

ON THE BLOW UP CRITERION OF 3D-NAVIER-STOKES EQUATION IN $\dot{H}^{5/2}$

JAMEL BENAMEUR AND HAJER ORF

ABSTRACT. In this paper, we prove two results about the blow up criterion of the three-dimensional incompressible Navier-Stokes equation in the sobolev space $\dot{H}^{5/2}$. The first one improves the result of [8]. The second deals with the relationship of the blow up in $\dot{H}^{5/2}$ and some critical spaces. Fourier analysis and standard techniques are used.

CONTENTS

1.	Introduction	1
2.	Notations and preliminary results	3
2.1.	Notations	3
2.2.	Preliminary results	3
2.3.	Remarks	7
3.	Proof of Theorem 1.1	7
4.	Blow up criterion in $\dot{H}^{5/2}$ with respect to $\dot{H}^{1/2}$ norm	8
5.	Blow up criterion in $\dot{H}^{5/2}$ with respect to \mathcal{X}^{-1} norm	10
6.	General remarks	11
	References	13

1. INTRODUCTION

The 3D incompressible Navier-Stokes equations are given by:

$$(NSE) \quad \begin{cases} \partial_t u - \nu \Delta u + u \cdot \nabla u &= -\nabla p \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u &= 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0, x) &= u^0(x) \text{ in } \mathbb{R}^3, \end{cases}$$

where $\nu > 0$ is the viscosity of fluid, $u = u(t, x) = (u_1, u_2, u_3)$ and $p = p(t, x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and $(u \cdot \nabla u) := u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u$, while $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is an initial given velocity. If u^0 is quite regular, the divergence free condition determines the pressure p .

In 1934, Leray [16] showed that there exists an absolute constant c such that, if $\|u(t)\|_{\dot{H}^{5/2}}$ continuous on $[0, T^*]$, and blow up at time T^* , then

$$\|u(t)\|_{\dot{H}^1} \geq C(T^* - t)^{-1/4}.$$

Moreover, he stated the bound for L^p norms for $3 < p < \infty$ (without proof)

$$\|u(t)\|_{L^p} \geq C(T^* - t)^{\frac{3-p}{2p}}.$$

2000 *Mathematics Subject Classification.* 35-xx, 35Bxx, 35Lxx.

Key words and phrases. Navier-Stokes Equations; Critical spaces; Long time decay.

More recently, there have been a number of papers that treat the problem of blow up in Sobolev spaces \dot{H}^s $s > 1/2$.

Benameur (2010) [4] showed that for $s > 5/2$,

$$\|u(t)\|_{\dot{H}^s} \geq C(T^* - t)^{-\frac{s}{3}},$$

which was improved by Robinson, Sadwiski, Silva (2012) [10]

$$\|u(t)\|_{\dot{H}^s} \geq \begin{cases} C(T^* - t)^{-(2s-1)/4} & 1/2 < s < 5/2, s \neq 3/2, \\ C(T^* - t)^{-s/5} & s > 5/2. \end{cases}$$

The border cases $s = 3/2$ and $s = 5/2$ required separate treatment. In [9], Cortissoz, Montero, and Pinilla (2014) proved lower bounds in $\dot{H}^{3/2}$ and $\dot{H}^{5/2}$ at the optimal rates but with logarithmic corrections,

$$\|u(t)\|_{\dot{H}^{3/2}} \geq \frac{c}{\sqrt{(T^* - t)|\log(T^* - t)|}}, \quad \|u(t)\|_{\dot{H}^{5/2}} \geq \frac{c'}{(T^* - t)|\log(T^* - t)|},$$

where in both cases c depends on $\|u_0\|_{L^2}$. Recently, in [7] the authors proved

$$\|u(t)\|_{\dot{H}^{3/2}} \geq \frac{c}{\sqrt{(T^* - t)}},$$

which we refer to as a strong blowup estimate, and

$$\limsup_{t \rightarrow T^*} (T^* - t) \|u(t)\|_{\dot{H}^{5/2}} \geq c,$$

which we refer to as a weak blowup estimate. They also show a strong blowup estimate in the Besov space $\dot{B}_{2,1}^{5/2}$, which has the same scaling as $\dot{H}^{5/2}$,

$$\|u(t)\|_{\dot{B}_{2,1}^{5/2}} \geq C(T^* - t)^{-1}.$$

The interesting and open question is the strong blow up estimate

$$\|u(t)\|_{\dot{H}^{5/2}} \geq C(T^* - t)^{-1}.$$

The some kind of question appears for Lei-Lin espace

$$\|u(t)\|_{\chi^1} \geq C(T^* - t)^{-1},$$

and,

$$\|\nabla u(t)\|_{L^\infty} \geq C(T^* - t)^{-1}$$

i.e., a bound with the optimal rate in a space with the same scaling as $\dot{H}^{5/2}$.

Our first result is the following:

Theorem 1.1. *Let $u \in \mathcal{C}([0, T^*), H^{5/2}(\mathbb{R}^3)$ be a maximal solution of Navier-Stokes system. If, T^* is finite then, there is a constant $c_0 = c_0(\nu, \|u^0\|_{L^2})$ such that*

$$(1.1) \quad \liminf_{t \nearrow T^*} (T^* - t) \sqrt{|\ln(T^* - t)|} \|u(t)\|_{\dot{H}^{5/2}} \geq c_0.$$

The second result is the following

Theorem 1.2. *Let $u \in \mathcal{C}([0, T^*), H^{5/2}(\mathbb{R}^3)$ be a maximal solution of Navier-Stokes system. If, T^* is finite then, there is a universal constant $c_1 > 0$ such that*

$$(1.2) \quad \liminf_{t \nearrow T^*} (T^* - t) \sqrt{|\ln(4(c\nu)^{-1} \|u(t)\|_{\dot{H}^{1/2}})|} \|u(t)\|_{\dot{H}^{5/2}} \geq c_1.$$

The last result is the following

Theorem 1.3. *Let $u \in \mathcal{C}([0, T^*), H^{5/2}(\mathbb{R}^3)$ be a maximal solution of Navier-Stokes system. If, T^* is finite then, there is a universal constant $c_2 > 0$ such that*

$$(1.3) \quad \liminf_{t \nearrow T^*} (T^* - t) \sqrt{|\ln(8\nu^{-1} \|u(t)\|_{\chi^{-1}})|} \|u(t)\|_{\dot{H}^{5/2}} \geq c_2.$$

The paper is organized in the following way: In section 2, we give some notations and important preliminary results. Section 3 we prove the main result of this paper and we give some important remarks. The proof used standard Fourier techniques. In section 4 and 5 we give a proof respectively of theorem 1.2 and 1.3. In section 6, we give a simple proof of the explosion result in the space $\dot{H}^{3/2}$.

2. Notations and preliminary results

2.1. Notations. In this section, we collect some notations and definitions that will be used later.

- The Fourier transformation is normalized as

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix \cdot \xi) f(x) dx, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

- The inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(i\xi \cdot x) g(\xi) d\xi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

- The convolution product of a suitable pair of function f and g on \mathbb{R}^3 is given by

$$(f * g)(x) := \int_{\mathbb{R}^3} f(y) g(x - y) dy.$$

- If $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ are two vector fields, we set

$$f \otimes g := (g_1 f, g_2 f, g_3 f),$$

and

$$\operatorname{div}(f \otimes g) := (\operatorname{div}(g_1 f), \operatorname{div}(g_2 f), \operatorname{div}(g_3 f)).$$

- Let $(B, \|\cdot\|)$, be a Banach space, $1 \leq p \leq \infty$ and $T > 0$. We define $L_T^p(B)$ the space of all measurable functions $[0, t] \ni t \mapsto f(t) \in B$ such that $t \mapsto \|f(t)\| \in L^p([0, T])$.
- The Sobolev space $H^s(\mathbb{R}^3) = \{f \in \mathcal{S}'(\mathbb{R}^3); (1 + |\xi|^2)^{s/2} \widehat{f} \in L^2(\mathbb{R}^3)\}$.
- The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3) = \{f \in \mathcal{S}'(\mathbb{R}^3); \widehat{f} \in L_{loc}^1 \text{ and } |\xi|^s \widehat{f} \in L^2(\mathbb{R}^3)\}$.
- The Lei-Lin space $\mathcal{X}^\sigma(\mathbb{R}^3) = \{f \in \mathcal{S}'(\mathbb{R}^3); \widehat{f} \in L_{loc}^1 \text{ and } |\xi|^\sigma \widehat{f} \in L^1(\mathbb{R}^3)\}$.

2.2. Preliminary results. In this section, we recall some classical results and we give new technical lemmas.

Lemma 2.1. *We have $\mathcal{X}^{-1}(\mathbb{R}^3) \cap \mathcal{X}^1(\mathbb{R}^3) \hookrightarrow \mathcal{X}^0(\mathbb{R}^3)$. Precisely, we have*

$$(2.1) \quad \|f\|_{\mathcal{X}^0(\mathbb{R}^3)} \leq \|f\|_{\mathcal{X}^{-1}(\mathbb{R}^3)}^{1/2} \|f\|_{\mathcal{X}^1(\mathbb{R}^3)}^{1/2}, \quad \forall f \in \mathcal{X}^{-1}(\mathbb{R}^3) \cap \mathcal{X}^1(\mathbb{R}^3).$$

Proof. We can write

$$\begin{aligned} \|f\|_{\mathcal{X}^0} &= \int |\widehat{f}| d\xi \\ &\leq \int |\xi|^{1/2} |\widehat{f}|^{1/2} \frac{|\widehat{f}|^{1/2}}{|\xi|^{1/2}} d\xi. \end{aligned}$$

Cauchy Schwartz inequality gives the result.

Lemma 2.2. *We have $\mathcal{X}^{-1}(\mathbb{R}^3) \cap \dot{H}^{5/2}(\mathbb{R}^3) \hookrightarrow \mathcal{X}^0(\mathbb{R}^3)$. Precisely, there is a constant $C_1 > 0$ such that*

$$(2.2) \quad \|f\|_{\mathcal{X}^0(\mathbb{R}^3)} \leq C_1 \|f\|_{\mathcal{X}^{-1}(\mathbb{R}^3)}^{1/2} \|f\|_{\dot{H}^{5/2}(\mathbb{R}^3)}^{1/2}, \quad \forall f \in \mathcal{X}^{-1}(\mathbb{R}^3) \cap \dot{H}^{5/2}(\mathbb{R}^3).$$

Proof. For $R > 0$ we have

$$\|f\|_{\chi^0} \leq \|f1_{|D|<R}\|_{\chi^0} + \|f1_{|D|>R}\|_{\chi^0}.$$

Cauchy Schwartz inequality gives

$$\begin{aligned} \|f1_{|D|<R}\|_{\chi^0} &= \int_{|\xi|<R} |\widehat{f}(\xi)| d\xi \\ &\leq \int_{|\xi|<R} |\xi| \frac{|\widehat{f}(\xi)|}{|\xi|} d\xi \\ &\leq R \|f\|_{\chi^{-1}}, \end{aligned}$$

and

$$\begin{aligned} \|f1_{|D|>R}\|_{\chi^0} &= \int_{|\xi|>R} |\widehat{f}(\xi)| d\xi \\ &\leq \int_{|\xi|>R} |\xi|^{\frac{5}{2}} |\widehat{f}(\xi)| |\xi|^{-\frac{5}{2}} d\xi \\ &\leq \left(\int_{|\xi|>R} |\xi|^{-5} \right)^{1/2} \|f\|_{\dot{H}^{5/2}} \\ &\leq \sqrt{\pi} R^{-1} \|f\|_{\dot{H}^{5/2}}, \end{aligned}$$

To conclude, it suffices to take $R = \left(\frac{\|f\|_{\dot{H}^{5/2}}}{\|f\|_{\chi^{-1}}} \right)^{1/2}$

Lemma 2.3. *We have $\dot{H}^{1/2}(\mathbb{R}^3) \cap \mathcal{X}^1(\mathbb{R}^3) \hookrightarrow \mathcal{X}^0(\mathbb{R}^3)$. Precisely, there is a constant $C_2 > 0$ such that*

$$(2.3) \quad \|f\|_{\mathcal{X}^0(\mathbb{R}^3)} \leq C_2 \|f\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^{1/2} \|f\|_{\mathcal{X}^1(\mathbb{R}^3)}^{1/2}, \quad \forall f \in \dot{H}^{1/2}(\mathbb{R}^3) \cap \mathcal{X}^1(\mathbb{R}^3).$$

Proof. For $R > 0$ we have

$$\|f\|_{\chi^0} \leq \|f1_{|D|<R}\|_{\chi^0} + \|f1_{|D|>R}\|_{\chi^0}.$$

Cauchy Schwartz inequality gives

$$\begin{aligned} \|f1_{|D|<R}\|_{\chi^0} &= \int_{|\xi|<R} |\widehat{f}(\xi)| d\xi \\ &\leq \int_{|\xi|<R} |\xi|^{-1/2} |\xi|^{1/2} |\widehat{f}(\xi)| d\xi \\ &\leq \left(\int_{|\xi|<R} |\xi|^{-1} d\xi \right)^{1/2} \|f\|_{\dot{H}^{1/2}} \\ &\leq \sqrt{2\pi} R \|f\|_{\dot{H}^{1/2}} \end{aligned}$$

and

$$\begin{aligned} \|f1_{|D|>R}\|_{\chi^0} &= \int_{|\xi|>R} |\widehat{f}(\xi)| d\xi \\ &\leq \int_{|\xi|>R} |\xi|^{-1} |\xi| |\widehat{f}(\xi)| d\xi \\ &\leq R^{-1} \|f\|_{\chi^1}. \end{aligned}$$

To conclude, it suffices to take $R = \left(\frac{\|f\|_{\chi^1}}{\|f\|_{\dot{H}^{1/2}}} \right)^{1/2}$

Lemma 2.4. *For $s \geq 0$. If $f \in \dot{H}^s(\mathbb{R}^3) \cap \chi^1(\mathbb{R}^3)$, then, there exist a constant $C = C(s) > 0$ such that*

$$(2.4) \quad |\langle f \cdot \nabla f, f \rangle_{\dot{H}^s}| \leq C \|f\|_{\chi^1} \|f\|_{\dot{H}^s}.$$

Proof. If $\operatorname{div} f = 0$ then, $\langle f \cdot \nabla(|D|^s f), |D|^s f \rangle_{L^2} = 0$, which yields:

$$\begin{aligned} |\langle f \cdot \nabla f, f \rangle_{\dot{H}^s}| &\leq |\langle |D|^s(f \cdot \nabla f), |D|^s f \rangle_{L^2}| \\ &\leq |\langle |D|^s(f \cdot \nabla f) - f \cdot \nabla(|D|^s f), |D|^s f \rangle_{L^2}| \\ &\leq \| |D|^s(f \cdot \nabla f) - f \cdot \nabla(|D|^s f) \|_{L^2} \| |D|^s f \|_{L^2} \\ &\leq \| |D|^s(f \cdot \nabla f) - f \cdot \nabla(|D|^s f) \|_{L^2} \| f \|_{\dot{H}^s}. \end{aligned}$$

we will estimate the first norm, we obtain:

$$\begin{aligned} \| |D|^s(f \cdot \nabla f) - f \cdot \nabla(|D|^s f) \|_{L^2} &\leq \left(\int \| |\xi|^s \mathcal{F}(f \cdot \nabla f) - \mathcal{F}(f \cdot \nabla(|D|^s f)) \|^2 d\xi \right)^{1/2} \\ &\leq \left(\int \left| \int |\xi|^s \widehat{f}(\xi - \eta) \widehat{\nabla f}(\eta) - \widehat{f}(\xi - \eta) |\eta|^s \widehat{\nabla f}(\eta) d\eta \right|^2 d\xi \right)^{1/2} \\ &\leq \left(\int \left| \int (|\xi|^s - |\eta|^s) \widehat{f}(\xi - \eta) \widehat{\nabla f}(\eta) d\eta \right|^2 d\xi \right)^{1/2}. \end{aligned}$$

We have:

$$\begin{aligned} \left| |\xi|^s - |\eta|^s \right| &\leq s c^{s-1} |\xi - \eta| \leq s |\xi - \eta| (|\xi|^{s-1} + |\eta|^{s-1}), \\ |\xi|^{s-1} &\leq 2^{s-1} |\xi - \eta|^{s-1} + 2^{s-1} |\eta|^{s-1} \leq C_s (|\xi - \eta|^{s-1} + |\eta|^{s-1}). \end{aligned}$$

Then, we get:

$$\begin{aligned} \| |D|^s(f \cdot \nabla f) - f \cdot \nabla(|D|^s f) \|_{L^2} &\leq C \left(\int \left(\int |\xi - \eta| |\xi - \eta|^{s-1} |\widehat{f}(\xi - \eta)| |\widehat{\nabla f}(\eta)| d\eta \right. \right. \\ &\quad \left. \left. + \int |\xi - \eta| |\widehat{f}(\xi - \eta)| |\eta|^{s-1} |\widehat{\nabla f}(\eta)| d\eta \right)^2 d\xi \right)^{1/2} \\ &\leq C \left(\int \left(\int |\xi - \eta|^s |\widehat{f}(\xi - \eta)| |\widehat{\nabla f}(\eta)| d\eta + \int |\widehat{\nabla f}(\xi - \eta)| |\eta|^s |\widehat{f}(\eta)| d\eta \right)^2 d\xi \right)^{1/2} \\ &\leq C \left(\int \left(\int |\xi - \eta|^s |\widehat{f}(\xi - \eta)| |\widehat{\nabla f}(\eta)| d\eta \right)^2 d\xi \right)^{1/2} \\ &\leq C \| (|\cdot|^s |\mathcal{F}(f)|) * |\mathcal{F}(\nabla f)| \|_{L^2}, \end{aligned}$$

Young lemma yields,

$$\begin{aligned} \| |D|^s(f \cdot \nabla f) - f \cdot \nabla(|D|^s f) \|_{L^2} &\leq C \| (|\cdot|^s |\mathcal{F}(f)|) \|_{L^2} \| \mathcal{F}(\nabla f) \|_{L^1} \\ &\leq C \| f \|_{\dot{H}^s} \| f \|_{\mathcal{X}^1}. \end{aligned}$$

Then, the proof is finished.

Lemma 2.5. For $f \in H^{7/2}(\mathbb{R}^3)$. There is a constant $C > 0$ such that for all $0 < \alpha < \beta < \infty$, we have

$$(2.5) \quad \| f \|_{\mathcal{X}^1} \leq C \sqrt{4\pi} \left[\alpha^{5/2} \| f \|_{L^2} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \| f \|_{\dot{H}^{5/2}} + \frac{1}{\beta} \| f \|_{\dot{H}^{7/2}} \right].$$

$$(2.6) \quad \| f \|_{\mathcal{X}^1} \leq C \sqrt{4\pi} \left[\alpha^2 \| f \|_{\dot{H}^{1/2}} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \| f \|_{\dot{H}^{5/2}} + \frac{1}{\beta} \| f \|_{\dot{H}^{7/2}} \right].$$

$$(2.7) \quad \| f \|_{\mathcal{X}^1} \leq C \sqrt{4\pi} \left[\alpha^2 \| f \|_{\mathcal{X}^{-1}} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \| f \|_{\dot{H}^{5/2}} + \frac{1}{\beta} \| f \|_{\dot{H}^{7/2}} \right].$$

Proof.

Let $0 < \alpha < \beta < \infty$ as:

$$\| f \|_{\mathcal{X}^1} = \int |\xi| |\widehat{f}(\xi)| d\xi = I_\alpha + J_{\alpha, \beta} + K_\beta,$$

where:

$$\begin{aligned} I_\alpha &= \int_{|\xi| < \alpha} |\xi| |\widehat{f}(\xi)| d\xi, \\ J_{\alpha, \beta} &= \int_{\alpha < |\xi| < \beta} |\xi| |\widehat{f}(\xi)| d\xi, \\ K_\beta &= \int_{|\xi| > \beta} |\xi| |\widehat{f}(\xi)| d\xi. \end{aligned}$$

Cauchy Schwartz inequality gives:

$$\begin{aligned} I_\alpha &= \int_{|\xi| < \alpha} |\xi| |\widehat{f}(\xi)| d\xi \\ &\leq \left(\int_{|\xi| < \alpha} |\xi|^2 d\xi \right)^{1/2} \|f\|_{L^2} \\ &\leq \sqrt{\frac{4\pi}{5}} \alpha^{5/2} \|f\|_{L^2}, \\ J_{\alpha, \beta} &= \int_{\alpha < |\xi| < \beta} |\xi| |\widehat{f}(\xi)| d\xi \\ &= \int_{\alpha < |\xi| < \beta} |\xi|^{-3/2} |\xi|^{5/2} |\widehat{f}(\xi)| d\xi \\ &\leq \sqrt{4\pi} \left(\ln \frac{\beta}{\alpha} \right)^{1/2} \|f\|_{\dot{H}^{5/2}}, \end{aligned}$$

and,

$$\begin{aligned} K_\beta &= \int_{|\xi| > \beta} |\xi| |\widehat{f}(\xi)| d\xi \\ &= \int_{|\xi| > \beta} |\xi|^{-5/2} |\xi|^{7/2} |\widehat{f}(\xi)| d\xi \\ &\leq \sqrt{4\pi} \left(\frac{1}{2\beta^2} \right)^{1/2} \|f\|_{\dot{H}^{7/2}} \\ &\leq \sqrt{2\pi} \frac{1}{\beta} \|f\|_{\dot{H}^{7/2}}. \end{aligned}$$

Then, we can deduce (2.5). For the second estimate, we can write

$$\begin{aligned} I_\alpha &= \int_{|\xi| < \alpha} |\xi| |\widehat{f}(\xi)| d\xi \\ &= \int_{|\xi| < \alpha} |\xi|^{1/2} |\xi|^{1/2} |\widehat{u}(\xi)| d\xi \\ &\leq \left(\int_{|\xi| < \alpha} |\xi| d\xi \right)^{1/2} \|f\|_{\dot{H}^{1/2}} \\ &\leq \sqrt{4\pi} \left(\frac{\alpha^4}{4} \right)^{1/2} \|f\|_{\dot{H}^{1/2}} \\ &\leq \sqrt{\pi} \alpha^2 \|f\|_{\dot{H}^{1/2}}, \end{aligned}$$

which gives the inequality (2.6). (2.7) deduce from

$$\begin{aligned} I_\alpha &= \int_{|\xi| < \alpha} |\xi| |\widehat{f}(\xi)| d\xi \\ &= \int_{|\xi| < \alpha} |\xi|^2 |\xi|^{-1} |\widehat{f}(\xi)| d\xi \\ &\leq \alpha^2 \|f\|_{\chi^{-1}}, \end{aligned}$$

Then, the desired result is proved, and the proof of Lemma 2.5 is finished.

2.3. Remarks.

(i) Leray showed that if the maximal data T^* is finite then

$$\|u(t)\|_{\dot{H}^1} \geq C(T^* - t)^{-1/4}$$

Interpolation inequality gives:

$$\|u(t)\|_{\dot{H}^1} \leq \|u^0\|_{L^2}^{3/5} \|u(t)\|_{\dot{H}^{5/2}}^{2/5}$$

then,

$$C(T^* - t)^{-1/4} \leq \|u^0\|_{L^2}^{3/5} \|u(t)\|_{\dot{H}^{5/2}}^{2/5}$$

which implies that

$$\lim_{t \rightarrow T^*} \|u(t)\|_{\dot{H}^{5/2}} = +\infty$$

if $T^* < \infty$ we can suppose that

$$(2.8) \quad \|u(t)\|_{\dot{H}^{5/2}} > 1 \quad \forall t \in [0, T^*).$$

(ii) if there is a time $t_0 \in [0, T^*)$ such that $\|u(t_0)\|_{\dot{H}^{1/2}} < c\nu$ then $T^* = +\infty$.
Particulary, if $T^* < +\infty$, then

$$(2.9) \quad \|u\|_{\dot{H}^{1/2}} \geq c\nu \quad t \in [0, T^*).$$

(iii) if there is a time $t_0 \in [0, T^*)$ such that $\|u(t_0)\|_{\chi^{-1}} < \nu$ then $T^* = +\infty$.
Particulary, if $T^* < +\infty$, then

$$(2.10) \quad \|u\|_{\chi^{-1}} \geq \nu \quad t \in [0, T^*).$$

3. Proof of Theorem 1.1

Let u be a maximal solution of Navier-Stokes system in the space $C([0, T^*), \dot{H}^{5/2}(\mathbb{R}^3)) \cap L^2([0, T^*), \dot{H}^{7/2}(\mathbb{R}^3))$.

Suppose that $u \in \dot{H}^{5/2}(\mathbb{R}^3) \cap \chi^1(\mathbb{R}^3)$. Taking the norm of $\dot{H}^{5/2}(\mathbb{R}^3)$ and using lemma 2.4, we get:

$$\begin{aligned} \partial_t \|u\|_{\dot{H}^{5/2}}^2 + 2\nu \|u\|_{\dot{H}^{7/2}}^2 &\leq |\langle u, \nabla u, u \rangle_{\dot{H}^{5/2}}| \\ &\leq C \|u\|_{\chi^1} \|u\|_{\dot{H}^{5/2}}^2 \end{aligned}$$

Using inequality (2.5) we obtain:

$$\begin{aligned} \partial_t \|u\|_{\dot{H}^{5/2}}^2 + 2\nu \|u\|_{\dot{H}^{7/2}}^2 &\leq C \|u\|_{\dot{H}^{5/2}}^2 \|u\|_{\chi^1} \\ &\leq C \|u\|_{\dot{H}^{5/2}}^2 \left[\alpha^{2/5} \|u^0\|_{L^2} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \|u\|_{\dot{H}^{5/2}} + \frac{1}{\beta} \|u\|_{\dot{H}^{7/2}} \right] \\ &\leq C \|u\|_{\dot{H}^{5/2}}^2 \left[\alpha^{2/5} \|u^0\|_{L^2} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \|u\|_{\dot{H}^{5/2}} \right] + \frac{4\pi C \|u\|_{\dot{H}^{5/2}}^4}{\nu\beta^2} + \frac{\nu}{2} \|u\|_{\dot{H}^{7/2}}^2. \end{aligned}$$

Then, we have:

$$\partial_t \|u\|_{\dot{H}^{5/2}}^2 \leq C \|u\|_{\dot{H}^{5/2}}^2 \left[\alpha^{5/2} \|u^0\|_{L^2} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \|u\|_{\dot{H}^{5/2}} + \frac{\|u\|_{\dot{H}^{5/2}}^2}{\nu\beta^2} \right].$$

Put

$$t_0 = \inf\{t \in [0, T^*), \|u(t)\|_{\dot{H}^{5/2}} = 2\|u^0\|_{\dot{H}^{5/2}}\},$$

then, we get:

$$4\|u^0\|_{\dot{H}^{5/2}}^2 \leq \|u^0\|_{\dot{H}^{5/2}}^2 + CT^*\|u^0\|_{\dot{H}^{5/2}}^2 \left[\alpha^{5/2}\|u^0\|_{L^2} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)}\|u^0\|_{\dot{H}^{5/2}} + \frac{\|u^0\|_{\dot{H}^{5/2}}^2}{\nu\beta^2} \right].$$

By taking $\|u^0\|_{\dot{H}^{5/2}} > 1$ by remark 2.8, we can choose:

$$\alpha = \|u^0\|_{\dot{H}^{5/2}}^{2/5} < \beta = \sqrt{\|u^0\|_{\dot{H}^{5/2}}},$$

we obtain:

$$\begin{aligned} 1 &\leq CT^* \left[\|u^0\|_{\dot{H}^{5/2}}\|u^0\|_{L^2} + \sqrt{\ln(\|u^0\|_{\dot{H}^{5/2}})}\|u^0\|_{\dot{H}^{5/2}} + \frac{\|u^0\|_{\dot{H}^{5/2}}}{\nu} \right] \\ &\leq CT^*\|u^0\|_{\dot{H}^{5/2}} \left[\|u^0\|_{L^2} + \sqrt{\ln(\|u^0\|_{\dot{H}^{5/2}})} + \frac{1}{\nu} \right]. \end{aligned}$$

Let $t_1 \in [0, T^*)$ be a instant such that,

$$\sqrt{\ln(\|u(t)\|_{\dot{H}^{5/2}})} \geq 2(\|u^0\|_{L^2} + \frac{1}{\nu}), \quad \forall t \in [t_1, T^*)$$

which yields:

$$1 \leq C(T^* - t_1)\|u(t_1)\|_{\dot{H}^{5/2}} \sqrt{\ln(\|u(t_1)\|_{\dot{H}^{5/2}})}.$$

By changing t_1 with any $t \in [t_1, T^*)$, we get the following estimate:

$$\frac{1}{T^* - t} \leq C\|u(t)\|_{\dot{H}^{5/2}} \sqrt{\ln(\|u(t)\|_{\dot{H}^{5/2}})}, \quad \forall t \in [t_1, T^*).$$

Then, there is a constant $C_1 > 0$ for all $0 \leq t < T^*$ we have:

$$\frac{1}{C_1(T^* - t)\sqrt{|\ln(T^* - t)|}} \leq \|u(t)\|_{\dot{H}^{5/2}}.$$

In fact: Let

$$\begin{aligned} f &: [4, +\infty[\rightarrow [4\sqrt{\ln(4)}, +\infty[\\ x &\mapsto f(x) = x\sqrt{\ln(x)} \end{aligned}$$

be a continues bijection.

Then, we have:

$$y = x\sqrt{\ln(x)} \implies \ln(y) = \ln(x) + \ln(\sqrt{\ln(x)}) \implies \ln(y) \underset{x \rightarrow +\infty}{\sim} \ln(x)$$

which implies

$$x \underset{y \rightarrow +\infty}{\sim} \frac{y}{\sqrt{\ln(y)}}.$$

4. Blow up criterion in $\dot{H}^{5/2}$ with respect to $\dot{H}^{1/2}$ norm

In this section we give a proof of Theorem 1.2.

Inequality (2.6) implies

$$\|u\|_{\chi^1} \leq C \left[\alpha^2\|u\|_{\dot{H}^{1/2}} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)}\|u\|_{\dot{H}^{5/2}} + \frac{1}{\beta}\|u\|_{\dot{H}^{7/2}} \right].$$

Then, we get

$$\partial_t \|u\|_{\dot{H}^{5/2}}^2 + 2\nu\|u\|_{\dot{H}^{7/2}}^2 \leq C\|u\|_{\dot{H}^{5/2}}^2 \left[\alpha^2\|u\|_{\dot{H}^{1/2}} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)}\|u\|_{\dot{H}^{5/2}} \right] + \frac{C\|u\|_{\dot{H}^{5/2}}^4}{\nu\beta^2} + \frac{\nu}{2}\|u\|_{\dot{H}^{7/2}}^2.$$

Moreover, we have:

$$\partial_t \|u\|_{\dot{H}^{5/2}}^2 \leq C \|u\|_{\dot{H}^{5/2}}^2 \left[\alpha^2 \|u\|_{\dot{H}^{1/2}} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \|u\|_{\dot{H}^{5/2}} + \frac{\|u\|_{\dot{H}^{5/2}}^2}{\nu\beta^2} \right].$$

Let $t_0 = \inf\{t \in [0, T^*), \|u(t)\|_{\dot{H}^{5/2}} = 2\|u^0\|_{\dot{H}^{5/2}}\}$, then we get:

$$\|u^0\|_{\dot{H}^{5/2}}^2 \leq CT^* \|u^0\|_{\dot{H}^{5/2}}^2 \left[\alpha^2 \sup_{0 \leq s \leq t_0} \|u(s)\|_{\dot{H}^{1/2}} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \|u^0\|_{\dot{H}^{5/2}} + \frac{\|u^0\|_{\dot{H}^{5/2}}^2}{\nu\beta^2} \right].$$

Using remark (2.9), we can choose α and β such that:

$$\alpha = \sqrt{\frac{c\nu \|u^0\|_{\dot{H}^{5/2}}}{\sup_{0 \leq t \leq s} \|u(t)\|_{\dot{H}^{1/2}}}} < \beta = \sqrt{\|u^0\|_{\dot{H}^{5/2}}}.$$

Then, we obtain:

$$\|u^0\|_{\dot{H}^{5/2}}^2 \leq CT^* \|u^0\|_{\dot{H}^{5/2}}^2 \left[\|u^0\|_{\dot{H}^{5/2}} + \sqrt{\ln\left(\frac{\sup_{0 \leq t \leq s} \|u(t)\|_{\dot{H}^{1/2}}}{c\nu}\right)} \|u^0\|_{\dot{H}^{5/2}} + \frac{\|u^0\|_{\dot{H}^{5/2}}}{\nu} \right].$$

Consequently, we get:

$$\begin{aligned} 1 &\leq CT^* \|u^0\|_{\dot{H}^{5/2}} \left[1 + \sqrt{\ln\left(\frac{\sup_{0 \leq t \leq s} \|u(t)\|_{\dot{H}^{1/2}}}{c\nu}\right)} + \frac{1}{\nu} \right] \\ &\leq CT^* \|u^0\|_{\dot{H}^{5/2}} \left[C_\nu + \sqrt{\ln\left(\frac{\sup_{0 \leq t \leq s} \|u(t)\|_{\dot{H}^{1/2}}}{c\nu}\right)} \right]. \end{aligned}$$

On the other hand, by interpolation, we have:

$$\begin{aligned} \partial_t \|u\|_{\dot{H}^{1/2}}^2 + 2\nu \|u\|_{\dot{H}^{3/2}} &\leq \|u \otimes u\|_{\dot{H}^{1/2}} \|u\|_{\dot{H}^{3/2}} \\ &\leq C_0 \|u\|_{\dot{H}^{1/2}} \|u\|_{\dot{H}^{3/2}}^2 \\ &\leq C_0 \|u\|_{\dot{H}^{1/2}}^2 \|u\|_{\dot{H}^{5/2}}. \end{aligned}$$

Then,

$$\frac{\frac{\partial_t \|u\|_{\dot{H}^{1/2}}^2}{c\nu}}{\frac{\|u\|_{\dot{H}^{1/2}}^2}{c\nu}} \leq C_0 \|u\|_{\dot{H}^{5/2}}.$$

Integrating on $[0, t)$, we obtain:

$$\ln\left(\frac{\|u\|_{\dot{H}^{1/2}}^2}{c\nu}\right) \leq \ln\left(\frac{\|u^0\|_{\dot{H}^{1/2}}^2}{c\nu}\right) + C_0 \int_0^t \|u(s)\|_{\dot{H}^{5/2}} ds.$$

Taking the sup over $[0, t_0]$, we get:

$$\begin{aligned} \ln\left(\frac{\sup_{0 \leq s \leq t_0} \|u(s)\|_{\dot{H}^{1/2}}}{c\nu}\right) &\leq \ln\left(\frac{\|u^0\|_{\dot{H}^{1/2}}^2}{c\nu}\right) + 2C_0 t_0 \|u^0\|_{\dot{H}^{5/2}} \\ &\leq \ln\left(\frac{\|u^0\|_{\dot{H}^{1/2}}^2}{c\nu}\right) + 2C_0 T^* \|u^0\|_{\dot{H}^{5/2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &\leq C_1 T^* \|u^0\|_{\dot{H}^{5/2}} \left[C_\nu + \sqrt{\ln\left(\frac{\|u^0\|_{\dot{H}^{1/2}}^2}{c\nu}\right)} + \sqrt{2T^* \|u^0\|_{\dot{H}^{5/2}}} \right] \\ &\leq C_\nu T^* \|u^0\|_{\dot{H}^{5/2}} \left[1 + \sqrt{\ln\left(\frac{\|u^0\|_{\dot{H}^{1/2}}^2}{c\nu}\right)} + 2T^* \|u^0\|_{\dot{H}^{5/2}} \right]. \end{aligned}$$

Put $X = T^* \|u^0\|_{\dot{H}^{5/2}}$ and $a = 1 + \sqrt{\ln\left(\frac{\|u^0\|_{\dot{H}^{1/2}}^2}{c\nu}\right)}$, we get:

$$X(a + X) \geq \frac{1}{C_\nu}$$

$$P(X) = X^2 + aX - \frac{1}{C_\nu} \geq 0 \implies \Delta = a^2 + \frac{1}{C_\nu} > 0$$

The solution of P are:

$$\begin{cases} X_1 = \frac{-a + \sqrt{\Delta}}{2} > 0 \\ X_2 = \frac{-a - \sqrt{\Delta}}{2} < 0, \end{cases}$$

then, we have:

$$X = T^* \|u^0\|_{\dot{H}^{5/2}} \geq X_1$$

which implies,

$$\begin{aligned} C_\nu T^* \|u^0\|_{\dot{H}^{5/2}} (1 + a) &\geq 1 \\ C_\nu T^* \|u^0\|_{\dot{H}^{5/2}} \sqrt{\ln\left(\frac{\|u^0\|_{\dot{H}^{1/2}}^2}{c\nu}\right)} &\geq 1. \end{aligned}$$

We change the initial data with any $t \in [0, T^*)$, we obtain:

$$\liminf_{t \rightarrow T^*} (T^* - t) \|u(t)\|_{\dot{H}^{5/2}} \sqrt{\ln\left(\frac{\|u(t)\|_{\dot{H}^{1/2}}^2}{c\nu}\right)} \geq \frac{1}{C_\nu}.$$

5. Blow up criterion in $\dot{H}^{5/2}$ with respect to \mathcal{X}^{-1} norm

In this section we give a proof of Theorem 1.3.

Using (2.7), we obtain:

$$\partial_t \|u\|_{\dot{H}^{5/2}}^2 \leq C \|u\|_{\dot{H}^{5/2}}^2 \left[\alpha^2 \|u\|_{\mathcal{X}^{-1}} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \|u\|_{\dot{H}^{5/2}} + \frac{\|u\|_{\dot{H}^{5/2}}^2}{\nu\beta^2} \right].$$

Let

$$t_0 = \inf\{t \in [0, T^*), \|u(t)\|_{\dot{H}^{5/2}} = 2\|u^0\|_{\dot{H}^{5/2}}\}.$$

which yields:

$$4\|u^0\|_{\dot{H}^{5/2}}^2 \leq \|u^0\|_{\dot{H}^{5/2}}^2 + CT^* \|u^0\|_{\dot{H}^{5/2}}^2 \left[\alpha^2 \sup_{0 \leq s \leq t_0} \|u(s)\|_{\mathcal{X}^{-1}} + \sqrt{\ln\left(\frac{\beta}{\alpha}\right)} \|u^0\|_{\dot{H}^{5/2}} + \frac{\|u^0\|_{\dot{H}^{5/2}}^2}{\nu\beta^2} \right]$$

Using remark (2.10) we can choose, α and β such that:

$$\alpha = \sqrt{\frac{c\nu \|u^0\|_{\dot{H}^{5/2}}}{\sup_{0 \leq t \leq s} \|u(t)\|_{\mathcal{X}^{-1}}}} < \beta = \sqrt{\|u^0\|_{\dot{H}^{5/2}}},$$

we get:

$$\|u^0\|_{\dot{H}^{5/2}}^2 \leq CT^* \|u^0\|_{\dot{H}^{5/2}}^2 \left[\|u^0\|_{\dot{H}^{5/2}} + \sqrt{\ln\left(\frac{\sup_{0 \leq t \leq s} \|u(t)\|_{\mathcal{X}^{-1}}}{c\nu}\right)} \|u^0\|_{\dot{H}^{5/2}} + \frac{\|u^0\|_{\dot{H}^{5/2}}}{\nu} \right].$$

then, we have:

$$\begin{aligned} 1 &\leq CT^* \|u^0\|_{\dot{H}^{5/2}} \left[1 + \sqrt{\ln\left(\frac{\sup_{0 \leq t \leq s} \|u(t)\|_{\mathcal{X}^{-1}}}{c\nu}\right)} + \frac{1}{\nu} \right] \\ &\leq C_\nu T^* \|u^0\|_{\dot{H}^{5/2}} \left[1 + \sqrt{\ln\left(\frac{\sup_{0 \leq t \leq s} \|u(t)\|_{\mathcal{X}^{-1}}}{c\nu}\right)} \right]. \end{aligned}$$

On the other hand, we have:

$$\|u(t)\|_{\mathcal{X}^{-1}} \leq \|u^0\|_{\mathcal{X}^{-1}} + \int_0^t \|u(s)\|_{\mathcal{X}^{-1}} \|u(s)\|_{\dot{H}^{5/2}} ds.$$

By Granwall lemma, we obtain, for $0 \leq t < T^*$:

$$\|u(t)\|_{\mathcal{X}^{-1}} \leq \|u^0\|_{\mathcal{X}^{-1}} e^{\int_0^t \|u(s)\|_{\dot{H}^{5/2}} ds}.$$

By applying \ln function, we get:

$$\ln\left(\frac{\sup_{0 \leq s \leq t_0} \|u(s)\|_{X^{-1}}}{c\nu}\right) \leq \ln\left(\frac{\|u^0\|_{X^{-1}}^2}{c\nu}\right) + 2C't_0 \|u^0\|_{\dot{H}^{5/2}},$$

and

$$(5.1) \quad \ln\left(\frac{\sup_{0 \leq s \leq t_0} \|u(s)\|_{X^{-1}}}{c\nu}\right) \leq \ln\left(\frac{\|u^0\|_{X^{-1}}^2}{c\nu}\right) + 2C'T^* \|u^0\|_{\dot{H}^{5/2}}.$$

by using (5.1), we can deduce:

$$1 \leq C_\nu T^* \|u^0\|_{\dot{H}^{5/2}} \left[2 + \sqrt{\ln\left(\frac{\|u^0\|_{X^{-1}}}{c\nu}\right) + 2C'T^* \|u^0\|_{\dot{H}^{5/2}}} \right].$$

then we have:

$$C_\nu T^* \|u^0\|_{\dot{H}^{5/2}} \sqrt{\ln\left(\frac{\|u^0\|_{X^{-1}}}{c\nu}\right)} \geq 1.$$

By changing the initial data with any $t \in [0, T^*)$, we obtain:

$$\liminf_{t \rightarrow T^*} (T^* - t) \|u(t)\|_{\dot{H}^{5/2}} \sqrt{\ln\left(\frac{\|u(t)\|_{X^{-1}}}{c\nu}\right)} \geq \frac{1}{C_\nu}.$$

6. General remarks

In this section we give a simple proof of the explosion result in $\dot{H}^{3/2}(\mathbb{R}^3)$, we give a simple proof of the following theorem given in [7]

Proposition 6.1. *Let $u \in C([0, T^*), (H^{3/2}(\mathbb{R}^3))^3)$ be a maximale solution of Navier-Stokes system. If T^* is finite then,*

$$\|u(t)\|_{\dot{H}^{3/2}} \geq C\nu^{1/2} (T^* - t)^{-1/2}.$$

Proof. Taking the inner product in $\dot{H}^{3/2}$, we obtain:

$$\langle \partial_t u, u \rangle_{\dot{H}^{3/2}} - \nu \langle \Delta u, u \rangle_{\dot{H}^{3/2}} + \langle u \cdot \nabla u, u \rangle_{\dot{H}^{3/2}} = - \underbrace{\langle \nabla p, u \rangle_{\dot{H}^{3/2}}}_0.$$

• we start by prove: $\langle u \cdot \nabla u, u \rangle_{\dot{H}^{3/2}} \leq C \|u\|_{\dot{H}^{3/2}}^2 \|u\|_{\dot{H}^{5/2}}$.
we have:

$$\begin{aligned} \langle u \cdot \nabla u, u \rangle_{\dot{H}^{3/2}} &= \int_{\mathbb{R}^3} |\xi|^3 \mathcal{F}(u \cdot \nabla u)(\xi) \mathcal{F}(u)(-\xi) d\xi \\ &= \int_{\xi} \int_{\eta} |\xi|^{3/2} \widehat{u}(\xi - \eta) \widehat{\nabla u}(\eta) |\xi|^{3/2} \widehat{u}(-\xi) d\eta d\xi \end{aligned}$$

By using,

$$\langle f \cdot \nabla g, g \rangle_{L^2} = 0 \quad \text{si} \quad \text{div } f = 0,$$

we get,

$$\langle u \cdot \nabla u, u \rangle_{\dot{H}^{3/2}} = \int_{\xi} \int_{\eta} |\xi|^{3/2} \widehat{u}(\xi - \eta) \widehat{\nabla u}(\eta) |\xi|^{3/2} \widehat{u}(-\xi) - |\eta|^{3/2} \widehat{u}(\xi - \eta) \widehat{\nabla u}(\eta) |\xi|^{3/2} \widehat{u}(-\xi) d\eta d\xi.$$

Cauchy-Schawrtz inequality gives:

$$\left| \langle u \cdot \nabla u, u \rangle_{\dot{H}^{3/2}} \right| \leq \left(\int_{\xi} \left(\int_{\eta} (|\xi|^{3/2} - |\eta|^{3/2}) |\widehat{u}(\xi - \eta)| |\widehat{\nabla u}(\eta)| d\eta \right)^2 d\xi \right)^{1/2} \|u\|_{\dot{H}^{3/2}}.$$

For $\xi, \eta \in \mathbb{R}^3$, we have

$$\left| |\xi|^{3/2} - |\eta|^{3/2} \right| \leq \frac{3}{2} \max(|\xi|, |\eta|)^{1/2} |\xi - \eta| \leq \frac{3}{2} (|\xi|^{1/2} + |\eta|^{1/2}) |\xi - \eta|.$$

Or we have

$$|\xi| \leq |\xi - \eta| + |\eta| \leq 2 \max(|\xi - \eta|, |\eta|),$$

then,

$$|\xi|^{1/2} \leq \sqrt{2}(|\xi - \eta|^{1/2} + |\eta|^{1/2}),$$

which yields,

$$\begin{aligned} \left| |\xi|^{3/2} - |\eta|^{3/2} \right| &\leq \frac{3}{2} |\xi - \eta| (\sqrt{2} |\xi - \eta|^{1/2} + (1 + \sqrt{2}) |\eta|^{1/2}) \\ &\leq \frac{3(1 + \sqrt{2})}{2} (|\xi - \eta|^{3/2} + |\eta|^{1/2} |\xi - \eta|). \end{aligned}$$

By using this inequality, we get:

$$\begin{aligned} &\left(\int_{\xi} \left(\int_{\eta} \left| |\xi|^{3/2} - |\eta|^{3/2} \right| |\widehat{u}(\xi - \eta)| |\widehat{\nabla} u(\eta)| d\eta \right)^2 d\xi \right)^{1/2} \\ &\leq C \left[\left(\int_{\xi} \left(\int_{\eta} |\xi - \eta|^{3/2} |\widehat{u}(\xi - \eta)| |\widehat{\nabla} u(\eta)| d\eta \right)^2 d\xi \right)^{1/2} \right. \\ &\quad \left. + \left(\int_{\xi} \left(\int_{\eta} |\xi - \eta| |\widehat{u}(\xi - \eta)| |\eta|^{1/2} |\widehat{\nabla} u(\eta)| d\eta \right)^2 d\xi \right)^{1/2} \right] \\ &\leq I_1 + I_2, \end{aligned}$$

where

$$\begin{cases} I_1 &= \left(\int_{\xi} \left(\int_{\eta} |\xi - \eta|^{3/2} |\widehat{u}(\xi - \eta)| |\widehat{\nabla} u(\eta)| d\eta \right)^2 d\xi \right)^{1/2} = \|f_1 g_1\|_{\dot{H}^0}; \\ f_1 &= \mathcal{F}^{-1}(|\xi|^{3/2} |\widehat{u}(\xi)|) \\ g_1 &= \mathcal{F}^{-1}(|\widehat{\nabla} u(\xi)|), \end{cases}$$

and

$$\begin{cases} I_2 &= \left(\int_{\xi} \left(\int_{\eta} |\xi - \eta| |\widehat{u}(\xi - \eta)| |\eta|^{1/2} |\widehat{\nabla} u(\eta)| d\eta \right)^2 d\xi \right)^{1/2} = \|f_2 g_2\|_{\dot{H}^0}; \\ f_2 &= \mathcal{F}^{-1}(|\xi| |\widehat{u}(\xi)|) \\ g_2 &= \mathcal{F}^{-1}(|\xi|^{1/2} |\widehat{\nabla} u(\xi)|). \end{cases}$$

Applying the product lower of homogeneous Sobolev spaces, we obtain

$$\begin{aligned} I_1 &\leq C \|f_1 g_1\|_{\dot{H}^0} \\ &\leq C \|f_1\|_{\dot{H}^1} \|g_1\|_{\dot{H}^{1/2}} \\ &\leq C \|u\|_{\dot{H}^{5/2}} \|u\|_{\dot{H}^{3/2}} \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq C \|f_2 g_2\|_{\dot{H}^0} \\ &\leq C \|f_2\|_{\dot{H}^{1/2}} \|g_2\|_{\dot{H}^1} \\ &\leq C \|u\|_{\dot{H}^{3/2}} \|u\|_{\dot{H}^{5/2}}. \end{aligned}$$

then, we get:

$$\left| \langle u, \nabla u, u \rangle_{\dot{H}^{3/2}} \right| \leq C \|u\|_{\dot{H}^{3/2}}^2 \|u\|_{\dot{H}^{5/2}}.$$

which implied,

$$\partial_t \|u\|_{\dot{H}^{3/2}}^2 + 2\nu \|u\|_{\dot{H}^{5/2}}^2 \leq C \|u\|_{\dot{H}^{3/2}}^2 \|u\|_{\dot{H}^{5/2}}.$$

Inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$, gives:

$$\begin{aligned} \partial_t \|u\|_{\dot{H}^{3/2}}^2 + 2\nu \|u\|_{\dot{H}^{5/2}}^2 &\leq C\nu^{-1} \|u\|_{\dot{H}^{3/2}}^4 + \frac{\nu}{2} \|u\|_{\dot{H}^{5/2}}^2 \\ \implies \partial_t \|u\|_{\dot{H}^{3/2}}^2 &\leq C\nu^{-1} \|u\|_{\dot{H}^{3/2}}^4. \end{aligned}$$

integrating over $[t, T^*) \subset [0, T^*)$, we get:

$$\|u(t)\|_{\dot{H}^{3/2}}^2 \geq C\nu(T^* - t)^{-1}.$$

Then, the proof of theorem 1.2 is finished.

REFERENCES

- [1] H. Bahouri, J.Y Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer Verlag, 343p, 2011.
- [2] M. Cannone, *Harmonic analysis tools for solving the incompressible Navier-Stokes equations*, Diterot Editeur, Paris, 1995. 599-624.
- [3] H.Fujita,T.Kato, *On the Navier-Stokes initial value problem* , I.Arch.Ration.Mech.Anal.16 (1964) 269-315.
- [4] J. Benameur, *On the blow-up criterion of 3D NavierStokes equations.*,Journal of Mathematical Analysis and Applications. Volume 371, Issue 2, 15 November 2010, Pages 719-727, (2010).
- [5] J. Benameur, *On the blow-up criterion of the periodic incompressible fluids.*,Mathematical Methods in the Applied Sciences. Volume 36, Issue 2 30 January 2013 Pages 143153, (2013).
- [6] J. Benameur, *Long Time Decay to the Lei-Lin solution of 3D Navier Stokes equation.* J.Math.Anal.Appl.(2015).
- [7] DS. McCormick, EJ. Olson, JC. Robinson, Jose L. Rodrigo, Alejandro Vidal-Lpez, and Yi Zhou *Lower bounds on blowing-up solutions of the 3D Navier-Stokes equations in $\dot{H}^{3/2}$, $\dot{H}^{5/2}$, and $\dot{B}_{2,1}^{5/2}$.*, SIAM Journal on Mathematical Analysis. 2016, Vol. 48, No. 3, pp. 2119-2132, (2016).
- [8] A. Cheskidov and K. Zaya, *Lower bounds of potential blow-up solutions of three-dimentional Navier-Stokes equations in $\dot{H}^{3/2}$.* Journal of Mathematical Physics 57, 023101 (2016).
- [9] C. Cortissoz, J. A. Montero, and C. E. Pinilla , *On lower bounds for possible blow-up solutions to the periodic Navier-Stokes equation*, J. Math. Phys., 55, 033101. (2014)
- [10] J. C. Robinson, W. Sadowski, and R. P. Silva, *Lower bounds on blow up solutions of the three-dimensional Navier-Stokes equations in homogeneous Sobolev spaces*, J. Math. Phys, 53 (2012),115618.
- [11] T.Kato, *Quasi-Linear Equations of Evolution, With Application to Partial Differential Equations* , Lecture Notes in Math, vol.448,Sringer-Verlag,1975,pp. 25-70.
- [12] T.Kato. *L^p -solution of the Navier Stokes in \mathbb{R}^m . With applications to weak solutions*, Math.Z. 187 (4)(1984) 471-480.
- [13] H.Koch.D.Tataru. *Well-posedness for the Navier Stokes equations* , Adv.Math.157(1)(2001) 22-35.
- [14] Z.Lei.F.Lin, *Global mild solutions of Navier Stokes equations* , Comm.Pure Appl.Math.LXIV (2011) 1297 1304.
- [15] J.Leray. *Essai sur lr mouvement d'un liquide visqueux emplissant l'espace*, Acta Math.63 (1933) 22-25.
- [16] J.Leray. *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math.63 (1) (1934) 193-248. Indian University Mathematics Journal,vol.56,no. 3.pp.1157-1188,2007.

I.S.S.A.T. GABÈS, UNIVERSITY OF GABÈS, GABÈS, TUNISIA
E-mail address: jamelbenameur@gmail.com and jlbenameur@hotmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, GABÈS, GABÈS, TUNISIA
E-mail address: hajerorf17@gmail.com