Weights which respect support and NN-decoding

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Abstract—In this work we explore a family of metrics over finite fields which respect the support of vectors. We show how these metrics can be obtained from the edge-weighted Hamming cube and, based on this representation we give a description of a group of linear isometries (with respect to the metric). Next we introduce the concept of conditional sum of metrics and determine what conditions determine a metric respecting support, out of two such given metrics. Finally we introduce the labeled-poset block metrics, a new family of metrics which respects support of vector, filling a gap existing in the known such metrics. For this family we give a full description of the group of linear isometries and determine necessary and sufficient conditions for the existence of a MacWilliams identity.

I. INTRODUCTION

In coding theory, there are two main sources of decoding criteria: a probabilistic (Maximum Likelihood Decoding - MLD) and a metric (Minimum Distance Decoding - MDD). While the first one focuses on the properties of the channel and is the optimal criterion (in term of minimizing the error probability of the encoding-transmission-decoding process), the last generally has properties that may help in the implementation of decoding algorithms.

The most important instance of channel is the binary symmetric channel which MLD criterion matches the MDD criterion determined by the Hamming metric.The Hamming metric d_H has two important properties that are very valuable:

P1 _Weight condition:_ The metric d_H is determined by the Hamming weight wt_H, i.e., $d_H(u,v) = \mathrm{wt}_H(u-v)$.

P2 _Support condition:_ If the vectors $u = (u_1, \ldots, u_n)$ and $u' = (u'_1, \ldots, u'_n)$ are such that $u_i \neq 0$ whenever $u'_i \neq 0$, then $\mathrm{wt}_H(u) \geq \mathrm{wt}_H(u')$. In this case we say that the metrics _respects support_.

The first of this properties gives an important tool for implementing algorithms: the well known syndrome decoding may be performed for every metric determined by a weight. The second property makes it meaningful in the context of coding theory, in the sense that making extra errors can not improve the result. Altogether, we name these as our _basic decoding conditions_ (BDC, for short).

In the literature, matching between channels and metrics (that is, the maximum likelihood decoding coincides with nearest neighbor decoding) is not much explored. Despite the large number of channels that are studied and the large number of metrics described in the literature in the context of Coding Theory (see, for example, [Chapter 16] in [1], and a recent survey of Gabidulin [2]), there are a few examples of classical metrics and channels which are proved to be matched.

Although matching channels and metrics is not widely studied, we can find in the literature large families of metrics satisfying the basic decoding conditions. We cite, for example, the poset metrics of Brualdi [3], Gabidulin's combinatorial metrics [4], poset-block metrics [5] and digraph metrics [6]. [1]

All those generalize the Hamming metric and they represent very large families of metrics over a vector space $\mathbb{F}_q^n$ (large in the sense that each of those families grows exponentially with n). Nevertheless, those are not sufficient to determine all the MDD criteria satisfying the support condition. Example 1 illustrates such an affirmation for the smallest possible case, $n = 2$.

Before introducing the example, we should remark that different metrics may determine the same MMD, and in this case, we should consider such metrics to be equivalent. To be more precise: two metrics d_1 and d_2 over a space V are _decoding-equivalent_ if given any code $C \subset V$ and any received message $x \in V$, the MDDs determined by both metrics generate the same set of codewords, i.e., $\arg\min_{c \in C} d_1(x,c) = \arg\min_{c \in C} d_2(x,c)$, for all $x \in \mathbb{F}_q^n$. It is not difficult to prove that for metrics defined by weights, being equivalent means that, when ordering the vectors in $\mathbb{F}_q^n$ according to the two different weights we get the same ordering (see [7] for details).

Example 1: Let us consider the space $\mathbb{F}_2^2 = \{00, 10, 01, 11\}$. In this case we have 4 non decoding-equivalent criteria. In the table bellow we present these criteria and check each that can be determined by a metric in one of the large families we have mentioned: poset wt_P, poset-blocks wt_{PB}, combinatorial wt_C and digraph wt_D. It is worth to note that only the first one can be determined by any of this families of metrics.

Criterion	wt_H	wt_P	wt_{PB}	wt_C	wt_D
$\mathrm{wt}(10) = \mathrm{wt}(01) < \mathrm{wt}(11)$	✓	✓	✓	✓	✓
$\mathrm{wt}(10) = \mathrm{wt}(01) = \mathrm{wt}(11)$			✓	✓	✓
$\mathrm{wt}(10) < \mathrm{wt}(01) = \mathrm{wt}(11)$		✓	✓		✓
$\mathrm{wt}(10) < \mathrm{wt}(01) < \mathrm{wt}(11)$					

TABLE I
DECODING CRITERIA WHICH RESPECT SUPPORT IN $\mathbb{F}_2^2$

We stress that different metrics can be decoding-equivalent. In fact, even though the second criterion in the table ($\mathrm{wt}(u)$

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[1] Let $[n] = \{1, 2, ..., n\}$. Given a partial order (poset) $P = ([n], \preceq_P)$ on the set $[n]$, the poset weight $\mathrm{wt}_P(x)$ of a vector $x \in \mathbb{F}_q^n$ is defined as $|\langle \mathrm{supp}(x) \rangle_P|$, where $\mathrm{supp}(c)$ is the support of the vector, $\langle X \rangle_P$ is the smallest order-ideal containing X and $|A|$ is the cardinality of A. If $\mathcal{F} = \{A_1, A_2, \cdots A_r\}$ is a covering of $[n]$, the $\mathcal{F}$-combinatorial weight of a vector $x \in \mathbb{F}_q^n$ is $\min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{F}$ and $\mathcal{A}$ is a covering of $\mathrm{supp}(x)\}$. For more details on all those metrics, see [?].

is constant for $u \neq 0$) may be determined by a poset-block wt_{PB} and also by a digraph wt_D weight, simple computations shows that $\text{wt}_{PB}(u) = 1$ and $\text{wt}_D(u) = 2$, for $u \neq 0$. More important, we note that the last decoding criterion can not be determined by any metric belonging to one of these families.

This work aims to reduce the gap between the known and studied metrics satisfying the BDC and the space of all possible metrics satisfying the BDC.

In Section II we give the first systematic approach to the space of all metrics satisfying the BDC. We introduce a conditional operator on metrics and the main result is to establish what are the conditions that ensure that from a pair of metrics satisfying the BDC one gets another metric satisfying the BDC. This is a starting point to estimate how large the metrics obtained by a conditional sum of a poset, digraph and combinatorial metrics is in the space of a metrics satisfying the BDC. In other words, this section moves one step forward in a long term goal to develop an "approximation theory" of metrics in the context of coding theory.

The rest of the work is devoted to introduce a new family of metrics that generalizes both the digraph metrics and the poset-block metrics, introduced in [6] and [5], respectively. In Section III, we introduce the basic concepts and define the labeled-poset-block metrics. In Section III-A we characterize the group of linear isometries of a space endowed with a labeled-poset-block metric. In Section III-B we give necessary and sufficient conditions to ensure that every linear code admits a canonical decomposition and derive necessary condition to ensure the existence of a MacWilliams' Identity.

II. OPERATING WITH METRICS WHICH RESPECT SUPPORT

Since we are concerned with metrics determined by weights, we establish a condition on weights to ensure that the corresponding metric satisfies the MDC.

Definition 1: A function $\text{wt} : \mathbb{F}_2^n \to \mathbb{Z}$ is a *weight respecting support* (or simply S-weight) if the following holds:

1) $\text{wt}(u) \geq 0$ and equality implies $x = 0$;
2) If $\text{supp}(u) \subset \text{supp}(v)$, then $\text{wt}(u) \leq \text{wt}(v)$, where $\text{supp}(x) = \{i \in \{1, \ldots, n\} : x_i \neq 0\}$.

An S-weight determines a semi-metric, by defining $d(u, v) = \text{wt}(u - v)$, and two weights determine the same semi-metric if, and only if, they are equal. In order to guarantee the $d(u, v) = \text{wt}(u - v)$ is a metric it is required the triangle inequality which we ignore this work due to the fact that every semi-metric d can be rescaled to a decoding-equivalent metric d' as follows:

$$d'(u, v) = \begin{cases} 0, & \text{if} \quad u = v, \\ d(u, v) + \max_{x, y \in \mathbb{F}_q^n} d(x, y), & \text{if} \quad u \neq v. \end{cases}$$

So, to understand the space of all metrics satisfying BDC (conditions P1 and P2), it is enough to study the space of all S-weights up to the following equivalence:

Definition 2: We say that two S-weights wt_1 and wt_2 are *equivalent* (denoted by $\text{wt}_1 \sim \text{wt}_2$) if

$$\text{wt}_1(u) < \text{wt}_1(v) \iff \text{wt}_2(u) < \text{wt}_2(v), \forall u, v \in \mathbb{F}_q^n.$$

It is not difficult to see that two S-weights are equivalent if, and only if, they are decoding-equivalent (in the sense defined in Section I, see [7] for details).

Given these initial definitions, we split this section into two parts. Through the first part, the S-weight functions are naturally represented by an edge-weighted Hamming cube that allows us to give a partial description for the group of linear isometries. We introduce the conditional sum of known weights in order to generate new S-weight which are not decoding-equivalent to the former ones and are able, for example, to fill the gap in the last row of Table I.

We give necessary and sufficient conditions to guarantee that a conditional sum of two S-weights is also an S-weight. In addition, we prove that every S-weight is equivalent to an S-weight obtained by a finite conditional sum of poset, digraph and combinatorial weights (to be introduced in).

A. Weights respecting support

In order to explore properties of S-weights, our approach is to construct general S-weights in a way it can inherit the knowledge accumulated about poset, digraph and combinatorial metrics. For this purpose, we shall represent S-weights by labeling the edges of the Hamming cube. This approach allows us to obtain a partial description for the group of linear isometries which is a fundamental tool in the context of coding theory. Indeed, being in the same orbit of this group is the definition of code equivalence and the structure of these groups is used to determine whenever the MacWilliams' extension property is satisfied and so forth.

We start by considering the Hamming cube $\mathcal{H}^n$ as a directed graph where $\mathbb{F}_2^n$ is the set of vertices and (u, v) is a (directed) arc if, and only if, $d_H(u, v) = 1$ and $\text{wt}_H(u) < \text{wt}_H(v)$. In addition, to every arc is assigned a non-negative integer $\delta(u, v)$. The pair $(\mathcal{H}^n, \delta)$ is called a δ-*weighted Hamming cube*.

It is simple to see that given an S-weight $\text{wt} : \mathbb{F}_q^n \to \mathbb{N}$, by setting $\delta((u, v)) = \text{wt}(v) - \text{wt}(u)$ we have that $\text{wt}(w) = \sum_{(u,v) \in \tau} \delta((u, v))$ where τ is any *trail* from the null vector to w. By simplicity, we denote $\delta(\tau) = \sum_{(u,v) \in \tau} \delta((u, v))$ whenever no confusion may arise. This shows that every S-weight can be represented by a δ (weighted) Hamming cube. From here on, except if explicitly stated, we assume that every trail has 0 as its initial vertex.

However, not every $(\mathcal{H}^n, \delta)$ determines an S-weight. For this to happen, the δ function should avoid the situation depicted in the following example. Consider the δ-Hamming cube below.
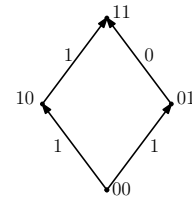
This δ function does not induce an S-weight because the sum of weights on the left and right trails from 00 to 11 are differ-

ent, i.e., $\delta((00, 10)) + \delta((10, 11)) \neq \delta((00, 01)) + \delta((01, 11))$. We avoid this situation imposing it as a necessary condition:

Lemma 1: The map $\mathrm{wt}(w) = \delta(\tau)$, where τ is a trail from 0 to w, is an S-weight if, and only if, $\delta(0, e_i) > 0$, for every $i \in \{1, \ldots, n\}$ and $\delta(\tau) = \delta(\tau')$, for any trails τ, τ' in $\mathcal{H}^n$ with same initial and final vertices.

Proof: The proof follows directly from the definitions and it is omitted due to lack of space. ∎

The next proposition characterizes the weight functions which determines a combinatorial metric, as introduced by Gabidulin, [4].

Proposition 1: Let δ be determined by an S-weight wt as before. Then, $d(u, v) = \mathrm{wt}(u - v)$ is a combinatorial metric if, and only if, $\delta((u, v)) \in \{0, 1\}$.

Proof: If $\delta((u, v)) \geq 2$, then $\mathrm{wt}(v) = \mathrm{wt}(u) + \delta((u, v)) \leq \mathrm{wt}(u) + \mathrm{wt}(v - u)$. This implies that $\mathrm{wt}(v - u) \geq \delta((u, v)) \geq 2$. Since $v - u = e_i$, for some $i \in [n]$, and $\mathrm{wt}(e_i) = 1$ for any combinatorial weight, we have that wt does not induce a combinatorial metric. The proof of the *if* part is constructive and will be omitted due to lack of space. ∎

We reserve the rest of this work to present a standard form for a weight wt, which will be used to prove some key coding results such as characterizing the group of linear isometries and the existence of a MacWilliams' Identity.

Definition 3: Let δ be obtained from an S-weight wt. We say that δ is in a *standard form* if, given a trail τ with $\delta(\tau) = k > 1$ there is a trail τ' such that $\delta(\tau') = k - 1$. We say that an S-weight wt *admits a standard form* if it is equivalent to an S-weight which determines a weight in a standard form.

Proposition 2: Every S-weight has a unique standard form.

Proof: The proof is obtained by following an algorithm which assign new values for δ which is a simple extension of the steps we detail in Example 2. ∎

Example 2: Consider the figure bellow. The δ-Hamming cube on the left is not in a standard form since $\delta(00, 01) = 3$ and there is no trail τ with $\delta(\tau) = 2$. In the middle, we assign the value $\delta((00, 01)) = 2$, and, since $\delta((00, 10)) + \delta((10, 11)) = 4$, Lemma 1 imposes $\delta((01, 11)) = 2$ to get a weight wich defines an S-weight. Now, on the right side, we repeat the procedure for the trail $\tau = \{(00, 01), (01, 11)\}$ which has $\delta(\tau) = 4$ while there is no trail τ' with $\delta(\tau') = 3$.
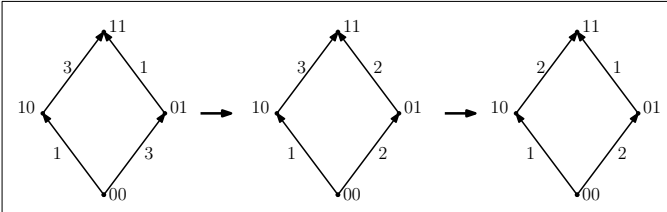
We remark that since the values of δ decrease at every step, the algorithm will have a stopping point.

From here on, we assume that every δ is in a standard form.

Now, we turn our attention to describe the group of linear isometries in a space $\mathbb{F}_q^n$ endowed with a metric determined by an S-weight wt, i.e.,

$$GL(n, q, \mathrm{wt}) = \{T : \mathbb{F}_q^n \to \mathbb{F}_q^n : T \text{ is linear},$$
$$d(u, v) = d(T(u), T(v))\}.$$

Given a linear map $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$, we say it *respects domination* if: *(i)* $T(e_i) = \sum_{j=1}^n \lambda_j e_j$, with $\lambda_i \neq 0$; *(ii)* $\delta(\tau) = 0$ if τ is a trail from $u \in \mathbb{F}_q^n$ to $u + T(e_i) - e_i$, with $u_i \neq 0$. We denote by $N(\mathcal{H}, \delta)$ the group of transformations preserving domination and by $Aut(\mathcal{H}, \delta)$ the group of automorphisms of $\mathcal{H}$ which preserve δ.

The group $GL(n, q, \mathrm{wt})$ is fully described by the following propositions, which proofs are omitted due lack of space.

Proposition 3: Let $\phi \in S_n$. The map $T_\phi : \mathbb{F}_q^n \to \mathbb{F}_q^n$, defined by $T_\phi(x_1, \ldots, x_n) = (x_{\phi(1)}, \ldots, x_{\phi(n)})$ is a linear isometry if, and only if, $T_\phi \in Aut(\mathcal{H}, \delta)$.

Proposition 4: Any map $T \in N(\mathcal{H}, \delta)$ is a linear isometry.

Theorem 1: Let $Aut(\delta) \ltimes N(\delta)$ be the semi-direct product. $Aut(\delta) \ltimes N(\delta) \subseteq GL(n, q, \mathrm{wt})$ and the equality holds for every $q > 2$.

B. Conditional sums

In previous section we showed how S-weights are related to δ-Hamming cubes in the most general setting. The S-weights are not studied in its most general setting, so we wish to approximate a general or obtain a general S-weights from particular families that are better understood, for the same reason that one approximate smooth functions by polynomials. The most studied such metrics are the families of poset and combinatorial metrics, It is worth to note that they are virtually complementary, in the sense that the Hamming metric is the unique metric that belongs to both the families.

So, we are left with two fundamental questions to be addressed in this section: How can we obtain a new S-weight out from two given S-weights? How large is the family of S-weights that can be constructed from a combination of poset and combinatorial weights?

We start by presenting a conditional sum which permits to obtain new S-weights out of given ones.

Proposition 5: Let wt_1 and wt_2 be S-weights. Then, the k-*sum*

$$(\mathrm{wt}_1 \oplus_k \mathrm{wt}_2)(u) = \begin{cases} \mathrm{wt}_1(u), & \text{if } \mathrm{wt}_1(u) < k, \\ \mathrm{wt}_1(u) + \mathrm{wt}_2(u), & \text{if } \mathrm{wt}_1(u) \geq k. \end{cases}$$

is a S-weight.

Proof: The proof follows straightforward from definitions. ∎

We remark that for $k = 0, 1$, $\mathrm{wt}_1 \oplus_k \mathrm{wt}_2$ is the usual sum $\mathrm{wt}_1 + \mathrm{wt}_2$. The previous proposition implies that the set of all S-weights endowed with directed sum or k-sum is a magma, i.e., the set of all weights is closed under "$\oplus_k$".

Remark 1: We stress that the conditional sum $\oplus_k$ may be replaced by a similar sum in which we consider different conditions *respecting the support*[2] of vectors $u \in \mathbb{F}_q^n$. For

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[2]A condition C on vectors of $\mathbb{F}_q^n$ *respects support* if $C(u) = \texttt{true}$ implies $C(v) = \texttt{true}$ for every v such that $\mathrm{supp}(u) \subset \mathrm{supp}(v)$.

instance, given two weights wt_1 and wt_2, let us consider the (H,k)-conditional sum $\mathrm{wt}_1 \oplus_{(H,k)} \mathrm{wt}_2$ defined as follows

$$(\mathrm{wt}_1 \oplus_{(H,k)} \mathrm{wt}_2)(u) = \begin{cases} \mathrm{wt}_1(u), & \text{if } \mathrm{wt}_H(u) < k, \\ \mathrm{wt}_1(u) + \mathrm{wt}_2(u), & \text{if } \mathrm{wt}_H(u) \geq k. \end{cases}$$

Example 3: Let wt_P be the poset weight that covers the criterion $\mathrm{wt}(00) < \mathrm{wt}(10) < \mathrm{wt}(01) = \mathrm{wt}(11)$ and wt_H be the Hamming weight. Then, the $(H,1)$-conditional sum $\mathrm{wt}_P \oplus_{(H,1)} \mathrm{wt}_H$ covers the remaining criterion in $\mathbb{F}_2^2$ given by $\mathrm{wt}(00) < \mathrm{wt}(10) < \mathrm{wt}(01) < \mathrm{wt}(11)$, the last row of Table I.

Not every conditional sum leads to non-equivalent metric. We wish to know under what conditions we have that wt_1, wt_2 and $\mathrm{wt}_1 \oplus_C \mathrm{wt}_2$ are all equivalent. It happens, for example, for $\mathrm{wt} \oplus_0 \mathrm{wt}$. We start with the following lemma:

Lemma 2: Let wt_1 and wt_2 be equivalent weights. Suppose that $\mathrm{wt}_1 \oplus_C \mathrm{wt}_2$ is also equivalent to wt_1 and wt_2 for a given condition C. Then,

1) If $\mathrm{wt}_1(u) = \mathrm{wt}_2(v)$ and $C(u) = \texttt{true}$ then $C(v) = \texttt{true}$.
2) If $\mathrm{wt}_1(u) < \mathrm{wt}_2(v)$ and $C(u) = \texttt{true}$, then either $C(v) = \texttt{true}$ or $2 \cdot \mathrm{wt}_1(u) < \mathrm{wt}_2(v)$.

Proof: The proof can be obtained by simple computations and it is omitted. ∎

The second part of the previous Lemma ensures that, if $\mathrm{wt}_1 \sim \mathrm{wt}_2 \sim \mathrm{wt}_1 \oplus_C \mathrm{wt}_2$ we can choose all of them to be equal and we get the following:

Proposition 6: Let wt be a S-weight. Then, $\mathrm{wt} \sim \mathrm{wt} \oplus_C \mathrm{wt}$ if, and only if, $\oplus_C = \oplus_k$, for some $k \in \mathbb{N}$.

Proof: It is enough to choose $k = \min\{\mathrm{wt}(u) : u \in \mathbb{F}_q^n \text{ satisfies } C\}$. ∎

Theorem 2: Every S-weight can be reached by a finite conditional sum of poset and combinatorial weights.

Proof: The proof is omitted due to lack of space. ∎

III. LABELED-POSET-BLOCK METRICS

In this section, we introduce a particular family which is a generalization for the digraph metrics which arises naturally from the reduced canonical form for directed graphs presented in [6]. This canonical form makes a contraction of each maximal cycle into a unique vertex and, then, such vertex is labeled by the number of vertices contained in the original cycle. If we allow this labeling to assume different values such extension also generalizes the poset-block metrics by labeling every maximal cycle with 1. The goal in this section, is to present an structured family of metrics that covers the remaining decoding criterion in Table I. Despite the generality of the approach, we also produce a description of group of linear isometries and determine conditions for a MacWilliams identity to be available.

Let $P = ([m], \preceq_P)$ be a partially ordered set (abbreviated as poset), where $\preceq_p$ is a partial order over $[m] := \{1, \ldots, m\}$. An *ideal* in $P = ([m], \preceq_P)$ is a subset $I \subseteq [m]$ such that, if $b \in I$ and $a \preceq_P b$, then $a \in I$. Given $A \subseteq [m]$, we denote by $\langle A \rangle_P$ the smallest ideal of P containing A and call it as the *ideal generated by* A. An element a of a set $A \subseteq [m]$ is called a *maximal* element of A if $a \preceq_P b$ for some $b \in A$ implies

$b = a$. The set of all maximal elements of A is denoted by $\mathcal{M}_P(A)$. Note that if $I \subseteq [m]$ is an ideal, then $\mathcal{M}_P(I)$ is the minimal set that generates I, i.e., $\langle \mathcal{M}_P(I) \rangle_P = I$.

Given two posets P and Q over $[m]$, a *poset isomorphism* is a bijection $\phi : [m] \to [m]$ such that $i \preceq_P j \iff \phi(i) \preceq_Q \phi(j)$. When $P = Q$, ϕ is called a *P-automorphism*. The set of all P-automorphisms is a group denoted by $Aut(P)$.

A *chain* in a poset P is a subset $X \subseteq [m]$ such that any two elements $a, b \in X$ are comparable, in the sense that $a \preceq_P b$ or $b \preceq_P a$. We remark that any (finite) chain has a unique maximal element. The *height* $h(a)$ of an element $a \in P$ is the cardinality of a largest chain having a as the maximal element. The *height* $h(P)$ *of the poset* is the maximal height of its elements, i.e., $h(P) = \max\{h(a) : a \in [m]\}$. The *i-th level* Γ_i^P of a poset P is the set of all elements with height i, i.e., $\Gamma_i^P = \{a \in [m] : h(a) = i\}$. A poset P is *hierarchical* if elements at different levels are always comparable, i.e., $a \in \Gamma_i^P$ and $b \in \Gamma_j^P$ implies $a \prec_P b$ for any $1 \leq i < j \leq h(P)$.

Let us consider a map $\pi : [n] \to [m]$ with $n \geq m$ (called a *block map*). A vector $u \in \mathbb{F}_q^n$ may be written as $u = (u_1, \ldots, u_m)$, where $u_i \in \mathbb{F}_q^{k_i}$, with $k_i = |\pi^{-1}(i)|$. The π-*support* is defined as

$$\mathrm{supp}_\pi(u) = \{i \in [m] : u_i \neq 0\}.$$

Given a block function $\pi : [n] \to [m]$, a poset $P = ([m], \preceq_P)$ and a *label function* $L : [m] \to \mathbb{N}$, the (P, π, L)-*weight* of u is defined as

$$\mathrm{wt}_{(P,\pi,L)}(u) = \sum_{i \in \langle \mathrm{supp}_\pi(u) \rangle_P} L(i).$$

For $u, v \in \mathbb{F}_q^n$, we define the *labeled-poset-block distance* by:

$$d_{(P,\pi,L)}(u, v) = \mathrm{wt}_{(P,\pi,L)}(u - v).$$

Proposition 7: If the label function L assumes only positive values, then $d_{(P,\pi,L)}(u, v)$ determines a metric over $\mathbb{F}_q^n$.

Proof: The proof follows straight from the definitions. ∎

A. (P, π, L)-linear isometries

Let $GL(P, \pi, L)_q$ be the group of linear isometries of the space $\mathbb{F}_q^n$ endowed with a (P, π, L)-metric. Our goal in this section is to characterize $GL(P, \pi, L)_q$.

To be more precise,

$$GL(P, \pi, L)_q = \{T : \mathbb{F}_q^n \to \mathbb{F}_q^n \ : \ T \text{ is linear,}$$
$$d_{(P,\pi,L)}(x, y) = d_{(P,\pi,L)}(T(x), T(y)), \forall\, x, y \in \mathbb{F}_q^n\}$$
$$= \{T : \mathbb{F}_q^n \to \mathbb{F}_q^n \ : \ T \text{ is linear,}$$
$$\mathrm{wt}_{(P,\pi,L)}(x) = \mathrm{wt}_{(P,\pi,L)}(T(x)), \forall\, x \in \mathbb{F}_q^n\}$$

Similarly to what happens in the case of posets, $GL(P, \pi, L)_q$ can be described as the semi-direct product of two subgroups. We start presenting one of them, which is a subgroup of the permutation group $[m]$ that preserves the involved structures: the order structure P, the block map π and the label function L.

Definition 4: A map $\phi : [m] \to [m]$ is a (P, π, L)-automorphism if it is a *P-automorphism* with $L(i) = L(\phi(i))$

and $k_i = k_{\phi(i)}$, for every $i \in [m]$. We denote by $Aut(P, \pi, L)$ the set of all (P, π, L)-automorphisms.

We remark that $Aut(P, \pi, L)$ is a group. The following proposition follows straight from the definition of $d_{(P,\pi,L)}$.

Proposition 8: Let ϕ be a (P, π, L)-automorphism. The linear map $T_\phi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ defined by $T_\phi(e_{ij}) = e_{\phi(i)j}$ is an isometry. Moreover, the map $\varphi : Aut(P, \pi, L) \to GL(P, \pi, L)$ that associates $\phi \mapsto T_\phi$ is an injective homomorphism of groups.

We denote by $\mathcal{A} := \{T_\phi \in GL(P, \pi, L); \phi \in Aut(P, \pi, L)\}$ the subgroup of isometries *induced by* (P, π, L)-*automorphisms*. The two next propositions are far from trivial, but the proofs are omitted, for the usual reason.

Proposition 9: Let $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ be a linear isomorphism satisfying the following condition: for every $u_i \in \mathbb{F}_q^{k_i} \setminus \{0\}$, there are $u_i' \in \mathbb{F}_q^{k_i}$ and $v_i \in \mathbb{F}_q^n$ with $\text{supp}_\pi(v_i) \subset \langle i \rangle_P \setminus \{i\}$ such that $T(u_i) = u_i' + v_i$. Then, $T \in GL(P, \pi, L)$.

We denote by $\mathcal{N}$ the set of all the (P, π, L)-isometries obtained as in Proposition 9. It is possible to prove that $\mathcal{N}$ is a normal subgroup of $GL(P, \pi, L)_q$.

Theorem 3: Every linear isometry S can be written in a unique way as a product $S = F \circ T_\phi$, where $F \in \mathcal{N}$ and $\phi \in Aut(P, \pi, L)$. Furthermore, $GL(P, \pi, L)_q$ is the semi-direct product $GL(P, \pi, L)_q = \mathcal{N} \rtimes \mathcal{A}$.

B. G-Canonical Decomposition of linear codes for hierarchical posets of directed cycles

Two linear codes $\mathcal{C}, \mathcal{C}' \subseteq \mathbb{F}_q^n$ are (P, π, L)-*equivalent* if there is $T \in GL(P, \pi, L)_q$ such that $T(\mathcal{C}) = \mathcal{C}'$.

A decomposition $\mathcal{C} = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{h(P)}$ of a code $\mathcal{C}$ is called (P, π, L)-*canonical decomposition* if $\text{supp}_\pi(\mathcal{C}_i) \subseteq \Gamma_i^P$. Working with such decompositions simplifies the computation of all metric invariants of a code. Naturally, not every code admits a (P, π, L)-*canonical decomposition*, but it may be equivalent to a code that has such a decomposition.

Definition 5: Let $P = ([m], \preceq)$ be a poset with $h(P)$ levels. We say that a linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$ admits a (P, π, L)-*canonical decomposition* if it is (P, π, L)-equivalent to a linear code $\tilde{\mathcal{C}} = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{h(P)}$, where $\text{supp}_\pi(\mathcal{C}_i) \subseteq \Gamma_i^P$.

The next theorem is a generalization of the P-canonical decomposition for poset metrics, determined in [8].

Theorem 4: The poset P is hierarchical if, and only if, any linear code $\mathcal{D}$ admits a (P, π, L)-canonical decomposition.

Proof: This proof can be obtained in a similar way present by Etzion in [6]. ∎

The existence of a (P, π, L)-canonical decomposition is a very useful tool, allowing to simplify the computation of many metric invariants (minimal distance, packing and covering radius) and also to determine conditions which ensure the validity of important results in coding theory, such as the MacWilliams' Extension Property. Just as an example, we show how it allows to determine a type of MacWilliams' Identity for linear codes.

Definition 6: A (P, π, L)-structure satisfies the *unique decomposition property* if, for $1 \le i \le h(P)$, given $S, S' \subseteq \Gamma_i^P$ such that

$$\sum_{a \in S} L(a) = \sum_{b \in S'} L(b),$$

there is a bijection $g : S \to S'$ such that $L(a) = L(g(a))$ and $|\pi^{-1}(a)| = |\pi^{-1}(g(a))|$ for all $a \in S$.

The (P, π, L)-*weight enumerator* of a code $\mathcal{C}$ is the polynomial

$$W_{\mathcal{C}}^{(P,\pi,L)}(X) = \sum_{i=0}^{n} A_i^{(P,\pi,L)}(\mathcal{C}) X^i$$

where $A_i^{(P,\pi,L)}(\mathcal{C}) = |\{c \in \mathcal{C} : \text{wt}_{(P,\pi,L)}(c) = i\}|$.

As we know, given a poset $P([n], \preceq_P)$ its *dual* is the poset $P^\perp([n], \preceq_{P^\perp})$ defined by the opposite relations

$$i \preceq_P j \in E \iff j \preceq_{P^\perp} i.$$

Definition 7: (The MacWilliams Identity) A (P, π, L)-weight *admits* a MacWilliams Identity if for every linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$, the (P, π, L)-weight enumerator $W_{\mathcal{C}}^{(P,\pi,L)}(X)$ of $\mathcal{C}$ determines the $(P^\perp, \pi, L)$-weight enumerator $W_{\mathcal{C}^\perp}^{(P^\perp,\pi,L)}$ of the dual code $\mathcal{C}^\perp$.

Theorem 5: Consider a (P, π, L)-weight with P is a hierarchical poset. The (P, π, L)-weight admits the MacWilliams Identity if, and only if, it satisfies the unique decomposition property.

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