Existence and profile of ground-state solutions to a 1-Laplacian problem in \mathbb{R}^N

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Abstract

In this work we prove the existence of ground state solutions for the following class of problems

$$\begin{cases} -\Delta_1 u + (1 + \lambda V(x)) \frac{u}{|u|} = f(u), \quad x \in \mathbb{R}^N, \\ u \in BV(\mathbb{R}^N), \end{cases}$$

where $\lambda > 0$, Δ_1 denotes the 1-Laplacian operator which is formally defined by $\Delta_1 u = \operatorname{div}(\nabla u/|\nabla u|), V : \mathbb{R}^N \to \mathbb{R}$ is a potential satisfying some conditions and $f : \mathbb{R} \to \mathbb{R}$ is a subcritical and superlinear nonlinearity. We prove that for $\lambda > 0$ large enough there exists ground-state solutions and, as $\lambda \to +\infty$, such solutions converges to a ground-state solution of the limit problem in $\Omega = \operatorname{int}(V^{-1}(\{0\})).$

Key Words. Bounded variation functions, 1-Laplacian operator, concentration results. AMS Classification. 35J62, 35J93.

1 Introduction

Let us consider the following class of quasilinear elliptic problem

$$\begin{cases} -\Delta_1 u + (1 + \lambda V(x)) \frac{u}{|u|} = f(u), \quad x \in \mathbb{R}^N, \\ u \in BV(\mathbb{R}^N), \end{cases}$$
(P)_{\lambda}

where $\lambda > 0$, $N \ge 2$, the operator Δ_1 is the well known 1–Laplacian operator, whose formal definition is given by $\Delta_1 u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$. On the nonlinearity f we assume the following conditions:

- $(f_1) f \in C(\mathbb{R});$
- $(f_2) f(s) = o(1) \text{ as } s \to 0;$
- (f₃) There exist constants $c_1, c_2 > 0$ and $p \in (1, 1^*)$ such that

$$|f(s)| \le c_1 + c_2 |s|^{p-1}, \quad \forall s \in \mathbb{R};$$

 (f_4) There exists $\theta > 1$ such that

$$0 < \theta F(s) \le f(s)s$$
, for $s \ne 0$,

where $F(s) = \int_0^s f(t)dt;$

 (f_5) f is increasing.

The potential V is going to be considered satisfying the following conditions:

- $(V_1) V(x) \ge 0, \forall x \in \mathbb{R}^N;$
- (V₂) There exists $M_0 > 0$ such that $|\{x \in \mathbb{R}^N; V(x) \le M_0\}| < +\infty$, where |A| denotes the Lebesgue measure of a mensurable set $A \subset \mathbb{R}^N$.
- $(V_3) \ \Omega = \operatorname{int}(V^{-1}(\{0\})) \neq \emptyset.$

Several recent studies have focused on the nonlinear Schrödinger equation with deep potential well

$$-\Delta u + (\lambda a(x) + b(x))u = |u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,$$
(1.1)

where a(x), b(x) are suitable continuous functions and $p \in (2, \frac{2N}{N-2})$ if $N \ge 3$; $p \in (1, \infty)$ if N = 1, 2. In [4], for b(x) = 1, Bartsch and Wang proved the existence of a least energy solution for λ large enough and that the sequence of solutions converges strongly to a least energy solution for a problem in a bounded domain. They also showed the existence of at least cat Ω positive solutions for large λ , where $\Omega = int(a^{-1}(0))$, and p is close to the critical exponent. In [9], Clapp and Ding study the existence of nodal solutions that change sign exactly once, considering the critical growth case. We also refer to [5] for nonconstant b(x) > 0, where the authors prove the existence of k solutions that may change sign for any k and λ large enough. For other results related to Schrödinger equations with deep potential well, we may refer the readers to [10, 19, 18, 21].

Motivated by the above references our intention is to prove that some of these results hold for problem $(P)_{\lambda}$. The main difficulties arise mainly because of the following facts:

- The lack of smoothness on the energy functional associated to $(P)_{\lambda}$;
- The lack of reflexiveness on $BV(\mathbb{R})$, which is the functional space we are going to work with;
- The difficulty in adapting well known technical results and estimates to our framework, taking into account the way in which we are going to define the sense of solutions.

We would like point out that there is in the literature few papers involving the 1-Laplacian operator in the whole \mathbb{R}^N . In fact the authors know only the papers due to Alves and Pimenta [1] and Figueiredo and Pimenta [13, 14]. In [1], Alves and Pimenta have studied the existence and concentration of solution for the following class of problem

$$\begin{cases} -\epsilon \Delta_1 u + V(x) \frac{u}{|u|} &= f(u) \quad \text{in } \mathbb{R}^N, \\ & u \in BV(\mathbb{R}^N), \end{cases}$$

where $\epsilon > 0$ and V, f are continuous functions that satisfy some technical conditions. Actually f has a subcritical growth and V verifies the condition

$$\liminf_{|z|\to\infty} V(z) > \inf_{z\in\mathbb{R}^N} V(z) = V_0 > 0.$$

In [13], Figueiredo and Pimenta has obtained the existence of radially symmetric solutions when V = 1, by working with the space of radially symmetric BV functions, which is proved to be embedded in $L^q(\mathbb{R}^N)$, for all $q \in (1, 1^*)$. In [14] the same authors shown the existence of ground-state bounded variation solutions for a problem involving the 1-Laplacian operator and vanishing potentials.

In this work our main result is the following.

Theorem 1. Suppose that f satisfies $(f_1) - (f_5)$ and that V satisfies $(V_1) - (V_3)$, then there exists $\lambda^* > 0$ such that $(P)_{\lambda}$ has a ground-state bounded variation solution u^{λ} for all $\lambda \geq \lambda^*$. Moreover, there exists $u_{\Omega} \in BV(\mathbb{R}^N)$ such that, if $\lambda_n \to +\infty$, up to a subsequence, $u_{\lambda_n} \to u_{\Omega}$ in $L^q_{loc}(\mathbb{R}^N)$, for $1 \leq q < 1^*$ and

$$||u_n||_{\lambda_n} - ||u_\Omega||_{\Omega} \to 0, \quad as \ n \to +\infty,$$

where $u_{\Omega} \equiv 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ and u_{Ω} is a bounded variation solution of

$$\begin{cases} -\Delta_1 u + \frac{u}{|u|} = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$
(1.2)

Some words about the limit problem (1.2) are in oder, mainly because the way in which we are going to consider the Dirichlet boundary condition. Note that $u \in BV(\Omega)$ is a bounded variation solution of (1.2) if

$$\|v\|_{\Omega} - \|u\|_{\Omega} \ge \int_{\Omega} f(u)(v-u), \quad \forall v \in BV(\Omega),$$
(1.3)

where $||u||_{\Omega} = \int_{\Omega} |Du| + \int_{\Omega} |u| dx + \int_{\partial\Omega} |u| d\mathcal{H}_{N-1}$. Since the trace operator from $BV(\Omega)$ into $L^1(\partial\Omega)$ does not have good properties of continuity w.r.t. the $L^q(\Omega)$ convergence, it is completely useless trying to impose the boundary condition in the space, working on $BV_0(\Omega) = \{u \in BV(\Omega); u = 0 \text{ on } \partial\Omega\}$. In fact our approach follows what is usually done in the literature on 1–Laplacian problems with Dirichlet boundary conditions, i.e., imposing the boundary conditions by considering the term $\int_{\partial\Omega} |u| d\mathcal{H}_{N-1}$ in the energy functional, where \mathcal{H}_{N-1} denotes the (N-1)-dimensional Housdorff measure in \mathbb{R}^N .

The paper is organized as follows. In Section 2, we recall some properties involving the space $BV(\mathbb{R}^N)$ and prove some properties of the energy functional associated with the problem. In Section 3, we prove the existence of ground state for λ large enough. In the last section we study the concentration arguments and the profile of the solutions as $\lambda \to +\infty$.

2 Preliminary results

Let us introduce the space of functions of bounded variation defined by

$$BV(\mathbb{R}^N) = \left\{ u \in L^1(\mathbb{R}^N); Du \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^N) \right\}.$$

It can be proved that $u \in BV(\mathbb{R}^N)$ is equivalent to $u \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |Du| := \sup\left\{\int_{\mathbb{R}^N} u \mathrm{div}\phi dx; \ \phi \in C_c^1(\mathbb{R}^N, \mathbb{R}^N), \text{ s.t. } |\phi|_\infty \le 1\right\} < +\infty.$$

The space $BV(\mathbb{R}^N)$ is a Banach space when endowed with the norm

$$||u|| := \int_{\mathbb{R}^N} |Du| + |u|_1,$$

which is continuously embedded into $L^r(\mathbb{R}^N)$ for all $r \in [1, 1^*]$.

As one can see in [3], the space $BV(\mathbb{R}^N)$ has different convergence and density properties than the usual Sobolev spaces. For example, $C_0^{\infty}(\mathbb{R}^N)$ is not dense in $BV(\mathbb{R}^N)$ with respect to the strong convergence, since the closure of $C_0^{\infty}(\mathbb{R}^N)$ in the norm of $BV(\mathbb{R}^N)$ is equal to $W^{1,1}(\mathbb{R}^N)$, which is a proper subspace of $BV(\mathbb{R}^N)$. This has motivated people to define a weaker sense of convergence in $BV(\mathbb{R}^N)$, called *intermediate convergence*. We say that $(u_n) \subset BV(\mathbb{R}^N)$ converge to $u \in BV(\mathbb{R}^N)$ in the sense of the intermediate convergence if

$$u_n \to u$$
, in $L^1(\mathbb{R}^N)$

and

$$\int_{\mathbb{R}^N} |Du_n| \to \int_{\mathbb{R}^N} |Du|,$$

as $n \to \infty$. Fortunately, with respect to the intermediate convergence, $C_0^{\infty}(\mathbb{R}^N)$ is dense in $BV(\mathbb{R}^N)$. This fact is going to be used later.

For a vectorial Radon measure $\mu \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^N)$, we denote by $\mu = \mu^a + \mu^s$ the usual decomposition stated in the Radon Nikodyn Theorem, where μ^a and μ^s are, respectively, the absolute continuous and the singular parts with respect to the N-dimensional

Lebesgue measure \mathcal{L}^N . We denote by $|\mu|$, the absolute value of μ , the scalar Radon measure defined as in [3][pg. 125]. By $\frac{\mu}{|\mu|}(x)$ we denote the usual Lebesgue derivative of μ with respect to $|\mu|$, given by

$$\frac{\mu}{|\mu|}(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}.$$

It can be proved that $\mathcal{J}: BV(\mathbb{R}^N) \to \mathbb{R}$, given by

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| dx, \qquad (2.4)$$

is a convex functional and Lipschitz continuous in its domain. It is also well know that \mathcal{J} is lower semicontinuous with respect to the $L^r(\mathbb{R}^N)$ topology, for $r \in [1, 1^*]$ (see [15] for example). Although non-smooth, the functional \mathcal{J} admits some directional derivatives. More specifically, as is shown in [2], given $u \in BV(\mathbb{R}^N)$, for all $v \in BV(\mathbb{R}^N)$ such that $(Dv)^s$ is absolutely continuous w.r.t. $(Du)^s$ and such that v is equal to 0 a.e. in the set where u vanishes, it follows that

$$\mathcal{J}'(u)v = \int_{\mathbb{R}^N} \frac{(Du)^a (Dv)^a}{|(Du)^a|} dx + \int_{\mathbb{R}^N} \frac{Du}{|Du|} (x) \frac{Dv}{|Dv|} (x) |(Dv)|^s + \int_{\mathbb{R}^N} \operatorname{sgn}(u) v dx, \quad (2.5)$$

where $\operatorname{sgn}(u(x)) = 0$ if u(x) = 0 and $\operatorname{sgn}(u(x)) = u(x)/|u(x)|$ if $u(x) \neq 0$. In particular, note that, for all $u \in BV(\mathbb{R}^N)$,

$$\mathcal{J}'(u)u = \mathcal{J}(u). \tag{2.6}$$

We have also that $BV(\mathbb{R}^N)$ is a *lattice*, i.e., if $u, v \in BV(\mathbb{R}^N)$, then $\max\{u, v\}, \min\{u, v\} \in BV(\mathbb{R}^N)$ and also

 $\mathcal{J}(\max\{u,v\}) + \mathcal{J}(\min\{u,v\}) \le \mathcal{J}(u) + \mathcal{J}(v), \quad \forall u, v \in BV(\mathbb{R}^N).$ (2.7)

Let us denote

$$E_{\lambda} = \left\{ u \in BV(\mathbb{R}^N); \int_{\mathbb{R}^N} (1 + \lambda V(x)) |u| dx < +\infty \right\},\$$

the subspace of $BV(\mathbb{R}^N)$ endowed with the following norm

$$\|v\|_{\lambda} := \int_{\mathbb{R}^N} |Dv| + \int_{\mathbb{R}^N} (1+\lambda)V(x)|v|dx.$$

Note that the embedding $E_{\lambda} \hookrightarrow BV(\mathbb{R}^N)$ is continuous in such a way that E_{λ} is a Banach space that is continuously embedded into $L^q(\mathbb{R}^N)$, for all $q \in [1, 1^*]$.

Let us define the functionals $\Phi_{\lambda}, \Psi_{\lambda}, \Phi_F : E_{\lambda} \to \mathbb{R}$ given by

$$\Phi_{\lambda}(u) = \|u\|_{\lambda},$$
 $\Phi_{F}(u) = \int_{\mathbb{R}^{N}} F(u) dx$

and

$$\Psi_{\lambda}(u) = \Phi_{\lambda}(u) - \int_{\mathbb{R}^N} F(v) dx.$$

Note that Ψ_{λ} is written as the difference of a convex locally Lipschitz functional Φ_{λ} , and a $C^{1}(E)$ one, Φ_{F} . Then we can use the theory of subdifferentials of Clarke [8] to say that u^{λ} is a critical point of Ψ_{λ} if $0 \in \partial \Psi_{\lambda}(u^{\lambda})$, where $\partial \Psi_{\lambda}(u^{\lambda})$ denotes the subdifferential of Ψ_{λ} in u^{λ} . This, in turn, is equivalent to $\Phi'_{F}(u^{\lambda}) \in \partial \Phi_{\lambda}(u^{\lambda})$, which is equivalent to

$$\|v\|_{\lambda} - \|u^{\lambda}\|_{\lambda} \ge \int_{\mathbb{R}^N} f(u^{\lambda})(v - u^{\lambda}), \quad \forall v \in E_{\lambda}.$$
(2.8)

2.1 The Euler-Lagrange equation

Since $(P)_{\lambda}$ contains expressions that doesn't make sense when $\nabla u = 0$ or u = 0, then it can be understood just as the formal version of the Euler-Lagrange equation associated to the functional Ψ_{λ} . In this section we present the precise form of an Euler-Lagrange equation satisfied by all bounded variation critical points of Ψ_{λ} . In order to do so we closely follow the arguments in [16], however we have introduced new ideas, because we are working in whole \mathbb{R}^N .

The first step is to consider the extension of the functionals Φ_{λ}, Φ_{F} and Ψ_{λ} to $X = L^{1}(\mathbb{R}^{N}) \cap L^{\frac{N}{N-1}}(\mathbb{R}^{N})$ endowed with the norm $||w||_{X} = |w|_{1} + |w|_{\frac{N}{N-1}}$, given respectively by $\overline{\Phi_{\lambda}}, \overline{\Phi_{F}}, \overline{\Psi_{\lambda}} : X \to \mathbb{R} \cup \{+\infty\}$, where

$$\overline{\Phi_{\lambda}}(v) = \begin{cases} \Phi_{\lambda}(v), & \text{if } v \in E_{\lambda}, \\ +\infty, & \text{if } v \in X \setminus E_{\lambda}, \end{cases}$$
$$\overline{\Phi_{F}}(u) = \int_{\mathbb{R}^{N}} F(u) dx$$

and $\overline{\Psi}_{\lambda} = \overline{\Phi_{\lambda}} - \overline{\Phi_{F}}$. It is easy to see that $\overline{\Phi_{F}}$ belongs to $C^{1}(X, \mathbb{R})$ and that $\overline{\Phi_{\lambda}}$ is a convex lower semicontinuous functional defined in X. Hence the subdifferential (in the sense of [20]) of $\overline{\Phi_{\lambda}}$, denoted by $\partial \overline{\Phi_{\lambda}}$, is well defined. The following is a crucial result in obtaining an Euler-Lagrange equation satisfied by the critical points of Ψ_{λ} .

Lemma 2. If $u^{\lambda} \in BV(\mathbb{R}^N)$ is such that $0 \in \partial \Psi_{\lambda}(u^{\lambda})$, then $0 \in \partial \overline{\Psi_{\lambda}}(u^{\lambda})$.

Proof. Suppose that $0 \in \partial \Psi_{\lambda}(u^{\lambda})$, i.e., that u^{λ} satisfies (2.8). We would like to prove that

$$\overline{\Phi_{\lambda}}(v) - \overline{\Phi_{\lambda}}(u^{\lambda}) \ge \overline{\Phi_F}'(u^{\lambda})(v - u^{\lambda}), \quad \forall v \in X.$$

To see why, consider $v \in X$ and note that:

• if $v \in E_{\lambda} \cap X$, then

$$\begin{aligned} \overline{\Phi_{\lambda}}(v) - \overline{\Phi_{\lambda}}(u^{\lambda}) &= \Phi_{\lambda}(v) - \Phi_{\lambda}(u^{\lambda}) \\ &\geq \Phi'_{F}(u^{\lambda})(v - u^{\lambda}) \\ &= \int_{\mathbb{R}^{N}} f(u^{\lambda})(v - u^{\lambda}) dx \\ &= \overline{\Phi_{F}}'(u^{\lambda})(v - u^{\lambda}); \end{aligned}$$

• if $v \in X \setminus E_{\lambda}$, since $\overline{\Phi_{\lambda}}(v) = +\infty$ and $\overline{\Phi_{\lambda}}(u^{\lambda}) < +\infty$, it follows that

$$\overline{\Phi_{\lambda}}(v) - \overline{\Phi_{\lambda}}(u^{\lambda}) = +\infty$$

$$\geq \overline{\Phi_{F}}'(u^{\lambda})(v - u^{\lambda}).$$

Therefore the result follows.

Let us assume that $u^{\lambda} \in BV(\mathbb{R}^N)$ is a bounded variation solution of $(P)_{\lambda}$, i.e., that u^{λ} satisfies (2.8). Since $0 \in \partial \Psi_{\lambda}(u^{\lambda})$, by the last result it follows that $0 \in \partial \overline{\Psi}_{\lambda}(u^{\lambda})$. Since $\overline{\Phi_{\lambda}}$ is convex and $\overline{\Phi_F}$ is smooth, it follows that $\overline{\Phi_F}'(u^{\lambda}) \in \partial \overline{\Phi_{\lambda}}(u^{\lambda})$. In what follows, we set $\overline{\Phi_{\lambda}^1}, \overline{\Phi_{\lambda}^2} : X \to \mathbb{R} \cup \{+\infty\}$ by

$$\overline{\Phi_{\lambda}^{1}}(v) := \begin{cases} \int_{\mathbb{R}^{N}} |Dv|, & \text{if } v \in BV(\mathbb{R}^{N}), \\ +\infty, & \text{if } v \in X \setminus BV(\mathbb{R}^{N}), \end{cases}$$

and

$$\overline{\Phi_{\lambda}^{2}}(v) := \int_{\mathbb{R}^{N}} (1 + \lambda V(x)) |v| \, dx.$$

Note that $\overline{\Phi_{\lambda}^2} \in C(X, \mathbb{R}), \ \overline{\Phi_{\lambda}^2} \in C(BV(\mathbb{R}^N), \mathbb{R})$ and

$$\overline{\Phi_{\lambda}}(v) = \overline{\Phi_{\lambda}^{1}}(v) + \overline{\Phi_{\lambda}^{2}}(v), \quad \forall v \in X.$$

Since $\overline{\Phi_{\lambda}^{1}}$ and $\overline{\Phi_{\lambda}^{2}}$ are convex, and $\overline{\Phi_{\lambda}^{2}}$ is finite and continuous in every point of E_{λ} , it follows from [3, Theorem 9.5.4] that

$$\overline{\Phi_F}'(u^{\lambda}) \in \partial \overline{\Phi_{\lambda}}(u^{\lambda}) = \partial \overline{\Phi_{\lambda}^1}(u^{\lambda}) + \partial \overline{\Phi_{\lambda}^2}(u^{\lambda}).$$

By using the same arguments explored in [6, Theorem 8.15], it follows that $X' \subset L_{\infty,N}(\mathbb{R}^N)$ where

$$L_{\infty,N}(\mathbb{R}^N) = \{g : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : ||g||_{\infty,N} < \infty\}$$

where

$$||g||_{\infty,N} = \sup_{|\phi|_1+|\phi|_{\frac{N}{N-1}} \le 1} \left| \int_{\mathbb{R}^N} g\phi \, dx \right|.$$

It is possible to prove that $\| \|_{\infty,N}$ is a norm in $L_{\infty,N}(\mathbb{R}^N)$. Moreover, the inclusion $L_{\infty,N}(\mathbb{R}^N) \hookrightarrow L^N(B_R(0))$ is continuous for all R > 0.

From the above commentaries, there are $z_1^*, z_2^* \in L_{\infty,N}(\mathbb{R}^N)$ such that $z_1^* \in \partial \overline{\Phi_{\lambda}^1}(u^{\lambda})$, $z_2^* \in \partial \overline{\Phi_{\lambda}^2}(u^{\lambda})$ and

$$\overline{\Phi_F}'(u^{\lambda}) = z_1^* + z_2^* \quad \text{in } L_{\infty,N}(\mathbb{R}^N).$$

Following the same arguments in [16, Proposition 4.23, pg. 529], we have that there exists $z \in L^{\infty}(\mathbb{R}^N, \mathbb{R}^N)$ such that $|z|_{\infty} \leq 1$,

$$-\operatorname{div} z = z_1^* \quad \text{in } L_{\infty,N}(\mathbb{R}^N)$$
(2.9)

and

$$-\int_{\mathbb{R}^N} u^{\lambda} \mathrm{div} z dx = \int_{\mathbb{R}^N} |Du^{\lambda}|, \qquad (2.10)$$

where the divergence in (2.9) has to be understood in the distributional sense. Moreover, the same result implies that z_2^* is such that

$$z_2^*|u^{\lambda}| = (1 + \lambda V(x))u^{\lambda}, \quad \text{a.e. in } \mathbb{R}^N.$$
(2.11)

Therefore, it follows from (2.9), (2.10) and (2.11) that u^{λ} satisfies

$$\exists z \in L^{\infty}(\mathbb{R}^{N}, \mathbb{R}^{N}), \ \|z\|_{\infty} \leq 1, \ \operatorname{div} z \in L_{\infty,N}(\mathbb{R}^{N}), \ -\int_{\mathbb{R}^{N}} u^{\lambda} \operatorname{div} z dx = \int_{\mathbb{R}^{N}} |Du^{\lambda}|,$$
$$\exists z_{2}^{*} \in L_{\infty,N}(\mathbb{R}^{N}), \ z_{2}^{*}|u^{\lambda}| = (1 + \lambda V(x))u^{\lambda} \quad \text{a.e. in } \mathbb{R}^{N},$$
$$-\operatorname{div} z + z_{2}^{*} = f(u^{\lambda}), \quad \text{a.e. in } \mathbb{R}^{N}.$$
$$(2.12)$$

Hence, (2.12) is the precise version of $(P)_{\lambda}$.

3 Existence of solution

Let us first verify that the geometrical conditions of the Mountain Pass Theorem are satisfied by Ψ_{λ} .

Lemma 3. There exist $\alpha, \rho > 0$ (uniform in λ) such that,

- i) $\Psi_{\lambda}(u) \geq \alpha$ for all $u \in E_{\lambda}$ such that $||u||_{\lambda} = \rho$, for all $\lambda > 0$;
- *ii)* For each $\lambda > 0$, there exists $e_{\lambda} \in E_{\lambda}$ such that $||e_{\lambda}||_{\lambda} > \rho$ and $\Psi_{\lambda}(e_{\lambda}) < 0$.

Proof. By $(f_2) - (f_3)$, it follows that for each $\eta > 0$, there exists $A_{\eta} > 0$ such that

$$|F(s)| \le \eta |s| + A_\eta |s|^p, \quad \forall s \in \mathbb{R}.$$
(3.13)

Note that, by (3.13) and the embeddings of E_{λ} ,

$$\Psi_{\lambda}(u) = \int_{\mathbb{R}^{N}} |Du| + \int_{\mathbb{R}^{N}} (1 + \lambda V(x))|u| dx - \int_{\mathbb{R}^{N}} F(u) dx$$

$$\geq ||u||_{\lambda} - \eta ||u||_{1} - A_{\eta} |u||_{p}^{p}$$

$$\geq ||u||_{\lambda} - \eta ||u||_{\lambda} - c_{3} ||u||_{\lambda}^{p}$$

$$= ||u||_{\lambda} \left(1 - \eta - c_{3} ||u||_{\lambda}^{p-1}\right)$$

$$\geq \alpha,$$

for all $u \in E_{\lambda}$, such that $||u||_{\lambda} = \rho$, where $0 < \eta < 1$ is fixed, $0 < \rho < \left(\frac{1-\eta}{c_3}\right)^{\frac{1}{p-1}}$ and $\alpha = \rho(1-\eta-c_3\rho^{p-1}).$

In order to verify *ii*) note that by (f_4) , there exist constants $d_1, d_2 > 0$ such that

$$F(s) \ge d_1 |s|^{\theta} - d_2, \quad \forall s \in \mathbb{R}.$$

$$(3.14)$$

If u is a function in $E_{\lambda} \setminus \{0\}$ with compact support, we derive that

$$\Psi_{\lambda}(tu) \le t \|u\|_{\lambda} - d_1 t^{\theta} |u|_{\theta}^{\theta} + d_2 |\operatorname{supp}(u)| \to -\infty,$$
(3.15)

as $t \to +\infty$. Since $\theta > 1$, we can choose $e_{\lambda} \in E_{\lambda}$ such that $\Psi(e_{\lambda}) < 0$.

By [13, Theorem 1.3] it follows that, for all $\lambda > 0$, there exists a sequence $(u_n^{\lambda}) \subset E_{\lambda}$ such that

$$\Psi_{\lambda}(u_n^{\lambda}) = c_{\lambda} + o_n(1) \tag{3.16}$$

and

$$\|v\|_{\lambda} - \|u_n^{\lambda}\|_{\lambda} \ge \int_{\mathbb{R}^N} f(u_n^{\lambda})(v - u_n^{\lambda}) - \tau_n \|v - u_n^{\lambda}\|_{\lambda}, \quad \forall v \in E_{\lambda},$$
(3.17)

where $\tau_n \to 0$ as $n \to +\infty$. The minimax value c_{λ} is given by

$$c_{\lambda} = \inf_{\gamma \in \Gamma_{\lambda}} \max_{t \in [0,1]} \Psi_{\lambda}(\gamma(t)),$$

where $\Gamma_{\lambda} = \{\gamma \in C([0,1], E_{\lambda}); \gamma(0) = 0 \text{ and } \Psi_{\lambda}(\gamma(1)) < 0\}$. Note that by Lemma 3,

$$c_{\lambda} \ge \alpha > 0, \quad \forall \lambda > 0.$$
 (3.18)

In our approach will be important the so called *Nehari set*, defined as

$$\mathcal{N}_{\lambda} = \{ u \in E_{\lambda} \setminus \{0\}; \ \Psi_{\lambda}'(u)u = 0 \}$$
$$= \left\{ u \in E_{\lambda} \setminus \{0\}; \ \|u\|_{\lambda} = \int_{\mathbb{R}^{N}} f(u)u \, dx \right\}.$$

This set is going to give us a better characterization of the minimax level c_{λ} . From (2.5), \mathcal{N}_{λ} is a set that contains all nontrivial bounded variation solutions of $(P)_{\lambda}$. Its definition is based on arguments that can be found in [12] which, in turn, are strongly influenced by those ones in [17]. More specifically, they consist in performing a study of the *fibering* maps $\gamma_u(t) := \Psi_{\lambda}(tu)$, by using $(f_1) - (f_5)$ to show that \mathcal{N}_{λ} is radially homeomorphic to the unit sphere in E_{λ} . In fact, for each $u \in E_{\lambda} \setminus \{0\}$, by (f_2) and (f_3) , it can be seen that there exists $t_0 > 0$ such that $\gamma_u(t_0) > 0$. On the other hand, (f_4) implies that $\gamma_u(t) \to -\infty$ as $t \to +\infty$. Then there exists $t_u > 0$ such that $\gamma_u(t_u) = \max_{t>0} \gamma_u(t)$ and then that $\gamma'_u(t_u) = 0$. But (f_5) implies that such t_u is unique. Then for each $u \in E_V \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_{\lambda}$. This establishes such a radial homeomorphism. Still with arguments presented in [17], one can prove that the minimax level c_{λ} satisfies

$$c_{\lambda} = \inf_{u \in E_{\lambda} \setminus \{0\}} \max_{t \ge 0} \Psi_{\lambda}(tu) = \inf_{u \in \mathcal{N}_{\lambda}} \Psi_{\lambda}(u).$$
(3.19)

Lemma 4. There exist constants $\alpha_0, \alpha_1 > 0$ which do not depend on $\lambda > 0$, such that

$$\alpha_0 \le c_\lambda \le \alpha_1, \quad \forall \lambda > 0.$$

Proof. By Lemma 3 it is enough to take $\alpha_0 \in (0, \alpha)$. In order to obtain α_1 , let us fix $\varphi \in C_0^{\infty}(\Omega)$. Then, for all t > 0, as in (3.15) we get

$$\Psi_{\lambda}(t\varphi) \leq t\left(\int_{\mathbb{R}^{N}} |D\varphi| + \int_{\mathbb{R}^{N}} |\varphi| dx\right) - d_{1}t^{\theta} |\varphi|_{\theta}^{\theta} + d_{2} |\operatorname{supp}(\varphi)| \to -\infty,$$

as $t \to +\infty$. Hence if $\alpha_1 =: \max_{t>0} \Psi_{\lambda}(t\varphi) > 0$, it follows from the definition of c_{λ} that

$$c_{\lambda} \leq \alpha_1, \quad \forall \lambda > 0.$$

Now let us study some more refined information about the sequence $(u_n)_{n \in \mathbb{N}}$.

Lemma 5. The sequence $(u_n^{\lambda})_{n \in \mathbb{N}}$ is bounded in E_{λ} .

Proof. Considering $v = 2u_n^{\lambda}$ in (3.17), we obtain

$$\|u_n^{\lambda}\|_{\lambda} \ge \int_{\mathbb{R}^N} f(u_n^{\lambda}) u_n^{\lambda} dx - \tau_n \|u_n^{\lambda}\|_{\lambda},$$

or equivalently

$$(1+\tau_n)\|u_n^\lambda\|_\lambda \ge \int_{\mathbb{R}^N} f(u_n^\lambda) u_n^\lambda dx.$$
(3.20)

Then, by (f_4) and (3.20),

$$\begin{aligned} c_{\lambda} + o_{n}(1) &\geq \Psi_{\lambda}(u_{n}^{\lambda}) \\ &= \|u_{n}^{\lambda}\|_{\lambda} + \int_{\mathbb{R}^{N}} \left(\frac{1}{\theta}f(u_{n}^{\lambda})u_{n}^{\lambda} - F(u_{n}^{\lambda})\right) dx - \int_{\mathbb{R}^{N}} \frac{1}{\theta}f(u_{n}^{\lambda})u_{n}^{\lambda} dx \\ &\geq \|u_{n}^{\lambda}\|_{\lambda} \left(1 - \frac{1}{\theta} - \frac{\tau_{n}}{\theta}\right) \\ &\geq C\|u_{n}^{\lambda}\|_{\lambda}, \end{aligned}$$

for some C > 0 that does not depend on $n \in \mathbb{N}$ nor $\lambda > 0$.

Remark 6. Note that by Lemmas 4 and 5, there exists a constant C > 0 that does not depend on λ , such that

$$\|u_n^{\lambda}\|_{\lambda} \le C, \quad \forall n \in \mathbb{N}.$$

By Lemma 5 and the compactness of the embeddings of BV(K) in $L^q(K)$ for $1 \le q < 1^*$ and $K \subset \mathbb{R}^N$ compact, there exists $u^{\lambda} \in BV_{loc}(\mathbb{R}^N)$ such that

$$u_n^{\lambda} \to u^{\lambda}$$
 in $L^q_{loc}(\mathbb{R}^N)$ for $1 \le q < 1^*$ (3.21)

and

$$u_n^{\lambda}(x) \to u^{\lambda}(x) \quad \text{a.e. } x \in \mathbb{R}^N,$$
 (3.22)

as $n \to +\infty$. Moreover u^{λ} belongs to $BV(\mathbb{R}^N)$ and then to E_{λ} (by using Fatou Lemma and the boundedness of the sequence $(||u_n||_{\lambda})_{n\in\mathbb{N}}$). In fact, if R > 0, by the semicontinuity of the norm in $BV(B_R(0))$ w.r.t. the $L^1(B_R(0))$ topology it follows that

$$\|u^{\lambda}\|_{BV(B_R(0))} \le \liminf_{n \to +\infty} \|u_n^{\lambda}\|_{BV(B_R(0))} \le \liminf_{n \to +\infty} \|u_n^{\lambda}\|_{BV(\mathbb{R}^N)} \le C,$$
(3.23)

where C does not depend on n nor on R. Since the last inequality holds for every R > 0, then $u^{\lambda} \in BV(\mathbb{R}^N)$.

The next result will help us to get some compactness properties involving the sequence (u_n^{λ}) .

Lemma 7. Fix $q \in [1, 1^*)$. Then, for a given $\epsilon > 0$, there exists $\lambda^* > 0$ and R > 0 such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} |u_n^\lambda|^q dx \le \epsilon, \tag{3.24}$$

for all $\lambda \geq \lambda^*$ and $n \in \mathbb{N}$.

Proof. In fact, for a given R > 0, let us define the sets

$$A(R) = \{ x \in \mathbb{R}^N; |x| > R \text{ and } V(x) \ge M_0 \}$$

and

$$B(R) = \{x \in \mathbb{R}^N; |x| > R \text{ and } V(x) < M_0\},\$$

where M_0 is given in (V_2) .

Note that, by Remark 6 and (V_2) ,

$$\int_{A(R)} |u_n^{\lambda}| dx \le \frac{1}{\lambda M_0 + 1} \|u_n\|_{\lambda} \le \frac{C}{\lambda M_0 + 1} < \frac{\epsilon}{2}, \quad \forall n \in \mathbb{N},$$
(3.25)

if $\lambda > \lambda^*$ where $\lambda^* \ge M_0^{-1} \left(\frac{2C}{\epsilon} - 1\right)$.

On the other hand, again by Remark 6, Hölder inequality and the embeddings of E_{λ} ,

$$\int_{B(R)} |u_n^{\lambda}| dx \le C |u_n^{\lambda}|_{1^*}^{1^*} |B(R)|^{\frac{1}{N}} \le C |B(R)|^{\frac{1}{N}} < \frac{\epsilon}{2}$$
(3.26)

if R > 0 is large enough, since by (V_2) , $|B(R)| \to 0$ as $R \to +\infty$.

Then, if $\lambda > \lambda^*$ and R > 0 is large enough, from (3.25) and (3.26) it follows the claim for q = 1. Now by Remark 6, the estimate for $q \in (1, 1^*)$ follows from interpolation in Lebesgue spaces since (u_n^{λ}) is bounded (uniformly in λ) in $L^{1^*}(\mathbb{R}^N)$.

The next result will be used to show that $u^{\lambda} \neq 0$.

Lemma 8.

$$\liminf_{n \to +\infty} \|u_n^{\lambda}\|_{\lambda} \ge \alpha_0 \quad \forall \lambda > 0.$$
(3.27)

Proof. Note that from (3.16) and Lemma 4,

$$\alpha_0 + o_n(1) \le c_\lambda + o_n(1) = \Psi_\lambda(u_n^\lambda) = \|u_n^\lambda\|_\lambda - \int_{\mathbb{R}^N} F(u_n^\lambda) dx \le \|u_n^\lambda\|_\lambda.$$

Lemma 9. For λ^* as in Lemma 7, it follows that $u^{\lambda} \neq 0$ for all $\lambda \geq \lambda^*$.

Proof. Considering in (3.17) $v = u_n^{\lambda} + t u_n^{\lambda}$ and taking the limit as $t \to 0^{\pm}$, we find

$$\Psi_{\lambda}'(u_n^{\lambda})u_n^{\lambda} = o_n(1),$$

which implies that

$$\|u_n^{\lambda}\|_{\lambda} = \int_{\mathbb{R}^N} f(u_n^{\lambda}) u_n^{\lambda} dx + o_n(1)$$

=
$$\int_{B_R(0)} f(u_n^{\lambda}) u_n^{\lambda} dx + \int_{\mathbb{R}^N \setminus B_R(0)} f(u_n^{\lambda}) u_n^{\lambda} dx + o_n(1).$$
(3.28)

From (f_3) ,

$$\int_{\mathbb{R}^N \setminus B_R(0)} f(u_n^\lambda) u_n^\lambda dx \le c_1 \int_{\mathbb{R}^N \setminus B_R(0)} |u_n^\lambda| dx + c_2 \int_{\mathbb{R}^N \setminus B_R(0)} |u_n^\lambda|^p dx.$$
(3.29)

By taking q = p and ϵ small enough in Lemma 7, the inequality (3.29) gives that

$$\limsup_{n \to +\infty} \int_{\mathbb{R}^N \setminus B_R(0)} f(u_n^\lambda) u_n^\lambda dx \le \frac{\alpha_0}{2},\tag{3.30}$$

where α_0 is as in Lemma 8.

From the compactness of the embeddings $BV(B_R(0)) \hookrightarrow L^q(B_R(0))$ for $q \in [1, 1^*)$ and (f_3) , we have that

$$\lim_{n \to +\infty} \int_{B_R(0)} f(u_n^{\lambda}) u_n^{\lambda} dx = \int_{B_R(0)} f(u^{\lambda}) u^{\lambda} dx.$$
(3.31)

Hence, from (3.27), (3.28), (3.30) and (3.31),

$$\begin{split} \int_{B_R(0)} f(u^{\lambda}) u^{\lambda} dx &= \lim_{n \to +\infty} \int_{B_R(0)} f(u^{\lambda}_n) u^{\lambda}_n dx \\ &\geq \lim_{n \to +\infty} \inf \left(\|u^{\lambda}_n\|_{\lambda} - \int_{\mathbb{R}^N \setminus B_R(0)} f(u^{\lambda}_n) u^{\lambda}_n dx \right) \\ &\geq \frac{\alpha_0}{2}, \end{split}$$

if $\lambda \geq \lambda^*$. This implies that $u^{\lambda} \neq 0$.

Lemma 10. $\Phi'_{\lambda}(u^{\lambda})u^{\lambda} \leq 0.$

Proof. Note that, if $\varphi \in C_0^{\infty}(\mathbb{R}^N)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_1(0)$ and $\varphi \equiv 0$ in $B_2(0)^c$, for $\varphi_R := \varphi(\cdot/R)$, it follows that for all $u \in BV(\mathbb{R}^N)$,

$$(D(\varphi_R u))^s$$
 is absolutely continuous w.r.t. $(Du)^s$. (3.32)

In fact, note that

$$D(\varphi_R u) = \nabla \varphi_R u + \varphi_R D u = \nabla \varphi_R u + \varphi_R D u^a + \varphi_R D u^s, \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Then it follows that

$$(D(\varphi_R u))^s = (\varphi_R (Du)^s)^s = \varphi_R (Du)^s.$$

Taking (3.32) into account, the fact that $\varphi_R u_n^{\lambda}$ is equal to 0 a.e. in the set where u_n^{λ} vanishes and also the fact that $\frac{\varphi_R \mu}{|\varphi_R \mu|} = \frac{\mu}{|\mu|}$ a.e. in $B_R(0)$, it follows that it is well defined $\Psi'_{\lambda}(u_n^{\lambda})(\varphi_R u_n^{\lambda})$. Moreover, by (2.5), it follows that

$$\begin{split} \Psi_{\lambda}'(u_{n}^{\lambda})(\varphi_{R}u_{n}^{\lambda}) &= \int_{\mathbb{R}^{N}} \frac{((Du_{n}^{\lambda})^{a})^{2}\varphi_{R} + u_{n}^{\lambda}(Du_{n}^{\lambda})^{a} \cdot \nabla\varphi_{R}}{|(Du_{n}^{\lambda})^{a}|} dx \\ &+ \int_{\mathbb{R}^{N}} \frac{Du_{n}^{\lambda}}{|Du_{n}^{\lambda}|} \frac{\varphi_{R}(Du_{n}^{\lambda})^{s}}{|\varphi_{R}(Du_{n}^{\lambda})^{s}|} |\varphi_{R}(Du_{n}^{\lambda})^{s}| + \\ &+ \int_{\mathbb{R}^{N}} (1 + \lambda V(x)) \mathrm{sgn}(u_{n}^{\lambda})(\varphi_{R}u_{n}^{\lambda}) dx - \int_{\mathbb{R}^{N}} f(u_{n}^{\lambda})\varphi_{R}u_{n}^{\lambda} dx \\ &= \int_{\mathbb{R}^{N}} \varphi_{R} |(Du_{n}^{\lambda})^{a}| dx + \int_{\mathbb{R}^{N}} \frac{u_{n}^{\lambda}(Du_{n}^{\lambda})^{a} \cdot \nabla\varphi_{R}}{|(Du_{n}^{\lambda})^{a}|} dx + \\ &+ \int_{\mathbb{R}^{N}} \frac{(Du_{n}^{\lambda})^{s}}{|(Du_{n}^{\lambda})^{s}|} \frac{\varphi_{R}(Du_{n}^{\lambda})^{s}}{|\varphi_{R}(Du_{n}^{\lambda})^{s}|} |\varphi_{R}(Du_{n}^{\lambda})^{s}| + \int_{\mathbb{R}^{N}} (1 + \lambda V(x))|u_{n}^{\lambda}|\varphi_{R} dx - \\ &- \int_{\mathbb{R}^{N}} f(u_{n}^{\lambda})\varphi_{R}u_{n}^{\lambda} dx. \end{split}$$

The last equality together with the lower semicontinuity of the norm in $BV(B_R(0))$ w.r.t. the $L^1(B_R(0))$ convergence and the fact that $\Psi'_{\lambda}(u_n^{\lambda})(\varphi_R u_n^{\lambda}) = o_n(1)$ (since $(\varphi_R u_n^{\lambda})$ is bounded in $BV(\mathbb{R}^N)$), imply that

$$\int_{B_R(0)} |Du^{\lambda}| + \liminf_{n \to \infty} \int_{\mathbb{R}^N} \frac{u_n^{\lambda} (Du_n^{\lambda})^a \cdot \nabla \varphi_R}{|(Du_n^{\lambda})^a|} dx + \int_{\mathbb{R}^N} (1 + \lambda V(x)) \varphi_R |u^{\lambda}| dx \le \int_{\mathbb{R}^N} f(u^{\lambda}) u^{\lambda} \varphi_R dx.$$
(3.33)

By doing $R \to +\infty$ in both sides of (3.33) we get that

$$\int_{\mathbb{R}^N} |Du^{\lambda}| + \int_{\mathbb{R}^N} (1 + \lambda V(x)) |u^{\lambda}| dx \le \int_{\mathbb{R}^N} f(u^{\lambda}) u^{\lambda} dx, \qquad (3.34)$$

and the proof is finished.

By the last result there exists $t_{\lambda} \in (0, 1]$ such that $t_{\lambda} u^{\lambda} \in \mathcal{N}_{\lambda}$.

Before to get more information about t_{λ} , let's just give a piece of information.

Lemma 11. Under (f_5) , f is such that $t \mapsto f(t)t - F(t)$ is increasing for $t \in (0, +\infty)$ and decreasing for $t \in (-\infty, 0)$.

Proof. Let $t_1 > t_2 > 0$, then

$$f(t_1)t_1 - F(t_1) = f(t_1)t_1 - F(t_2) - \int_{t_2}^{t_1} f(s)ds$$

> $f(t_1)t_1 - F(t_2) - f(t_1) \int_{t_2}^{t_1} ds$
> $f(t_2)t_2 - F(t_2).$

The case in which $t_1 < t_2 < 0$ is analogous.

Lemma 12. Let λ^* be as in Lemma 7. If $\lambda \geq \lambda^*$, then $t_{\lambda} = 1$,

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n^{\lambda}) u_n^{\lambda} dx = \int_{\mathbb{R}^N} f(u^{\lambda}) u^{\lambda} dx,$$
$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n^{\lambda}) dx = \int_{\mathbb{R}^N} F(u^{\lambda}) dx$$

and

$$\lim_{n \to +\infty} \|u_n^{\lambda}\|_{\lambda} = \|u^{\lambda}\|_{\lambda}.$$

Proof. Note that

$$c_{\lambda} + o_n(1) = \Psi_{\lambda}(u_n^{\lambda}) + o_n(1) = \Psi_{\lambda}(u_n^{\lambda}) - \Psi_{\lambda}'(u_n^{\lambda})u_n^{\lambda} = \int_{\mathbb{R}^N} \left(f(u_n^{\lambda})u_n^{\lambda} - F(u_n^{\lambda}) \right) dx.$$
(3.35)

Applying Fatou Lemma in the last inequality together with Lemma 11, we derive that

$$c_{\lambda} \geq \int_{\mathbb{R}^{N}} \left(f(u^{\lambda})u^{\lambda} - F(u^{\lambda}) \right) dx$$

$$\geq \int_{\mathbb{R}^{N}} \left(f(t_{\lambda}u^{\lambda})t_{\lambda}u^{\lambda} - F(t_{\lambda}u^{\lambda}) \right) dx$$

$$= \Psi_{\lambda}(t_{\lambda}u^{\lambda}) - \Psi_{\lambda}'(t_{\lambda}u^{\lambda})t_{\lambda}u^{\lambda}$$

$$= \Psi_{\lambda}(t_{\lambda}u^{\lambda})$$

$$\geq c_{\lambda}.$$

Hence, $t_{\lambda} = 1$, $\Phi_{\lambda}(u^{\lambda}) = c_{\lambda}$, and by (3.35),

$$f(u_n^{\lambda})u_n^{\lambda} - F(u_n^{\lambda}) \to f(u^{\lambda})u^{\lambda} - F(u^{\lambda}) \quad \text{in} \quad L^1(\mathbb{R}^N).$$
(3.36)

This limit together with (f_4) and (3.22) yield

$$f(u_n^{\lambda})u_n^{\lambda} \to f(u^{\lambda})u^{\lambda}$$
 in $L^1(\mathbb{R}^N)$ (3.37)

$$F(u_n^{\lambda}) \to F(u^{\lambda}) \quad \text{in } L^1(\mathbb{R}^N)$$
 (3.38)

and

$$\|u_n^{\lambda}\|_{\lambda} \to \|u^{\lambda}\|_{\lambda}. \tag{3.39}$$

Here, we have used the fact that (f_4) ensures that

$$0 \le (1 - 1/\theta) f(u_n^{\lambda}) v_n \le f(u_n^{\lambda}) v_n - F(u_n^{\lambda})$$

and

$$0 \le (\theta - 1)F(u_n^{\lambda}) \le f(u_n^{\lambda})u_n^{\lambda} - F(u_n^{\lambda}).$$

Then, by (3.36), we can apply the Lebesgue Dominated Convergence Theorem to get (3.37) and (3.38). Recalling that $||u^{\lambda}||_{\lambda} = \int_{\mathbb{R}^N} f(u^{\lambda})u^{\lambda}$ and $||u^{\lambda}_n||_{\lambda} = \int_{\mathbb{R}^N} f(u^{\lambda}_n)u^{\lambda}_n + o_n(1)$, the limit (3.37) implies in (3.39).

As a consequence of the last result, we see that u^{λ} is a bounded variation solution of $(P)_{\lambda}$. In fact, from (3.17), Lemma 12 and the lower semicontinuity of $\|\cdot\|_{\lambda}$ w.r.t. the $L^{1}(\mathbb{R}^{N})$ convergence, it follows that

$$\|v\|_{\lambda} - \|u^{\lambda}\|_{\lambda} \ge \int_{\mathbb{R}^N} f(u^{\lambda})(v - u^{\lambda})dx, \quad \forall v \in E_{\lambda},$$
(3.40)

and then u^{λ} is in fact a nontrivial solution of $(P)_{\lambda}$. Moreover, note that from (3.16)

$$c_{\lambda} \leq \Psi_{\lambda}(u^{\lambda})$$

$$= \Psi_{\lambda}(u^{\lambda}) - \Psi_{\lambda}'(u^{\lambda})u^{\lambda}$$

$$= \int_{\mathbb{R}^{N}} \left(f(u^{\lambda})u^{\lambda} - F(u^{\lambda}) \right) dx$$

$$\leq \liminf_{n \to \infty} \int_{\mathbb{R}^{N}} \left(f(u^{\lambda}_{n})u^{\lambda}_{n} - F(u^{\lambda}_{n}) \right) dx$$

$$= \Psi_{\lambda}(u^{\lambda}_{n}) + o_{n}(1)$$

$$= c_{\lambda},$$

which implies that

$$\Psi_{\lambda}(u^{\lambda}) = c_{\lambda}. \tag{3.41}$$

Since \mathcal{N}_{λ} contains all nontrivial bounded variation solutions of $(P)_{\lambda}$, from (3.41), in view of (3.19) it follows that u^{λ} is a ground-state solution of $(P)_{\lambda}$.

4 The concentration arguments

4.1 The behavior of the $(PS)_{c,\infty}$ sequences

First of all let us consider the following definition.

Definition 13. A sequence $(u_n) \subset BV(\mathbb{R}^N)$ is called a $(PS)_{c,\infty}$ -sequence for the family $(\Psi_{\lambda})_{\lambda \geq 1}$, if there is a sequence $\lambda_n \to \infty$ such that $u_n \in E_{\lambda_n}$ for $n \in \mathbb{N}$,

$$\Psi_{\lambda_n}(u_n) \to c$$

as $n \to +\infty$ and moreover

$$\|v\|_{\lambda_n} - \|u_n\|_{\lambda_n} \ge \int_{\mathbb{R}^N} f(u_n)(v - u_n) - \tau_n \|v - u_n\|_{\lambda_n}, \quad \forall v \in E_{\lambda_n}$$
(4.42)

where $\tau_n \to 0$ as $n \to +\infty$.

Before to proceed with other results, let us point out some facts about the limit problem (1.2). Note that (1.2) is the formal version of the Euler-Lagrange equation of the functional $\Phi_{\Omega}: BV(\Omega) \to \mathbb{R}$ given by

$$\Phi_{\Omega}(u) = \int_{\Omega} |Du| + \int_{\Omega} |u| dx + \int_{\partial \Omega} |u| d\mathcal{H}_{N-1} - \int_{\Omega} F(u) dx.$$

By the assumptions $(f_1) - (f_5)$, the Nehari set associated to Φ_{Ω} is well defined by

$$\mathcal{N}_{\Omega} = \{ u \in BV(\Omega) \setminus \{0\}; \ \Phi'_{\Omega}(u)u = 0 \},\$$

and let us define

$$c_{\Omega} = \inf_{\mathcal{N}_{\Omega}} \Phi_{\Omega}.$$

Note that c_{Ω} is well defined since from (f_4) , $\Phi_{\Omega}(u) \geq 0$ for all $u \in \mathcal{N}_{\Omega}$. In fact, since I_{Ω} satisfies the geometric conditions of the Mountain Pass Theorem, well known arguments imply that

$$c_{\Omega} = \inf_{\gamma \in \Gamma_{\Omega}} \max_{t \in [0,1]} I_{\Omega}(\gamma(t)),$$

where $\Gamma_{\Omega} = \{ \gamma \in C([0,1], BV(\Omega)); \ \gamma(0) = 0 \text{ and } I_{\Omega}(\gamma(1)) < 0 \}.$

Proposition 14. Let $(u_n) \subset BV(\mathbb{R}^N)$ be a $(PS)_{d,\infty}$ -sequence for $(\Psi_{\lambda})_{\lambda \geq 1}$, where $d \in \mathbb{R}$. Then either d = 0 or $d \geq c_{\Omega}$. In the last case, there exists $u_{\Omega} \in BV(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u_{\Omega}$ in $L^q_{loc}(\mathbb{R}^N)$, for $1 \leq q < 1^*$, $u_{\Omega} \equiv 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ and u_{Ω} is a bounded variation solution of (1.2). Moreover, if $d = c_{\Omega}$, then

$$||u_n||_{\lambda_n} - ||u_\Omega||_{\Omega} \to 0, \quad as \ n \to +\infty.$$

Proof. First of all note that the arguments in Lemma 5 imply that $d \ge 0$, since it holds

$$d + o_n(1) \ge C \|u_n\|_{\lambda_n}.$$
 (4.43)

It follows also from (4.43) that $(||u_n||_{\lambda_n})$ is a bounded sequence in \mathbb{R} and then that (u_n) is bounded in $BV(\mathbb{R}^N)$. By the Sobolev embeddings, there exists $u_{\Omega} \in BV_{loc}(\mathbb{R}^N)$ such that $u_n \to u_{\Omega}$ in $L^q_{loc}(\mathbb{R}^N)$, for $1 \leq q < 1^*$. Moreover, it is possible to argue as in the last section in order to show that in fact $u_{\Omega} \in BV(\mathbb{R}^N)$.

Now let us show that $u_{\Omega} \equiv 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. For each $m \in \mathbb{N}$, let us define $C_m = \{x \in \mathbb{R}^N; V(x) \ge 1/m\}$. Then note that

$$\begin{split} \int_{C_m} |u_n| dx &\leq \frac{m}{\lambda_n} \int_{C_m} \lambda_n V(x) |u_n| dx \\ &\leq \frac{m}{\lambda_n} \|u_n\|_{\lambda_n} \\ &= o_n(1), \end{split}$$

since $(||u_n||_{\lambda_n})$ is a bounded sequence. Then, Fatou Lemma and the last inequality implies that

$$\int_{C_m} |u_\Omega| dx = 0. \tag{4.44}$$

Hence, since $\mathbb{R}^N \backslash \Omega = \bigcup_{i=1}^{+\infty} C_m$, it follows that

$$\int_{\mathbb{R}^N \setminus \Omega} |u_{\Omega}| dx = 0$$

and then that $u_{\Omega} = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

If d = 0, then (4.43) imply that $||u_n||_{\lambda_n} \to 0$ as $n \to +\infty$ and nothing is left to prove. If d > 0, since

$$d + o_n(1) = \Psi_{\lambda_n}(u_n) \le ||u_n||_{\lambda_n},$$

the same arguments in Lemma 9 can be used to show that $u^{\lambda} \neq 0$.

Since $u_{\Omega} \neq 0$, there exists t > 0 such that $tu_{\Omega} \in \mathcal{N}_{\Omega}$. Let us prove that $t \in (0, 1]$, what is implied by the following claim.

Claim. $\Phi'_{\Omega}(u_{\Omega})u_{\Omega} \leq 0.$

For $\delta > 0$, let $\varphi_{\delta} \in C^{\infty}(\mathbb{R}^N)$ be such that $\varphi_{\delta} \equiv 1$ in Ω_{δ} , $\varphi_{\delta} \equiv 0$ in $(\Omega_{2\delta})^c$ and $|\nabla \varphi_{\delta}|_{\infty} \leq C/\delta$, where by Ω_{σ} we mean the σ -neighborhood of Ω , $\sigma > 0$. Let us define

$$\overline{u_{\Omega}}(x) = \begin{cases} u_{\Omega}(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$
(4.45)

Note that $\overline{u_{\Omega}} \in BV(\mathbb{R}^N)$ and, by the Green Formula for BV functions (see [3][Theorem 10.2.1] for instance),

$$\int_{\Omega_{\delta}} |D\overline{u_{\Omega}}| + \int_{\Omega_{\delta}} |\overline{u_{\Omega}}| dx = \int_{\Omega} |Du_{\Omega}| + \int_{\Omega} |u_{\Omega}| dx + \int_{\partial\Omega} |u_{\Omega}| d\mathcal{H}_{N-1}.$$
(4.46)

As in the proof of Lemma 10, note that $\Psi'_{\lambda_n}(u_n)(\varphi_{\delta}u_n)$ is well defined and, by using $u_n + t\varphi_{\delta}u_n$ as test function in (4.42) and doing $t \to 0^{\pm}$, since (u_n) is bounded in $BV(\mathbb{R}^N)$, it is possible to see that

$$\Psi_{\lambda_n}'(u_n)(\varphi_\delta u_n) = o_n(1). \tag{4.47}$$

Then by (2.5) it follows that

$$\Psi_{\lambda_{n}}'(u_{n})(\varphi_{\delta}u_{n}) = \int_{\mathbb{R}^{N}} \frac{((Du_{n})^{a})^{2}\varphi_{\delta} + u_{n}(Du_{n})^{a} \cdot \nabla\varphi_{\delta}}{|(Du_{n})^{a}|} dx + \int_{\mathbb{R}^{N}} \frac{Du_{n}}{|Du_{n}|} \frac{\varphi_{\delta}(Du_{n})^{s}}{|\varphi_{\delta}(Du_{n})^{s}|} |\varphi_{\delta}(Du_{n})^{s}| + + \int_{\mathbb{R}^{N}} (1 + \lambda_{n}V(x))\operatorname{sgn}(u_{n})(\varphi_{\delta}u_{n})dx - \int_{\mathbb{R}^{N}} f(u_{n})\varphi_{\delta}u_{n}dx = \int_{\mathbb{R}^{N}} \varphi_{\delta}|(Du_{n})^{a}|dx + \int_{\mathbb{R}^{N}} \frac{u_{n}(Du_{n})^{a} \cdot \nabla\varphi_{\delta}}{|(Du_{n})^{a}|}dx + + \int_{\mathbb{R}^{N}} \frac{(Du_{n})^{s}}{|(Du_{n})^{s}|} \frac{\varphi_{\delta}(Du_{n})^{s}}{|\varphi_{\delta}(Du_{n})^{s}|} |\varphi_{\delta}(Du_{n})^{s}| + \int_{\mathbb{R}^{N}} (1 + \lambda_{n}V(x))|u_{n}|\varphi_{\delta}dx - - \int_{\mathbb{R}^{N}} f(u_{n})\varphi_{\delta}u_{n}dx.$$

$$(4.48)$$

Since $u_n \to \overline{u_\Omega}$ in $L^q(\Omega_\delta)$ for $1 \le q < 1^*$, by the lower semicontinuity of $\|\cdot\|_{BV(\Omega_\delta)}$ w.r.t. the $L^q(\Omega_\delta)$ convergence, (4.46), (4.47) and (4.48), it follows that

$$\begin{aligned} \|u_{\Omega}\|_{\Omega} &\leq \liminf_{n \to +\infty} \left(\int_{\Omega_{\delta}} |Du_{n}| + \int_{\Omega_{\delta}} |u_{n}| dx \right) \\ &\leq \liminf_{n \to +\infty} \left(\int_{\mathbb{R}^{N}} \varphi_{\delta} |(Du_{n})^{a}| dx + \int_{\mathbb{R}^{N}} (1 + \lambda_{n} V(x)) |u_{n}| \varphi_{\delta} dx \\ &+ \int_{\mathbb{R}^{N}} \frac{(Du_{n})^{s}}{|(Du_{n})^{s}|} \frac{\varphi_{\delta} (Du_{n})^{s}}{|\varphi_{\delta} (Du_{n})^{s}|} |\varphi_{\delta} (Du_{n})^{s}| \right) \\ &\leq \limsup_{n \to +\infty} \left(\int_{\mathbb{R}^{N}} f(u_{n}) \varphi_{\delta} u_{n} dx - \int_{\Omega_{2\delta} \setminus \Omega_{\delta}} \frac{u_{n} (Du_{n})^{a} \cdot \nabla \varphi_{\delta}}{|(Du_{n})^{a}|} dx \right) \\ &= \int_{\Omega_{2\delta}} f(u_{\Omega}) \varphi_{\delta} u_{\Omega} dx, \end{aligned}$$
(4.49)

since

$$\int_{\Omega_{2\delta}\setminus\Omega_{\delta}} \frac{u_n (Du_n)^a \cdot \nabla \varphi_{\delta}}{|(Du_n)^a|} dx \leq \frac{C}{\delta} \int_{\Omega_{2\delta}\setminus\Omega_{\delta}} |u_n| dx = o_n(1),$$

By doing $\delta \to 0$ in (4.49) it follows that $\Phi'_{\Omega}(u_{\Omega})u_{\Omega} \leq 0$ and the Claim is proved.

Then there exists $t \in (0, 1]$ such that $tu_{\Omega} \in \mathcal{N}_{\Omega}$.

Note moreover that

$$d + o_n(1) = \Psi_{\lambda_n}(u_n) + o_n(1) = \Psi_{\lambda_n}(u_n) - \Psi'_{\lambda_n}(u_n)u_n = \int_{\mathbb{R}^N} \left(f(u_n)u_n - F(u_n) \right) dx.$$
(4.50)

Applying Fatou Lemma in the last inequality together with Lemma 11, we derive that

$$d \geq \int_{\Omega} (f(u_{\Omega})u_{\Omega} - F(u_{\Omega})) dx$$

$$\geq \int_{\Omega} (f(tu_{\Omega})tu_{\Omega} - F(tu_{\Omega})) dx$$

$$= \Phi_{\Omega}(tu_{\Omega}) - \Phi'_{\Omega}(tu_{\Omega})tu_{\Omega}$$

$$= \Phi_{\Omega}(tu_{\Omega})$$

$$\geq c_{\Omega}, \qquad (4.51)$$

what shows that $d \geq c_{\Omega}$.

Now let us suppose that $d = c_{\Omega}$. In this case, we can prove that t = 1, i.e., that $u_{\Omega} \in \mathcal{N}_{\Omega}$. This follows since in this case, from (4.51) and the fact that $d = c_{\Omega}$, it follows that t = 1, $u_{\Omega} \in \mathcal{N}_{\Omega}$, $\Phi_{\Omega}(u_{\Omega}) = c_{\Omega}$, and from (4.50),

$$f(u_n)u_n - F(u_n) \to f(u_\Omega)u_\Omega - F(u_\Omega)$$
 in $L^1(\mathbb{R}^N)$. (4.52)

This limit together with (f_4) imply that

$$f(u_n)u_n \to f(u_\Omega)u_\Omega \quad \text{in } L^1(\mathbb{R}^N)$$
 (4.53)

$$F(u_n) \to F(u_\Omega) \quad \text{in } L^1(\mathbb{R}^N)$$

$$(4.54)$$

and

$$\|u_n\|_{\lambda_n} \to \|u_\Omega\|_{\Omega}. \tag{4.55}$$

From (4.53), (4.55), by taking the $\limsup_{n\to+\infty}$ in (4.42) and noting that, for all $v \in BV(\Omega)$, if \overline{v} is defined as in (4.45),

$$\|v\|_{\Omega} = \|\overline{v}\|_{\lambda_n},$$

it follows that

$$\|v\|_{\Omega} - \|u_{\Omega}\|_{\Omega} \ge \int_{\Omega} f(u_{\Omega})(v - u_{\Omega}).$$

Then u_{Ω} is a bounded variation solution of (1.2).

4.2 Proof of Theorem 1

Let us consider a sequence $\lambda_n \to +\infty$ as $n \to +\infty$ and, for each $n \in \mathbb{N}$, $u_n := u^{\lambda_n}$ the bounded variation solution of $(P)_{\lambda_n}$ obtained in Section 3, which is such that $\Phi_{\lambda_n}(u_n) = c_{\lambda_n}$.

Note that, for a given $u \in BV(\Omega)$, denoting by \overline{u} its extension by zero outside Ω (as in (4.45)), it follows from Green Formula for BV functions that

$$\int_{\mathbb{R}^N} |D\overline{u}| + \int_{\mathbb{R}^N} |\overline{u}| dx = \int_{\Omega} |Du| + \int_{\Omega} |u| dx + \int_{\partial\Omega} |u| d\mathcal{H}_{N-1}.$$
 (4.56)

Hence, if $u \in BV(\Omega)$, then $\overline{u} \in E_{\lambda}$ and $\Phi_{\Omega}(u) = \Psi_{\lambda}(\overline{u})$ for every $\lambda > 0$. Then, for each $\gamma \in \Gamma_{\Omega}$, it follows that $\overline{\gamma} \in \Gamma_{\lambda}$. Based on this fact, it is easy to see that

$$c_{\lambda} = \inf_{\gamma \in \Gamma_{\lambda}} \max_{t \in [0,1]} \Psi_{\lambda}(\gamma(t)) \le \inf_{\gamma \in \Gamma_{\Omega}} \max_{t \in [0,1]} \Phi_{\Omega}(\gamma(t)) = c_{\Omega}, \tag{4.57}$$

for every $\lambda > 0$.

Then it follows that

$$(c_{\lambda_n})_{n\in\mathbb{N}}\subset[0,c_\Omega],$$

which implies that, up to a subsequence, $\Psi_{\lambda_n}(u_{\lambda_n}) \to d \in [0, c_\Omega]$, as $n \to +\infty$. Since u_n satisfies (4.42) with $\tau_n = 0$, it follows that (u_n) is in fact a $(PS)_{d,\infty}$ sequence.

Note that by (3.18), d > 0. On the other hand, by Proposition 14, it holds that

$$d \ge c_{\Omega}.\tag{4.58}$$

Then, from (4.57) and (4.58) it follows that (u_n) is a $(PS)_{c_{\Omega},\infty}$ -sequence and then, again by Proposition 14 there exists $u_{\Omega} \in BV(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u_{\Omega}$ in $L^q_{loc}(\mathbb{R}^N)$, for $1 \leq q < 1^*$, $u_{\Omega} \equiv 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ and u_{Ω} is a bounded variation solution of (1.2). Moreover,

$$||u_n||_{\lambda_n} - ||u_\Omega||_{\Omega} \to 0, \text{ as } n \to +\infty$$

and Theorem 1 is proved.

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