

EXISTENCE OF WEAK SOLUTIONS TO FIRST-ORDER STATIONARY MEAN-FIELD GAMES WITH DIRICHLET CONDITIONS

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ABSTRACT. In this paper, we study first-order stationary monotone mean-field games (MFGs) with Dirichlet boundary conditions. While for Hamilton–Jacobi equations Dirichlet conditions may not be satisfied, here, we establish the existence of solutions of MFGs that satisfy those conditions. To construct these solutions, we introduce a monotone regularized problem. Applying Schaefer’s fixed-point theorem and using the monotonicity of the MFG, we verify that there exists a unique weak solution to the regularized problem. Finally, we take the limit of the solutions of the regularized problem and using Minty’s method, we show the existence of weak solutions to the original MFG.

1. INTRODUCTION

Mean-field games (MFGs) were introduced in the mathematical community in [27], [28], and [29] and, independently, around the same time in the engineering community in [25] and [26]. These games model the behavior of large populations of rational agents who seek to optimize an individual utility. Here, we consider the following first-order stationary MFG with a Dirichlet boundary condition.

Problem 1. *Suppose that $\Omega \subset \mathbb{R}^d$ is an open and bounded set with smooth boundary, $\partial\Omega$, and $\gamma > 1$. Let $V, \phi, h \in C^\infty(\overline{\Omega})$, $g \in C^\infty(\mathbb{R}_0^+)$, and $H \in C^\infty(\overline{\Omega} \times \mathbb{R}^d)$ be such that g is increasing, $\phi \geq 0$ in Ω , and $\int_\Omega \phi \, dx = 1$. Find $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ satisfying $m \geq 0$ and*

$$\begin{cases} -u - H(x, Du) + g(m) - V(x) = 0 & \text{in } \Omega, \\ m - \operatorname{div}(m D_p H(x, Du)) - \phi = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

In Problem 1, H is the Hamiltonian and its Legendre transform, given by

$$L(v) = \sup_p \{-p \cdot v - H(x, p)\},$$

gives the agent’s cost of movement at speed v ; the potential, V , determines spatial preferences of each agent, and the coupling, g , encodes the interactions between agents and the mean field. When agents leave the domain through a point $x \in \partial\Omega$, they incur a charge $h(x)$. Finally, the source, ϕ , represents an incoming flow of agents replacing the ones leaving through $\partial\Omega$ or due to the unit discount encoded in the term $-u$ in the Hamilton–Jacobi equation and in the term m in the transport equation. We note that m is the density of the distribution of the agents and $u(x)$ the value function of an agent in the state x .

In general, Hamilton–Jacobi equations with Dirichlet boundary conditions do not admit continuous solutions up to the boundary. This fact can be illustrated by the equation

$$u'(x) = 0$$

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for $0 < x < 1$ with $u(0) = 0$ and $u(1) = 1$. However, keeping the boundary conditions and coupling the previous equation with a transport equation

$$\begin{cases} u'(x) = m(x) \\ m'(x) = 0, \end{cases}$$

we obtain a model that has a unique solution continuous up to the boundary, $(u, m) = (x, 1)$. This model motivates our main result, the existence of solutions for Problem 1 satisfying the Dirichlet boundary condition in the trace sense.

MFGs have been studied intensively, see, for example, the monograph [22], the surveys [1], [24], the note [6], and the lectures [30]. Because there are few known explicit solutions for stationary MFGs [20], [19], and [14], a substantial effort has been undertaken to develop numerical methods [3] and [5], find special transformations [8], and to establish the existence of solutions. The existence of solution for second-order, stationary MFGs without congestion was investigated in [23], [21], [32], [4], and [10]; problems with congestion were examined in [13], [18]. The theory for first-order MFGs is less developed. The existence of solutions for first or second-order stationary MFGs was examined in [16] (also see [2]) using monotone operators and, using a variational approach, certain first-order MFGs with congestion were examined in [12].

For first-order MFGs and often for second-order MFGs, existing publications consider only periodic boundary conditions. However, in applications, MFGs with boundary conditions are quite natural: for example, Dirichlet boundary conditions arise in models where agents can leave the domain and are charged an exit fee. Here, our goal is to prove the existence of weak solutions for the first-order stationary monotone MFG with Dirichlet boundary conditions. Thus, we introduce a notion of weak solutions to Problem 1 similar to the one considered in [16]. Here, however, we account for the Dirichlet boundary conditions. We recall that in [16], the authors consider stationary monotone MFGs with periodic boundary conditions, and their notion of weak solutions is induced by monotonicity. Monotonicity plays an essential role in the uniqueness of solutions [30], and in its absence, the theory becomes substantially harder, see the examples in [20], and the non-monotone second-order MFGs in [11], [9], and [31].

Throughout this paper, $k \in \mathbb{N}$ is a fixed natural number such that $2k > \frac{d}{2} + 2$, and \mathcal{A} and $H_h^{2k}(\Omega)$ are the sets given, respectively, by

$$\mathcal{A} := \left\{ m \in H^{2k}(\Omega) \mid m \geq 0 \right\} \quad (1.1)$$

and

$$H_h^{2k}(\Omega) := \left\{ w \in H^{2k}(\Omega) \mid w - h \in H_0^{2k}(\Omega) \right\}. \quad (1.2)$$

Definition 1.1. *A weak solution to Problem 1 is a pair $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ satisfying*

(D1) $u = h$ on $\partial\Omega$, $m \geq 0$ in Ω ,

(D2) $\left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix} \right\rangle \geq 0$ for all $(\eta, v) \in \mathcal{A} \times H_h^{2k}(\Omega)$,

where, for $(\eta, v) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ fixed, $F[\eta, v] : L^1(\Omega) \times L^1(\Omega) \rightarrow \mathbb{R}$ is the functional given by

$$\begin{aligned} \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle &:= \int_{\Omega} (-v - H(x, Dv) + g(\eta) - V) w_1 \, dx \\ &+ \int_{\Omega} \left(\eta - \operatorname{div}(\eta D_p H(x, Dv)) - \phi \right) w_2 \, dx. \end{aligned} \quad (1.3)$$

Next, we state our main theorem that, under the assumptions detailed in Section 2, establishes the existence of weak solutions to Problem 1.

Theorem 1.2. *Consider Problem 1 and suppose that Assumptions 1–7 hold. Then, there exists a weak solution $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ to Problem 1 in the sense of Definition 1.1.*

In [16], the method of continuity was used to prove the existence of a weak solution to stationary monotone MFGs with periodic boundary conditions. Here, we use a different approach: we apply Schaeffer's fixed-point theorem and extend the results in [16] to Dirichlet boundary conditions.

To prove Theorem 1.2, we introduce a regularized problem, Problem 2, that we believe to be of interest on its own. This regularized problem preserves the monotonicity of the original MFG in the sense of Assumption 7 (see Section 2 and Lemma 7.2). Note that the choice of boundary conditions is critical to preserve monotonicity. Moreover, because of the regularizing terms (see the ϵ -terms in (1.4) below), it is simpler to prove the existence and uniqueness of weak solutions to this problem. Then, by letting $\epsilon \rightarrow 0$, we can construct a weak solution to Problem 1 (see Section 7).

Problem 2. *Let Ω be an open and bounded set with smooth boundary, $\partial\Omega$, and outward pointing unit normal vector \mathbf{n} . Let $V, \phi, \xi, h \in C^\infty(\bar{\Omega})$, $g \in C^\infty(\mathbb{R}_0^+)$, and $H \in C^\infty(\bar{\Omega} \times \mathbb{R}^d)$ be such that g is increasing, $\phi \geq 0$ in Ω , and $\int_\Omega \phi \, dx = 1$. Fix $\epsilon \in (0, 1)$. Find $(m, u) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfying $m \geq 0$ and*

$$\begin{cases} -u - H(x, Du) + g(m) - V(x) + \epsilon(m + \Delta^{2k}m) = 0 & \text{in } \Omega, \\ m - \operatorname{div}(mD_p H(x, Du)) - \phi + \epsilon(u + \xi + \Delta^{2k}(u + \xi)) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial \mathbf{n}} \partial^\alpha \Delta^i m = 0 & \text{on } \partial\Omega \text{ for all } \alpha \in \mathbb{N}_0^d \text{ and } i \in \mathbb{N}_0 \text{ such that } |\alpha| + i = 2k - 1, \\ \partial^\beta u = \partial^\beta h & \text{on } \partial\Omega \text{ for all } \beta \in \mathbb{N}_0^d \text{ and such that } |\beta| \leq 2k - 1. \end{cases} \quad (1.4)$$

In the preceding problem, ξ is a technical term used to cancel the boundary conditions in u so that we can work with vanishing Dirichlet boundary conditions, see Section 6. Since the regularizing terms are of order greater than two, we cannot apply the maximum principle. Thus, Problem 2 may not have classical solutions with $m \geq 0$. Hence, in the following definition, we introduce a notion of weak solution to Problem 2 that requires positivity and relaxes the equality in the Hamilton–Jacobi equation. This definition is related to the ones in [7] and in [16], where u is only required to be a subsolution of the Hamilton–Jacobi equation.

Definition 1.3. *A weak solution to Problem 2 is a pair $(m, u) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfying, for all $w \in \mathcal{A}$ and $v \in H_0^{2k}(\Omega)$,*

$$\begin{aligned} \text{(E1)} \quad & u \in H_h^{2k}(\Omega), \quad m \geq 0 \text{ in } \Omega, \\ \text{(E2)} \quad & \int_\Omega (-u - H(x, Du) + g(m) - V)(w - m) \, dx \\ & + \int_\Omega \left[\epsilon m(w - m) + \epsilon \sum_{|\alpha|=2k} \partial^\alpha m (\partial^\alpha w - \partial^\alpha m) \right] \, dx \geq 0, \\ \text{(E3)} \quad & \int_\Omega (m - \operatorname{div}(mD_p H(x, Du)) - \phi)v \, dx \\ & + \int_\Omega \left[\epsilon (uv + \sum_{|\alpha|=2k} \partial^\alpha u \partial^\alpha v) + \epsilon (\xi + \Delta^{2k}\xi)v \right] \, dx = 0. \end{aligned}$$

Remark 1.4. Assume that (m, u) is a weak solution to Problem 2. Let $\Omega' := \{x \in \Omega \mid m(x) > 0\}$, and fix $w_1 \in C_c^\infty(\Omega')$. For all $\tau \in \mathbb{R}$ with $|\tau|$ sufficiently small, we have $w = m + \tau w_1 \in \mathcal{A}$. Then, from (E2), we obtain

$$\tau \int_\Omega (-u - H(x, Du) + g(m) - V)w_1 \, dx + \tau \int_\Omega \left(\epsilon m w_1 + \epsilon \sum_{|\alpha|=2k} \partial^\alpha m \partial^\alpha w_1 \right) \, dx \geq 0.$$

Because the sign of τ is arbitrary, we conclude that m satisfies

$$-u - H(x, Du) + g(m) - V(x) + \epsilon(m + \Delta^{2k}m) = 0 \quad \text{pointwise in } \Omega'.$$

Moreover, let $w_2 \in C_c^\infty(\Omega)$ be such that $w_2 \geq 0$; taking $w = m + w_2 \in \mathcal{A}$ in (E2) and integrating by parts, we obtain

$$\tau \int_{\Omega} (-u - H(x, Du) + g(m) - V(x))w_2 \, dx + \tau \int_{\Omega} (\epsilon m w_2 + \epsilon \Delta^{2k} m w_2) \, dx \geq 0.$$

Thus,

$$-u - H(x, Du) + g(m) - V(x) + \epsilon(m + \Delta^{2k} m) \geq 0 \quad \text{in the sense of distributions in } \Omega.$$

Also, by (E3), we have

$$m - \operatorname{div}(m D_p H(x, Du)) - \phi + \epsilon(u + \xi + \Delta^{2k}(u + \xi)) = 0 \quad \text{in the sense of distributions in } \Omega.$$

Under the same assumptions of Theorem 1.2, we prove the existence and uniqueness of the weak solutions to Problem 2.

Theorem 1.5. Consider Problem 2 and suppose that Assumptions 1–5 and 7 hold for some $\gamma > 1$. Then, there exists a unique weak solution $(m, u) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ to Problem 2 in the sense of Definition 1.3.

We begin by addressing Theorem 1.5. First, in Section 3, we prove a priori estimates for classical and weak solutions to Problem 2. Second, in Sections 4 and 5, we introduce two auxiliary problems: a variational problem whose Euler–Lagrange equation is the first equation in Problem 2 and a problem associated with a bilinear form corresponding to the second equation in that problem. In each of these two sections, we show that there exists a unique solution. Finally, in Section 6, we prove Theorem 1.5 using Schaefer’s fixed-point theorem together with the results established in Sections 3–5. Finally, the proof of Theorem 1.2 is given in Section 7 using Minty’s method.

2. ASSUMPTIONS

To prove our main results, we need the following assumptions on the functions that arise in Problems 1 and 2. The first three assumptions prescribe standard growth conditions on the Hamiltonian, H . For instance,

$$H(x, p) = a(x)(1 + |p|^2)^{\frac{\gamma}{2}} + b(x) \cdot p,$$

where $a \in C(\overline{\Omega})$, $a(x) > 0$ for all $x \in \overline{\Omega}$, and $b : \overline{\Omega} \rightarrow \mathbb{R}^d$ is a C^∞ function satisfies our assumptions.

Assumption 1. *There exists a constant, $C > 0$, such that, for all $(x, p) \in \Omega \times \mathbb{R}^d$,*

$$-H(x, p) + D_p H(x, p) \cdot p \geq \frac{1}{C}|p|^\gamma - C.$$

Assumption 2. *There exists a constant, $C > 0$, such that, for all $(x, p) \in \Omega \times \mathbb{R}^d$,*

$$H(x, p) \geq \frac{1}{C}|p|^\gamma - C.$$

Assumption 3. *There exists a constant, $C > 0$, such that, for all $(x, p) \in \Omega \times \mathbb{R}^d$,*

$$|D_p H(x, p)| \leq C|p|^{\gamma-1} + C.$$

The next three assumptions impose growth conditions on g . For example, these are satisfied for $g(m) = m^\alpha$, $\alpha > 0$, or by $g(m) = \ln m$ (apart from small changes).

Assumption 4. *The function g is increasing in \mathbb{R}_0^+ . Moreover, for all $\delta > 0$, there exists a constant, $C_\delta > 0$, such that, for all $m \in L^1(\Omega)$,*

$$\max \left\{ \int_{\Omega} |g(m)| \, dx, \int_{\Omega} m \, dx \right\} \leq \delta \int_{\Omega} m g(m) \, dx + C_\delta.$$

Assumption 5. *There exists a constant, $C > 0$, such that, for all $m \in L^1(\Omega)$,*

$$\int_{\Omega} mg(m) \, dx \geq -C.$$

Assumption 6. *If $\{m_j\}_{j=1}^{\infty} \subseteq L^1(\Omega)$ is a sequence satisfying*

$$\sup_{j \in \mathbb{N}} \int_{\Omega} m_j g(m_j) \, dx < +\infty,$$

then there exists a subsequence of $\{m_j\}_{j=1}^{\infty}$ that converges weakly in $L^1(\Omega)$.

Remark 2.1. If g is an increasing function with $g \geq 0$ and $\lim_{t \rightarrow \infty} g(t) = \infty$, then g satisfies Assumption 6. This fact is a consequence of the *De la Vallée Poussin* lemma together with the Dunford–Pettis theorem; see, for instance, Theorems 2.29 and 2.54 in [17].

Our final assumption concerns the monotonicity of the functional F introduced in Definition 1.1. Monotonicity is the key ingredient in the proof of Theorem 1.2 from Theorem 1.5 through Minty’s method.

Assumption 7. *The functional F introduced in Definition 1.1 is monotone with respect to the $L^2 \times L^2$ -inner product; that is, for all $(\eta_1, v_1), (\eta_2, v_2) \in \mathcal{A} \times H_h^2(\Omega)$, F satisfies*

$$\left\langle F \begin{bmatrix} \eta_1 \\ v_1 \end{bmatrix} - F \begin{bmatrix} \eta_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} \eta_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} \eta_2 \\ v_2 \end{bmatrix} \right\rangle \geq 0.$$

3. PROPERTIES OF WEAK SOLUTIONS

In this section, we investigate properties of weak solutions, (m, u) , to Problem 2. First, we prove an a priori estimate for classical solutions. Then, we verify that this a priori estimate also holds for weak solutions. Finally, we show that u is bounded in $W^{1,\gamma}(\Omega)$, and that $(\sqrt{\epsilon}m, \sqrt{\epsilon}u)$ is bounded in $H^{2k}(\Omega) \times H^{2k}(\Omega)$. In Section 6, we combine these estimates with Schaefer’s fixed-point theorem to prove the existence of weak solutions to Problem 2.

To simplify the notation, throughout this section, we use the same letter C to denote any positive constant that depends only on the problem data; that is, may depend on Ω , d , γ , H , V , ϕ , ξ , and h , on the constants in the Assumptions 1–5, or on universal constants such as the constant in Morrey’s theorem or Gagliardo–Nirenberg interpolation inequality. In particular, these constants do not depend on the particular choice of solutions to Problem 2 nor on ϵ .

Proposition 3.1. Consider Problem 2 and suppose that Assumptions 1–4 hold for some $\gamma > 1$. Then, there exists a positive constant, C , that depends only on the problem data such that for any solution (m, u) to Problem 2 in the classical sense, we have

$$\int_{\Omega} \left(mg(m) + \frac{1}{C} m |Du|^{\gamma} + \frac{1}{C} \phi |Du|^{\gamma} \right) dx + \epsilon \int_{\Omega} \left[m^2 + u^2 + \sum_{|\alpha|=2k} ((\partial^{\alpha} m)^2 + (\partial^{\alpha} u)^2) \right] dx \leq C. \quad (3.1)$$

Proof. Multiplying the first equation in (1.4) by $(m - \phi)$ and the second one by $(u - h)$, adding and integrating over Ω , and then using integration by parts and the boundary conditions,

we have

$$\begin{aligned}
& \int_{\Omega} \left[mg(m) + m(-H(x, Du) + D_p H(x, Du) \cdot Du) + \phi H(x, Du) \right. \\
& \qquad \qquad \qquad \left. + \epsilon \left(m^2 + u^2 + \sum_{|\alpha|=2k} ((\partial^\alpha m)^2 + (\partial^\alpha u)^2) \right) \right] dx \\
&= \int_{\Omega} \left[\phi g(m) + (\epsilon \phi + V + h)m + (\epsilon \xi h - V \phi - \phi h) + m D_p H(x, Du) \cdot Dh \right. \\
& \qquad \qquad \qquad \left. + \epsilon \left(u(h - \xi) + \sum_{|\alpha|=2k} (\partial^\alpha \phi \partial^\alpha m + \partial^\alpha u (\partial^\alpha h - \partial^\alpha \xi) + \partial^\alpha \xi \partial^\alpha h) \right) \right] dx.
\end{aligned} \tag{3.2}$$

By Assumptions 1–3, Young’s inequality, and the positivity of m and ϕ , we have

$$\begin{aligned}
& \int_{\Omega} m(-H(x, Du) + D_p H(x, Du) \cdot Du) dx \geq \int_{\Omega} \left(\frac{m|Du|^\gamma}{C} - Cm \right) dx, \\
& \int_{\Omega} \phi H(x, Du) dx \geq \int_{\Omega} \left(\frac{\phi|Du|^\gamma}{C} - C\phi \right) dx, \\
& \int_{\Omega} m D_p H(x, Du) \cdot Dh dx \leq \int_{\Omega} Cm(|Du|^{\gamma-1} + 1) \leq \int_{\Omega} \left(\frac{m|Du|^\gamma}{2C} + Cm \right) dx.
\end{aligned}$$

Therefore, from Assumption 4, Young’s inequality, and (3.2), we obtain

$$\begin{aligned}
& \int_{\Omega} \left[mg(m) + \frac{m|Du|^\gamma}{C} + \frac{\phi|Du|^\gamma}{C} + \frac{\epsilon}{2} \left(m^2 + u^2 + \sum_{|\alpha|=2k} ((\partial^\alpha m)^2 + (\partial^\alpha u)^2) \right) \right] dx \\
& \leq \int_{\Omega} \left(\phi g(m) + Cm + \frac{m|Du|^\gamma}{2C} \right) dx + C \leq \frac{1}{2} \int_{\Omega} \left(mg(m) + \frac{m|Du|^\gamma}{C} \right) dx + C,
\end{aligned}$$

from which Proposition 3.1 follows. \square

Corollary 3.2. Consider Problem 2 and suppose that Assumptions 1–4 hold for some $\gamma > 1$, and let (m, u) be a weak solution to Problem 2 in the sense of Definition 1.3. Then, (m, u) satisfies the estimate (3.1).

Proof. Let (m, u) be a weak solution to Problem 2 in the sense of Definition 1.3. Taking $v = u - h \in H_0^{2k}(\Omega)$ and $w = \phi \in \mathcal{A}$ in (E2) and (E3) in Definition 1.3, and then summing the resulting inequalities, we obtain (3.2) with “=” replaced by “ \leq ”. Thus, arguing as in Proposition 3.1, we conclude that (m, u) satisfies (3.1). \square

Corollary 3.3. Consider Problem 2 and suppose that Assumptions 1–5 hold for some $\gamma > 1$. Then, there exists a positive constant, C , that depends only on the problem data such that for any weak solution (m, u) to Problem 2, we have $\|u\|_{W^{1,\gamma}(\Omega)} \leq C$.

Proof. By Corollary 3.2, the positivity of m and ϕ , and Assumption 5, we have

$$\left| \int_{\Omega} mg(m) dx \right| \leq C.$$

On the other hand, using (E2) in Definition 1.3 with $w := m+1 \in \mathcal{A}$, $\epsilon \leq 1$, Assumption 4, the previous estimate, and a weighted Young’s inequality, we obtain

$$\int_{\Omega} H(x, Du) dx \leq \int_{\Omega} (\epsilon m - u + g(m) - V) dx \leq \sigma \int_{\Omega} |u|^\gamma dx + C_\sigma,$$

where $\sigma > 0$ is arbitrary. Moreover, because $u - h = 0$ on $\partial\Omega$, Poincaré’s inequality yields

$$\int_{\Omega} |u|^\gamma dx \leq C \int_{\Omega} |Du|^\gamma dx + C. \tag{3.3}$$

Then, invoking Assumption 2 and using the estimates above with σ chosen appropriately, we obtain

$$\int_{\Omega} |Du|^\gamma dx \leq C.$$

Using (3.3) once more, we conclude that $\|u\|_{W^{1,\gamma}(\Omega)} \leq C$, where C is a positive constant that depends only on the problem data. \square

Corollary 3.4. Consider Problem 2 and suppose that Assumptions 1–5 hold for some $\gamma > 1$. Then, there exists a positive constant, C , that depends only on the problem data such that for any weak solution (m, u) to Problem 2, we have $\|\sqrt{\epsilon}m\|_{H^{2k}(\Omega)} + \|\sqrt{\epsilon}u\|_{H^{2k}(\Omega)} \leq C$.

Proof. Using Corollary 3.2, Assumption 5, and the positivity of m and ϕ , we obtain

$$\epsilon \int_{\Omega} \left[m^2 + u^2 + \sum_{|\alpha|=2k} ((\partial^\alpha m)^2 + (\partial^\alpha u)^2) \right] dx \leq C,$$

where C is a positive constant that depends only on the problem data. The conclusion follows from the Gagliardo–Nirenberg interpolation inequality. \square

4. AN AUXILIARY VARIATIONAL PROBLEM

Here, we introduce a variational problem whose Euler–Lagrange equation is the first equation in (1.4). We prove the existence and uniqueness of a minimizer, m , to this variational problem. Then, we investigate properties of m that enable us to prove the existence and uniqueness of a weak solution to Problem 2 in Section 6.

Given $(m, u) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m \geq 0$, let $I_{(m,u)} : H^{2k}(\Omega) \rightarrow \mathbb{R}$ be the functional defined, for $w \in H^{2k}(\Omega)$, by

$$I_{(m,u)}[w] := \int_{\Omega} \left[\frac{\epsilon}{2} \left(w^2 + \sum_{|\alpha|=2k} (\partial^\alpha w)^2 \right) + (-u - H(x, Du) + g(m) - V)w \right] dx. \quad (4.1)$$

Note that by Morrey’s embedding theorem, $H^{2k-2}(\Omega)$ is compactly embedded in $C^{0,l}(\overline{\Omega})$ for some $l \in (0, 1)$. In particular, there exists a positive constant, $C = C(\Omega, k, d, l)$, such that for all $\vartheta \in H^{2k-2}(\Omega)$, we have

$$\|\vartheta\|_{C^{0,l}(\overline{\Omega})} \leq C \|\vartheta\|_{H^{2k-2}(\Omega)}. \quad (4.2)$$

Next, we fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geq 0$, and set $I_0 = I_{(m_0, u_0)}$. We address the variational problem finding $m \in \mathcal{A}$ such that

$$I_0[m] = \inf_{w \in \mathcal{A}} I_0[w], \quad (4.3)$$

where \mathcal{A} is defined in (1.1).

Proposition 4.1. Let H, g , and V be as in Problem 2, and fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geq 0$. Then, there exists a unique $m \in \mathcal{A}$ satisfying (4.3).

Proof. Let $\{m_n\}_{n=1}^\infty \subset \mathcal{A}$ be a minimizing sequence for (4.3), and fix $\delta \in (0, 1)$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$I_0[m_n] < \inf_{w \in \mathcal{A}} I_0[w] + \delta \leq I_0[0] + 1 = 1. \quad (4.4)$$

By (4.2), we have $m_0 \in C^{0,l}(\overline{\Omega})$ and $u_0 \in C^{1,l}(\overline{\Omega})$ for some $l \in (0, 1)$, and $C_0 := \|-u_0 - H(x, Du_0) + g(m_0) - V\|_{L^\infty(\Omega)} \in \mathbb{R}$. Then, using Young’s inequality and (4.4), for all $n \geq N$, we have

$$\frac{\epsilon}{2} \int_{\Omega} \left[m_n^2 + \sum_{|\alpha|=2k} (\partial^\alpha m_n)^2 \right] dx \leq \int_{\Omega} C_0 m_n dx + 1 \leq \frac{\epsilon}{4} \int_{\Omega} m_n^2 dx + \frac{C_0^2}{\epsilon} + 1. \quad (4.5)$$

By the Gagliardo–Nirenberg interpolation inequality, we have

$$\|\partial^\alpha m_n\|_{L^2(\Omega)}^2 \leq C (\|m_n\|_{L^2(\Omega)}^2 + \|D^{2k} m_n\|_{L^2(\Omega)}^2), \quad (4.6)$$

where α is any multi-index such that $|\alpha| \leq 2k$. Hence, from (4.5) and (4.6), we conclude that $\{m_n\}_{n=1}^\infty$ is bounded in $H^{2k}(\Omega)$. Consequently, $m_n \rightharpoonup m$ weakly in $H^{2k}(\Omega)$ for some $m \in H^{2k}(\Omega)$, extracting a subsequence if necessary. Because $m_n \geq 0$, also $m \geq 0$; so, $m \in \mathcal{A}$.

Moreover, $m_n \rightarrow m$ in $L^2(\Omega)$ and $\|D^{2k}m\|_{L^2(\Omega)}^2 \leq \liminf_{n \rightarrow \infty} \|D^{2k}m_n\|_{L^2(\Omega)}^2$; hence, $I_0[m] \leq \liminf_n I_0[m_n] = \inf I_0[w] \leq I_0[m]$, which shows that m is a minimizer of I_0 over \mathcal{A} .

We now prove uniqueness. Assume that $m, \tilde{m} \in \mathcal{A}$ are minimizers of I_0 over \mathcal{A} with $m \neq \tilde{m}$. Then, $\frac{m+\tilde{m}}{2} \in \mathcal{A}$ and, recalling that $m - \tilde{m} \in C^0(\bar{\Omega})$, $\int_{\Omega} (m - \tilde{m})^2 dx > 0$. Moreover,

$$\begin{aligned}
I_0 \left[\frac{m + \tilde{m}}{2} \right] &= \int_{\Omega} \left[\frac{\epsilon}{2} \left(\left(\frac{m + \tilde{m}}{2} \right)^2 + \sum_{|\alpha|=2k} \left(\frac{\partial^\alpha m + \partial^\alpha \tilde{m}}{2} \right)^2 \right) \right. \\
&\quad \left. + (-u_0 - H(x, Du_0) + g(m_0) - V) \left(\frac{m + \tilde{m}}{2} \right) \right] dx \\
&= \frac{1}{2} \int_{\Omega} \left[\frac{\epsilon}{2} (m^2 + \sum_{|\alpha|=2k} (\partial^\alpha m)^2) + (-u_0 - H(x, Du_0) + g(m_0) - V)m \right] dx \\
&\quad + \frac{1}{2} \int_{\Omega} \left[\frac{\epsilon}{2} (\tilde{m}^2 + \sum_{|\alpha|=2k} (\partial^\alpha \tilde{m})^2) + (-u_0 - H(x, Du_0) + g(m_0) - V)\tilde{m} \right] dx \\
&\quad - \frac{\epsilon}{8} \int_{\Omega} \left[(m - \tilde{m})^2 + \sum_{|\alpha|=2k} (\partial^\alpha m - \partial^\alpha \tilde{m})^2 \right] dx \\
&< \frac{1}{2} I_0[m] + \frac{1}{2} I_0[\tilde{m}] = \min_{w \in \mathcal{A}} I_0[w],
\end{aligned} \tag{4.7}$$

which contradicts the fact that $\frac{m+\tilde{m}}{2} \in \mathcal{A}$. Thus, $m = \tilde{m}$. \square

Corollary 4.2. Let H, g , and V be as in Problem 2, fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geq 0$, and let $m \in \mathcal{A}$ be the unique solution to (4.3). Set $C_0 := \|u_0 - H(x, Du_0) + g(m_0) - V\|_{L^\infty(\Omega)}$. Then, there exists a positive constant, C , depending only on the problem data and on C_0 , such that $\|m\|_{H^{2k}(\Omega)} \leq C$.

Proof. Because $I_0[m] \leq I_0[0]$, (4.5) and (4.6) hold with m_n replaced by m , from which Corollary 4.2 follows. \square

Proposition 4.3. Let H, g , and V be as in Problem 2, fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geq 0$, and let $m \in \mathcal{A}$ be the unique solution to (4.3). Then, for all $w \in \mathcal{A}$, m satisfies

$$\begin{aligned}
&\int_{\Omega} (-u_0 - H(x, Du_0) + g(m_0) - V)(w - m) dx \\
&\quad + \int_{\Omega} \left[\epsilon m(w - m) + \epsilon \sum_{|\alpha|=2k} \partial^\alpha m (\partial^\alpha w - \partial^\alpha m) \right] dx \geq 0.
\end{aligned} \tag{4.8}$$

Proof. Let $w \in \mathcal{A}$. If $\tau \in [0, 1]$, then $m + \tau(w - m) = (1 - \tau)m + \tau w \in \mathcal{A}$; hence, the mapping $i : [0, 1] \rightarrow \mathbb{R}$ given by

$$i[\tau] := I_0[m + \tau(w - m)]$$

is a well-defined C^1 -function.

Because $i(0) \leq i(\tau)$ for all $0 \leq \tau \leq 1$, we have $i'(0) \geq 0$. On the other hand, for $0 < \tau \leq 1$, we have

$$\begin{aligned}
\frac{1}{\tau} (i(\tau) - i(0)) &= \int_{\Omega} (-u_0 - H(x, Du_0) + g(m_0) - V)(w - m) dx \\
&\quad + \epsilon \int_{\Omega} \left[m(w - m) + \sum_{|\alpha|=2k} \partial^\alpha m (\partial^\alpha w - \partial^\alpha m) \right] dx \\
&\quad + \tau \frac{\epsilon}{2} \int_{\Omega} \left[(w - m)^2 + \sum_{|\alpha|=2k} (\partial^\alpha w - \partial^\alpha m)^2 \right] dx.
\end{aligned}$$

Consequently, letting $\tau \rightarrow 0^+$ in this equality and using $i'(0) \geq 0$, we obtain (4.8). \square

Proposition 4.4. Let H, g , and V be as in Problem 2, fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geq 0$, and let m be the unique solution of (4.3). Set $\Omega_1 = \{x \in \Omega \mid m(x) > 0\}$; then, m satisfies

$$-u_0 - H(x, Du_0) + g(m_0) - V + \epsilon(m + \Delta^{2k}m) = 0 \quad \text{pointwise in } \Omega_1$$

and

$$-u_0 - H(x, Du_0) + g(m_0) - V + \epsilon(m + \Delta^{2k}m) \geq 0 \quad \text{in the sense of distributions in } \Omega.$$

Proof. To prove Proposition 4.4, it suffices to argue as in Remark 1.4, invoking (4.8) in place of (E2) and recalling the embedding $H^{2k-2}(\Omega) \hookrightarrow C^{0,l}(\overline{\Omega})$ for some $l \in (0, 1)$. \square

5. A PROBLEM GIVEN BY A BILINEAR FORM

Here, we consider an auxiliary problem determined by a bilinear form related to the second equation in (1.4). Using the Lax–Milgram Theorem, we prove the existence and uniqueness of a solution, u , to this auxiliary problem, and we establish a uniform bound on u . These results are used in Section 6 to study the existence and uniqueness of a weak solution to Problem 2.

With H, ϕ , and ξ as in Problem 2, given $(m, u) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m \geq 0$, we define a bilinear form, $B : H_0^{2k}(\Omega) \times H_0^{2k}(\Omega) \rightarrow \mathbb{R}$, and a linear functional, $f_{(m,u)} : L^2(\Omega) \rightarrow \mathbb{R}$, by setting, for $v_1, v_2 \in H_0^{2k}(\Omega)$ and $v \in L^2(\Omega)$,

$$\begin{aligned} B[v_1, v_2] &:= \int_{\Omega} \epsilon \left(v_1 v_2 + \sum_{|\alpha|=2k} \partial^\alpha v_1 \partial^\alpha v_2 \right) dx, \\ \langle f_{(m,u)}, v \rangle &:= \int_{\Omega} \left[-m + \operatorname{div} (m D_p H(x, Du)) + \phi - \epsilon(\xi + \Delta^{2k}\xi) \right] v dx. \end{aligned} \quad (5.1)$$

Fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geq 0$, and set $f_0 := f_{(m_0, u_0)}$. We address next the problem of finding $u \in H_0^{2k}(\Omega)$ satisfying

$$B[u, v] = \langle f_0, v \rangle \quad \text{for all } v \in H_0^{2k}(\Omega). \quad (5.2)$$

Proposition 5.1. Let H, ϕ , and ξ be as in Problem 2, and fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geq 0$. Then, there exists a unique solution, $u \in H_0^{2k}(\Omega)$, to (5.2).

Proof. Because $(m_0, u_0) \in (H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)) \cap (C^{0,l}(\overline{\Omega}) \times C^{1,l}(\overline{\Omega}))$ for some $l \in (0, 1)$ (see (4.2)), we have $(-m_0 + \operatorname{div} (m_0 D_p H(x, Du_0)) + \phi - \epsilon(\xi + \Delta^{2k}\xi)) \in L^2(\Omega)$. Hence, by Hölder's inequality, f_0 is bounded in $L^2(\Omega)$.

Using Hölder's and Poincaré's inequalities, we have $|B[v_1, v_2]| \leq c \|v_1\|_{H_0^{2k}(\Omega)} \|v_2\|_{H_0^{2k}(\Omega)}$ for all $v_1, v_2 \in H_0^{2k}(\Omega)$, where $c > 0$ is a constant independent of v_1 and v_2 . Moreover, we clearly have $B[v_1, v_1] \geq \epsilon \|v_1\|_{H_0^{2k}(\Omega)}^2$ for all $v_1 \in H_0^{2k}(\Omega)$.

Therefore, by the Lax–Milgram Theorem, there exists a unique $u \in H_0^{2k}(\Omega)$ satisfying (5.2). \square

Lemma 5.2. Let H, ϕ , and ξ be as in Problem 2, fix $(m_0, u_0) \in H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ with $m_0 \geq 0$, and let u be the unique solution to (5.2) in $H_0^{2k}(\Omega)$. Then, there exists a positive constant, C , depending only on the problem data and on $\|m_0\|_{H^{2k-2}(\Omega)}$ and $\|u_0\|_{H^{2k-1}(\Omega)}$, such that $\|u\|_{H^{2k}(\Omega)} \leq C$.

Proof. Arguing as in the proof of the Proposition 5.1, we have $c_0 := \|-m_0 + \operatorname{div} (m_0 D_p H(x, Du_0)) + \phi - \epsilon(\xi + \Delta^{2k}\xi)\|_{L^2(\Omega)}^2 \in \mathbb{R}_0^+$. Then, using Young's inequality, we obtain

$$\epsilon \left(\|u\|_{L^2(\Omega)}^2 + \|u\|_{H_0^{2k}(\Omega)}^2 \right) = B[u, u] = \langle f_0, u \rangle \leq \|u\|_{L^2(\Omega)}^2 + \frac{c_0}{4\epsilon}.$$

Hence, $\|u\|_{H_0^{2k}(\Omega)}^2 \leq c_0/(4\epsilon^2)$, and the conclusion follows by Poincaré's inequality. \square

6. PROOF OF THEOREM 1.5

This section is devoted to the proof of Theorem 1.5. First, using Schaefer's fixed-point theorem, we prove existence and uniqueness of a weak solution to (1.4) with $h \equiv 0$. Next, we apply this result to address the case of an arbitrary $h \in C^\infty(\bar{\Omega})$.

Let $\tilde{\mathcal{A}}$ be the subset of \mathcal{A} (see (1.1)) given by

$$\tilde{\mathcal{A}} := \{w \in H^{2k-2}(\Omega) \mid w \geq 0\},$$

and consider the mapping $A : \tilde{\mathcal{A}} \times H^{2k-1}(\Omega) \rightarrow \tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$ defined, for $(m_0, u_0) \in \tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$, by

$$A \begin{bmatrix} m_0 \\ u_0 \end{bmatrix} := \begin{bmatrix} m_0^* \\ u_0^* \end{bmatrix}, \quad (6.1)$$

where $m_0^* \in \mathcal{A}$ is the unique solution to (4.3) and $u_0^* \in H_0^{2k}(\Omega)$ is the unique solution to (5.2). In the following proposition, we show that A is continuous and compact.

Proposition 6.1. Let H, g, V, ϕ , and ξ be as in Problem 2. Then, the mapping $A : \tilde{\mathcal{A}} \times H^{2k-1}(\Omega) \rightarrow \tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$ defined by (6.1) is continuous and compact.

Proof. We start by proving that A is continuous. Let $(m_0, u_0), (m_n, u_n) \in \tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$ be such that $m_n \rightarrow m_0$ in $H^{2k-2}(\Omega)$ and $u_n \rightarrow u_0$ in $H^{2k-1}(\Omega)$. We want to show that $m_n^* \rightarrow m_0^*$ in $H^{2k-2}(\Omega)$ and $u_n^* \rightarrow u_0^*$ in $H^{2k-1}(\Omega)$, where

$$\begin{bmatrix} m_0^* \\ u_0^* \end{bmatrix} = A \begin{bmatrix} m_0 \\ u_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} m_n^* \\ u_n^* \end{bmatrix} = A \begin{bmatrix} m_n \\ u_n \end{bmatrix}.$$

Recalling (4.1) and (5.1), we set $I_n := I_{(m_n, u_n)}$ and $f_n := f_{(m_n, u_n)}$. By the definition of A , we have that (m_0^*, u_0^*) and (m_n^*, u_n^*) belong to $\mathcal{A} \times H_0^{2k}(\Omega)$ and satisfy, for all $v \in L^2(\Omega)$,

$$I_0[m_0^*] = \min_{w \in \mathcal{A}} I_0[w], \quad I_n[m_n^*] = \min_{w \in \mathcal{A}} I_n[w], \quad B[u_0^*, v] = \langle f_0, v \rangle, \quad B[u_n^*, v] = \langle f_n, v \rangle.$$

Also, using (4.2), there exists a positive constant, $c > 0$, independent of $n \in \mathbb{N}$, such that

$$\sup_{n \in \mathbb{N}} (\|m_0\|_{L^\infty(\Omega)} + \|m_n\|_{L^\infty(\Omega)} + \|u_0\|_{W^{1,\infty}(\Omega)} + \|u_n\|_{W^{1,\infty}(\Omega)}) < c. \quad (6.2)$$

Then, because $H, D_p H$, and g are locally Lipschitz functions, we have

$$\tilde{c} := \max \{Lip(H; \Omega \times B(0, c)), Lip(D_p H; \Omega \times B(0, c)), Lip(g; B(0, c))\} \in \mathbb{R}_0^+. \quad (6.3)$$

Using the fact that m_n^* and m_0^* are minimizers, we have

$$I_0[m_0^*] + I_n[m_n^*] \leq I_0 \left[\frac{m_0^* + m_n^*}{2} \right] + I_n \left[\frac{m_0^* + m_n^*}{2} \right].$$

Then, exploiting the second equality in (4.7) in the preceding estimate first, and then using Young's inequality and (6.3), we obtain

$$\begin{aligned} & \int_{\Omega} \frac{\epsilon}{4} \left[(m_0^* - m_n^*)^2 + \sum_{|\alpha|=2k} (\partial^\alpha m_0^* - \partial^\alpha m_n^*)^2 \right] dx \\ & \leq \int_{\Omega} \frac{1}{2} |m_0^* - m_n^*| (|u_0 - u_n| + |H(x, Du_0) - H(x, Du_n)| + |g(m_0) - g(m_n)|) dx \quad (6.4) \\ & \leq \int_{\Omega} \left[\frac{\epsilon}{8} (m_0^* - m_n^*)^2 + \frac{6}{\epsilon} ((u_0 - u_n)^2 + \tilde{c}^2 |Du_0 - Du_n|^2 + \tilde{c}^2 (m_0 - m_n)^2) \right] dx. \end{aligned}$$

Because $m_n \rightarrow m_0$ in $H^{2k-2}(\Omega)$ and $u_n \rightarrow u_0$ in $H^{2k-1}(\Omega)$, (6.4) yields

$$\lim_{n \rightarrow \infty} \|m_0^* - m_n^*\|_{L^2(\Omega)} = 0, \quad \lim_{n \rightarrow \infty} \sum_{|\alpha|=2k} \|\partial^\alpha m_0^* - \partial^\alpha m_n^*\|_{L^2(\Omega)} = 0.$$

Then, by the Gagliardo–Nirenberg interpolation inequality, we have $m_n^* \rightarrow m_0^*$ in $H^{2k-2}(\Omega)$.

Next, we show that u_n^* converges to u_0^* in $H^{2k}(\Omega)$. By (5.1), (6.2), and (6.3), we have

$$\begin{aligned}
& \epsilon \left(\|u_0^* - u_n^*\|_{L^2(\Omega)}^2 + \sum_{|\alpha|=2k} \|\partial^\alpha u_0^* - \partial^\alpha u_n^*\|_{L^2(\Omega)}^2 \right) \\
&= B[u_0^* - u_n^*, u_0^* - u_n^*] = \langle f_0 - f_n, u_0^* - u_n^* \rangle \\
&\leq \int_{\Omega} [|m_n - m_0| |u_0^* - u_n^*| + |m_0 D_p H(x, Du_0) - m_n D_p H(x, Du_n)| |Du_0^* - Du_n^*|] dx \\
&\leq \int_{\Omega} [|m_n - m_0| |u_0^* - u_n^*| + \tilde{c} |m_0 - m_n| |Du_0^* - Du_n^*| + c |Du_0 - Du_n| |Du_0^* - Du_n^*|] dx.
\end{aligned} \tag{6.5}$$

Using Gagliardo–Nirenberg interpolation inequality together with Young’s inequality, we obtain from (6.5) that

$$\|u_0^* - u_n^*\|_{L^2(\Omega)}^2 + \sum_{|\alpha|=2k} \|\partial^\alpha u_0^* - \partial^\alpha u_n^*\|_{L^2(\Omega)} \leq C (\|m_0 - m_n\|_{L^2(\Omega)}^2 + \|Du_0 - Du_n\|_{L^2(\Omega)}^2)$$

for some constant $C > 0$ independent of $n \in \mathbb{N}$. Arguing as before, we conclude that $u_n^* \rightarrow u_0^*$ in $H^{2k}(\Omega)$.

Finally, we address the compactness of A . We want to show that if $\{(m_n, u_n)\}_{n=1}^\infty$ is a bounded sequence in $\tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$, then $\{A(m_n, u_n)\}_{n=1}^\infty$ is pre-compact in $\tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$. This is an immediate consequence of (4.2), Corollary 4.2, Lemma 5.2, and the compact embedding $H^{2k}(\Omega) \times H^{2k}(\Omega) \hookrightarrow H^{2k-2}(\Omega) \times H^{2k-1}(\Omega)$ due to the Rellich–Kondrachov theorem. \square

As we mentioned before, the existence of weak solutions to Problem 2 follows from Schaefer’s fixed-point Theorem. We state next the version of this result that we use here, whose proof is a straightforward adaptation of the proof of Theorem 4, Section 9.2.2, in [15].

Theorem 6.2. *Let X be a convex and closed subset of a Banach space with the property that $\lambda w \in X$ whenever $w \in X$ and $\lambda \in [0, 1]$. Assume that $A : X \rightarrow X$ is a continuous and compact mapping such that the set*

$$\{w \in X \mid w = \lambda A[w] \text{ for some } \lambda \in [0, 1]\}$$

is bounded. Then, A has a fixed point.

Proposition 6.3. Consider Problem 2, let A be the mapping defined in (6.1), and suppose that Assumptions 1–5 hold for some $\gamma > 1$. Then, there exists a unique weak solution, $(m, u) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$, to Problem 2 with $h = 0$ in the sense of Definition 1.3.

Proof. (Existence) Fix $\lambda \in [0, 1]$, and let $(m_\lambda, u_\lambda) \in \tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$ be such that

$$\begin{bmatrix} m_\lambda \\ u_\lambda \end{bmatrix} = \lambda A \begin{bmatrix} m_\lambda \\ u_\lambda \end{bmatrix}.$$

If $\lambda = 0$, then $(m_\lambda, u_\lambda) = (0, 0)$. Assume that $0 < \lambda \leq 1$; then, by the definition of A , Proposition 4.1, Corollary 4.3, and Proposition 5.1, we have $\frac{m_\lambda}{\lambda} \in \mathcal{A}$, $\frac{u_\lambda}{\lambda} \in H_0^{2k}(\Omega)$, and

$$\begin{aligned}
& \int_{\Omega} \lambda (-u_\lambda - H(x, Du_\lambda) + g(m_\lambda) - V)(w - m_\lambda) dx \\
& \quad + \int_{\Omega} \epsilon m_\lambda (w - m_\lambda) + \epsilon \sum_{|\alpha|=2k} \partial^\alpha m_\lambda (\partial^\alpha w - \partial^\alpha m_\lambda) dx \geq 0, \\
& \int_{\Omega} [\lambda (m_\lambda - \operatorname{div}(m_\lambda D_p H(x, Du_\lambda)) - \phi)v] dx \\
& \quad + \int_{\Omega} \left[\epsilon (u_\lambda v + \sum_{|\alpha|=2k} \partial^\alpha u_\lambda \partial^\alpha v) + \epsilon (\xi + \Delta^{2k} \xi)v \right] dx = 0,
\end{aligned}$$

for all $w \in \mathcal{A}$ and $v \in H_0^{2k}(\Omega)$. Hence, arguing as in Corollary 3.2, we have

$$\begin{aligned} & \int_{\Omega} \lambda [m_{\lambda} g(m_{\lambda}) + m_{\lambda} |Du_{\lambda}|^{\gamma} + \phi |Du_{\lambda}|^{\gamma}] \, dx \\ & + \epsilon \int_{\Omega} \left[m_{\lambda}^2 + u_{\lambda}^2 + \sum_{|\alpha|=2k} ((\partial^{\alpha} m_{\lambda})^2 + (\partial^{\alpha} u_{\lambda})^2) \right] \, dx \leq C, \end{aligned}$$

where C is a positive constant independent of λ . Consequently, by Assumption 5 and the positivity of m_{λ} and ϕ , we have

$$\epsilon \int_{\Omega} \left[m_{\lambda}^2 + u_{\lambda}^2 + \sum_{|\alpha|=2k} ((\partial^{\alpha} m_{\lambda})^2 + (\partial^{\alpha} u_{\lambda})^2) \right] \, dx \leq C$$

where C is another positive constant independent of λ . Invoking the Gagliardo–Nirenberg interpolation inequality, we conclude that $(m_{\lambda}, u_{\lambda})$ is uniformly bounded in $H^{2k}(\Omega) \times H^{2k}(\Omega)$ with respect to λ . This fact and Proposition 6.1 allow us to use Theorem 6.2 to conclude that A has a fixed point, $(m, u) \in \tilde{\mathcal{A}} \times H^{2k-1}(\Omega)$. Finally, as before, using the definition of A , Proposition 4.1, Corollary 4.3, and Proposition 5.1, we conclude that (m, u) belongs to $H^{2k}(\Omega) \times H^{2k}(\Omega)$ and satisfies (E1)–(E3) with $h = 0$.

(*Uniqueness*) Assume that there are two fixed points, (m_1, u_1) and (m_2, u_2) . Taking $w = m_2$ in (E2) for (u_1, m_1) and $w = m_1$ in (E2) for (u_2, m_2) , and then summing the resulting inequalities, we have

$$\begin{aligned} & \int_{\Omega} [u_1 - u_2 + H(x, Du_1) - H(x, Du_2) - (g(m_1) - g(m_2))](m_1 - m_2) \, dx \\ & - \int_{\Omega} \left[\epsilon(m_1 - m_2)^2 + \epsilon \sum_{|\alpha|=2k} (\partial^{\alpha} m_1 - \partial^{\alpha} m_2)^2 \right] \, dx \geq 0. \end{aligned} \quad (6.6)$$

Because $u_1 - u_2 \in H_0^{2k}(\Omega)$, choosing $v = u_1 - u_2$ in (E3) for (u_1, m_1) and (u_2, m_2) , and then subtracting the resulting equalities, we obtain

$$\begin{aligned} & \int_{\Omega} \left(m_1 - m_2 - \operatorname{div} (m_1 D_p H(x, Du_1) - m_2 D_p H(x, Du_2)) \right) (u_1 - u_2) \, dx \\ & + \int_{\Omega} \left[\epsilon(u_1 - u_2)^2 + \epsilon \sum_{|\alpha|=2k} (\partial^{\alpha} u_1 - \partial^{\alpha} u_2)^2 \right] \, dx = 0. \end{aligned} \quad (6.7)$$

Subtracting (6.6) from (6.7) and using Assumption 7, we have

$$\begin{aligned} 0 & \geq \int_{\Omega} \left[\epsilon(m_1 - m_2)^2 + \epsilon \sum_{|\alpha|=2k} (\partial^{\alpha} m_1 - \partial^{\alpha} m_2)^2 + \epsilon(u_1 - u_2)^2 + \epsilon \sum_{|\alpha|=2k} (\partial^{\alpha} u_1 - \partial^{\alpha} u_2)^2 \right] \, dx \\ & + \left\langle F \begin{bmatrix} m_1 \\ u_1 \end{bmatrix} - F \begin{bmatrix} m_2 \\ u_2 \end{bmatrix}, \begin{bmatrix} m_1 \\ u_1 \end{bmatrix} - \begin{bmatrix} m_2 \\ u_2 \end{bmatrix} \right\rangle \geq 0. \end{aligned}$$

Invoking Assumption 7 once more, we conclude that the integral in the preceding estimate is equal to zero, from which we the identity $(m_1, u_1) = (m_2, u_2)$ follows. \square

Proof of Theorem 1.5. Define, for $x \in \Omega$ and $p \in \mathbb{R}^d$, $\widehat{H}(x, p) := H(x, p + Dh(x))$, $\widehat{V}(x) := V(x) + h(x)$, and $\widehat{\xi}(x) := \xi(x) + h(x)$.

Note that H satisfies Assumptions 1–3 for some $\gamma > 1$ if and only if \widehat{H} satisfies Assumptions 1–3 for the same γ . Moreover, $(u, m) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfies (E1)–(E3) if and only if $(\hat{u}, m) := (u - h, m) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfies (E1)–(E3) with $h = 0$ and with H , V , and ξ replaced by \widehat{H} , \widehat{V} , and $\widehat{\xi}$, respectively.

To conclude, it suffices to invoke Proposition 6.3, which gives existence and uniqueness of a pair $(\hat{u}, m) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ satisfying (E1)–(E3) with $h = 0$ and with H , V , and ξ replaced by \widehat{H} , \widehat{V} , and $\widehat{\xi}$, respectively. \square

7. PROOF OF THEOREM 1.2

Here, we prove Theorem 1.2. First, we study a compactness property of the unique weak solution to Problem 2. Then, we introduce a linear functional, F_ϵ , corresponding to the equations (1.4) in Problem 2 and address its monotonicity. Finally, using Minty's method, we prove the existence of a weak solution to Problem 1.

Lemma 7.1. Consider Problem 2 and suppose that Assumptions 1–6 hold for some $\gamma > 1$. Let $(m_\epsilon, u_\epsilon) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ be the unique weak solution to Problem 2. Then, there exists $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ such that $m \geq 0$, $u = h$ on $\partial\Omega$ in the sense of traces, and (m_ϵ, u_ϵ) converges to (m, u) weakly in $L^1(\Omega) \times W^{1,\gamma}(\Omega)$ as $\epsilon \rightarrow 0$, extracting a subsequence if necessary.

Proof. The existence of $u \in W^{1,\gamma}(\Omega)$ as stated follows from the fact that $u_\epsilon = h$ on $\partial\Omega$ in the sense of traces and, by Corollary 3.3, u_ϵ is uniformly bounded in $W^{1,\gamma}(\Omega)$ with respect to ϵ .

On the other hand, by Corollary 3.2 and the positivity of m_ϵ and ϕ , we have

$$\sup_{\epsilon \in (0,1)} \int_{\Omega} m_\epsilon g(m_\epsilon) \, dx < \infty.$$

Therefore, by Assumption 6, there exists $m \in L^1(\Omega)$ such that $m_\epsilon \rightharpoonup m$ in $L^1(\Omega)$ as $\epsilon \rightarrow 0$, extracting a subsequence if necessary. Because $m_\epsilon \geq 0$, we have $m \geq 0$. \square

Fix $(\eta, v) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$, let $F[\eta, v]$ be the functional introduced in (1.3), and let $F_\epsilon[\eta, v] : H^{2k}(\Omega) \times H^{2k}(\Omega) \rightarrow \mathbb{R}$ be the linear functional given by

$$\begin{aligned} \left\langle F_\epsilon \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle &:= \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle + \int_{\Omega} \left(\epsilon \eta w_1 + \epsilon \sum_{|\alpha|=2k} \partial^\alpha \eta \partial^\alpha w_1 \right) \, dx \\ &+ \int_{\Omega} \left[\epsilon (v + \xi) w_2 + \epsilon \sum_{|\alpha|=2k} \partial^\alpha (v + \xi) \partial^\alpha w_2 \right] \, dx. \end{aligned} \quad (7.1)$$

Next, we prove the monotonicity of F_ϵ over $\mathcal{A} \times H_h^{2k}(\Omega)$, where, we recall, \mathcal{A} and $H_h^{2k}(\Omega)$ are given by (1.1)–(1.2).

Lemma 7.2. Let H, g, V, ϕ, ξ , and h be as in Problem 2, let F_ϵ be given by (7.1), and suppose that Assumption 7 holds. Then, for any $(\eta_1, v_1), (\eta_2, v_2) \in \mathcal{A} \times H_h^{2k}(\Omega)$, we have

$$\left\langle F_\epsilon \begin{bmatrix} \eta_1 \\ v_1 \end{bmatrix} - F_\epsilon \begin{bmatrix} \eta_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} \eta_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} \eta_2 \\ v_2 \end{bmatrix} \right\rangle \geq 0.$$

Proof. Let $(\eta_1, v_1), (\eta_2, v_2) \in \mathcal{A} \times H_h^{2k}(\Omega)$. Then, $v_1 - v_2 \in H_0^{2k}(\Omega)$; thus, using Assumption 7 and integrating by parts, we obtain

$$\begin{aligned} &\left\langle F_\epsilon \begin{bmatrix} \eta_1 \\ v_1 \end{bmatrix} - F_\epsilon \begin{bmatrix} \eta_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} \eta_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} \eta_2 \\ v_2 \end{bmatrix} \right\rangle \\ &\geq \int_{\Omega} \epsilon \left[(\eta_1 - \eta_2)^2 + (v_1 - v_2)^2 + \sum_{|\alpha|=2k} ((\partial^\alpha \eta_1 - \partial^\alpha \eta_2)^2 + (\partial^\alpha v_1 - \partial^\alpha v_2)^2) \right] \, dx \geq 0. \quad \square \end{aligned}$$

Proof of Theorem 1.2. Let $(m_\epsilon, u_\epsilon) \in H^{2k}(\Omega) \times H^{2k}(\Omega)$ be the unique weak solution to Problem 2 in the sense of Definition 1.3. Fix $(\eta, v) \in \mathcal{A} \times H_h^{2k}(\Omega)$. By (E2) and (E3), we have

$$\left\langle F_\epsilon \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle \geq 0.$$

Thus, by Lemma 7.2,

$$0 \leq \left\langle F_\epsilon \begin{bmatrix} \eta \\ v \end{bmatrix} - F_\epsilon \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle \leq \left\langle F_\epsilon \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle = \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle + c_\epsilon, \quad (7.2)$$

where

$$c_\epsilon := \int_{\Omega} \left(\epsilon \eta (\eta - m_\epsilon) + \epsilon \sum_{|\alpha|=2k} \partial^\alpha \eta \partial^\alpha (\eta - m_\epsilon) \right) dx \\ + \int_{\Omega} \left[\epsilon (v + \xi) (v - u_\epsilon) + \epsilon \sum_{|\alpha|=2k} \partial^\alpha (v + \xi) \partial^\alpha (v - u_\epsilon) \right] dx.$$

By Hölder's inequality and Corollary 3.4, we conclude that

$$\lim_{\epsilon \rightarrow 0} c_\epsilon = 0. \quad (7.3)$$

On the other hand, by Lemma 7.1, there exists $(m, u) \in L^1(\Omega) \times W^{1,\gamma}(\Omega)$ satisfying (D1) and such that (m_ϵ, u_ϵ) converges to (m, u) weakly in $L^1(\Omega) \times W^{1,\gamma}(\Omega)$ as $\epsilon \rightarrow 0$, extracting a subsequence if necessary. Then, using the definition of $F[\eta, v]$ (see (1.3)), we get

$$\lim_{\epsilon \rightarrow 0} \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} \right\rangle = \left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix} \right\rangle. \quad (7.4)$$

From (7.2), (7.3), and (7.4), we conclude that

$$\left\langle F \begin{bmatrix} \eta \\ v \end{bmatrix}, \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix} \right\rangle \geq 0;$$

that is, (m, u) also satisfies (D2). Hence, (m, u) is a weak solution of Problem 1 in the sense of Definition 1.1. \square

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