# A CATEGORICAL MODEL FOR THE HOPF FIBRATION

#### BJÖRN GOHLA

Abstract. We give a description up to homeomorphism of  $S^3$  and  $S^2$  as classifying spaces of small categories, such that the Hopf map  $S^3 \longrightarrow S^2$  is the realization of a functor.

## 1. Introduction

Madahar and Sarkaria [2000] describe simplicial complices whose realizations are  $S^3$  and  $S^2$  such that the Hopf map is the realization of a simplicial map. Quillen [1973] Introduced the classifying space of a category in order to study algebraic K-theory. In particular he defined homotopy groups of (additive) categories as homotopy groups of their nerves.

We on the other hand were interested in seeing if one can engineer categories whose nerves would have interesting an geometry. The first thing one notices is that such categories should not contain non-trivial endomorphisms or isomorphisms, since this would immediately make the nerve very large, and arguably ungeometric.

So one is led to focus on the following class of categories and their functors:

**Definition 1.** A category A is called **skeletal** if for all arrows  $x \xrightarrow{f} y$  in A we have that f being an isomorphism implies x = y.

Note that this does not mean that f is an identity.

**Definition 2.** A category A is called **progressive** if all endomorphisms  $x \xrightarrow{f} x$  in A are identities.

A category that is both skeletal and progressive is in some ways similar to a poset. These categories were called **acyclic** and studied by Kozlov [2008], in particular their quotients with respect to group actions. We will only have reason to use finite such categories.

We would like to note that the existence of a categorical model of the Hopf map itself is clear, since one can take the simplicial model  $\eta: S_{12}^3 \longrightarrow S_4^2$  of Madahar and Sarkaria [2000] and consider the posets of simplices ordered by inclusion, and the induced order preserving map between them  $U(\eta): U(S_{12}^3) \longrightarrow U(S_4^2)$ . These posets are by implication acyclic categories. The realization has the right homeomorphism type, but  $U(S_{12}^3)$  is a category with 168 objects,  $U(S_4^2)$  with 14. By comparison, the acyclic categories in our construction have 10 and 4 objects. This seems to be a minimum.

1

BG was supported by the Portuguese Science Foundation (FCT) under the post-doc grant SFRH/BPD/99060/2013.

Finally we remark, that one could define a categorical model R of  $\mathbb{C}P^2$  from our model H of the Hopf map by forming the pushout

$$(1) \qquad P \xrightarrow{i_0} M_P \xrightarrow{0} 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

where  $M_{P \longrightarrow \underline{0}}$  is the cone over P, i.e., a model of the 4-ball. We did not pursue this avenue any further.

## 2. From Categories to Spaces

Let  $\Delta_+$  be the category of finite non-empty ordinals, i.e., categories of the form  $\underline{n}=\{0\longrightarrow 1\longrightarrow \cdots\longrightarrow n\}$ , and the order preserving maps between them, i.e., functors.  $\Delta_+$  is obviously a subcategory of the category Cat of all small categories by means of the functor  $I:\Delta_+\longrightarrow \mathsf{Cat}$ .

Simplicial sets are the objects of the pre-sheaf category  $SSet = \widehat{\Delta_+} = CAT(\Delta_+^{op}, Set)$ . The nerve functor  $N: Cat \longrightarrow SSet$  is defined by  $NC = Cat(I_-, C)$ . Typically we will write  $(NC)_i = Cat(I_i, C)$ . By left Kan extension there is a left adjoint  $\tau_1 \dashv N$ ; [see Riehl, 2014] or [see Mac Lane, 1998]. We mention this only because this guarantees that N preserves all limits.

To be explicit, the 0-simplices of NC are the objects of C; 1-simplices are the arrows of C, i-simplices are strings of i composable arrows. Faces are obtained by composing or dropping elements, degeneracies result by inserting identity arrows into a string.

There is a well-known functor, confusingly also called  $\Delta$ , that goes  $\widehat{\Delta_+} \longrightarrow \mathsf{Top}$ . It takes every ordinal  $\underline{n}$  to the affine n-simplex. Similar to the situation with categories this defines, a functor  $S:\mathsf{Top} \longrightarrow \mathsf{SSet}$  defined as  $SX = \mathsf{Top}(\Delta_-, X)$ . Again, by Kan extension, there is a left adjoint  $|\_| \dashv S$  that gives the topological realization |X| of a simplicial set. Note that as a left adjoint  $|\_|$  preserves all colimits.

By combining nerve and realization one can now define the so called classifying space functor of small categories  $B = |N_{\perp}|$  introduced by Quillen [1973]. We shall call BC the topological realization of C.

The realization of an ordinal  $\underline{n}$  is of course simply the *n*-simplex  $B\underline{n}$ . Further simple examples can be found at the beginning of Section 3.1. A category whose realization is  $S^2$  can be found in (Eq. (11)).

### 3. The Hopf Fibration

If we define  $S^3$  as the set  $\{(z_0,z_1)\in\mathbb{C}^2:z_0\overline{z_0}+z_1\overline{z_1}=1\}$  there is a natural action  $(z_0,z_1)\longmapsto (wz_0,wz_1)$  on  $S^3$  by  $w\in U(1)$ . The quotient by this action happens to be  $S^2$  and the quotient map  $h:S^3\longrightarrow S^2$  is known as the Hopf map. Its homotopy class is characterized by the fact that any two distinct fibres are simply linked circles.

One can decompose the Hopf map by decomposing  $S^2$  along the equator, and  $S^3$  along the pre-image of the equator, which is a torus. So h appears as a pushout in the arrow category  $\mathsf{Top}^{\rightarrow}$  of two circle bundles over the 2-disk, see (Eq. (12)).

Madahar and Sarkaria [2000] present these circle bundles as simplicial complices. We show that instead one can describe these circle bundles using acyclic categories and functors.

3.1. Categorical Models. We start by describing small categories that model the circle and the disk. Let S be the small category given by the following diagram:

$$(2) S = \begin{cases} A \underbrace{\int_{g}^{f}} B \end{cases}$$

with no relations. Next, the category D is given by

$$(3) D = \begin{cases} A \underbrace{\int_{g}^{f} B \xrightarrow{t} X} \end{cases}$$

with tf = tg. S is obviously a full subcategory of D.

The central piece of our construction is the torus T given the by the diagram

$$T = \begin{cases} A_0 \xrightarrow{f_0} B_0 \xleftarrow{g_0} A_0 \\ P_A & B_1 \\ P_A & P_A \\ P_A &$$

with all squares commuting. There are three projection  $F_M, F_N, G: T \longrightarrow S$ . Where

(5) 
$$F_{M}: A_{0}, B_{0} \longmapsto A$$

$$A_{1}, B_{3}, B_{2}, B_{1} \longmapsto B$$

$$f_{0}, g_{0} \longmapsto \mathrm{id}_{Fx}$$

$$f_{3}, p_{B_{3}}, q_{B_{1}}, g_{1} \longmapsto \mathrm{id}_{Fx}$$

$$q_{A}, q_{B_{3}} \longmapsto f$$

$$p_{A}, p_{B_{1}} \longmapsto g$$

and

(6) 
$$F_{N}: A_{x} \longmapsto A$$

$$B_{x} \longmapsto B$$

$$p_{x}, q_{x} \longmapsto \mathrm{id}_{Fx}$$

$$f_{i} \longmapsto f$$

$$g_{i} \longmapsto g.$$

These functors can be thought to be the vertical and the horizontal projections of the diagram (Eq. (4)), respectively, where the vertical and horizontal zig-zags need to be seen as straightened in the appropriate way.

Finally,  $G: T \longrightarrow S$  projects in the top-left, bottom-right direction. Explicitly:

(7) 
$$G: A_0, B_2 \longmapsto A$$

$$A_1, B_3, B_0, B_1 \longmapsto B$$

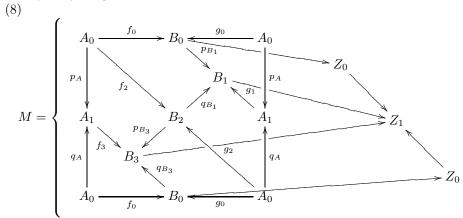
$$f_3, q_{B_3}, f_2, g_2, p_{B_1}, g_1 \longmapsto \mathrm{id}_{Fx}$$

$$q_A, f_0 q_{B_1} \longmapsto f$$

$$p_A, p_{B_3}, g_0 \longmapsto g.$$

In analogy with Madahar and Sarkaria [2000] we define two different presentations of the solid torus, and then identify their boundaries. We define M as the categorical mapping cylinder of  $F_M$ , and N as the categorical mapping cylinder of  $F_N$ . In essence for a functor  $F:A\longrightarrow B$  this construction introduces an arrow  $x\longrightarrow Fx$  for each object x of A in a way that is compatible with the existing arrows of A and B. The construction is detailed in Section 3.3.

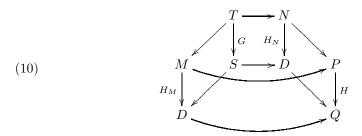
Explicitly we get



where all the squares commute.

The universal property of the strict lax pushout determines two unique extensions of the projection  $G, H_M: M \longrightarrow D$  and  $H_N: N \longrightarrow D$ .

These  $G, H_M, H_N$  with the obvious inclusions assemble into a cospan in the arrow category  $\mathsf{Cat}^{\to}$ :



We define  $H: P \longrightarrow Q$  as the pushout of this diagram in  $\mathsf{Cat}^{\to}$ . P is given by the amalgamation of the two diagrams (Eq. (8)) and (Eq. (9)) along the square. Q is the amalgamation of two copies of (Eq. (3)) along the left-hand part; explicitly

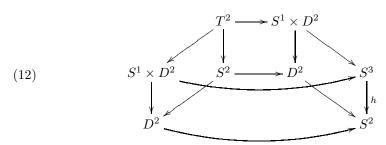
$$Q = \begin{cases} A & B \\ A & B \\ Z & Z \end{cases}$$

3.2. **Realization.** Let  $B = |N_{-}|$  be the topological realization functor. We apply it to the categories described in Section 3.1. It is easy to see that we obtain  $BS = S^{1}$ ,  $BT = T^{2}$ ,  $BD^{2}$ ,  $BQ = S^{2}$ . If we let  $T^{2} = ([0,1] \times [0,1])/\sim$ , where  $(0,s) \sim (1,s)$  and  $(r,0) \sim (r,1)$ , then the realizations of the projections  $H_{M}, H_{N}, G$  can be described as  $BH_{M}(r,s) = r$ ,  $BH_{N}(r,s) = s$ , HG(r,s) = r + s.

It is now easy to see that BM is the mapping cylinder of  $BF_M$ , as BN is the mapping cylinder of  $BF_N$ .

We note that the nerve N as a right adjoint can not be expected to preserve colimits in general. Direct inspection however shows that the pushout in (Eq. (10)) is preserved.  $|\_|$  is a left adjoint and preserves all colimits.

**Theorem 3.** Applying the classifying space functor  $B = |N_{-}|$  to (10) gives a pushout diagram in  $\mathsf{Top}^{\rightarrow}$ 



Hence BH = h is the Hopf map.

*Proof.* B preserves the mapping cylinders and pushouts since by Proposition 4 N does so, and | | does by being a left adjoint.

3.3. Mapping Cylinders and Pushouts. The categorical mapping cylinder of a functor  $F: A \longrightarrow B$  in Cat is given as the strict lax pushout

(13) 
$$A \xrightarrow{\operatorname{Id}_A} A \\
F \downarrow \stackrel{a}{\swarrow} \bigvee_{i_A} i_A \\
B \xrightarrow{i_B} M_F$$

where for any

(14) 
$$A \xrightarrow{\operatorname{Id}_A} A \\
\downarrow r \qquad \qquad \downarrow H \\
B \xrightarrow{K} X$$

there is a unique  $G: M_F \longrightarrow X$  such that  $Gi_A = H$ ,  $Gi_B = K$  and Ga = r.

This means that  $M_F$  has objects those of A and those of B. The arrows of  $M_F$  are of three kinds: those in A, those in B, and those of the form  $ga_xf$  where  $g \in B_1$ ,  $f \in A_1$  (which we shall tacitly assume for the rest of this section) and  $a_x$  is the appropriate component of the natural transformation a.

Note that this is the same as the ordinary pushout

$$\begin{array}{ccc}
A & \xrightarrow{i_1} & A \times \underline{2} \\
F & \downarrow & \downarrow \\
B & \xrightarrow{} & M_F
\end{array}$$

where  $\underline{2}$  is the ordinal  $((01):0\longrightarrow 1)$ .

The nerve  $NM_F$  of  $M_F$  has the following types of simplices: in dimension 0 we have  $(NM_F)_0 = A_0 \cup B_0$ . In dimension 1, the elements of  $(NM_F)_1$  are of the form f, g, or  $ga_xf$ , where  $f \in A_0, g \in B_0, a_x : x \longrightarrow Fx$ . In dimension 2 simplices are of the form  $(f, f'), (g, g'), (ga_xf, f')$  or  $(g, g'a_xf)$ . In dimension  $n = i + j + 1 \ge 3$  the simplices are of the form  $(f_1, \ldots, f_n), (g_1, \ldots, g_n)$ , or  $(g_1, \ldots, g_{i+1}a_xf_1, \ldots, f_{j+1})$ . If  $Ff_k = g_k$  for some  $1 \le k \le i + 1$  we have of course

$$(g_1, \ldots, g_{i+1}, \ldots, g_n) = (g_1, \ldots, Ff_{i+1}, \ldots, g_n)$$

and

$$(g_1,\ldots,g_{i+1}a_xf_1,\ldots,f_{j+1})=(g_1,\ldots,Ff_k,\ldots,g_{i+1}a_xf_1,\ldots,f_{j+1}).$$

The simplicial mapping cylinder of NF is of course defined as the pushout of simplicial sets

(16) 
$$\begin{array}{c}
NA \xrightarrow{i_1} N(A \times \underline{2}) \\
NF \downarrow \qquad \qquad \downarrow \\
NB \xrightarrow{} M_{NF}
\end{array}$$

The 0-simplices of  $M_{NF}$  are the same as those of  $NM_F$ ; 1-simplices are  $(f, id_0)$ ,  $(f, (01)), (f, id_1)$  or g where  $g = (f, id_1)$  if Ff = g; 2-simplices are  $((f, id_0), (f', id_0)), (g, g'), ((f, (01)), (f', id_0))$  or  $((f, id_1), (f', (01)))$ .

The universal property of (Eq. (16)) induces a comparison map  $k: M_{NF} \longrightarrow NM_F$ ; it maps 1-simplices thus

(17) 
$$k: g \longmapsto g$$
$$(f, id_0) \longmapsto f$$
$$(f, id_1) \longmapsto Ff$$
$$(f, (01)) \longmapsto a_x f.$$

2-simplices are mapped

(18) 
$$k:((f, \mathrm{id}_0), (f', \mathrm{id}_0)) \longmapsto (f, f')$$
$$(g, g') \longmapsto (g, g)$$
$$((f, (01)), (f', \mathrm{id}_0)) \longmapsto (Ffa_x, f') = (a_y f, f')$$
$$((f, \mathrm{id}_1), (f', (01))) \longmapsto (Ff, Ff'a_x) = (Ff, a_y f').$$

 $k:((f_1,\mathrm{id}_0),\ldots,(f_n,\mathrm{id}_0))\longmapsto(f_1,\ldots,f_n)$ 

Higher simplices are mapped similarly

$$(g_1, \dots, g_n) \longmapsto (g_1, \dots, g_n)$$

$$\begin{pmatrix} (f_1, \mathrm{id}_1), \dots, (f_i, \mathrm{id}_1), \\ (f_{i+1}, (01)), \\ (f_{i+2}, \mathrm{id}_0), \dots, (f_n, \mathrm{id}_1) \end{pmatrix} \longmapsto (Ff_1, \dots, Ff_i, Ff_{i+1}a_x, f_{i+2}, \dots, f_n)$$

$$= (Ff_i, \dots, Ff_i, a, f_i, a, f_i$$

Note in particular that  $(g, a_x) \in (NM_F)_2$  only has a pre-image under k if g is in the image of F.

We can not always expect the nerve to preserve the structure of the mapping cylinder: Taking for example the mapping cylinder of  $F: \underline{1} \longrightarrow \underline{2}$  with F(0) = 0 we get a category

(20) 
$$M_F = \begin{cases} 0 \xrightarrow{a_0} F0 \\ 0 \downarrow 0 \downarrow \\ 1 \end{cases} .$$

The nerve of the mapping cylinder  $NM_F$  has a 2-simplex ((01),  $a_0$ ), that does not exist in the simplicial mapping cylinder of NF.

**Proposition 4.** The comparison map  $k: M_{NF} \longrightarrow NM_F$  is an isomorphism of simplicial sets if the functor  $F: A \longrightarrow B$  has the property that  $g: Fx \longrightarrow y$  in B implies g = Ff for some f in A. Hence the nerve functor  $N: \mathsf{Cat} \longrightarrow \mathsf{SSet}$  preserves the mapping cylinder  $M_F$ .

*Proof.* The condition ensures that all the simplices of  $NM_F$  are in the image of the comparison map k explicitly described in (Eq. (17)), (Eq. (18)), and (Eq. (19)). Finally, k is obviously injective.

**Remark 5.** Proposition 4 clearly applies to the functors  $F_N$ ,  $F_N$ , G defined in Section 3.1.

### 4. Acknowledgments

We would like to thank John Huerta for suggesting to look at the Hopf map as an example to consider. Many thanks also are due to Benjamín Alarcón Heredia for intense discussions.

#### References

- Dimitry Kozlov. Combinatorial algebraic topology. Berlin: Springer, 2008. ISBN 978-3-540-71961-8/hbk; 978-3-540-73051-4/pbk.
- Saunders Mac Lane. Categories for the working mathematician. 2nd ed., volume 5. New York, NY: Springer, 2nd ed edition, 1998. ISBN 0-387-98403-8/hbk.
- K.V. Madahar and K.S. Sarkaria. A minimal triangulation of the Hopf map and its application. Geom. Dedicata, 82(1-3):105–114, 2000. ISSN 0046-5755; 1572-9168/e. doi: 10.1023/A:1005102800486.
- Daniel Quillen. Higher algebraic K-theory. I. Algebr. K-Theory I, Proc. Conf. Battelle Inst. 1972, Lect. Notes Math. 341, 85-147 (1973)., 1973.
- Emily Riehl. Categorical homotopy theory. Cambridge: Cambridge University Press, 2014. ISBN 978-1-107-04845-4/hbk; 978-1-107-26145-7/ebook. doi: 10. 1017/CBO9781107261457.

E-mail address, Björn Gohla: b.gohla@gmx.de

Grupo de Física Matemática, Faculdade de Ciências da Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal