

# A NONLINEAR PROBLEM WITH A WEIGHT AND A NONVANISHING BOUNDARY DATUM

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ABSTRACT. We consider the problem:

$$\inf_{u \in H_q^1(\Omega), \|u\|_q=1} \int_{\Omega} p(x) |\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 4$ ,  $p : \bar{\Omega} \rightarrow \mathbb{R}$  is a given positive weight such that  $p \in H^1(\Omega) \cap C(\bar{\Omega})$ ,  $0 < c_1 \leq p(x) \leq c_2$ ,  $\lambda$  is a real constant and  $q = \frac{2n}{n-2}$  and  $g$  a given positive boundary data. The goal of this present paper is to show that minimizers do exist. We distinguish two cases, the first is solved by a convex argument while the second is not so straightforward and will be treated using the behavior of the weight near its minimum and the fact that the boundary datum is not zero.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  of class  $C^1$ ,  $n \geq 3$ . Let us consider the minimization problem

$$(1.1) \quad S_0(p, g) = \inf_{u \in H_q^1(\Omega), \|u\|_q=1} \int_{\Omega} p(x) |\nabla u(x)|^2 dx$$

where

$$H_g^1(\Omega) = \{u \in H^1(\Omega) \text{ s.t. } u = g \text{ on } \partial\Omega\},$$

$g \in H^{\frac{1}{2}}(\partial\Omega) \cap C(\partial\Omega)$  is a given boundary datum and  $q = \frac{2n}{n-2}$  is the critical Sobolev exponent.

Note that it is well known that  $H^1(\Omega) \hookrightarrow L^r(\Omega)$  is continuous for any  $1 \leq r \leq \frac{2n}{n-2}$ . Moreover this embedding is compact for  $1 \leq r < \frac{2n}{n-2}$ .

We suppose that the weight  $p : \bar{\Omega} \rightarrow \mathbb{R}$  is a smooth function such that  $0 < c_1 \leq p(x) \leq c_2 \forall x \in \bar{\Omega}$  and  $p$  is in  $H^1(\Omega) \cap C(\bar{\Omega})$ .

In this paper, we ask the question whenever the problem (1.1) has a minimizer. Note that if the infimum (1.1) is achieved by some  $u$  then we have

$$(1.2) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \Lambda u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

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where  $\Lambda \in \mathbb{R}$  is the Lagrange multiplier associated to the problem (1.1).

These kind of problems, which are known to bear features of noncompactness are studied by many authors. First existence results for the problem with a linear perturbation are due to Brezis-Nirenberg. Set

$$(1.3) \quad S_\lambda(p, g) = \inf_{u \in H_g^1(\Omega), \|u\|_q=1} \int_{\Omega} p(x) |\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx$$

They showed that if  $g = 0$  and  $p = 1$ , then  $S_\lambda(1, 0)$  is attained as soon as  $S_\lambda(1, 0) < S$  and this is the case if  $n \geq 4$ ,  $0 < \lambda < \lambda_1$ , or  $n = 3$  and  $0 < \lambda^* < \lambda < \lambda_1$  where  $\lambda_1$  is the first eigenvalue of  $-\Delta$  and  $\lambda^*$  depends on the domain, (see [6]). They showed also that if  $g \neq 0$ ,  $\lambda = 0$  and  $p = 1$  then the infimum in (1.1) is achieved, (see [7]). Our approach uses their method.

In the case of  $p = 1$  and  $g = 0$ , Coron, Bahri and Coron exploited the topology of the domain. They proved that equation

$$\begin{cases} -\Delta u = u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution provided that the domain has nontrivial topology, (see [8] and [3]).

We refer to [13], [14] for the study of existence and multiplicity solutions of problem (1.1) with the presence of a smooth and positive weight and with homogeneous Dirichlet boundary condition. Nevertheless, in [12], it is shown that if  $p$  is discontinuous then a solution of  $S_0(p, 0)$  still exists.

In [10], the authors studied the minimization problem on compact manifolds in the case  $\lambda = 0$  with many variants.

For more general weights, depending on  $x$  and on  $u$ , in a recent paper written with Vigneron, we showed that in the case of homogeneous Dirichlet boundary condition and in the presence of a linear perturbation the corresponding minimizing problem possesses a solution. The model of the weight is  $p(x, u) = \alpha + |x|^\beta |u|^k$  with positive parameters  $\alpha$ ,  $\beta$  and  $k$ . Note that in this case natural scalings appear and the answer depends on the ratio  $\frac{\beta}{k}$ . For more details, we refer to [2] and [15].

To motivate our problem, we briefly recall that it is inspired by the study of the classical Yamabe problem which has been the source of a large literature, (see for example [1], [3], [6], [8], [10] and [16]), we refer to [15] and the references therein for many recent developments in quasi-linear elliptic equations.

In this paper, we will assume that if  $g \neq 0$  having a constant sign and the weight  $p$  has a global minimum  $a \in \Omega$  such that satisfies:

$$(1.4) \quad p(x) \leq p_0 + \gamma |x - a|^\alpha \quad \forall x \in B(a, R) \subset \Omega,$$

for constants  $\alpha > 1$ ,  $\gamma > 0$  and  $R > 0$ .

The following auxiliary linear Dirichlet problem will play an important role in this paper:

$$(1.5) \quad \begin{cases} -\operatorname{div}(p\nabla v) = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

**1.1. Statement of the main result.** Our main result is the following:

**Theorem 1.1.** *Let us assume that the dimension  $n \geq 3$  and  $g \in H^{\frac{1}{2}}(\partial\Omega) \cap C(\partial\Omega)$  is a given boundary datum. Let  $v$  be the unique solution of (1.5). We have*

- (1) *Let  $\|v\|_q < 1$  and let assume that  $g \not\equiv 0$  and having a constant sign. Assume that  $p$  has a global minimum  $a \in \Omega$  that satisfies (1.4). Then for every  $n \in [3, 2\alpha + 2[$  the infimum  $S_0(p, g)$  is achieved in  $H_g^1(\Omega)$ .*
- (2) *If  $\|v\|_q \geq 1$  then for every  $n \geq 3$  the infimum  $S_0(p, g)$  is achieved in  $H_g^1(\Omega)$ .*

The next proposition tell us that one has  $\Sigma_g = \{u \in H_g^1(\Omega), \|u\|_q = 1\} \neq \emptyset$  which ensures that  $S_0(p, g)$  is well defined:

**Proposition 1.2.** *Let  $g \in H^{\frac{1}{2}}(\partial\Omega) \cap C(\partial\Omega)$  be given boundary datum and  $v$  be the unique solution of (1.5), we have*

- *If  $\|v\|_q < 1$ , then there is a bijection between  $\Sigma_0$  and  $\Sigma_g$ .*
- *If  $\|v\|_q \geq 1$ , then  $\Sigma_g \neq \emptyset$ .*

Our problem depends on  $\|v\|_q$ . More precisely, we will use a convex argument to show that if  $\|v\|_q \geq 1$  then the infimum (1.1) is achieved, while the case where  $\|v\|_q < 1$  is not so straightforward and will be treated using the behavior of  $p$  near its minimum and the fact that  $g$  has a constant sign. We will argue by contradiction, supposing that minimizing sequence converges weakly to some limit  $u$ . The fact that the boundary datum is not 0 will give us that  $u$  is not identically 0. Then, by using a suitable test functions, we will show equality (4.2) below which is due to term of order 0. After precise computations, we get strict inequality in (4.29) which is due to the next term in the same expansion, which is lead to a contradiction.

Since the nonlinearity of the problem is as stronger as  $n$  is low, it is rather surprising that the infimum is achieved for lower dimensions  $n \in [3, 2\alpha + 2[$ . Note that the presence of  $p$  is more significative if  $\alpha > 0$  is low. The compromise is that  $n \in [3, 2\alpha + 2[$ . Remark that if  $\alpha = 0$  then infimum of  $p = p_0 + \gamma$  is not  $p_0$ .

For general boundary data  $g$ , we do not have control over the normal derivative of a solution of (1.2) on the boundary of  $\Omega$  and then, standard Pohozaev identity cannot be used.

**1.2. Structure of the paper.** The paper is structured as follows: In section 2 we give the notations and some preliminary results.

In the next section, we state two results related to our main result namely, Theorem 3.1 which gives the sign of the Lagrange-multiplier associated to

minimizers of  $S_0(p, g)$  given by Theorem 1.1 and Theorem 3.2 which generalizes our main result in case of the presence of a linear perturbation.

In section 4, we will focus on the proof of Theorem 1.1, which is the main result of this paper, it will be proved by a contradiction argument that spans the whole of this section.

In section 5, we give the proof of Theorem 3.1.

The last section is dedicated to the problem of existence of minimizer in the presence of a linear perturbation and the proof of Theorem 3.2.

## 2. NOTATIONS AND PRELIMINARY RESULTS

Sobolev inequality says that there exist  $M > 0$  such that

$$\int_{\Omega} p(x) |\nabla \phi|^2 dx \geq M \left( \int_{\Omega} |\phi|^q dx \right)^{\frac{2}{q}} \quad \text{for all } \phi \in H_0^1(\Omega).$$

The best constant is defined by

$$S_0(p, 0) = \inf_{u \in H_0^1(\Omega), \|u\|_q = 1} \int_{\Omega} p(x) |\nabla u|^2 dx.$$

Set

$$S = S_0(1, 0) = \inf_{u \in H_0^1(\Omega), \|u\|_q = 1} \int_{\Omega} |\nabla u|^2 dx.$$

We know that when the domain is  $\mathbb{R}^n$ , the constant  $S_0(1, 0)$  is achieved by the functions:

$$U_{x_0, \varepsilon}(x) = \left( \frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}, \quad x \in \mathbb{R}^n$$

where  $x_0 \in \mathbb{R}^n$  and  $\varepsilon > 0$ , (see [1], [6], [16]). Let us denote by

$$(2.1) \quad u_{x_0, \varepsilon}(x) = U_{x_0, \varepsilon}(x) \psi(x)$$

where  $\psi \in C^\infty(\mathbb{R}^n)$ ,  $\psi \equiv 1$  in  $B(x_0, r)$   $\psi \equiv 0$  on  $B(x_0, 2r) \subset \Omega$ ,  $r > 0$ . We have

$$(2.2) \quad \int_{\Omega} p(x) |\nabla u_{x_0, \varepsilon}|^2 dx = p(x_0) K_1 + O(\varepsilon^{n-2}),$$

$$(2.3) \quad \int_{\Omega} |u_{x_0, \varepsilon}|^q dx = K_2 + O(\varepsilon^n),$$

where  $K_1$  and  $K_2$  are positive constants with  $\frac{K_1}{K_2^{\frac{q}{2}}} = S$ .

We have also

$$\begin{aligned} u_{x_0, \varepsilon} &\rightharpoonup 0 \quad \text{in } H_0^1(\Omega). \\ -\Delta U_{x_0, \varepsilon} &= c_n U_{x_0, \varepsilon}^{q-1} \quad \text{in } \mathbb{R}^n. \end{aligned}$$

It is well known that  $S$  is never achieved for bounded domain, (see [6]).

In the the presence of the weight  $p$  we have

**Proposition 2.1.** *Suppose that  $a \in \Omega$  be a global minimum of  $p$ . Set  $p_0 = p(a)$ . If  $g = 0$ , we have  $S_0(p, 0)$  is never achieved and*

$$S_0(p, 0) = p_0 S_0(1, 0) = p_0 S.$$

**Proof.** When  $g = 0$ , the functions  $\frac{u_{a,\varepsilon}}{\|u_{a,\varepsilon}\|_q}$  are admissible test functions for  $S_0(p, 0)$  and we have as  $\varepsilon \rightarrow 0$

$$\begin{aligned} p_0 S \leq S_0(p, 0) &\leq \int_{\Omega} p(x) \left| \nabla \frac{u_{a,\varepsilon}}{\|u_{a,\varepsilon}\|_q} \right|^2 dx \\ &= p_0 S + \int_{\Omega} (p(x) - p_0) \left| \nabla \frac{u_{a,\varepsilon}}{\|u_{a,\varepsilon}\|_q} \right|^2 dx + o(1) \\ &= p_0 S + o(1). \end{aligned}$$

Passing to the limit  $\varepsilon \rightarrow 0$  state that  $S_0(p, 0) = p_0 S$ .

This implies that  $S_0(p, 0)$  is not achieved. Indeed, let us suppose that  $S_0(p, 0)$  is achieved by some  $u$ . Using the fact that  $S$  is never achieved in bounded domains, we obtain

$$p_0 S < p_0 \int_{\Omega} |\nabla u|^2 dx \leq \int_{\Omega} p(x) |\nabla u|^2 dx = p_0 S.$$

This leads to a contradiction. ■

**2.1. The auxiliary Dirichlet problem.** The linear Dirichlet problem (1.5) has a unique solution which solves the following problem

$$(2.4) \quad \min_{v \in H_g^1(\Omega)} \int_{\Omega} p(x) |\nabla v(x)|^2 dx.$$

Let us give now the proof of Proposition 1.2: Recall that

$$\Sigma_g = \{u \in H_g^1(\Omega), \|u\|_q = 1\}$$

. In the first case we can construct a bijection between  $\Sigma_0$  and  $\Sigma_g$ . Indeed, let us define, for  $t$  in  $\mathbb{R}$  and  $u \in \Sigma_0$  the function

$$(2.5) \quad f(t) = \int_{\Omega} |tu + v|^q$$

since  $f$  is smooth,  $f''(t) = q(q-1) \int_{\Omega} |tu+v|^{q-2} u^2$ ,  $f(0) < 1$  and  $\lim_{t \rightarrow \infty} f(t) = \infty$ , using the intermediate value theorem and the convexity of  $f$ , we obtain, for every  $u$  in  $\Sigma_0$ , the existence of a unique  $t(u) > 0$  such that  $\|t(u)u + v\|_q = 1$ .

Let us denote by  $\varphi : \Sigma_0 \rightarrow \Sigma_g$  the function defined by  $\varphi(u) = t(u)u + v$ . Let  $u_1$  and  $u_2$  in  $\Sigma_0$  such that  $\varphi(u_1) = t(u_1)u_1 + v = \varphi(u_2) = t(u_2)u_2 + v$ , we have necessarily  $\|t(u_1)u_1\|_q = \|t(u_2)u_2\|_q$ , this implies that  $t(u_1) = t(u_2)$  and  $u_1 = u_2$ . Therefore we have that  $\varphi$  is one to one function. Let  $w \in \Sigma_g$ ,

$w \neq v$ , set  $u = \frac{w-v}{\|w-v\|_q}$ , we have  $t(u) = \|w-v\|_q$  and  $\varphi(\frac{w-v}{\|w-v\|_q}) = w$ . Thus,  $\varphi$  is a bijection.

Suppose  $\|v\|_q \geq 1$ , let  $\zeta \in C_c^\infty(\Omega)$  is such that  $\|v - \zeta v\|_q < 1$ . Observe that  $v - \zeta v = g$  on boundary. The same argument as above gives  $t > 0$  such that  $\|v - t\zeta v\|_q = 1$ .  $\blacksquare$

### 3. STATEMENT OF FURTHER RESULTS

**3.1. The sign of the Euler-Lagrange.** Let  $u$  be a minimizer for the problem (1.1), then, it satisfies the following Euler-Lagrange equation

$$(3.1) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \Lambda u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \\ \|u\|_q = 1, \end{cases}$$

where  $\Lambda \in \mathbb{R}$  is the Lagrange multiplier associated to the problem (1.1), let  $v$  be defined by (1.5). The sign of  $\Lambda$  is given by the following:

**Theorem 3.1.** *The sign of  $\Lambda$  is as the following: If  $\|v\|_q < 1$  then  $\Lambda > 0$ , if  $\|v\|_q > 1$  then  $\Lambda < 0$  and if  $\|v\|_q = 1$  then  $\Lambda = 0$ .*

**3.2. Presence of a linear perturbation.** Over the course of the proof of Theorem 1.1, one also reaps the following compactness result.

**Theorem 3.2.** *We assume that  $p, g$  and  $v$  satisfy the same conditions as in Theorem 1.1. Assume that  $\|v\|_q < 1$ . Let us denote by  $\lambda_1$  the first eigenvalue of the operator  $-\operatorname{div}(p\nabla \cdot)$  with homogeneous Dirichlet boundary condition. Then for  $\lambda < \lambda_1$  we have the infimum in  $S_\lambda(p, g)$  is achieved in the following cases:*

- (1)  $\lambda > 0$ ,  $\alpha > 2$  and  $n \geq 3$ ,
- (2)  $\lambda > 0$ ,  $\alpha \leq 2$  and  $n \in [3, 2\alpha + 2[$ .
- (3)  $\lambda < 0$ ,  $n = 3$  or  $4$  with  $\alpha > 1$  and  $n = 5$  with  $\alpha > \frac{3}{2}$ .

In the presence of a linear perturbation, we will highlight a competition between three quantities, the dimension  $n$ , the exponent  $\alpha$  in (1.4) and the term of the linear perturbation. As we will see and as in Theorem 1.1 the behavior of  $p$  near its minimum plays an important role. The exponent  $\alpha = 2$  is critical in the case  $\lambda \neq 0$ .

### 4. PROOF OF THEOREM 1.1.

Let us start by proving the first part of Theorem 1.1. Suppose that  $\|v\|_q < 1$ . Since the function  $u$  is a solution of  $S_0(p, -g)$  if and only if  $-u$  is solution of  $S_0(p, g)$ , it suffices to consider the case  $g \geq 0$ .

Let  $(u_j)$  be a minimizing sequence for  $S_0(p, g)$ , that is,

$$\int_{\Omega} p(x)|\nabla u_j(x)|^2 dx = S_0(p, g) + o(1)$$

and

$$\|u_j\|_q = 1, \quad u_j = g \quad \text{in } \partial\Omega.$$

Since  $g \geq 0$ , we may always assume that  $u_j \geq 0$ , indeed,  $(|u_j|)$  is also a minimizing sequence. Since  $(u_j)$  is bounded in  $H^1$  we may extract a subsequence still denoted by  $(u_j)$  such that  $(u_j)$  converges weakly in  $H^1$  to a function  $u \geq 0$  a.e.,  $(u_j)$  converges strongly to  $u$  in  $L^2(\Omega)$ , and  $(u_j)$  converges to  $u$  a.e. on  $\Omega$  with  $u = g$  on  $\partial\Omega$ .

Using a standard lower semicontinuity argument, we infer that  $\|u\|_q \leq 1$ . To show that our infimum is achieved it suffices to prove that  $\|u\|_q = 1$ . Arguing by contradiction, let us assume that

$$\|u\|_q < 1.$$

We will prove that this is not possible with the assistance of several lemmas. We start by giving the first-order term of the energy  $\int_{\Omega} p(x)|\nabla u(x)|^2 dx$ , next, we show that  $u$  satisfies some kind Euler-Lagrange equation and then it is smooth. Finally, we compute the second-order term and highlight a contradiction.

#### 4.1. The first-order term.

**Lemma 4.1.** *For every  $w \in H_g^1(\Omega)$  such that  $\|w\|_q < 1$ , we have*

$$(4.1) \quad S_0(p, g) - \int_{\Omega} p(x)|\nabla w(x)|^2 dx \leq p_0 S \left( 1 - \int_{\Omega} |w|^q \right)^{\frac{2}{q}},$$

*For the weak limit  $u$ , we have equality:*

$$(4.2) \quad S_0(p, g) - \int_{\Omega} p(x)|\nabla u(x)|^2 dx = p_0 S \left( 1 - \int_{\Omega} |u|^q \right)^{\frac{2}{q}}.$$

**Proof.** Let  $w \in H_g^1(\Omega)$  such that  $\|w\|_q < 1$ . Therefore we can find a constant  $c_{\varepsilon, a} > 0$  such that

$$\|w + c_{\varepsilon, a} u_{\varepsilon, a}\|_q = 1.$$

Using Brezis-Lieb Lemma (see [4]), we obtain

$$(4.3) \quad c_{\varepsilon, a}^q = \frac{1}{K_2} \left( 1 - \int_{\Omega} |w|^q \right) + o(1)$$

where  $K_2$  is defined in (2.3). Careful expansion as  $\varepsilon \rightarrow 0$  shows that (see [13]), for  $n \geq 4$

$$(4.4) \quad \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx \leq \begin{cases} p_0 K_1 + O(\varepsilon^{n-2}) & \text{if } \begin{cases} n \geq 4 \text{ and} \\ n - 2 < \alpha, \end{cases} \\ p_0 K_1 + A_1 \varepsilon^\alpha + o(\varepsilon^\alpha) & \text{if } \begin{cases} n \geq 4 \text{ and} \\ n - 2 > \alpha, \end{cases} \\ p_0 K_1 + A_2 \varepsilon^{n-2} |\log \varepsilon| + o(\varepsilon^{n-2} |\log \varepsilon|) & \text{if } \begin{cases} n \geq 4 \text{ and} \\ \alpha = n - 2, \end{cases} \end{cases}$$

with

$$K_1 = (n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy$$

and where  $A_1$ ,  $A_2$  and  $A_3$  are positive constants depending only on  $n$ ,  $\gamma$  and  $\alpha$ , and for  $n = 3$  and for  $\alpha > 1$  we have as  $\varepsilon \rightarrow 0$ ,

$$\int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx = p_0 K_1 + [\omega_3 \int_0^R (p_0 + \gamma r^\alpha) |\psi'(r)|^2 dr + \omega_3 k \alpha \int_0^R |\psi|^2 r^{\alpha-2} dr] \varepsilon + o(\varepsilon).$$

where  $\psi$  is defined as in (2.1). Therefore for  $n = 3$  and  $\alpha > 1$  we obtain

$$(4.5) \quad \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx = p_0 K_1 + A_4 \varepsilon + o(\varepsilon).$$

where  $A_4$  is a positive constant.

Remark that regardless of dimension  $n$  as long as  $n \geq 3$  and for  $\alpha > 1$  we have

$$(4.6) \quad \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx \leq p_0 K_1 + o(1).$$

Using  $w_\varepsilon = w + c_{\varepsilon,a} u_{\varepsilon,a}$  as testing function in  $S_0(p, g)$  we obtain

$$S_0(p, g) \leq \int_{\Omega} p(x) |\nabla w(x)|^2 dx + c_{\varepsilon,a}^2 \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 + o(1)$$

Using (4.3), the fact that  $\frac{K_1}{K_2^{\frac{q}{2}}} = S$  and taking into account (4.6) we get the first assertion of the Lemma 4.1.

For the second part, thanks to (4.1), it suffices to prove one inequality for  $u$ .

$$(4.7) \quad S(p, g) - \int_{\Omega} p(x) |\nabla u(x)|^2 dx \geq p_0 S \left( 1 - \int_{\Omega} |u|^q \right)^{\frac{2}{q}}.$$

Set  $v_j = u_j - u$  so that  $v_j = 0$  in  $\partial\Omega$  and  $(v_j)$  converges weakly to 0 in  $H_0^1$  and *a.e.* We have by Sobolev inequality

$$(4.8) \quad \int_{\Omega} p(x)|\nabla v_j|^2 \geq p_0 S \|v_j\|_q^2.$$

On the other hand, we have (see [4])

$$(4.9) \quad 1 = \int_{\Omega} |v_j|^q + \int_{\Omega} |u|^q + o(1).$$

Since  $(u_j)$  is a minimizing sequence we have

$$(4.10) \quad S_0(p, g) = \int_{\Omega} p(x)|\nabla v_j|^2 + \int_{\Omega} p(x)|\nabla u|^2 + o(1),$$

hence, combining (4.8), (4.9) and (4.10) we obtain the desired conclusion. ■

We will now use the fact that  $g$  is not identically zero. A consequence of the above lemma is the following:

**Lemma 4.2.** *The function  $u$  satisfies*

$$(4.11) \quad \begin{cases} -\operatorname{div}(p\nabla u) &= p_0 S \left(1 - \int_{\Omega} |u|^q\right)^{\frac{2-q}{q}} |u|^{q-2} u & \text{in } \Omega \\ u &= g & \text{on } \partial\Omega \end{cases}$$

Moreover,  $u$  is smooth,  $u \in L^\infty(\Omega)$  and  $u > 0$  in  $\Omega$ .

**Proof.** Applying (4.1) to  $w = u + t\varphi$ ,  $\varphi \in C_0^\infty(\Omega)$  and  $|t|$  small enough, we have

$$\begin{aligned} S_0(p, g) &\leq \int_{\Omega} p(x)|\nabla u|^2 - 2t \int_{\Omega} p(x)\nabla u\nabla\varphi + o(t) + \\ &\quad p_0 S \left(1 - \int_{\Omega} |u|^q - qt \int_{\Omega} |u|^{q-2} u\varphi + o(t)\right)^{\frac{2}{q}}, \end{aligned}$$

thus

$$\begin{aligned} S_0(p, g) &\leq \int_{\Omega} p(x)|\nabla u|^2 - 2t \int_{\Omega} p(x)\nabla u\nabla\varphi + \\ &\quad p_0 S \left(1 - \int_{\Omega} |u|^q\right)^{\frac{2}{q}} \left(1 - 2t \frac{\int_{\Omega} |u|^{q-2} u\varphi}{1 - \int_{\Omega} |u|^q} + o(t)\right). \end{aligned}$$

Hence, by using (4.7) we obtain for every  $\varphi \in C_0^\infty(\Omega)$

$$(4.12) \quad - \int_{\Omega} p(x)\nabla u\nabla\varphi - \left(1 - \int_{\Omega} |u|^q\right)^{\frac{2-q}{q}} \int_{\Omega} |u|^{q-2} u\varphi = 0.$$

Since  $u = g$  on  $\partial\Omega$  we obtain (4.11).

For proving the regularity of  $u$ , it suffices, in view of the standard elliptic regularity theory to show that  $u$  is in  $L^t(\Omega)$  for all  $t < \infty$ . To see this, we shall apply Lemma A1 of [5], then,  $u$  is as smooth as the regularity of  $p$  and  $g$  permits.

By using the strong maximum principle, and the fact that  $g \geq 0$ ,  $g \not\equiv 0$  we get

$$(4.13) \quad u > 0 \quad \text{in} \quad \Omega.$$

■

**4.2. The second-order term.** Now, we need a refined version of (4.1). Similarly as in the proof of (4.1), let  $c_{\varepsilon,a}$  be defined by  $1 = \int_{\Omega} |u + c_{\varepsilon,a}u_{\varepsilon,a}|^q$ . We can write

$$(4.14) \quad c_{\varepsilon,a} = c_0(1 - \delta(\varepsilon))$$

with

$$(4.15) \quad c_0^q = \frac{1}{K_2} \left( 1 - \int_{\Omega} |u|^q \right) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0.$$

**Lemma 4.3.** *We have*

$$(4.16) \quad \delta(\varepsilon)K_2c_0^q \geq p_0\varepsilon^{\frac{n-2}{2}} \left( c_0 \int_{\Omega} u^{q-1} \frac{\psi}{|x-a|^{n-2}} c_0^q (q-1) Du(a) \right) \\ + \frac{q-1}{2} c_0^q K_2 \delta^2(\varepsilon) + o(\delta^2(\varepsilon)) + o(\varepsilon^{\frac{n-2}{2}}).$$

where  $D$  is a positive constant.

**Proof.**

First case  $q \geq 3$ . We need the following inequality, for all  $a \geq 0$  and  $b \geq 0$  we have

$$(4.17) \quad (a+b)^q \geq a^q + qa^{q-1}b + qab^{q-1} + b^q$$

which follows from

$$\frac{t^q + qt^{q-1} + qt + 1}{(1+t)^q} \leq 1$$

for  $t$  such that  $t = \frac{b}{a}$  if  $a \neq 0$ .

Using (4.17) and the fact that  $u > 0$  we get

$$1 = \int_{\Omega} |u + c_{\varepsilon,a}u_{\varepsilon,a}|^q \\ \geq \int_{\Omega} u^q + qc_{\varepsilon,a}^{q-1} \int_{\Omega} uu_{\varepsilon,a}^{q-1} + qc_{\varepsilon,a} \int_{\Omega} u^{q-1}u_{\varepsilon,a} + c_{\varepsilon,a}^q \int_{\Omega} u_{\varepsilon,a}^q.$$

and thus

$$(4.18) \quad 1 \geq \int_{\Omega} u^q + qc_{\varepsilon,a}^{q-1} \int_{\Omega} uu_{\varepsilon,a}^{q-1} + qc_0(1 - \delta(\varepsilon)) \int_{\Omega} u^{q-1}u_{\varepsilon,a} \\ + qc_0^q \left( 1 - q\delta(\varepsilon) + \frac{q(q-1)}{2} \delta^2(\varepsilon) + o(\delta^2(\varepsilon)) \right) \int_{\Omega} u_{\varepsilon,a}^q.$$

On the other hand we have

$$(4.19) \quad \int_{\Omega} uu_{\varepsilon,a}^{q-1} = \varepsilon^{\frac{n-2}{2}} Du(a) + o(\varepsilon^{\frac{n-2}{2}})$$

where  $D$  is a positive constant, and

$$(4.20) \quad \int_{\Omega} u^{q-1}u_{\varepsilon,a} = \varepsilon^{\frac{n-2}{2}} \int_{\Omega} u^{q-1} \frac{\psi}{|x-a|^{n-2}} + o(\varepsilon^{\frac{n-2}{2}}).$$

Combining (2.3), (4.18), (4.19) and (4.20) we obtain (4.16).

Second case  $2 < q < 3$ . In what follows  $C$  denote a positive constant independent of  $\varepsilon$ . The keys are the two following inequalities, we have for all  $a \geq 0$  and  $b \geq 0$

$$(4.21) \quad |(a+b)^q - (a^q + qa^{q-1}b + qab^{q-1} + b^q)| \leq Ca^{q-1}b \quad \text{if } a \leq b$$

and

$$(4.22) \quad |(a+b)^q - (a^q + qa^{q-1}b + qab^{q-1} + b^q)| \leq Cab^{q-1} \quad \text{if } a \geq b$$

which follows respectively from

$$(4.23) \quad \frac{|(1+t)^q - (t^q + qt^{q-1} + qt + 1)|}{t} \leq C$$

for  $t \geq 1$  and

$$(4.24) \quad \frac{|(1+t)^q - (t^q + qt^{q-1} + qt + 1)|}{t^{q-1}} \leq C$$

for  $t \leq 1$  for  $t$  such that  $t = \frac{b}{a}$  if  $a \neq 0$ .

Using (4.21) and (4.22) we get

$$(4.25) \quad \begin{aligned} 1 &= \int_{\Omega} |u + c_{\varepsilon,a}u_{\varepsilon,a}|^q \\ &= \int_{\Omega} u^q + qc_{\varepsilon,a}^{q-1} \int_{\Omega} uu_{\varepsilon,a}^{q-1} + qc_{\varepsilon,a} \int_{\Omega} u^{q-1}u_{\varepsilon,a} + c_{\varepsilon,a}^q \int_{\Omega} u_{\varepsilon,a}^q \\ &+ R_{\varepsilon}^{(1)} + R_{\varepsilon}^{(2)}. \end{aligned}$$

where

$$R_{\varepsilon}^{(1)} \leq C \int_{\{x, u \geq c_{\varepsilon,a}\psi U_{a,\varepsilon}\}} u |\psi U_{a,\varepsilon}|^{q-1}$$

and

$$R_{\varepsilon}^{(2)} \leq C \int_{\{x, u < c_{\varepsilon,a}\psi U_{a,\varepsilon}\}} u^{q-1} \psi U_{a,\varepsilon}.$$

We claim that the remainders terms  $R_\varepsilon^{(1)}$  and  $R_\varepsilon^{(2)}$  verify

$$(4.26) \quad R_\varepsilon^{(1)} = o(\varepsilon^{\frac{n-2}{2}}) \quad \text{and} \quad R_\varepsilon^{(2)} = o(\varepsilon^{\frac{n-2}{2}}).$$

Let us justify the first assertion in (4.26). In the set  $\Omega \setminus B(a, r)$  we have  $U_{a,\varepsilon}^{q-1} \leq C\varepsilon^{\frac{n+2}{2}}$  and in the set  $B(a, r) \cap \{x, u \geq c_{\varepsilon,a}\psi U_{a,\varepsilon}\}$  we have  $U_{a,\varepsilon} \leq C$  and then necessarily  $|x - a| \geq C\varepsilon^{\frac{1}{2}}$ , therefore

$$(4.27) \quad R_\varepsilon^{(1)} \leq C \int_{\{x, C\varepsilon^{\frac{1}{2}} < |x-a| \leq r\}} \left( \frac{\varepsilon}{\varepsilon^2 + |x-a|^2} \right)^{\frac{n+2}{2}} dx = o(\varepsilon^{\frac{n-2}{2}}).$$

Let us verify that  $R_\varepsilon^{(2)} = o(\varepsilon^{\frac{n-2}{2}})$ . In the set  $A_{a,\varepsilon} = \{x, u < c_{\varepsilon,a}\psi U_{a,\varepsilon}\}$  we

have  $\psi > 0$  and consequently, since  $u$  is smooth, there exists  $\delta > 0$  such that  $u > \delta$  in  $A_{a,\varepsilon}$  thus  $U_{a,\varepsilon} \geq C$ . This implies that  $|x - a| \leq C\varepsilon^{\frac{1}{2}}$ . We have

$$(4.28) \quad R_\varepsilon^{(2)} \leq C \int_{\{x, |x-a| \leq C\varepsilon^{\frac{1}{2}}\}} \left( \frac{\varepsilon}{\varepsilon^2 + |x-a|^2} \right)^{\frac{n-2}{2}} dx = o(\varepsilon^{\frac{n-2}{2}}).$$

Combining (4.25), (4.19), (4.20), and (4.26) we obtain that  $\delta(\varepsilon) = O(\varepsilon^{\frac{n-2}{2}})$  and (4.16).  $\blacksquare$

We are able to prove now:

**Lemma 4.4.** *If  $n \geq 3$  and  $\alpha > 1$  then we have for every  $3 \leq n < 2\alpha + 2$  we have*

$$(4.29) \quad S_0(p, g) - \int_{\Omega} p(x) |\nabla u(x)|^2 dx < p_0 S \left( 1 - \int_{\Omega} |u|^q \right)^{\frac{2}{q}}.$$

Let us postpone the proof of Lemma 4.4 and complete the first part of the proof of Theorem 1.1. Combining (4.29) and (4.2) this leads to a contradiction and then we obtain that  $\|u\|_q = 1$  and therefore the infimum  $S_0(p, g)$  is achieved.

**Proof.** of the first part of Theorem 1.1. Let us chose  $w_\varepsilon = u + c_{\varepsilon,a}u_{\varepsilon,a}$  as testing function in  $S_0(p, g)$ , we obtain

$$(4.30) \quad S_0(p, g) \leq \int_{\Omega} p |\nabla(u + c_{\varepsilon,a}u_{a,\varepsilon})|^2.$$

By (4.30) and (4.14) it is easy to see

$$\begin{aligned} S_0(p, g) &\leq \int_{\Omega} p |\nabla u|^2 - 2c_0 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} (\operatorname{div}(p \nabla u)) \frac{\psi}{|x-a|^{n-2}} \\ &\quad + c_0^2 (1 - 2\delta(\varepsilon) + \delta^2(\varepsilon)) \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx + o(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Now, using (4.16) and the fact that  $\delta(\varepsilon) = o(1)$  we infer

$$(4.31) \quad S_0(p, g) \leq \int_{\Omega} p |\nabla u|^2 + p_0 K_1 c_0^2 - 2c_0 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \operatorname{div}(p \nabla u) \frac{\psi}{|x-a|^{n-2}} \\ - 2c_0^2 \left[ \frac{\varepsilon^{\frac{n-2}{2}}}{K_2 c_0^q} \left( c_0 \int_{\Omega} u^{q-1} \frac{\psi}{|x-a|^{n-2}} + c_0^{q-1} D u(a) \right) + \frac{q-1}{2} \delta^2(\varepsilon) + o(\delta^2(\varepsilon)) \right] \\ \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx + c_0^2 \delta^2(\varepsilon) \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx + o(\varepsilon^{\frac{n-2}{2}}).$$

Since  $\int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx = K_2 + o(1)$  we obtain

$$S_0(p, g) \leq \int_{\Omega} p |\nabla u|^2 + c_0^2 \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx - (q-2) c_0^2 \delta^2(\varepsilon) + o(\delta^2(\varepsilon)) - \\ 2c_0 \left[ \int_{\Omega} \operatorname{div}(p \nabla u) \frac{\psi}{|x-a|^{n-2}} + \left( \frac{c_0^{2-q}}{K_2} \int_{\Omega} u^{q-1} \frac{\psi}{|x-a|^{n-2}} + \frac{D}{K_2} u(a) \right) (K_2 + o(1)) \right] \varepsilon^{\frac{n-2}{2}} \\ + o(\varepsilon^{\frac{n-2}{2}}).$$

This leads to

$$S_0(p, g) \leq \int_{\Omega} p |\nabla u|^2 + c_0^2 \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx - (q-2) c_0^2 K_2 \delta^2(\varepsilon) + o(\delta^2(\varepsilon)) \\ - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}).$$

We know that  $\delta^2(\varepsilon) = o(1)$  thus

$$(4.32) \quad S_0(p, g) \leq \int_{\Omega} p |\nabla u|^2 + c_0^2 \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx \\ - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}).$$

We are now able to give a precise asymptotic behavior of the RHS of (4.30). This will be possible thanks to the fact that  $u(a) \neq 0$ , namely  $u(a) > 0$ . One needs to distinguish between dimensions and the parameter  $\alpha$ . Four cases follow from (4.4) and (4.32):

- The case when  $n \geq 4$  and  $n < \alpha + 2$ . We have

$$S_0(p, g) \leq \int_{\Omega} p |\nabla u|^2 + c_0^2 (p_0 K_1 + o(\varepsilon^{n-2})) - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}).$$

Consequently, we have

$$(4.33) \quad S_0(p, g) \leq \int_{\Omega} p |\nabla u|^2 + p_0 c_0^2 K_1 - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}).$$

- The case when  $n \geq 4$  and  $n > \alpha + 2$ . We have

$$\begin{aligned} S_0(p, g) &\leq \int_{\Omega} p|\nabla u|^2 + c_0^2 (p_0 K_1 + A_2 \varepsilon^\alpha + o(\varepsilon^\alpha)) \\ &\quad - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Therefore, we have

$$S_0(p, g) \leq \int_{\Omega} p|\nabla u|^2 + p_0 c_0^2 K_1 - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + A_2 c_0^2 \varepsilon^\alpha + o(\varepsilon^\alpha) + o(\varepsilon^{\frac{n-2}{2}}).$$

Hence, if  $n < 2\alpha + 2$  then

$$(4.34) \quad S_0(p, g) \leq \int_{\Omega} p|\nabla u|^2 - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}).$$

- The case when  $n \geq 4$  and  $\alpha = n - 2$ . We have

$$\begin{aligned} S_0(p, g) &\leq \int_{\Omega} p|\nabla u|^2 + c_0^2 (p_0 K_1 + A_2 \varepsilon^{n-2} |\log \varepsilon| + o(\varepsilon^{n-2} |\log \varepsilon|)) \\ &\quad - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

thus we get

$$(4.35) \quad S_0(p, g) \leq \int_{\Omega} p|\nabla u|^2 + p_0 c_0^2 K_1 - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}).$$

- 4 The case when  $n = 3$  and  $\alpha > 1$ . We have

$$S_0(p, g) \leq \int_{\Omega} p|\nabla u|^2 + c_0^2 [p_0 K_1 + A_4 \varepsilon + o(\varepsilon)] - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}}).$$

Hence we have

$$(4.36) \quad S_0(p, g) \leq \int_{\Omega} p|\nabla u|^2 + p_0 c_0^2 K_1 - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}}).$$

Now, thanks to (4.33), (4.34), (4.35), (4.36) and the fact that  $u(a) > 0$  we obtain the estimates in Lemma 4.4.  $\blacksquare$

**4.3. The case  $\|v\|_q \geq 1$ .** For the proof of the second part of Theorem 1.1 we set

$$\alpha := \inf_{u \in H_g^1(\Omega), \|u\|_q = 1} \int_{\Omega} p(x) |\nabla u(x)|^2 dx$$

and

$$\beta := \inf_{u \in H_g^1(\Omega), \|u\|_q \leq 1} \int_{\Omega} p(x) |\nabla u(x)|^2 dx.$$

Indeed using the convexity of the problem  $\beta$ , it is clear that the infimum in  $\beta$  is achieved by some function  $w \in H_g^1(\Omega)$  satisfying  $\|w\|_q \leq 1$ . Necessarily we have equality. Let us reason by contradiction, if we had  $\|w\|_q < 1$ , let

$\zeta \in_c^\infty(\Omega)$ , for  $t$  real and small such that we have  $\|w + t\zeta\|_q < 1$ , using  $w + t\zeta$  as test function in  $\beta$  we obtain that  $w$  would be the unique solution of the following Euler-Lagrange equation:

$$(4.37) \quad \begin{cases} -\operatorname{div}(p\nabla w) = 0 & \text{in } \Omega, \\ w = g & \text{on } \partial\Omega. \end{cases}$$

that is mean  $w$  and  $v$  coincide, this leads to a contradiction since  $\|v\|_q \geq 1$ . Therefore  $\alpha$  is achieved.

Since  $\|w\|_q = 1$  we have  $\int_\Omega p(x)|\nabla w(x)|^2 dx = \beta \leq \alpha \leq \int_\Omega p(x)|\nabla w(x)|^2 dx$ . Thus  $\alpha = \beta$ .  $\blacksquare$

### 5. THE SIGN OF THE EULER-LANGANGE MULTIPLIER. PROOF OF THEOREM 3.1

We follow an idea of [11]. Let  $u$  be a minimizer for the problem (1.1) and  $v$  be defined by (1.5), using the fact that problem (1.5) has a unique solution which minimizes (2.4), we remark that we have  $\|v\|_q \neq 1$  if and only if we have  $\Lambda \neq 0$ .

Using (3.1) and (1.5) we obtain

$$(5.1) \quad \begin{cases} -\operatorname{div}(p(x)\nabla(u-v)) = \Lambda u^{q-1} & \text{in } \Omega, \\ u-v = 0 & \text{on } \partial\Omega. \end{cases}$$

First, suppose that  $\|v\|_q < 1$ . Multiplying (5.1) by  $u-v$  and integrating we obtain

$$(5.2) \quad \Lambda(\|u\|_q^q - \int_\Omega |u|^{q-1}v) = \int_\Omega p(x)|\nabla(u-v)|^2.$$

From Hölder inequality and the fact that  $\|u\|_q = 1$  we obtain

$$(5.3) \quad \|u\|_q^q - \int_\Omega |u|^{q-1}v \geq 1 - \|v\|_q > 0.$$

Putting together (5.2) and (5.3) and using the fact that  $u \neq v$  we see that  $\Lambda > 0$ . Suppose now that  $\|v\|_q > 1$ . For  $t \in \mathbb{R}$ , let us define the function  $f$  by

$$f(t) = \int_\Omega |tu + (1-t)v|^q dx.$$

Note that the function  $f$  is smooth and convex since  $f''(t) = q(q-2) \int_\Omega tu + (1-t)v|^{q-1}(u-v)^2 \geq 0$  and we have

$$(5.4) \quad f(0) = \|v\|_q^q > 1 \quad \text{and} \quad f(1) = \|u\|_q^q = 1.$$

We may use the following:

**Lemma 5.1.** *For all  $t \in [0, 1[$  we have  $f(t) > 1$ .*

**Proof.** Arguing by contradiction, since  $f$  is continuous, by the intermediate value theorem there exists  $t_0 \in [0, 1[$  such that  $f(t_0) = 1$ . Using  $t_0u + (1 - t_0)v \in \Sigma_g$  as testing function in  $S_0(p, g)$  we have

$$(5.5) \quad S_0(p, g) = \int_{\Omega} p(x)|\nabla u|^2 \leq \int_{\Omega} p(x)|\nabla(t_0u + (1 - t_0)v)|^2$$

Multiplying (1.5) by  $u - v$  and integrating we obtain

$$(5.6) \quad \int_{\Omega} p|\nabla v|^2 = \int_{\Omega} p\nabla u \nabla v$$

Using (5.5), (5.6) and the fact that  $t_0 < 1$  we obtain

$$\int_{\Omega} p|\nabla u|^2 \leq \int_{\Omega} p|\nabla v|^2$$

Since  $v$  is the unique solution of (1.5) we obtain that  $u = v$  which clearly contradicts (5.4). This complete the proof of Lemma 5.1.  $\blacksquare$

By the convexity of  $f$  and Lemma 5.1 we deduce that  $f'(1) \leq 0$ . But  $f'(1) = q \int_{\Omega} |u|^{q-1}(u - v)$  and then by (5.1) we have  $f'(1) = \frac{q}{\Lambda} \int_{\Omega} p(x)|\nabla(u - v)|^2$ . We conclude that  $\Lambda < 0$ .  $\blacksquare$

## 6. EXISTENCE OF MINIMIZER IN THE PRESENCE OF A LINEAR PERTURBATION: PROOF OF THEOREM 3.2

First, we claim that if problem (1.3) has a solution then  $\lambda < \lambda_1$ . Indeed, let  $u$  be a solution of (1.1) and  $v$  satisfying (1.5), we have

$$(6.1) \quad \begin{cases} -\operatorname{div}(p(x)\nabla(u - v)) = \Lambda(\lambda, u)u^{q-1} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u - v = 0 & \text{on } \partial\Omega. \end{cases}$$

where  $\Lambda(\lambda, u)$  is a Euler-Lagrange multiplier. Since  $\|v\|_q < 1$ , using section 5, we find that  $\Lambda(\lambda, u) > 0$ . Let  $\varphi_1$  be the eigenfunction of the operator  $-\operatorname{div}(p\nabla \cdot)$  with homogeneous Dirichlet boundary condition corresponding to  $\lambda_1$ . Multiplying (6.1) by  $\varphi_1$  and integrating we obtain

$$\begin{aligned} - \int_{\Omega} \operatorname{div}(p(x)\nabla(u - v))\varphi_1 &= \lambda_1 \int_{\Omega} (u - v)\varphi_1 \\ &= \Lambda(\lambda, u) \int_{\Omega} u^{q-1}\varphi_1 + \lambda \int_{\Omega} u\varphi_1. \end{aligned}$$

Then we get

$$(\lambda_1 - \lambda) \int_{\Omega} (u - v)\varphi_1 \geq \lambda_1 \int_{\Omega} v\varphi_1$$

and thus  $\lambda < \lambda_1$ .

The proof of Theorem 1.1 is similar to the one of Theorem 1.1 so that we briefly outline it. We need only to take into account the linear perturbation

term. We will then follow exactly all the steps in the proof of Theorem 1.1 until (4.32), we just need to account the linear perturbation. We get

$$(6.2) \quad S_\lambda(p, g) \leq \int_\Omega p |\nabla u|^2 + c_0^2 \left( \int_\Omega p(x) |\nabla u_{a,\varepsilon}|^2 dx - \lambda \int_\Omega |u_{a,\varepsilon}|^2 dx \right) - 2c_0 \frac{DK_1}{K_2} u(a) \varepsilon^{\frac{n-2}{2}} + o(\varepsilon^{\frac{n-2}{2}}).$$

From [6] we have

$$(6.3) \quad \|u_{a,\varepsilon}\|_2^2 = \begin{cases} K_3 \varepsilon^2 + O(\varepsilon^{n-2}) & \text{if } n \geq 5, \\ C_1 \varepsilon^2 |\log \varepsilon| + O(\varepsilon^2) & \text{if } n = 4, \\ C_2 \varepsilon + O(\varepsilon^2) & \text{if } n = 3 \end{cases}$$

where  $C_1$ ,  $C_2$  and  $C_3$  are positive constants. Using (4.4), (6.2) and (6.3) and the fact that  $u(a) > 0$  we conclude the proof of Theorem 3.2. ■

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