

# MOTION OF LEVEL SETS BY INVERSE ANISOTROPIC MEAN CURVATURE

FRANCESCO DELLA PIETRA, NUNZIA GAVITONE, AND CHAO XIA

**Abstract.** In this paper we consider the weak formulation of the inverse anisotropic mean curvature flow, in the spirit of Huisken-Ilmanen [8]. By using approximation method involving Finsler- $p$ -Laplacian, we prove the existence and uniqueness of weak solutions.

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## 1. INTRODUCTION

Let  $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$  be a Minkowski norm in  $\mathbb{R}^n$ , i.e.,

- (i)  $F$  is a norm in  $\mathbb{R}^n$ , i.e.,  $F$  is a convex, even, 1-homogeneous function satisfying  $F(\xi) > 0$  when  $\xi \neq 0$ ;
- (ii)  $F$  satisfies a uniformly elliptic condition:  $D^2(\frac{1}{2}F^2)$  is positive definite in  $\mathbb{R}^n \setminus \{0\}$ .

Let  $X(\cdot, t): M \times [0, T) \rightarrow \mathbb{R}^n$  be a family of smooth embeddings from a closed manifold  $M$  in  $\mathbb{R}^n$  satisfying the evolution equation

$$\frac{\partial}{\partial t} X(x, t) = \frac{1}{H_F(x, t)} \nu_F(x, t), \quad (1)$$

where  $H_F(x, t) > 0$  is the anisotropic mean curvature function of the hypersurface  $N_t = X(M, t)$  and  $\nu_F(x, t)$  is the unit anisotropic outer normal.

When  $F$  is the Euclidean norm,  $\nu_F$  and  $H_F$  reduce to the unit outer normal and the mean curvature respectively, and in turn (1) reduces to the classical inverse mean curvature flow (IMCF). When  $F$  is a general Minkowski norm, (1) is so-called inverse anisotropic mean curvature flow (IAMCF).

Gerhardt [6] and Urbas [15] proved that the classical IMCF which initiated from a star-shaped and strictly mean convex hypersurface exists for all time and converge to a round sphere after rescaling. For general initial data, the IMCF may develop singularity. Huisken-Ilmanen [8] has developed a theory of weak solutions for the IMCF of hypersurfaces in Riemannian manifolds by its level-set formulation and applied it to show the validity of the Riemannian Penrose inequality.

For the anisotropic counterpart, recently the third author [13] has studied the IAMCF which initiated from a star-shaped and strictly  $F$ -mean convex hypersurface and proved the long time existence and convergence result analogous to Gerhardt and Urbas' result. The aim of this paper is to study Huisken-Ilmanen type weak solutions for the IAMCF by its level-set formulation.

Suppose the evolving hypersurfaces  $N_t$  are given by level sets of a function  $u: \mathbb{R}^n \rightarrow \mathbb{R}$ , that is

$$E_t = \{x \in \mathbb{R}^n : u(x) < t\}, \quad N_t = \partial E_t.$$

If  $u$  is smooth and  $\nabla u \neq 0$ , then (1) is equivalent to the degenerate elliptic equation

$$\operatorname{div}(F_\xi(\nabla u)) = F(\nabla u). \quad (2)$$

See Section 2. When  $F$  is Euclidean, it is clear (2) reduces to

$$\operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) = |\nabla u|. \quad (3)$$

As Huisken-Ilmanen [8], we define a weak solution of (2) by the following minimization principle.

**Definition 1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set. A function  $u \in C_{loc}^{0,1}(\Omega)$  is called a weak solution to (2) if*

$$J_{F,u}(u) \leq J_{F,u}(\varphi) \quad (4)$$

for every precompact set  $K \subset \Omega$  and for every test function  $\varphi \in C_{loc}^{0,1}(\Omega)$  with  $\varphi = u$  in  $\Omega \setminus K$ , and where

$$J_{F,u}(\varphi) := \int_K [F(\nabla \varphi) + \varphi F(\nabla u)] dx. \quad (5)$$

Moreover,  $u$  is a proper solution if in addition

$$\lim_{|x| \rightarrow +\infty} u(x) = +\infty.$$

Our main result of this paper is the following existence result.

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set with smooth boundary such that  $\Omega^c = \mathbb{R}^n \setminus \Omega$  is bounded. There exists a unique proper weak solution  $u \in C_{loc}^{0,1}(\overline{\Omega})$  of (2), in the sense of Definition 1, such that  $u = 0$  on  $\partial\Omega$ . Moreover,  $u$  satisfies*

$$F(\nabla u(x)) \leq \sup_{\partial\Omega} H_F^+, \quad x \in \overline{\Omega}, \quad (6)$$

$$F(\nabla u(x)) \leq H_F^+(x), \quad x \in \partial\Omega, \quad (7)$$

where  $H_F^+(x) = \max\{H_F(x), 0\}$  and  $H_F$  is the anisotropic mean curvature of  $\partial\Omega$ .

Huisken-Ilmanen's approach in the classical IMCF case to prove the existence is studying an approximate equation of (3), known as elliptic regularization. One of the key feature of this elliptic regularization is that it corresponds to a family of translating graphs which solves the IMCF in  $\mathbb{R}^n \times \mathbb{R}$ . It seems such elliptic regularization is not available in the anisotropic case. Later, Moser [10] found another approximate equation of (3) involving the  $p$ -Laplacian. It turns out that this approximate equation is also effective to prove the existence of weak solutions for IAMCF.

Inspired by Moser's approach, we consider the approximate equation of (2) involving the Finsler- $p$ -Laplacian, that is,

$$\begin{cases} \operatorname{div} (F^{p-1}(\nabla u) F_\xi(\nabla u)) = F(\nabla u)^p & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \\ u \rightarrow \infty & \text{as } x \rightarrow \infty. \end{cases} \quad (8)$$

We have the following

**Theorem 1.2.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set with smooth boundary such that  $\Omega^c = \mathbb{R}^n \setminus \Omega$  is bounded. For every  $p > 1$ , there exists a unique solution  $u_p \in C_{loc}^{1,\alpha}(\overline{\Omega})$ . Moreover, for every  $\varepsilon > 0$ , there exists  $p_0 = p_0(\varepsilon) > 1$  such that if  $u_p \in C_{loc}^{1,\alpha}(\overline{\Omega})$  is the solution to (8) for  $1 < p \leq p_0$ , then*

$$F(\nabla u_p(x)) \leq \sup_{\partial\Omega} H_F^+ + \varepsilon, \quad x \in \overline{\Omega} \quad (9)$$

$$F(\nabla u_p(x)) \leq H_F^+(x) + \varepsilon, \quad x \in \partial\Omega. \quad (10)$$

Theorem 1.1 follows from Theorem 1.2 by approximation.

The rest of this paper is organized as follows. In Section 2, we recall some fundamentals on anisotropic functions and anisotropic mean curvature. In Section 3, we study Huisken-Ilmanen type weak formulation of IAMCF and its properties. In Section 4, we study the approximate equation involving the Finsler- $p$ -Laplacian and show the gradient estimate and the existence of weak solution of IAMCF.

## 2. NOTATION AND PRELIMINARIES

### 2.1. Minkowski norm and Wulff shape.

Let  $F$  be a Minkowski norm on  $\mathbb{R}^n$ . The polar function  $F^o: \mathbb{R}^n \rightarrow [0, +\infty[$  of  $F$ , defined as

$$F^o(x) = \sup_{\xi \neq 0} \frac{\langle \xi, x \rangle}{F(\xi)},$$

is again a Minkowski norm on  $\mathbb{R}^n$ . Furthermore,

$$F(\xi) = \sup_{x \neq 0} \frac{\langle \xi, x \rangle}{F^o(x)}.$$

Denote

$$\mathcal{W} = \{x \in \mathbb{R}^n : F^o(x) < 1\}.$$

This is the so-called Wulff shape centered at the origin. More generally, we denote by  $\mathcal{W}_r(x_0)$  the set  $r\mathcal{W} + x_0$ , that is the Wulff shape centered at  $x_0$  with radius  $r$  and  $\mathcal{W}_r = \mathcal{W}_r(0)$ .

The following properties of  $F$  and  $F^o$  hold true: for any  $x, \xi \in \mathbb{R}^n \setminus \{0\}$ ,

$$\begin{aligned} \langle F_\xi(\xi), \xi \rangle &= F(\xi), & \langle F_x^o(x), x \rangle &= F^o(x) \\ F(F_x^o(x)) &= F^o(F_\xi(\xi)) = 1, \\ F^o(x)F_\xi(F_x^o(x)) &= x, & F(\xi)F_x^o(F_\xi(\xi)) &= \xi. \end{aligned}$$

See e.g. [14], Chapter 2.

### 2.2. Anisotropic mean curvature and anisotropic area functional.

Let  $N$  be a smooth closed hypersurface in  $\mathbb{R}^n$  and  $\nu$  be the unit Euclidean outer normal of  $N$ . The anisotropic outer normal of  $N$  is defined by

$$\nu_F = F_\xi(\nu).$$

The anisotropic mean curvature of  $N$  is defined by

$$H_F = \operatorname{div}(\nu_F).$$

Here  $\operatorname{div}$  is the divergence operator with respect to the Euclidean metric. In this paper we are interested in the case when  $N$  is given by a level set of a smooth function  $u$ , namely,

$$N = N_t = \partial E_t, \text{ where } E_t = \{x \in \mathbb{R}^n : u(x) < t\}.$$

When  $\nabla u \neq 0$ , it is clear that  $\nu = \frac{\nabla u}{|\nabla u|}$  and

$$\nu_F = F_\xi(\nabla u), \quad H_F = \operatorname{div}(F_\xi(\nabla u)). \quad (11)$$

If  $N_t$  satisfies the IAMCF, we see that  $u(x(t)) = t$  and by taking derivative about  $t$ , we get

$$\left\langle \nabla u, \frac{1}{H_F} \nu_F \right\rangle = 1.$$

By virtue of (11), we arrive at (2).

The anisotropic area functional of  $N$  is defined as

$$\sigma_F(N) = \int_N F(\nu) d\mathcal{H}^{n-1},$$

where  $\mathcal{H}^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure.

It is well-known that a variational characterization of  $H_F$  is given by the first variational formula of  $\sigma_F$ , see for instance [11, 2, 13]. More precisely, we have

**Proposition 2.1** (Reilly [11], Bellettini-Novaga-Riey [2]).

Let  $N$  be a smooth closed hypersurface given by an embedding  $X_0 : M \rightarrow \mathbb{R}^n$ . Let  $N_s$  be a variation of  $N$  given by  $X(\cdot, s) : M \rightarrow \mathbb{R}^n$ ,  $s \in (-\varepsilon, \varepsilon)$ , whose variational vector field  $\frac{\partial}{\partial s}|_{s=0} X(\cdot, s) = V$ . Then

$$\frac{d}{ds} \Big|_{s=0} \sigma_F(N_s) = \int_N \operatorname{div}_{F,N}(V) F(\nu) d\mathcal{H}^{n-1} = \int_N H_F(X_0) \langle V, \nu \rangle d\mathcal{H}^{n-1}, \quad (12)$$

where

$$\operatorname{div}_{F,N}(V) := \operatorname{div} V - \left\langle \nabla_{\nu_F} V, \frac{\nu}{F(\nu)} \right\rangle.$$

*Proof.* We refer to [13] for the proof of the second equality. For completeness, we prove the first equality here. We denote  $\nu_s$  and  $d\sigma_s$  be the unit outer normal and the area element of  $N_s$  respectively. It is well-known that

$$\frac{d}{ds} \Big|_{s=0} \nu_s = -\langle \nu, \nabla_{e_i} V \rangle e_i,$$

where  $\{e_i\}$  is an orthonormal basis of  $TN$ .

$$\frac{d}{ds} \Big|_{s=0} d\sigma_s = \operatorname{div}_N(V) d\sigma.$$

Thus

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} F(\nu_s) d\sigma_s &= \langle F_\xi(\nu), -\langle \nu, \nabla_{e_i} V \rangle e_i \rangle d\sigma + F(\nu) \operatorname{div}_N(V) d\sigma \\ &= [-\langle \nu, \nabla_{e_i} V \rangle \langle F_\xi(\nu), e_i \rangle + (\operatorname{div} V - \langle \nabla_\nu V, \nu \rangle) F(\nu)] d\sigma \\ &= \left[ \operatorname{div} V - \left\langle \nabla_{\nu_F} V, \frac{\nu}{F(\nu)} \right\rangle \right] F(\nu) d\sigma \end{aligned}$$

The last line follows from

$$\nu_F = F_\xi(\nu) = F(\nu)\nu + \langle F_\xi(\nu), e_i \rangle e_i.$$

□

### 3. IAMCF: A VARIATIONAL FORMULATION

In this section, we review the weak formulation of IAMCF developed by Huisken-Ilmanen [8] by using a minimizing principle. We follow closely Huisken-Ilmanen's strategy in [8], Section 1.

**3.1. Weak formulation of IAMCF.** Recall (Definition 1) that  $u$  is called a weak solution (subsolution, supersolution resp.) of (2) in  $\Omega$  if  $u \in C_{\text{loc}}^{0,1}(\Omega)$  and  $J_{F,u}(u) \leq J_{F,u}(\varphi)$  for every precompact set  $K \subset \Omega$  and for every test function  $\varphi \in C_{\text{loc}}^{0,1}(\Omega)$  ( $\varphi \leq u$ ,  $\varphi \geq u$  resp.) with  $\varphi = u$  in  $\Omega \setminus K$ , where  $J_{F,u}$  is defined in (5).

The fact that

$$J_{F,u}(\min\{\varphi, u\}) + J_{F,u}(\max\{\varphi, u\}) = J_{F,u}(\varphi) + J_{F,u}(u)$$

whenever  $\{u \neq \varphi\}$  is precompact implies  $u$  is a solution if and only if it is both a weak supersolution and a weak subsolution.

There is an equivalent weak formulation by set functional. For  $K \subseteq \Omega$  and  $u \in C_{loc}^{0,1}(\Omega)$ , define

$$J_{F,u}(G) = J_{F,u}^K(G) := \int_{\partial^* G \cap K} F(\nu) d\mathcal{H}^{n-1} - \int_{G \cap K} F(\nabla u) dx,$$

for a set  $G$  of locally finite perimeter, and  $\partial^* G$  denotes the reduced boundary of  $G$ .

**Definition 2.** We say that  $E$  minimizes  $J_{F,u}$  in a set  $A$  (minimizes on the outside, minimizes on the inside, resp.) if

$$J_{F,u}(E) \leq J_{F,u}(G)$$

for any  $G$  such that  $E \Delta G \subset\subset A$  ( $G \supseteq E$ ,  $G \subseteq E$  resp.) and any compact set  $K$  containing  $E \Delta G$ . Here  $E \Delta G = (E \setminus G) \cup (G \setminus E)$ .

The fact that

$$J_{F,u}(E \cap G) + J_{F,u}(E \cup G) \leq J_{F,u}(E) + J_{F,u}(G)$$

whenever  $E \Delta G$  is precompact guarantees that  $E$  minimizes  $J_{F,u}$  in  $\Omega$  if and only if  $E$  minimizes  $J_{F,u}$  both on the inside and on the outside in  $\Omega$ .

The Definitions 1 and 2 are equivalent in the following sense.

**Proposition 3.1.** Let  $\Omega$  be an open set and  $u \in C_{loc}^{0,1}(\Omega)$ , then  $u$  is a weak solution of (2) in  $\Omega$  if and only if for each  $t$ ,  $E_t = \{x \in \Omega : u < t\}$  minimizes  $J_{F,u}$  in  $\Omega$ .

*Proof.* By the co-area formula, we have for a choice of  $a < b$  such that  $a < u < b$  and  $a < \varphi < b$  in  $K$ , that

$$\begin{aligned} J_{F,u}(\varphi) &= \int_K (F(\nabla \varphi) + \varphi F(\nabla u)) dx = \\ &= \int_a^b dt \int_{(\partial\{\varphi < t\}) \cap K} \left( F\left(\frac{\nabla \varphi}{|\nabla \varphi|}\right) + \frac{t}{|\nabla \varphi|} F(\nabla u) \right) d\sigma \quad (13) \\ &= \int_a^b \int_{(\partial\{\varphi < t\}) \cap K} F(\nu) d\sigma - \int_K \int_a^b \chi_{\{\varphi < t\}} F(\nabla u) d\sigma + \\ &\quad + b \int_K F(\nabla u) dx \\ &= \int_a^b J_u^K(\{\varphi < t\}) + b \int_K F(\nabla u) dx. \end{aligned}$$

Then, if for any  $t$ ,  $E_t$  is a minimizer of the set functional  $J_{F,u}$ , then

$$J_{F,u}(\varphi) \geq J_{F,u}(u),$$

that gives the minimality of  $u$ .

The viceversa can be proved exactly as in the proof of Lemma 1.1 in [8].  $\square$

Next we study the weak formulation of IAMCF with initial condition.

**Definition 3.** Let  $E_0$  be an open set with smooth boundary. Let  $\{E_t\}_{t>0}$  be a nested family of open sets in  $\mathbb{R}^n$ .

- (i)  $u$  is called a weak solution of (2) with initial condition  $E_0$  if  $u \in C_{loc}^{0,1}(\mathbb{R}^n)$ ,  $E_0 = \{u < 0\}$  and  $u$  is a weak solution of (2) in  $\mathbb{R}^n \setminus \overline{E_0}$ .
- (ii) Define  $u$  by the characterization  $E_t = \{u < t\}$ .  $\{E_t\}_{t>0}$  is called a weak solution of (1) with initial condition  $E_0$  if  $u \in C_{loc}^{0,1}(\mathbb{R}^n)$  and  $E_t$  minimizes  $J_{F,u}$  in  $\mathbb{R}^n \setminus E_0$  for each  $t > 0$ .

From Proposition 3.1, it is easy to see the above two definitions are also equivalent.

**Proposition 3.2.**  *$u$  is a weak solution of (2) with initial condition  $E_0$  if and only if  $\{u < t\}_{t>0}$  is a weak solution of (1) with initial condition  $E_0$ .*

The weak solution is unique.

**Proposition 3.3** (Uniqueness of the weak solutions). (1) *Let  $u$  and  $v$  be weak solutions to (2) in  $\Omega$  in the sense of Definition 1, and  $\{v > u\} \subset\subset \Omega$ . Then  $v \leq u$  in  $\Omega$ ;*  
 (2) *if  $\{E_t\}_{t>0}$  and  $\{F_t\}_{t>0}$  solve (1) in the sense of Definition 3, with initial data  $E_0, F_0$  respectively, and  $E_0 \subseteq F_0$ , then  $E_t \subseteq F_t$  as long as  $E_t$  is precompact. In particular, for a given  $E_0$  there exists at most one solution  $\{E_t\}_{t>0}$  of (1) such that  $E_t$  is precompact.*

*Proof.* See Huisken-Ilmanen [8], page 377-378.  $\square$

### 3.2. Properties of weak IAMCF.

**Definition 4.** *Let  $\Omega$  be an open set.*

(i) *A set  $E$  is called an  $F$ -minimizing hull in  $\Omega$  if*

$$\sigma_F(\partial^* E \cap K) \leq \sigma_F(\partial^* G \cap K) \quad (14)$$

*for any  $G$  containing  $E$  such that  $G \setminus E \subset\subset \Omega$  and any compact set  $K$  containing  $G \setminus E$ .*

(ii) *A set  $E$  is called a strictly  $F$ -minimizing hull in  $\Omega$  if it is an  $F$ -minimizing hull in  $\Omega$  and equality holds in (14) if and only if*

$$G \cap \Omega = E \cap \Omega \text{ a.e.}$$

(iii) *Given a measurable set  $E$ , the set  $E'$  is defined to be the intersection of all the strictly  $F$ -minimizing hulls in  $\Omega$  that contain  $E$ .*

One has the following properties for weak solutions of IAMCF.

**Proposition 3.4.** *Let  $u$  be a weak solution of (2) with initial condition  $E_0$ . Set*

$$E_t = \{u < t\}, \quad E_t^+ = \text{int}\{u \leq t\}.$$

*Then*

- (i) *For  $t > 0$ ,  $E_t$  is an  $F$ -minimizing hull in  $\mathbb{R}^n$ ;*
- (ii) *For  $t \geq 0$ ,  $E_t^+$  is a strictly  $F$ -minimizing hull in  $\mathbb{R}^n$  and  $E_t' = E_t^+$  if it is precompact;*
- (iii) *For  $t > 0$ ,  $\sigma_F(\partial E_t) = \sigma_F(\partial E_t^+)$  provided  $E_t^+$  is precompact. This holds true for  $t = 0$  if  $E_0$  is an  $F$ -minimizing hull.*
- (iv)  *$\sigma_F(E_t) = e^t \sigma_F(E_0)$  provided  $E_0$  is an  $F$ -minimizing hull.*

*Proof.* See Huisken-Ilmanen [8], page 372-373.  $\square$

Analog with the classical case, we define the weak anisotropic mean curvature by the first variational formula, Proposition 2.1.

**Definition 5.** *Let  $N \subset \mathbb{R}^n$  be a hypersurface of  $C^1$  or  $C^1$  with a small singular set and locally finite Hausdorff measure. A locally integrable function  $H_F$  on  $N$  is called weak anisotropic mean curvature provided it satisfies the second equality in (12) for every  $V \in C_c^\infty(\mathbb{R}^n)$ .*

For smooth IAMCF given by  $\{u = t\}$ , one sees  $H_F = F(\nabla u)$ . We show next weak solutions of (2) still have this property.

**Proposition 3.5.** *Let  $u$  be a weak solution of (2) with initial condition  $E_0$  and let  $N_t = \partial E_t = \partial\{u < t\}$ . Then for a.e.  $t$ , the weak anisotropic mean curvature  $H_F$  of  $N_t$  satisfies*

$$H_F = F(\nabla u) \text{ a.e. } x \in N_t.$$

*Proof.* Let  $V \in C_c^\infty(\mathbb{R}^n)$  and  $\Phi^s : \mathbb{R}^n \rightarrow \mathbb{R}^n, s \in (-\varepsilon, \varepsilon)$ , be the flow of diffeomorphisms generated by  $V$  and  $\Phi^0 = Id$ . Let  $W$  be any precompact open set containing  $\text{supp}(V)$ .

Because  $u$  be a weak solution of (2) in  $\mathbb{R}^n \setminus \overline{E_0}$ , we see  $J_{F,u}(u \circ \Phi^s) \leq J_{F,u}(u)$ . Thus  $\frac{d}{ds} \Big|_{s=0} J_{F,u}(u \circ \Phi^s) = 0$ . Next we derive  $\frac{d}{ds} \Big|_{s=0} J_{F,u}(u \circ \Phi^s)$ .

First, we assume  $u$  is smooth. Then

$$\begin{aligned}
 & \frac{d}{ds} \Big|_{s=0} \int_K F(\nabla(u \circ \Phi^s)) dx \\
 &= \int_W F_{\xi_i}(\nabla u) \nabla_i \left( \frac{d}{ds} \Big|_{s=0} (u \circ \Phi^s) \right) dx \\
 &= \int_W F_{\xi_i}(\nabla u) \nabla_i (\nabla_j u V^j) dx \\
 &= \int_W F_{\xi_i}(\nabla u) \nabla_{j_i}^2 u V^j + F_{\xi_i}(\nabla u) \nabla_j u \nabla_i V^j dx \\
 &= \int_W -F_{\xi_i \xi_k}(\nabla u) \nabla_{k_j}^2 u \nabla_i u V^j - F_{\xi_i}(\nabla u) \nabla_i u \nabla_j V^j + F_{\xi_i}(\nabla u) \nabla_j u \nabla_i V^j dx \\
 &= \int_W -F(\nabla u) \text{div} V + \langle \nabla u, \nabla_{F_\xi(\nabla u)} V \rangle dx.
 \end{aligned}$$

By co-area formula,

$$\begin{aligned}
 & \int_W -F(\nabla u) \text{div} V + \langle \nabla u, \nabla_{F_\xi(\nabla u)} V \rangle dx \\
 &= \int_{-\infty}^{\infty} \int_{N_t \cap W} -\text{div} V F(\nu) + \langle \nu, \nabla_{\nu_F} V \rangle d\sigma_t dt \\
 &= \int_{-\infty}^{\infty} \int_{N_t \cap W} -\text{div}_{F,N} V F(\nu) d\sigma_t dt.
 \end{aligned}$$

Thus

$$\frac{d}{ds} \Big|_{s=0} \int_K F(\nabla(u \circ \Phi^s)) dx = \int_{-\infty}^{\infty} \int_{N_t \cap W} -\text{div}_{F,N} V F(\nu) d\sigma_t dt. \quad (15)$$

By an approximation argument, we see that the formula (15) is still true for  $u$  only locally Lipschitz.

On the other hand, it is easy to see

$$\begin{aligned}
 \frac{d}{ds} \Big|_{s=0} \int_K (u \circ \Phi^s) F(\nabla u) dx &= \int_W \langle \nabla u, V \rangle F(\nabla u) dx \\
 &= \int_{-\infty}^{\infty} \int_{N_t \cap W} \langle \nu, V \rangle F(\nabla u) d\sigma_t dt. \quad (16)
 \end{aligned}$$

Combining (15) and (16), we get

$$\begin{aligned}
 0 &= \frac{d}{ds} \Big|_{s=0} J_{F,u}(u \circ \Phi^s) \\
 &= \int_{-\infty}^{\infty} \int_{N_t \cap W} -\text{div}_{F,N} V F(\nu) + \langle \nu, V \rangle F(\nabla u) d\sigma_t dt. \quad (17)
 \end{aligned}$$

Finally, by the definition of the weak anisotropic mean curvature, we conclude from (17) that

$$H_F = F(\nabla u) \text{ a.e. } x \in N_t \text{ a.e. } t.$$

□

## 4. EXISTENCE OF SOLUTIONS AND GRADIENT ESTIMATES

For any  $p > 1$ , we will consider the following auxiliary problem

$$\begin{cases} \operatorname{div} (F^{p-1}(\nabla v) F_\xi(\nabla v)) = 0 & \text{in } \Omega, \\ v = 1 & \text{in } \Omega^c. \\ v \rightarrow 0 & \text{as } x \rightarrow \infty. \end{cases} \quad (18)$$

**Proposition 4.1.** *If  $1 < p < n$ , then there exists a unique positive solution  $v_p \in C_{loc}^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \setminus \{\nabla v_p = 0\})$  of (18). If  $\mathcal{W}_r(x_0) \subset \Omega^c \subset \mathcal{W}_s(y_0)$ , then*

$$\left( \frac{r}{F^o(x-x_0)} \right)^{\frac{n-p}{p-1}} \leq v_p(x) \leq \left( \frac{s}{F^o(x-y_0)} \right)^{\frac{n-p}{p-1}}, \quad \forall x \in \Omega \setminus \{y_0\}; \quad (19)$$

Moreover,  $v_p$  verifies

$$\lim_{|x| \rightarrow \infty} \frac{F(\nabla v_p)}{v_p} = 0. \quad (20)$$

*Proof.* The proof of existence, uniqueness, regularity, as well as (19), follow by nowadays standard arguments; we refer the reader to [3, Theorem 3.3] for the general anisotropic case we consider.

Finally we prove (20). We argue as in [10]. Let  $\eta \in C_0^\infty(\Omega)$  be a suitable cut-off function. Taking  $\psi = v_p \eta^p$  as test function in the weak formulation for (18) and using the Hölder inequality, it easily follows that:

$$\int_{\Omega} \eta^p F(\nabla v_p)^p dx \leq p^p \int_{\Omega} v_p^p F(\nabla \eta)^p dx.$$

By Harnack inequality (see for instance [12]) we get

$$r^{p-n} \int_{B_{r/4}(x_0)} F(\nabla v_p)^p dx \leq C(n, p) \inf_{B_{r/2}(x_0)} v_p^p,$$

where  $C(n, p)$  is a positive constant depending on  $n$  and  $p$ . By applying the result contained in [5], we have

$$\|F(\nabla v_p)\|_{L^\infty(B_{r/8}(x_0))} \leq \frac{C(n, p)}{r} \inf_{B_{r/2}(x_0)} v_p,$$

which implies (20), and the proof is completed.  $\square$

It is direct to see that

$$u_p = (1-p) \log v_p \in C^{1,\alpha}(\overline{\Omega})$$

solves (8) (we refer the reader, e.g., also to [4] for problems involving equations as in (8)). Next we show the gradient estimate in Theorem 1.2, which is based on the following Lemma.

**Lemma 4.2.** *Let  $1 < p < n$  and  $u_p \in C_{loc}^{1,\alpha}(\overline{\Omega})$  be a solution to (8), then*

$$\sup_{\overline{\Omega}} F(\nabla u_p) = \sup_{\partial\Omega} F(\nabla u_p). \quad (21)$$

*Proof.* We omit the subscript  $p$  in  $u_p$  in the proof. Let  $\tau = \sup_{\partial\Omega} F(\nabla u)$ . We consider the following set

$$\Omega_\beta = \{x \in \Omega : F(\nabla u) > \beta\},$$

with  $\beta > \tau \geq 0$ . From (20) we see  $F(\nabla u)$  vanishes at infinity by (20), then  $\Omega_\beta$  is a bounded, open set such that  $\overline{\Omega}_\beta \cap \partial\Omega = \emptyset$  and  $F(\nabla u) = \beta$  on  $\partial\Omega_\beta$ .

In order to prove (21), we will prove that  $\Omega_\beta = \emptyset$ .

Note that in  $\overline{\Omega}_\beta$ ,  $\nabla u \neq 0$  and hence  $u \in C^\infty(\overline{\Omega}_\beta)$ . By writing

$$G(\xi) = \frac{1}{2} F^2(\xi),$$



the equation in (8) becomes

$$\operatorname{div} \left( G^{\frac{p}{2}-1} (\nabla u) G_\xi (\nabla u) \right) = G^{\frac{p}{2}} (\nabla u). \quad (22)$$

Hereafter we will adopt the Einstein convention on the repeated indices, and use the notations

$$\begin{aligned} G &= G(\nabla u), \quad G_i = G_{\xi_i}(\nabla u), \quad G_{ij} = G_{\xi_i \xi_j}(\nabla u), \\ u_i &= u_{x_i}, \quad u_{ij} = u_{x_i x_j}, \dots \end{aligned}$$

Differentiating (22) with respect to  $x_i$ , we get

$$\partial_{x_k} \left( \partial_{x_i} [G^{\frac{p}{2}-1} G_k] \right) = \frac{p}{2} G^{\frac{p}{2}-1} G_j u_{ij},$$

and

$$\partial_{x_k} \left( G_i \partial_{x_i} [G^{\frac{p}{2}-1} G_k] \right) = \frac{p}{2} G^{\frac{p}{2}-1} G_i G_j u_{ij} + G_{il} u_{lk} \partial_{x_i} [G^{\frac{p}{2}-1} G_k].$$

Then

$$\begin{aligned} \partial_{x_k} \left( \left[ \frac{p-2}{2} G^{\frac{p}{2}-2} G_j G_k + G^{\frac{p}{2}-1} G_{kj} \right] G_i u_{ij} \right) &= \\ &= \frac{p}{2} G^{\frac{p}{2}-1} G_i G_j u_{ij} + \left[ \frac{p-2}{2} G^{\frac{p}{2}-2} G_j G_k + G^{\frac{p}{2}-1} G_{kj} \right] G_{il} u_{lk} u_{ij}; \end{aligned}$$

hence

$$\begin{aligned} \partial_{x_k} \left( \left[ \frac{p-2}{2} G^{\frac{p}{2}-2} G_i G_j G_k + G^{\frac{p}{2}-1} G_i G_{kj} \right] u_{ij} \right) &= \\ &= \frac{p}{2} G^{\frac{p}{2}-1} G_i G_j u_{ij} + \left[ \frac{p-2}{2} G^{\frac{p}{2}-2} G_j G_k G_{il} u_{lk} u_{ij} + G^{\frac{p}{2}-1} G_{kj} G_{il} u_{lk} u_{ij} \right] = \\ &= \frac{p}{2} G^{\frac{p}{2}-1} G_i G_j u_{ij} + \frac{p-1}{2} G^{\frac{p}{2}-2} G_j G_k G_{il} u_{lk} u_{ij} + \\ &\quad + G^{\frac{p}{2}-2} \left[ -\frac{1}{2} G_j G_k G_{il} u_{lk} u_{ij} + G G_{kj} G_{il} u_{lk} u_{ij} \right]. \quad (23) \end{aligned}$$

The Kato type inequality (see [17, Lemma 2.2]) implies that

$$G G_{il} G_{jk} u_{ij} u_{kl} \geq \frac{1}{2} G_{il} G_j G_k u_{ij} u_{kl}.$$

Hence, from (23) we get

$$\begin{aligned} \partial_{x_k} \left( \left[ \frac{p-2}{2} G^{\frac{p}{2}-2} G_i G_j G_k + G^{\frac{p}{2}-1} G_i G_{kj} \right] u_{ij} \right) &\geq \\ &\geq \frac{p}{2} G^{\frac{p}{2}-1} G_i G_j u_{ij} + \frac{p-1}{2} G^{\frac{p}{2}-2} G_j G_k G_{il} u_{lk} u_{ij}. \end{aligned}$$

The above inequality can be read as

$$\begin{aligned} \operatorname{div} \left[ G^{\frac{p}{2}-1} \left( G_{\xi\xi} \nabla_x G + \frac{p-2}{2} \frac{(G_\xi \cdot \nabla_x G)}{G} G_\xi \right) \right] - \frac{p}{2} G^{\frac{p}{2}-1} (G_\xi \cdot \nabla_x G) &\geq \\ &\geq \frac{p-1}{2} (G_{\xi\xi} \nabla_x G) \cdot \nabla_x G, \end{aligned}$$

that is

$$\operatorname{div} \left[ G^{\frac{p}{2}-1} A \nabla_x G \right] - \frac{p}{2} G^{\frac{p}{2}-1} G_\xi \cdot \nabla_x G \geq \frac{p-1}{2} G^{\frac{p}{2}-2} (G_{\xi\xi} \nabla_x G) \cdot \nabla_x G \geq 0, \quad (24)$$

where

$$A = G_{\xi\xi} + \frac{p-2}{2} \frac{G_\xi \otimes G_\xi}{G}$$

is a uniformly positive definite matrix. Hence, the functional in the left-hand side of (24) can be seen as a linear elliptic operator acting on  $G(\nabla u)$ , and by the maximum principle we have that

$$\sup_{\Omega_\beta} G(\nabla u) \leq \sup_{\partial\Omega_\beta} G(\nabla u) = \frac{\beta^2}{2}.$$

This implies that  $\Omega_\beta = \left\{x \in \Omega : G(\nabla u) > \frac{\beta^2}{2}\right\}$  is empty, and the proof is completed.  $\square$

*Proof of Theorem 1.2.* We are remained to prove the boundary gradient estimate (10). The global gradient estimate (9) follows from Lemma 4.2 and (10).

Let  $x \in \partial\Omega$  such that  $\mathcal{W}_r(x_0) \subset \Omega^c$ . Since  $u_p = 0$  on  $\partial\Omega$ , hence if  $\nabla u_p(x) \neq 0$ , then  $\nu_F(x) = F_\xi(\nabla u_p(x))$  and  $F(\nabla u_p(x)) = \frac{\partial u_p}{\partial \nu_F}(x)$ . On the other hand, since  $\mathcal{W}_r(x_0)$  and  $\partial\Omega$  are tangent at  $x$ , we see  $x - x_0 = r\nu_F(x)$ . It follows that

$$\begin{aligned} F(\nabla u_p(x)) &= \frac{\partial u_p}{\partial \nu_F}(x) = \lim_{t \rightarrow 0} \frac{u_p(x + t\nu_F)}{t} \\ &\leq (n-p) \lim_{t \rightarrow 0} \frac{\log F^0(x + t\nu_F - x_0) - \log r}{t} \\ &= \frac{n-p}{r}. \end{aligned}$$

Thus if we define

$$R := \sup\{r > 0 : \forall x \in \partial\Omega, \exists \mathcal{W}_r(x_0) \subset \Omega^c \text{ such that } x \in \partial\mathcal{W}_r(x_0)\}. \quad (25)$$

then

$$\|F(\nabla u_p)\|_{L^\infty(\partial\Omega)} \leq \frac{n-p}{R}.$$

It follows from Lemma 4.2 that

$$\|F(\nabla u_p)\|_{L^\infty(\bar{\Omega})} \leq \frac{n-p}{R}. \quad (26)$$

Next prove the estimate (10). We argue as in [8, 9]. Let  $\varepsilon > 0$ . Choose  $\bar{w} \in C^\infty(\bar{\Omega})$  such that

- i)  $\bar{w} = 0$  on  $\partial\Omega$  and  $\bar{w} > 0$  in  $\Omega$ ;
- ii)  $H_F^+ < F(\nabla \bar{w}) \leq H_F^+ + \varepsilon$  on  $\partial\Omega$ .

Denote

$$\mathcal{Q}_p[\varphi] := \operatorname{div}(F^{p-1}(\nabla \varphi) F_\xi(\nabla \varphi)) - F(\nabla \varphi)^p. \quad (27)$$

Since  $F(\nabla \bar{w}) > 0$  and  $\bar{w} = 0$  on  $\partial\Omega$ , the anisotropic mean curvature of  $\partial\Omega$  is given by

$$H_F(x) = \operatorname{div}(F_\xi(\nabla \bar{w})).$$

Thus

$$\mathcal{Q}_1[\bar{w}](x) = H_F(x) - F(\nabla \bar{w}(x)) < 0 \text{ for } x \in \partial\Omega.$$

Let  $\delta > 0$  and denote by  $U_\delta$  the components of the set  $\{0 \leq \bar{w} < \delta\}$  containing  $\partial\Omega$ . If we choose  $\delta > 0$  small enough, we may have  $F(\nabla \bar{w}) > 0$  and  $\mathcal{Q}_1[\bar{w}] < 0$  in  $U_\delta$ .

Define  $w \in C^\infty(U_\delta)$  by

$$w = \frac{\bar{w}}{1 - \frac{\bar{w}}{\delta}}.$$

Then

$$\nabla w = \frac{\nabla \bar{w}}{\left(1 - \frac{\bar{w}}{\delta}\right)^2}.$$

A simple computation gives

$$\mathcal{Q}_1[w] = \mathcal{Q}_1[\bar{w}] + \left(1 - \frac{1}{\left(1 - \frac{\bar{w}}{\delta}\right)^2}\right) F(\nabla \bar{w}) < 0 \quad \text{on } U_\delta.$$

By (26), there exists a constant  $C = C(\delta) > 0$ , such that  $u_p \leq C$  in  $U_\delta$ . Denoted by  $\tilde{U}_{C+1}$  the component of the set  $\{0 \leq w \leq C + 1\}$  in  $U_\delta$ . Since  $u_p = w = 0$  on  $\partial\Omega$ , we see

$$u_p \leq w \quad \text{on } \partial\tilde{U}_{C+1}. \quad (28)$$

In order to compare  $u_p$  and  $w$  in  $\tilde{U}_{C+1}$ , we compute

$$\mathcal{Q}_p[w] = F^{p-1}(\nabla w) \left( \mathcal{Q}_1[w] + (p-1) \frac{w_{ik} F_{\xi_i}(\nabla w) F_{\xi_k}(\nabla w)}{F(\nabla w)} \right) \text{ in } \tilde{U}_{C+1}.$$

Since  $\mathcal{Q}_1[w] < 0$  in  $\tilde{U}_{C+1}$ , one may choose  $p-1$  small enough, depending on  $\|\bar{w}\|_{C^2(\tilde{U}_{C+1})}$ ,  $\inf_{\tilde{U}_{C+1}} F(\nabla \bar{w})$  and  $C$ , such that

$$\mathcal{Q}_p[w] < 0 \text{ in } \tilde{U}_{C+1}.$$

Then since  $\mathcal{Q}_p[u_p] = 0 > \mathcal{Q}_p[w]$  in  $\tilde{U}_{C+1}$  and (28), by the comparison principle, we have

$$u_p \leq w \text{ on } \tilde{U}_{C+1}.$$

It follows that

$$F(\nabla u_p) = \frac{\partial u_p}{\partial \nu_F} \leq \frac{\partial w}{\partial \nu_F} = \frac{\partial \bar{w}}{\partial \nu_F} = F(\nabla \bar{w}) \leq H_F^+ + \varepsilon \quad \text{on } \partial\Omega.$$

The proof of Theorem 1.2 is completed.  $\square$

Now we are ready to prove Theorem 1.1.

*Proof of the Theorem 1.1.* Let  $u_p$  be the solution of (8) in Theorem 1.2. Then for any precompact set  $K \subset \Omega$ ,  $u_p$  is also a minimizer of the functional

$$J_w^p(\varphi) = \int_K \left[ \frac{1}{p} F(\nabla \varphi)^p + \varphi F(\nabla w)^p \right] dx, \quad (29)$$

in the sense that

$$J_{u_p}^p(u_p) \leq J_{u_p}^p(\varphi), \quad \forall \varphi \in W_{\text{loc}}^{1,p}(\Omega) \text{ such that } \varphi = u_p \text{ in } \Omega \setminus K. \quad (30)$$

Indeed, being  $u_p - \varphi = 0$  outside  $K$ , and using it as a test function for (8), we get that

$$\begin{aligned} \int_K F^p(\nabla u_p)(u_p - \varphi) dx &= \int_K F^{p-1}(\nabla u_p) F_\xi(\nabla u_p) \cdot \nabla(\varphi - u_p) dx \\ &\leq \frac{1}{p} \int_K (F(\nabla \varphi)^p - F(\nabla u_p)^p) dx. \end{aligned}$$

The inequality above follows from the convexity of  $F(\xi)^p$ .

On the other hand, from (19), we know  $u_p$  has uniform upper bound on any compact set in  $\Omega$ . Since we also have uniform global gradient estimate (9) for  $\nabla u_p$ , by Arzela-Ascoli's theorem, we get that, there exists a subsequence  $p_k \rightarrow 1^+$  and  $u \in C_{\text{loc}}^{0,1}(\bar{\Omega})$  such that

$$u_{p_k} \rightarrow u \text{ uniformly in any compact sets of } \bar{\Omega}. \quad (31)$$

If we can show  $u$  is a weak solution of (2), then since the weak solution of (2) is unique, we will get  $u_p \rightarrow u$  uniformly convergence in any compact sets of  $\bar{\Omega}$  as  $p \rightarrow 1$ .

Next we show that  $u$  is a proper weak solution to problem (2), in the sense of Definition 1. To this aim, we need to prove that

$$|\nabla u_{p_k}|^{p_k} \rightarrow |\nabla u| \quad \text{in } L^1_{loc}(\Omega) \text{ for a subsequence } p_k \rightarrow 1^+. \quad (32)$$

Indeed from (31) and (32) we can pass to the limit in (29) obtaining that

$$J_{u_{p_k}}^K(\varphi, p_k) \rightarrow J_u^K(\varphi) \quad \text{and} \quad J_{u_{p_k}}^K(u_{p_k}, p_k) \rightarrow J_u^K(u)$$

In order to prove (32), we argue as in [10, 8]. Let  $K \subset \Omega$  be a precompact set and consider the following test function  $\psi$  in (30)

$$\psi = \eta\varphi + (1 - \eta)u_p,$$

where  $\eta \in C^\infty(\Omega)$  is a cut-off function such that  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  in  $K$ , and  $\varphi \in C^{0,1}_{loc}(\Omega)$ . Then we get

$$\begin{aligned} & \int_{\text{supp } \eta} \left( \frac{F^p(\nabla u_p)}{p} + \eta(u_p - \varphi) F^p(\nabla u_p) \right) dx \leq \\ & \leq \frac{1}{p} \int_{\text{supp } \eta} F^p(\nabla(\eta\varphi + (1 - \eta)u_p)) dx \leq \\ & \leq \frac{1}{p} \int_{\text{supp } \eta} [(\varphi - u_p)F(\nabla\eta) + \eta F(\nabla\varphi) + (1 - \eta)F(\nabla u_p)]^p dx \leq \\ & \leq \frac{3^{p-1}}{p} \int_{\text{supp } \eta} [|\varphi - u_p|^p F^p(\nabla\eta) + \eta^p F^p(\nabla\varphi) + (1 - \eta)^p F^p(\nabla u_p)] dx. \end{aligned}$$

Choosing  $\varphi = u$ , and letting  $p_k \rightarrow 1^+$ , we obtain

$$\limsup_{p_k \rightarrow 1^+} \int_{\Omega} \eta F(\nabla u_{p_k})^{p_k} dx \leq \int_{\Omega} \eta F(\nabla u) dx,$$

which together with Fatou's Lemma gives (32).

The properness of  $u$  follows directly from (19). The estimate follows from that in Theorem 1.2. The proof of Theorem 1.1 is completed.  $\square$

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UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II, DIPARTIMENTO DI MATEMATICA E APPLICAZIONI “R. CACCIOPPOLI”, VIA CINTIA, MONTE S. ANGELO - 80126 NAPOLI, ITALIA.  
*E-mail address:* [f.dellapietra@unina.it](mailto:f.dellapietra@unina.it)

UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II, DIPARTIMENTO DI MATEMATICA E APPLICAZIONI “R. CACCIOPPOLI”, VIA CINTIA, MONTE S. ANGELO - 80126 NAPOLI, ITALIA.  
*E-mail address:* [nunzia.gavitone@unina.it](mailto:nunzia.gavitone@unina.it)

SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, 361005, XIAMEN, CHINA  
*E-mail address:* [chaoxia@xmu.edu.cn](mailto:chaoxia@xmu.edu.cn)