smooth up to the poles $z = \pm 1$, the v_{h_k} in [\(5.2\)](#page-18-1) must be of the form

$$
v_{h_k}(z) = f(z)^{m_{h_k}} \cdot v_{h_k}^{reg}(z), \quad v_{h_k}^{reg} \in C^{\infty}([-1, 1]).
$$

Thus, since $m_{h_k} \neq 0$ here $(m_{h_k} \to \infty)$ and $f(\pm 1) = 0$, it follows that the v_{h_k} in [\(5.2\)](#page-18-1) satisfy the Dirichlet boundary conditions $v_{h_k}(\pm 1) = 0$.

We note that this is precisely what happens in the case of standard spherical harmonics, where $V = 0, E = 1, f(z) = \sqrt{1 - z^2}$ and the v_{h_k} are the associated Legendre polynomials.

Next, we make the radial change of variables $s \to r(s) = \int_0^s$ $f(\tau)$ $\frac{J(\tau)}{w(\tau)}d\tau$. In view of the assumption on profile function above (ie. $f^2 \in C^\infty[-1,1]$ and $\mp (f^2)'(\pm 1) > 0$), it follows that near $r = 1$, we have $f(r) \sim c_1(1 - r)^{1/2}$ and $f'(r) \sim -c_2(1 - r)^{-1/2}$ with $c_j > 0, j = 1, 2$. Thus, $\frac{dr}{ds} = \frac{f(r)}{\sqrt{1 + |f'|}}$ $\frac{f(r)}{1+|f'(r)|^2} \sim c_3(1-r)$ and so, $s(r) \sim -\log(1-r)$ as $r \to 1^-$. Consequently, for the inverse function $r \mapsto s(r)$, we have $s(1) = \infty$. Similar reasoning at $r = -1$ shows that $s(-1) = -\infty$.

Setting $\tilde{v}_{h_k}(s) := v_{h_k}(r(s))$, the equation [\(5.2\)](#page-18-1) becomes

$$
\left(-h_k^2 \frac{d^2}{ds^2} + f^2(r(s))(V(r(s)) - E(h)) + 1\right)\tilde{v}_{h_k}(s) = 0, \quad s \in \mathbb{R},\tag{5.4}
$$

where $\tilde{v}_{h_k}(\pm \infty) = 0$.

Consider the effective potential

$$
V_{eff}(r) := f^{2}(r)(V(r) - E) + 1,
$$

where $r \in [-1, 1]$.

We assume here that $V(r = 0) = E$. Then, since $V_{eff}(0) = 1$, it follows that there exists $\alpha > 0$ and a corresponding annulus $A(-\alpha, \alpha) = \{r; -\alpha < r < \alpha\}$ centered around ${r = 0} \subset {V = E}$ disjoint from the poles, so that

$$
V_{eff}(r) \ge \frac{1}{4}, \quad \forall r \in A(-\alpha, \alpha). \tag{5.5}
$$

Consider the annulus $A(-\varepsilon_0, \varepsilon_0) = \{r; -\varepsilon_0 < r < \varepsilon_0\}$ where $\varepsilon_0 \ll \alpha$. Then, in view of [\(5.5\)](#page-19-0), it then follows by the standard Agmon-Lithner estimate applied to [\(5.4\)](#page-19-1) that for any $\delta > 0$,

$$
\|e^{(1-\delta)\int_{r^{-1}(-\alpha)}^s \sqrt{V_{eff}(r(s))} ds/h_k} \tilde{v}_{h_k}(s)\|_{L^2(A(r^{-1}(-\varepsilon_0), r^{-1}(\varepsilon_0)))} = O_\delta(1).
$$

Making the change of variables $s \to r(s)$ then gives

$$
||e^{(1-\delta)\int_{-\alpha}^r \frac{\sqrt{V_{eff}(\tau)}}{\partial s\tau}d\tau/h_k} v_{h_k}(r)||_{L^2(A(-\varepsilon_0,\varepsilon_0))} = O_\delta(1),
$$
\n(5.6)

In view of [\(5.5\)](#page-19-0) and the fact that $0 < \partial_s r = \frac{f(r(s))}{w(r(s))} \leq C_1$ for all $r(s) \in A(-\alpha, \alpha)$ follows from [\(5.6\)](#page-19-2) that

$$
\|e^{\frac{(1-\delta)}{2C_1}(\alpha+r)/h_k}v_{h_k}(r)\|_{L^2(A(-\varepsilon_0,\varepsilon_0))}=O_{\delta}(1). \tag{5.7}
$$

But then, for $r \in A(-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small (i.e. $\varepsilon_0 \ll \alpha$), the inequality [\(5.7\)](#page-19-3) contradicts the control condition in Definition [1;](#page-1-2) indeed, the eigenfunctions already decay exponentially in h in the allowed region $A(-\varepsilon_0, 0)$.

We note that since $d_E(r) = O(|V(r) - E|^{3/2}) = O(\varepsilon_0^{3/2})$ $\binom{5}{0}$ for $r \in A(0, \varepsilon_0)$, it follows that for $\varepsilon_0 > 0$ sufficiently small, in the forbidden region where $r \in A(0, \varepsilon_0)$,

$$
\frac{1}{2C_1}(\alpha + r) > \tau_0 d_E(r).
$$

In this case, the exponential decay is therefore more pronounced than in Theorem [3.](#page-11-0) This is due to the presence of the effective potential term $m_k h_k \sim 1$ which in turn appears because of the particular choice of the sequence of Fourier modes in [\(5.1\)](#page-18-0). This is consistent with our results, since as we have already shown, the control condition is violated for this particular sequence of eigenfunctions.

5.2. Examples of eigenfunction sequences satisfying control. We consider precisely the same example of a Schrödinger operator on a convex surface of rotation as above but choose the quantum number $m = const. \neq 0$ so that $mh_k = O(h_k)$ as $h_k \to 0$. Then, the ODE in [\(5.4\)](#page-19-1) becomes

$$
\left(-h_k^2 \frac{d^2}{ds^2} + f^2(r(s))(V(r(s)) - E(h)) + O(h_k^2)\right) v_{h_k}(r(s)) = 0.
$$
\n(5.8)

The fact that the corresponding eigenfunctions $\varphi_h(r,\theta) = v_h(r)\psi_h(\theta)$ satisfy the control assumption is then an immediate consequence of standard WKB theory applied to the semi-classical ODE [\(5.8\)](#page-20-0). Indeed, writing $\Phi(r) := \int_{r_0}^r$ $f(r)$ $\frac{f(r)}{\partial s^r}(E-V(r))^{1/2} dr$, it follows by WKB asymptotics that for $r \in [-1, 1]$ satisfying $E - 2\varepsilon < V(r) < E - \varepsilon$,

$$
v_h(r) \sim_{h \to 0^+} e^{i\Phi(r)/h} c_1(h) a_1(r; h) + e^{-i\Phi(r)/h} c_2(h) a_2(r; h), \tag{5.9}
$$

where for $k = 1, 2, a_k(r; h) \sim \sum_{j=0}^{\infty} a_{k,j}(r) h^j$ and

$$
|c_1(h)|^2 + |c_2(h)|^2 \ge C_1 > 0, \quad |a_k(r;h)| \ge C_2(\varepsilon) > 0; \ k = 1, 2.
$$

Consequently, from [\(5.9\)](#page-20-1) we get that for any $\varepsilon > 0$,

$$
\int_{-2\varepsilon \le V(r)-E<-\varepsilon} \int_0^{2\pi} |\varphi_h(r,\theta)|^2 \, dr d\theta = \int_{-2\varepsilon \le V(r)-E<-\varepsilon} \int_0^{2\pi} |v_h(r)|^2 \, |e^{im\theta}|^2 \, dr d\theta
$$
\n
$$
= \int_{-2\varepsilon \le V(r)-E<-\varepsilon} \int_0^{2\pi} |v_h(r)|^2 \, dr d\theta \ge C(\varepsilon) > 0.
$$

In the last estimate, to control mixed terms, we have used that by an integration by parts,

$$
\int_{-2\varepsilon \langle V(r)-E\langle -\varepsilon \rangle} e^{\pm 2i\Phi(r)/h} a_1(r;h) a_2(r;h) dr = O_{\varepsilon}(h).
$$

As a result, this particular sequence clearly satisfies the control assumption in Definition [1](#page-1-2) with $N = 0$.

REFERENCES

- [CS81] R. Carmona and B. Simon. Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems. V. Lower bounds and path integrals. Comm. Math. Phys., 80(1):59–98, 1981.
- [CT16] Y. Canzani and J. A. Toth. Nodal sets of Schrödinger eigenfunctions in forbidden regions. Ann. Henri Poincaré, 17(11):3063-3087, 2016.
- [E] A. Erdélyi. Asymptotic expansions, Dover Books in Mathematics. Dover, New York, 1955.
- [Gr] A. Gray. Tubes, Progress in Mathematics (vol. 221). Birkh¨auser-Verlag, Basel-Boston-Berlin, 2004.
- [Hel88] B. Helffer. Semi-classical analysis for the Schrödinger operator and applications, volume 1336 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1988.
- [HHHZ] X. Han, A. Hassell, H. Hezari and S. Zelditch. Completeness of boundary traces of eigenfunctions. Proc. of London Math. Soc., 111(3): 749-773, 2015.
- [HS84] B. Helffer and J. Sjöstrand. Multiple wells in the semiclassical limit. I. Comm. Partial Differential Equations, 9(4):337–408, 1984.
- [Sim84] B. Simon. Semiclassical analysis of low lying eigenvalues. II. Tunneling. Ann. of Math. (2), 120(1):89- 18, 1984.
- [TZ09] J. A. Toth and S. Zelditch. Counting nodal lines which touch the boundary of an analytic domain. J. Differential Geom., 81(3):649–686, 2009.
- [TZ03] J. A. Toth and S. Zelditch. Norms of modes and quasimodes revisited. Contemporary Math. vol. 320 AMS, Providence, RI, 2003.
- [Wh] H. Whitney. Analytic extensions of functions defined in closed sets. Transactions of the AMS, American Mathematical Society, 36 (1): 63-89, 1934.
- [Wit82] E. Witten. Supersymmetry and Morse theory. J. Differential Geom., 17(4):661–692 (1983), 1982.
- [Zwo12] M. Zworski. Semiclassical analysis, volume 138 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012.

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